ELEMENTARY ALGEBRA


An' the formoola we uses is algebra---invented by a fella name of "Al."
Inasmuch as he done such a good job, the formoola was named after him---bein' called Al-gebra 'cause he was a zebra.
Al-Zebra?!
Zebra don't got no "G" in it. So how could---
Who said it do?
An' algebra don't got no "Z."
Algebra got nothin' but "Z's" an "X's" an' stuff in it!
True. true.

ELEMENTARY ALGEBRA

Harold R. Jacobs

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Supplements to Elementary Algebra

The *Teacher's Guide* which includes a convenient daily schedule, Set III exercises for additional practice, and a complete set of tests, including chapter tests, mid-year and final exams and answer keys.

The *Solutions Manual* with complete solutions to all of the Set I–IV exercises.

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Foreword

I am just an algebra teacher, more interested in how students learn, and how to open doors of opportunity for people than I am a mathematician. Harold Jacobs, through his texts, has taught me something about how to be a mathematician.

You will find that this book is far more than a text. It is unusual in that it has withstood the test of time, not because it is a reference for the subject, but by the popular demand of people who want these things from any text: the thoughtful, meticulous exposition of a subject as well as a genuine love for the subject, the understanding of the student, and the appreciation for the instructor. It also helps if the author has fun doing it and can invite the student along for the ride.

Every time I open this book to teach a new lesson I gain another insight, another 'Aha!' moment. I am a perpetual student of Harold Jacobs, even though I was never in his classroom. Harold Jacobs's books have that effect on people.

I am grateful to God that the book will continue to be published! It is a strong tool in the hands of just an adequate instructor like myself, attempting to crank open brains and doors of opportunity.

Molly Crocker
Adequate Instructor and Independent Educator
October 2015

Photograph by Roy Bishop

A Letter to the Student

The English philosopher and scientist Roger Bacon once wrote: "Mathematics is the gate and key of the sciences. . . . Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of this world."

In turn, algebra is the gate and key of mathematics. For this reason, colleges and universities require mastery of algebra in preparation for studying not only the sciences, but also such subjects as engineering, medicine, architecture, philosophy, psychology, and law.

Although many problems that can be solved by algebra can also be worked out by common sense, their translation into algebraic form generally makes them easier to deal with. Because of this, algebra has become the language of science. The goal of this course is to learn how to use this language.

Success in algebra depends on a combination of talent and effort. A few people are so gifted in mathematics that they can succeed with very little effort. For most people, however, diligent practice is the key to success. Like developing ability in a sport, becoming good at algebra takes practice. It is my hope that this book will help you both to enjoy the subject and to be successful in your studies.

Harold R. Jacobs

ELEMENTARY
ALGEBRA

INTRODUCTION
A Number Trick

Think of a number from one to ten. Add seven to it. Multiply the result by two. Subtract four. Divide by two. Subtract the number that you first thought of. Is your answer five?

Number tricks such as this have long been popular. That the final result can be known by someone who doesn't know which number was originally chosen is surprising.

How does the trick work? If we make a table (like the one at the top of the next page) showing what happens when it is done with each number from one to ten, some patterns appear.

Would these patterns continue if the table were extended to include other numbers? If we began by thinking of eleven, would the answer at the end still be five? What if we began with one hundred? Would we get five at the end if we began with zero? Do you think it is correct to assume that the trick will work for *any* number you might think of?

Even though you may feel that the answer to every one of these questions is yes, *how* the trick works is still not clear. Merely doing arithmetic with a series of different numbers cannot reveal the secret of why they all lead to the same result.

The number thought of:	1	2	3	4	5	6	7	8	9	10
Add seven:	8	9	10	11	12	13	14	15	16	17
Multiply by two:	16	18	20	22	24	26	28	30	32	34
Subtract four:	12	14	16	18	20	22	24	26	28	30
Divide by two:	6	7	8	9	10	11	12	13	14	15
Subtract the number first thought of:	5	5	5	5	5	5	5	5	5	5

There is a simple way, however, to discover the secret. Instead of writing down a specific number at the start, we will use a symbol to represent whatever number might be chosen. We will begin with a box.

□

Throughout the trick this box will represent the number originally chosen.

The next step in the trick is to add seven. We will represent numbers we know with sets of circles, and so seven will look like this:

○ ○ ○ ○ ○ ○ ○

To show the result of adding seven to the number, we draw seven circles beside the box.

□ ○ ○ ○ ○ ○ ○ ○

If we illustrate the entire trick in this way, it looks like this:

The number thought of:	□
Add seven:	□ ○ ○ ○ ○ ○ ○ ○
Multiply by two:	□ ○ ○ ○ ○ ○ ○ ○ □ ○ ○ ○ ○ ○ ○ ○
Subtract four:	□ ○ ○ ○ ○ ○ □ ○ ○ ○ ○ ○
Divide by two:	□ ○ ○ ○ ○ ○
Subtract the number first thought of:	○ ○ ○ ○ ○

The pictures make it easy to see why, no matter what number we start with, the answer at the end of the trick is always five. The box representing the original number disappears in the last step, leaving five circles.

Doing arithmetic with symbols rather than specific numbers is the basis of algebra. The explanation with the boxes and circles of what is happening throughout the number trick is an example of this. One of our goals in learning algebra will be to learn how to set up and solve problems using symbols such as these.

Exercises

1. Here are directions for another number trick and part of a table to show what happens when the trick is done with each number from one to five.

Think of a number:	1	2	3	4	5
Double it:	2	4			
Add six:	8				
Divide by two:	4				
Subtract the number that you first thought of:	3				

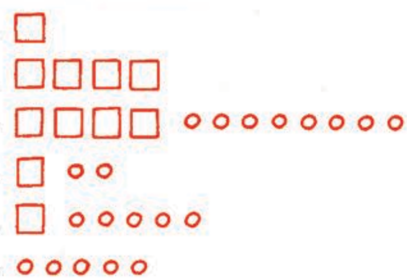
a) Copy and complete the table.
b) Does your table prove that the trick will work for *any* number?
c) Show how the trick works by illustrating the steps with boxes and circles. The first two steps are shown below.

Think of a number:
Double it:

d) Do your drawings prove that the trick will work for *any* number?

2. The pictures below illustrate the steps of another number trick. Tell what is happening in each step in words.

Step 1.
Step 2.
Step 3.
Step 4.
Step 5.
Step 6.



3. In the next number trick, we will study the effect of changing some of the directions.

Step 1. Think of a number.
Step 2. Add four.
Step 3. Multiply by two.
Step 4. Subtract four.
Step 5. Divide by two.
Step 6. Subtract the number that you first thought of.

a) What is the result at the end of this trick?

b) Suppose that the second step were changed as shown below.

Step 1.	Think of a number.
Step 2.	Add six.
Step 3.	Multiply by two.
Step 4.	Subtract four.
Step 5.	Divide by two.
Step 6.	Subtract the number that you first thought of.

The trick will still work, even though the result at the end is changed. How is it changed?

c) Suppose instead that the fourth step were changed as shown below.

Step 1.	Think of a number.
Step 2.	Add four.
Step 3.	Multiply by two.
Step 4.	Subtract six.
Step 5.	Divide by two.
Step 6.	Subtract the number that you first thought of.

What effect does this have on the trick?

d) Suppose instead that the third step were changed as shown below.

Step 1.	Think of a number.
Step 2.	Add four.
Step 3.	Multiply by four.
Step 4.	Subtract four.
Step 5.	Divide by two.
Step 6.	Subtract the number that you first thought of.

What effect does this have on the trick?

4. Here is the beginning of a number trick. Can you make up more steps so that it will give the same answer for any number a person might choose?

Think of a number.
Triple it.
Add twelve.

Chapter 1

FUNDAMENTAL OPERATIONS

"Can you do Addition?" the White Queen asked. "What's one and one and one and one and one and one and one and one and one and one?"
"I don't know," said Alice. "I lost count."

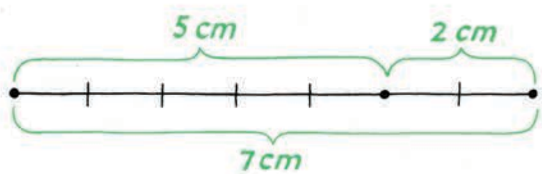
LEWIS CARROLL, *Through The Looking Glass*

LESSON 1
Addition

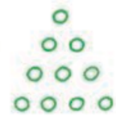
Soon after a child is able to count, he learns how to add. The two operations are closely connected, as anyone who has ever added by counting on his fingers knows. Consider the problem of adding the numbers represented by these two sets of circles:

At first a child finds the answer by counting all of the circles. Then he learns the fact that $5 + 2 = 7$.

Another way to picture addition is by lengths along a line. This figure also illustrates the fact that $5 + 2 = 7$.



The result of adding two or more numbers, called their **sum,** does not depend on either the order of the numbers or the order in which they are added. To find



the number of circles in the pattern above, for example, we could add the numbers of circles in the four rows from top to bottom:

$$1 + 2 + 3 + 4$$

or from bottom to top:

$$4 + 3 + 2 + 1$$

Either way, we get the same number: 10.

In algebra, it is often necessary to indicate the sum of two or more numbers without actually being able to add them. For example, in illustrating the number trick that appears in the introduction to this book, we used a box to represent the original number and a set of circles to represent the number seven:

Original number　　　　Seven

To represent their sum, we drew the seven circles beside the box:

The sum of the original
number and seven

Instead of bothering to draw pictures like this, it is easier to represent the original number with a letter, such as x, and simply write

$$x + 7$$

The expression $x + 7$ means "the sum of x and 7." If we replace x with 1, $x + 7 = 1 + 7 = 8$. If we replace x with 2, $x + 7 = 2 + 7 = 9$, and so forth. Because x can be replaced by various numbers, it is called a **variable**.

If we know both numbers being added, such as 4 and 5, we can write their sum as a number, 9. If we know only one number or neither one, the best that we can do is to write an expression such as $x + 2$ or $x + y$. The length of the line segment below, for example, is the sum of the lengths of the three marked segments.

To indicate this sum, we can write $3 + x + 1$ or, more briefly, $x + 4$. Without knowing the length labeled x, we cannot simplify this answer any further.

Exercises

TEACHER: Haven't you finished adding up those numbers yet?
STUDENT: Oh, yes. I've added them up ten times already.
TEACHER: Excellent! I like a student who is thorough.
STUDENT: Thank you. Here are the ten answers.*

Set I

Find each of the following sums.

1. $1000 + 700 + 70 + 6$
2. $999 + 99 + 9$
3. $1 + 0.9 + 0.08 + 0.004$
4. $20 + 0.2 + 0.002$
5. $1 + 12 + 123 + 1234$
6. $1111 + 222 + 33 + 4$
7. $1 + 1.2 + 1.23 + 1.234$
8. $1.111 + 2.22 + 3.3 + 4$
9. $0.7 + 0.70 + 0.700 + 0.7000$
10. $0.5 + 0.55 + 0.555$

Set II

11. Write a number or expression for each of the following.

a) The sum of 10 and 7.
b) The sum of x and 7.
c) The sum of 10 and y.
d) The sum of x and y.
e) Four added to 8.
f) Four added to z.
g) The sum of 2, 5, and 1.
h) The sum of x, 5, and 1.
i) The sum of 2, y, and 1.
j) The sum of x, y, and 1.

*Alan Wayne, in *Mathematical Circles Revisited* by Howard W. Eves. © Copyright Prindle, Weber & Schmidt, Inc. 1971.

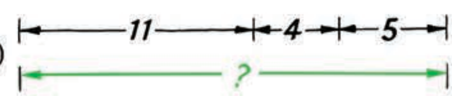
12. In the figures below, the box represents any number and the sets of circles represent specific numbers.

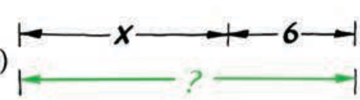
Figure 1

Figure 2

a) What addition problem is illustrated by Figure 1?
b) What is the answer to the problem?
c) Write an algebraic expression to represent the addition problem illustrated by Figure 2.
d) What is the answer to the problem if the box represents 2?
e) What is the answer to the problem if the box represents 4?

13. What is the length marked with a question mark in each of these figures?

a)


b)


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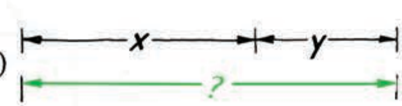
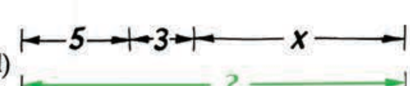
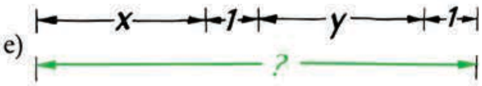
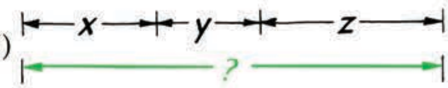


d)


e)


f)


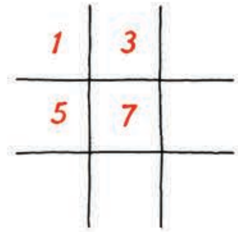
14. The figure below can be used to show that $3 + 7$ and $7 + 3$ are the same number, depending on whether the figure is read from left to right or from right to left.

Draw boxes and circles to show that
a) $x + 6$ and $6 + x$ mean the same thing.
b) $2 + x + 5$ and $x + 7$ mean the same thing.
c) $x + 4 + x$ and $4 + x + x$ mean the same thing.

15. The expression $x + y + 2$ represents the sum of x, y, and 2. If x is 1, it can be written as $1 + y + 2$ or $y + 3$. How can $x + y + 2$ be written if
a) x is 8?
b) x is 9?
c) y is 3?
d) y is 0?
e) x is 6 and y is 2?

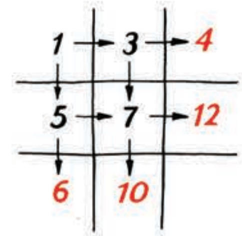
16. Mr. Benny is 39 years old.
a) How old will he be in 5 years?
b) How old will he be in x years?
c) How old will he be 6 years after that?
Mrs. Benny is x years old.
d) How old will she be in 5 years?
e) How old will she be in y years?
f) How old will she be z years after that?

Set IV A Number Puzzle



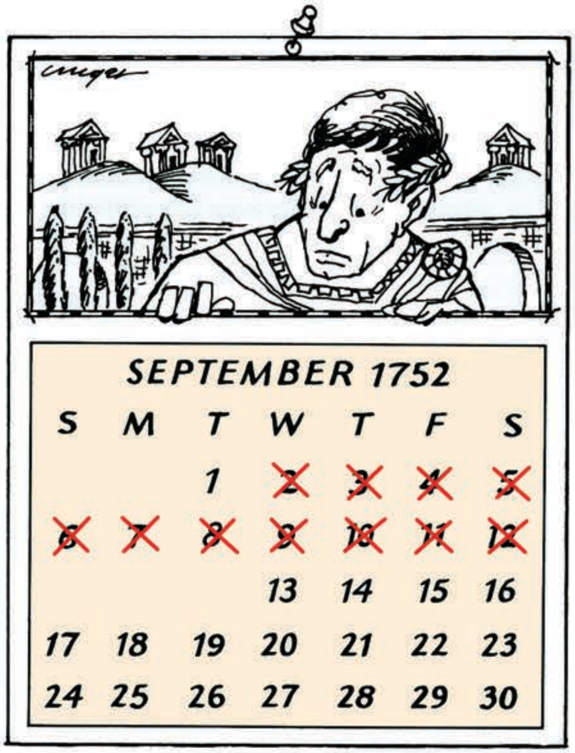
Numbers have been written in four spaces in this tic-tac-toe design. If we add across the rows and down the columns, we get the sums shown in the second figure. If we now add across the bottom row and down the last column, the answers are the same number:

$$6 + 10 = 16 \quad \text{and} \quad 4 + 12 = 16$$



Is this just a coincidence or would it happen if we started with *any* set of four numbers?

Draw a tic-tac-toe design and, in the same spaces as those in the example above, write four numbers of your own choosing. Add the rows and columns and see what happens. Can you explain why?



LESSON 2
Subtraction

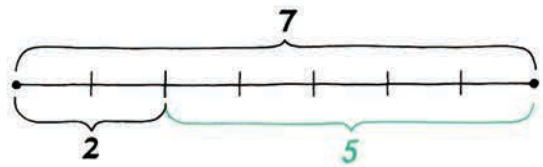
The month of September 1752 was one of the strangest months in history. The day following September 1 was September 13!

This was done to bring the calendar back into line with the seasons. The calendar established by Julius Caesar in 45 B.C. had as its basis a standard year of 365 days with every fourth year, "leap year," having 366. This resulted in the average length of a year being 365.25 days, whereas the earth in fact travels once around the sun in about 365.24 days. For a short period, this error didn't amount to much, but after many centuries it became so great that it had to be corrected.

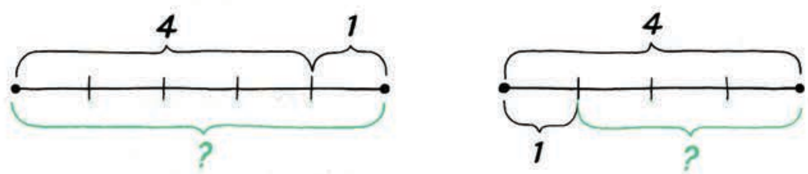
The number of days left in the month of September 1752 can be found by subtraction: $30 - 11 = 19$. Subtraction is the opposite of addition because we are "taking away" rather than "adding to." The two operations are closely related, however, because to every subtraction problem there corresponds an addition problem: $30 - 11 = 19$ because $19 + 11 = 30$.

To represent a subtraction problem such as $7 - 2$ by means of circles, we might draw seven circles from which two have been "taken away" by being crossed out.

Subtraction can also be pictured by lengths along a line. The figure below is another way of showing that $7 - 2 = 5$.



Although addition and subtraction are closely related, there is an important difference between the two operations. The sum of two numbers does not depend on the order of the numbers. The length marked with a question mark in the figure at the left below can be written either as $4 + 1$ or $1 + 4$.



The result of subtracting one number from another, called their **difference,** does depend on the order of the numbers. The length marked with a question mark in the figure at the right is $4 - 1$, not $1 - 4$. When we refer to the difference between two numbers, we mean the number that results from subtracting the second number from the first.

Exercises

Set I

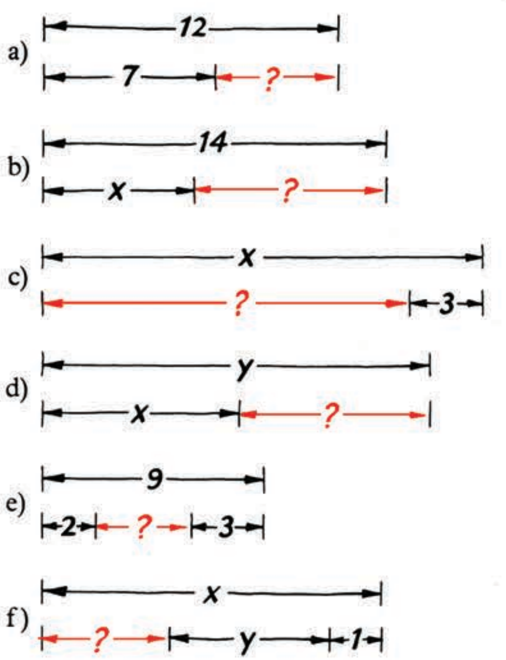
Find each of the following differences.

1. $22222 - 2000$
2. $666 - 77$
3. $1000 - 123$
4. $4.321 - 1$
5. $4.321 - 0.1$
6. $3.1416 - 3.1416$
7. $1 - 0.9$
8. $1 - 0.99$
9. $1812 - 18.12$
10. $181.2 - 1.812$

Set II

11. Write a number or expression for each of the following.
    a) The difference between 10 and 7.
    b) Six decreased by x.
    c) Six taken away from x.
    d) Three less than 11.
    e) One less than x.
    f) The difference between x and y.
    g) The result of subtracting x from 4.
    h) Four subtracted from x.

12. What is the length marked with a question mark in each of these figures?



13. Find the value of each of the following expressions for the numbers given.
    a) $x - 4$ if x is 6.
    b) $x - 4$ if x is 7.
    c) $x - 4$ if x is 14.
    d) What happens to the value of $x - 4$ as x gets larger?
    e) $15 - x$ if x is 3.
    f) $15 - x$ if x is 4.
    g) $15 - x$ if x is 10.
    h) What happens to the value of $15 - x$ as x gets larger?

14. Find the value of each of the following for the numbers given.
    The sum of x and $y - 3$
    a) if x is 7 and y is 4.
    b) if x is 2 and y is 11.
    The difference between $x + y$ and 3
    c) if x is 7 and y is 4.
    d) if x is 2 and y is 11.
    e) Can you explain why the answers to parts c and d are the same as those to parts a and b?

15. The sum of the numbers on any two opposite faces of a die is 7. Suppose that a die is thrown.
    a) If the number showing on the top of it is 3, what is the number on the bottom?
    b) If the number showing on the top of it is x, what is the number on the bottom?
    Suppose that two dice are thrown.
    c) If the sum of the two numbers showing on top is 8, what is the sum of the two numbers on the bottom?
    d) If the sum of the two numbers showing on top is y, what is the sum of the two numbers on the bottom?

16. Babar weighs 7,000 pounds.
    a) If he loses x pounds, how much will he weigh?
    b) If he gains y pounds, how much will he weigh?

17. The amount of profit that Shirley Feeney makes selling sandwiches depends on how much they cost her and how much she sells them for.

a) If peanut butter sandwiches cost her 21 cents each and she sells them for 45 cents, how much profit does she make on each one?

b) If jelly sandwiches cost her x cents each and she sells them for y cents, how much profit does she make on each one?

c) If egg sandwiches cost her x cents each and she wants to make a profit of 30 cents, how much should she sell them for?

d) If she sells ham sandwiches for 95 cents each and makes a profit of y cents on each one, how much do they cost her?

Set IV

"Forty-eight, forty-nine, fifty, seventy-five, nine, ten, twenty."

This seems like a strange way to count and yet clerks in stores do it all the time. What is going on? Can you tell what problem is being solved? Is the problem being solved by addition or subtraction?

"Six times six is 54! Don't they teach you anything at that school?"

LESSON 3
Multiplication

Learning the multiplication table is not an easy task. When you first learned how to multiply, you did it by adding. For example, the problem 3×5 can be illustrated by three sets of circles with five circles in each set.

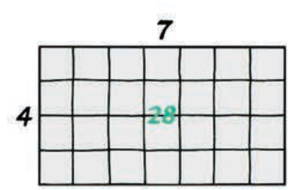
The circles can also be arranged in three rows to form a rectangle.

Both patterns show that $3 \times 5 = 5 + 5 + 5 = 15$. In learning the multiplication table, you memorized the answers to problems such as this so that pictures and adding became unnecessary.

The result of multiplying two or more numbers is called their **product.** Another way to picture a product is by means of area. The rectangle at the right, for example, is divided into 4 rows of squares with 7 squares in each row: it contains

$$4 \times 7$$

squares in all. The area of the rectangle, 28, is the product of its dimensions, 4 and 7.



Something that helps in learning the multiplication table is the fact that if

$$4 \times 7 = 28$$

then it is also true that

$$7 \times 4 = 28$$

The product of two or more numbers, like their sum, does not depend on either their order or the order in which they are multiplied.

Each of the number tricks that we considered in the introductory lesson included a step consisting of multiplication. For example, if we are told to think of a number and multiply it by four, the result might be illustrated by a set of four boxes:

If we use a letter, such as x, to represent the number thought of, we might write:

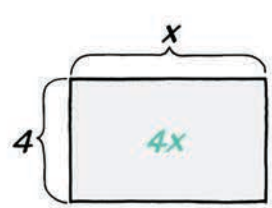
$$4 \times x$$

Because the symbol for multiplication used in arithmetic looks so much like the letter x, however, it is not ordinarily used in algebra. Instead, we simply write $4x$ with the understanding that this means "4 times x." We can't indicate the product of two numbers such as 3 and 5 this way because 35 means "thirty-five," not "three times five." To indicate that the 3 and 5 are two separate numbers, we can either enclose them in parentheses, (3)(5), or insert a raised dot between them, $3 \cdot 5$.

In this lesson we have observed that the product of two numbers, such as $4x$, can be interpreted either as repeated addition,

$$x + x + x + x$$

or as the area of a rectangle whose dimensions are 4 and x.



In the next lesson, we will see how these ideas can be applied to division.

Exercises

Set I

Find each of the following products.

1. $100 \cdot 360$
2. $(5)(142857)$
3. $271 \cdot 287$
4. $(0.05)(20)$
5. $(1.5)(8.23)$
6. $(8.23)(1.5)$
7. $(0.7)(1.1)(1.3)$
8. $(7)(1.1)(1.3)$
9. $(7)(11)(1.3)$
10. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5 \cdot 5$

Set II

11. Draw figures as indicated.
    a) A figure with circles to show that $4 \cdot 3$ and $3 \cdot 4$ are the same number.
    b) A figure with boxes to illustrate $5x$ if each box represents x.
    c) A rectangle divided into squares to illustrate $2 \cdot 7$.

12. Write a number or expression for each of the following.
    a) The product of 5 and 6.
    b) The sum of 5 and 6.
    c) The product of 5 and x.
    d) The sum of 5 and x.
    e) The product of x and y.
    f) The sum of x and y.
    g) The product of x and x.
    h) Eight multiplied by x.
    i) Eight subtracted from x.
    j) The sum of 2, 7, and x.
    k) The product of 2, 7, and x.
    l) The sum of 10, y, and 3.
    m) The product of 10, y, and 3.
    n) The sum of 4, x, and y.
    o) The product of 4, x, and y.

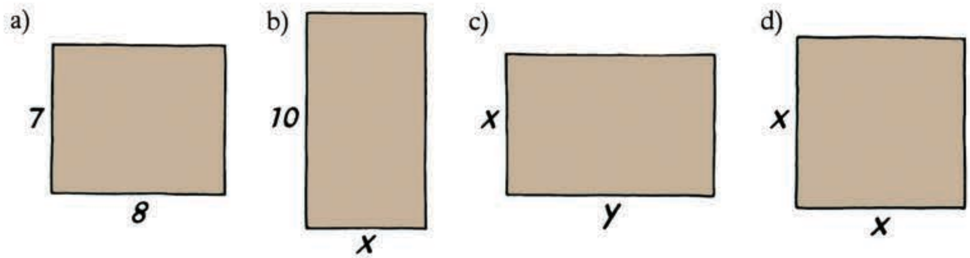
13. The multiplication problem $4 \cdot 3$ and the addition problem $3 + 3 + 3 + 3$ are equivalent. Write a multiplication problem equivalent to each of the following addition problems.

    a) $2 + 2 + 2 + 2 + 2 + 2$
    b) $6 + 6$
    c) $x + x + x + x + x$
    d) $\underbrace{7 + 7 + \cdots + 7}_{\text{11 of them}}$
    e) $\underbrace{7 + 7 + \cdots + 7}_{x \text{ of them}}$
    f) $\underbrace{y + y + \cdots + y}_{x \text{ of them}}$

    Write an addition problem equivalent to each of the following multiplication problems.

    g) $3 \cdot 17$
    h) $4x$
    i) $y \cdot 2$
    j) yz

14. The area of a rectangle is the product of its length and width. What is the area of each of these rectangles?



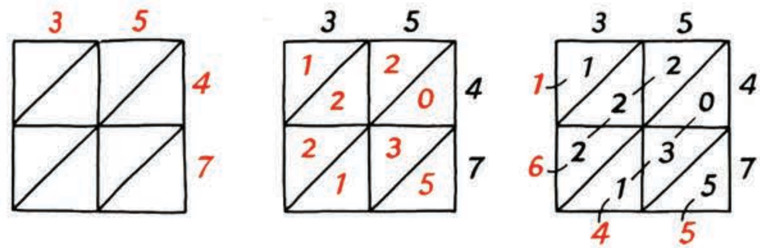
15. Although the name suggests that they have 100 legs, some centipedes have only 28 legs, whereas others have as many as 354.
    a) How many legs do 5 centipedes have altogether if each one has 28 legs?
    b) How many legs do x centipedes have altogether if each one has 354 legs?

16. Because there are 60 minutes in an hour, there are $60x$ minutes in x hours.
    a) How many days are there in x weeks?
    b) How many hours are there in x days?
    c) How many minutes are there in one day?
    d) How many minutes are there in x days?
    e) How many minutes are there in x weeks?
    f) How many years are there in x centuries?
    g) How many months are there in x centuries?

17. Miss Haversham's Hupmobile gets about 11 miles per gallon.
    a) Approximately how many miles should she be able to travel on a full tank of 15 gallons?
    b) Approximately how many miles can she travel on x gallons of gas?

Set IV

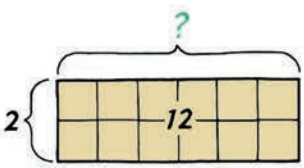
An old-fashioned method for multiplying two numbers is illustrated in the drawings shown here.



The numbers to be multiplied, 35 and 47, are written above and to the right of the figure as shown in the first drawing. Each digit of one number is multiplied by each digit of the other, $3 \cdot 4 = 12$, $5 \cdot 4 = 20$, $3 \cdot 7 = 21$, $5 \cdot 7 = 35$, and the answers written in the boxes as shown in the second drawing. The digits in each slanting column are added and their sums written below and to the left as shown in the third drawing. The answer is found by reading these digits in order from the upper left: $35 \cdot 47 = 1645$.

Try this method on the following problems. Does it give the correct answer in each case?

1. $52 \cdot 76$
2. $83 \cdot 29$



LESSON 4
Division

Problem: $\frac{12}{2} = ?$

Calculation:

$$\begin{array}{rl} 12 & \\ -\underline{2} & 1 \\ 10 & \\ -\underline{2} & 2 \\ 8 & \\ -\underline{2} & 3 \\ 6 & \\ -\underline{2} & 4 \\ 4 & \\ -\underline{2} & 5 \\ 2 & \\ -\underline{2} & 6 \text{ times} \\ 0 & \end{array}$$

Answer: $\frac{12}{2} = 6$

If two people share a dozen clams so that each one gets the same number, how many will each one get? This is such an easy division problem that we know the answer immediately. One way to illustrate it is shown below.

Twelve circles have been separated into groups of two ("one clam for you and one for me, one for you and one for me," and so forth); the answer, six, can be found by counting the number of groups.

The method that some mechanical calculators use to divide is illustrated at the left. The calculator subtracts 2 from 12, 2 from the result, 2 from that result, and so on until it arrives at 0. The number of times 2 has been subtracted is the answer.

Although this may seem like a peculiar way to divide, it is related to the way that we have been picturing multiplication as repeated addition. The calculator is doing division by repeated subtraction.

Division can also be interpreted in terms of multiplication. The answer to the problem of dividing 12 by 2 is the number that must be multiplied by 2 to give 12. This interpretation can also be pictured by means of the relationship of the area of a rectangle to its dimensions, as the figure at the right illustrates.

The result of dividing one number by another is called their **quotient.** By the quotient of the numbers x and y, we mean the result of dividing x by y and write it as $\frac{x}{y}$. The quotient of two numbers, like their difference, depends on the order of the numbers. The quotient of 3 and 6, $\frac{3}{6}$, for example, is not the same number as the quotient of 6 and 3, $\frac{6}{3}$. When we refer to the quotient of two numbers, we mean the number that results from dividing the first number by the second.

Although quotients are found by *division*, they can be checked by *multiplication*. To check that $\frac{80}{5} = 16$, for example, we multiply 5 by 16 to see if the result is 80. In general, the quotient $\frac{x}{y}$ is the number that must be multiplied by y to give x.

Exercises

Set I

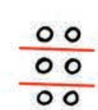
Find each of the following quotients.

1. $\frac{500}{10}$
2. $\frac{10}{500}$
3. $\frac{7404}{6}$
4. $\frac{111111}{37}$
5. $\frac{4}{2.5}$
6. $\frac{40}{2.5}$
7. $\frac{2.5}{4}$
8. $\frac{2.5}{40}$

Set II

9. Write a number or expression for each of the following.
   a) The quotient of 12 and 3.
   b) The difference between 12 and 3.
   c) Seven divided by x.
   d) Seven divided into x.
   e) The quotient of x and 2.
   f) The product of x and 2.
   g) The result of dividing 10 by x.
   h) The result of subtracting x from 10.
   i) The quotient of x and y.
   j) The product of x and y.

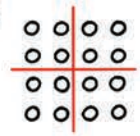
10. The figure below illustrates two division problems: $\frac{6}{2} = 3$ and $\frac{6}{3} = 2$.



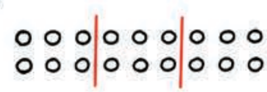
What division problems are illustrated by the following figures?

a)

b)



c)



11. A common way in which to check division is to multiply the answer by the dividing number to see if the result is equal to the number divided. For example, if $\frac{12}{6} = 2$, then $6 \cdot 2 = 12$. Write the multiplication problem that "checks" each of these division problems.
    a) $\frac{15}{5} = 3$
    b) $\frac{92}{23} = 4$
    c) $\frac{0}{12} = 0$
    d) $\frac{7.5}{7.5} = 1$
    e) $\frac{x}{10} = 7$
    f) $\frac{36}{x} = 12$
    g) $\frac{20}{4} = x$
    h) $\frac{x}{y} = 2$

12. Find the value of each of the following expressions for the numbers given.
    a) $9x$ if x is 5.
    b) $9x$ if x is 7.
    c) $9x$ if x is 12.
    d) What happens to the value of $9x$ as x gets larger?
    e) $\frac{x}{4}$ if x is 4.
    f) $\frac{x}{4}$ if x is 20.
    g) $\frac{x}{4}$ if x is 100.
    h) What happens to the value of $\frac{x}{4}$ as x gets larger?
    i) $\frac{30}{x}$ if x is 2.
    j) $\frac{30}{x}$ if x is 5.
    k) $\frac{30}{x}$ if x is 60.
    l) What happens to the value of $\frac{30}{x}$ as x gets larger?

13. A band of pirates has 300 doubloons which the pirates plan to share equally.
    a) If there are 15 pirates in the band, how many doubloons will each one get?
    b) If there are x pirates in the band, how many doubloons will each one get?

14. Mr. Vanderbilt bought some gold at \$170 per ounce.
    a) How much did he pay if he bought x ounces?
    b) How many ounces did he get if he paid \$102,000?
    c) How many ounces would he get for x dollars?

15. The common flea is capable of covering 12 inches in one jump.
    a) How far can a flea travel if it makes x jumps?
    b) How many jumps would a flea have to make in order to cover 600 inches?
    c) How many jumps would it have to make in order to cover x inches?

16. Miss Haversham drove her Hupmobile 159 miles.
    a) If it used 15 gallons of gas, how many miles per gallon did she get?
    b) If it used x gallons of gas, how many miles per gallon did she get?

Set IV The Pilgrims and the Loaves of Bread

Two pilgrims stopped by the side of the road to eat. One had seven loaves of bread and the other had five loaves. A third traveler arrived before they had begun their meal and asked them to share their food with him. They agreed and the three shared the bread equally.

After they had finished, the third traveler got up, thanked the two pilgrims for the bread, and left twelve silver pieces in payment for his meal. The pilgrim who originally had seven loaves of bread thought that he should receive seven of the coins and his fellow pilgrim should receive five, in the same numbers as their original loaves of bread. The other pilgrim, however, thought that the coins should be split six and six, because the bread had been shared equally.

They could not agree, and so they asked a local wise man what to do. The wise man decided that the pilgrim who originally had seven loaves of bread should receive nine silver pieces and the one who originally had five loaves should receive only three.*

Can you explain why this is fair?

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*Puzzles of this sort date from Roman times. An example similar to the one given here can be found in *Mathematical Recreations,* second edition, by Maurice Kraitchik (Dover, 1953).

LESSON 5
Raising to a Power

It is an amusing Speculation to look back, and compute what Numbers of Men and Women among the Ancients, clubb'd their Endeavours to the Production of a Single Modern.

BENJAMIN FRANKLIN, *Poor Richard's Almanack,* 1751

If you traced your family tree through ten generations, how many ancestors would there be in the tenth generation back? Because you are descended from two parents, each of whom had two parents, each of whom had two parents, and so on, the numbers in each generation back are:

$$
\begin{aligned}
2 &\text{ parents,}\\
2 \cdot 2 &\text{ grandparents,}\\
2 \cdot 2 \cdot 2 &\text{ great-grandparents,}\\
2 \cdot 2 \cdot 2 \cdot 2 &\text{ great-great-grandparents,}\\
2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 &\text{ great-great-great-grandparents,}\\
&\text{ and so on.}
\end{aligned}
$$

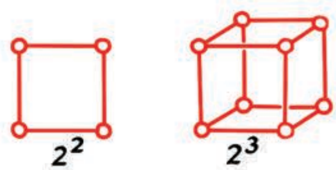
It can be seen from this pattern that a person has $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ ancestors in the tenth generation preceding his or her own.

A simpler way to write this number is 2^{10}. The 10, called an *exponent,* indicates that 10 twos are to be multiplied:

$$2^{10} = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 1{,}024$$

In the tenth generation back, you may have more than a thousand ancestors!

The solution of this problem requires repeated multiplication by the same number. Such an operation is called *raising to a power*. The number of grandparents that a person has is 2^2, or "2 raised to the second power." The number of a person's great grandparents is 2^3, or "2 raised to the third power." In each of these cases, we can represent the number by a geometric pattern of circles. Because 2^2 can be pictured as a square, it is also referred to as "2 squared." Because 2^3 can be pictured as a cube, it is also referred to as "2 cubed."



Space as we know it is limited to three dimensions, and so powers higher than the third do not have special names.

It is important to understand the difference in meaning between *multiplication* and *raising to a power* and in the symbols used to represent each of these operations. The following examples should make this difference clear.

$3 \cdot 4$ means "3 times 4" or $4 + 4 + 4$
4^3 means "4 to the third power" or $4 \cdot 4 \cdot 4$
$3x$ means "3 times x" or $x + x + x$
x^3 means "x to the third power" or $x \cdot x \cdot x$
nx means "n times x" or the sum of n x's
x^n means "x to the nth power" or the product of n x's

Exercises

Set I

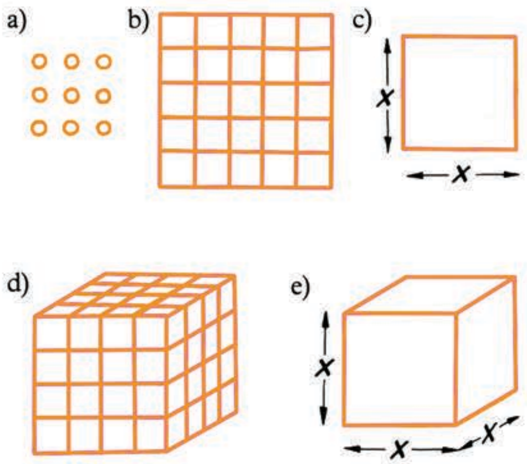
Find each of the following powers.

1. 5^2
2. 2^5
3. 10^3
4. 10^7
5. 1^3
6. 1^7
7. $(1.3)^2$
8. $(3.1)^2$
9. $(0.4)^3$
10. $(0.4)^6$

Set II

11. The expression x^2 can be named in more than one way.
    a) Write two different names for it.
    b) What is the 2 called?

12. What numbers or expressions do these figures represent? Express each as a power.



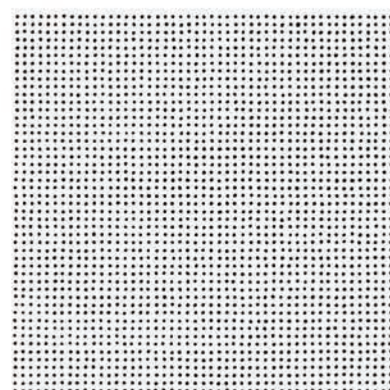
13. Write each of the following in symbols.
    a) Seven squared.
    b) Two raised to the sixth power.
    c) The number x cubed.
    d) The eighth power of x.
    e) Three raised to the xth power.
    f) The yth power of x.

14. The raising-to-a-power problem 3^5 and the multiplication problem $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$ are equivalent. Write a power problem equivalent to each of the following.
    a) $7 \cdot 7 \cdot 7 \cdot 7$
    b) $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$
    c) $x \cdot x \cdot x \cdot x \cdot x \cdot x$
    d) $\underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{12 \text{ of them}}$
    e) $\underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{x \text{ of them}}$
    f) $\underbrace{x \cdot x \cdot \ldots \cdot x}_{y \text{ of them}}$

    Write a multiplication problem equivalent to each of the following.

    g) 8^5 i) 3^x
    h) x^3 j) y^x

15. The figure below contains 7^4 dots.



    a) How many dots are there?
    b) Because the dots are arranged in a square pattern, their number can also be written as a certain number squared. What is it?

16. The number 243 can be written as a power of 3. To find out what power it is, we can make a list of powers of 3 until we come to 243:

$$3^2 = 9,\ 3^3 = 27,\ 3^4 = 81,\ 3^5 = 243$$

    Express each of the following numbers as a power of the number given.
    a) 729 as a power of 3.
    b) 64 as a power of 2.
    c) 64 as a power of 4.

d) 64 as a power of 8.
e) 10,000 as a power of 10.
f) 1,000,000,000 as a power of 10.
g) It is impossible to express 5 as a power of 1. Explain why.

17. If we know that 4^5 is 1,024, we can find 4^6 by observing that

$$4^6 = 4 \cdot \underbrace{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4}_{4^5} \quad \text{so} \quad 4^6 = 4 \cdot 1{,}024 = 4{,}096$$

Use a similar method to find each of the following.
a) 2^9 if $2^8 = 256$.
b) 11^4 if $11^3 = 1{,}331$.
c) 3^7 if $3^5 = 243$.
d) 5^8 if $5^5 = 3{,}125$.
e) What would you have to multiply x^6 by in order to get x^7?
f) What would you have to multiply x^{10} by in order to get x^{12}?

Set IV

After fooling around all summer, Obtuse Ollie didn't want to work very hard in the first few weeks of school. He decided to study algebra one minute the first week, two minutes the second week, four minutes the third week, and so on, doubling the amount of time each succeeding week.

If he sticks to this plan and the semester contains twenty weeks, how many minutes will Ollie study algebra in the last week?

A visual paradox: How many objects is the hand holding?

Zero is the first of ten symbols—the digits—with which we are able to represent any of an infinitude of numbers. Zero is also the first of the numbers which we must represent. Yet zero, first of the digits, was the last to be invented; and zero, first of the numbers, was the last to be discovered.

CONSTANCE REID, *From Zero to Infinity*

LESSON 6
Zero and One

Although the Alexandrian astronomer Ptolemy used the symbol o, an abbreviation of a word meaning "nothing," as a digit in his work, it was not until many centuries later that the idea of zero as a number was accepted. Because numbers originated with counting and it doesn't seem natural to count with zero, it was not considered to be a number. The *counting numbers,* also called the *natural numbers,* begin with one. Although zero is never used in counting, it is sometimes used to answer the same question that the counting numbers answer, the question of how many.

The behavior of the number zero in calculations differs from that of all other numbers in several basic ways. It is the only number that can be added to or subtracted from another number without changing that number.

▶ For every number x, $x + 0 = x$ (also, $0 + x = x$) and $x - 0 = x$.

It is the only number that, regardless of what number it is multiplied by, always gives the same result: zero.

▶ For every number x, $x \cdot 0 = 0$.

If x is a counting number, such as 5, it is easy to see why: $5 \cdot 0 = 0 + 0 + 0 + 0 + 0 = 0$. Assuming that the product of two numbers does not depend on the order in which they are multiplied, it is also true that

$$0x = 0$$

Strange as it may seem, it is easy to divide zero by another number, *yet dividing a number by zero makes no sense at all!* Remember that the quotient of two numbers x and y, $\frac{x}{y}$, is the number that must be multiplied by y to give x. For example, $\frac{6}{2} = 3$ because $3 \cdot 2 = 6$. Now dividing zero by another number is okay: $\frac{0}{x} = 0$, because, as we have observed above, $x \cdot 0 = 0$.

On the other hand, dividing a number by zero leads to trouble. If we tried dividing 3 by 0, for example, the number $\frac{3}{0}$ would be the number that must be multiplied by 0 to give 3. But there is *no such number;* every number multiplied by 0 gives *zero* as the result.

Dividing zero by itself leads to trouble of a different sort. Suppose that $\frac{0}{0}$ is equal to some number x: if $\frac{0}{0} = x$, then it must be true that $x \cdot 0 = 0$. But this is true for *every number* x. Hence $\frac{0}{0} = 0$, $\frac{0}{0} = 1$, $\frac{0}{0} = 2$, and so forth. Because $\frac{0}{0}$ can mean anything, it is meaningless.

The number one plays the same role in multiplication and division that the number zero plays in addition and subtraction: it does not change the number that it is multiplying or dividing.

▶ For every number x, $x \cdot 1 = x$ (also, $1x = x$) and $\frac{x}{1} = x$.

Exercises

Beginning with this lesson, the exercises in Set I will review ideas from earlier lessons.

Set I

1. Show how the following number trick works by drawing boxes and circles to illustrate the steps.
   a) Think of a number.
   b) Add four.
   c) Multiply by three.
   d) Subtract nine.
   e) Divide by three.
   f) Subtract the number that you first thought of.
   The result is one.

2. Write a number for each of the following:
   a) The product of 3 and x.
   b) The sum of 3 and x.
   c) The difference between 3 and x.
   d) The quotient of 3 and x.
   e) The third power of x.
   f) The xth power of 3.

3. Write another expression equivalent to each of the following.
   a) $a + a$
   b) $5b$
   c) $c \cdot c \cdot c$
   d) d^4
   e) $\underbrace{e + e + \cdots + e}_{x \text{ of them}}$
   f) $\underbrace{f \cdot f \cdot \cdots \cdot f}_{y \text{ of them}}$

Set II

4. What do you know about the following?
   a) The sum of any number and zero.
   b) The difference between any number and zero.
   c) The product of any number and zero.
   d) The product of any number and one.
   e) The quotient of zero and any number.
   f) The quotient of any number and zero.
   g) The quotient of any number and one.

5. Sometimes it is easier to multiply than to add. Figure out each of the following:
   a) $0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$
   b) $0 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9$
   c) $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$
   d) $1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$

6. The following questions are about powers of one.
   a) What is the value of 1^2? Why?
   b) What is the value of 1^7?
   c) What is the value of 1^x, in which x is a counting number larger than one?

7. The following remark appeared in a French arithmetic book published in 1485:

   "The digits are no more than ten different figures, of which nine have value and the tenth is worth nothing in itself but gives a higher value to the others."

   a) What digit "is worth nothing in itself"?
   b) Give an example of how it "gives a higher value" to another digit.

8. If possible, simplify each of the following.
   a) $1x$
   b) $0x$
   c) $x + 0$
   d) $x + 1$
   e) $x - 0$
   f) $\frac{0}{x}$
   g) $\frac{x}{0}$
   h) $\frac{x}{1}$

9. Each of the following expressions contains two unknown numbers, x and y. Simplify each expression as much as you can. You may assume that neither x nor y is zero.
   a) $1x + 1y$
   b) $1x - 0y$
   c) $0x + 1y$
   d) $0x + 0y$
   e) $\frac{x}{1} + \frac{y}{1}$
   f) $\frac{0}{x} + \frac{0}{y}$
   g) $\frac{x}{1} - \frac{0}{y}$
   h) $\frac{0}{x} + \frac{y}{1}$

10. When an *even* number is divided by two, the remainder is zero. For example,

$$\begin{array}{r} 6 \\ 2\overline{)12} \\ -12 \\ \hline 0 \end{array}$$

   a) What is the remainder when an *odd* number is divided by two?
   b) What is the remainder when *zero* is divided by two?
   c) Is zero *even* or *odd*?

Set IV

We have been using numbers larger than 1 as exponents to indicate repeated multiplication.

x^2 means $x \cdot x$,
x^3 means $x \cdot x \cdot x$,
x^4 means $x \cdot x \cdot x \cdot x$,
and so forth.

What would 1 or 0 mean if we used them as exponents? It seems rather obvious from the pattern above that

x^1 means x.

What do you think x^0 should mean? Rather than just making a guess, make a conclusion from the information in the table below.

x	x^4		x^3		x^2		x^1		x^0
4	256	→	64	→	16	→	4	→	?
3	81	→	27	→	9	→	3	→	?
2	16	→	8	→	4	→	2	→	?
1	1	→	1	→	1	→	1	→	?
0	0	→	0	→	0	→	0	→	?

LESSON 7
Several Operations

What do you think is the correct value for the following expression?

$$2 \times 12 + 3 \times 10$$

It all depends on what you are trying to find. For example, suppose that Mrs. Naugatuck wants to buy 2 pounds of porcupine at 12 cents a pound and 3 pounds of iguana at 10 cents a pound. How much will the order cost?

To answer this question, we have to find

$$2 \times 12 + 3 \times 10$$

It is obvious from the situation that both multiplications should be done before the addition:

$$\begin{array}{c} 2 \times 12 + 3 \times 10 = \\ 24 \quad + \quad 30 \quad = \\ 54 \end{array}$$

The order will cost 54 cents.

Now consider this problem. Mrs. Naugatuck wants to buy 2 dozen duck eggs and 3 buffalo sausages. If they cost 10 cents each, how much will she have to spend?

The answer to this question is also

$$2 \times 12 + 3 \times 10$$

In this case, however, the operations are done in a different order. Multiplying 12 by 2, adding 3, and multiplying the result by 10, we get

$$\begin{aligned} 2 \times 12 + 3 \times 10 &= \\ 24 \quad + 3 \times 10 &= \\ 27 \quad \times 10 &= \\ 270 & \end{aligned}$$

She will have to spend \$2.70.

The fact that the answer to a problem that requires several operations can depend on the order in which they are done has led mathematicians to make rules for dealing with such problems. The rules are:

First, figure out the powers if there are any.
Then do the multiplications and divisions in order from left to right.
Finally, do the additions and subtractions in order from left to right.

According to these rules, the answer to the problem written as

$$2 \times 12 + 3 \times 10$$

is 54. If we want to change the order of operations, as in the second problem, we use parentheses. It would be written as

$$(2 \times 12 + 3) \times 10$$

We will learn in the next lesson how to use parentheses to change the order of operations.

Examples of how the rules for order of operations are used are given on the next page.

EXAMPLE 1

Find the value of $5^2 - 2 \cdot 3^2 + 4$.

SOLUTION

Figuring out the powers first, we get

$$25 - 2 \cdot 9 + 4$$

(Notice that the 3 is squared before it is multiplied by 2 in the next step.) Doing the multiplication next, we get

$$25 - 18 + 4$$

Finally, doing the addition and subtraction in order from left to right, we get

$$7 + 4 =$$
$$11$$

EXAMPLE 2

Find the value of $3 \cdot 4^3 + 7 \cdot 5 - 11^2$.

SOLUTION

$$3 \cdot 4^3 + 7 \cdot 5 - 11^2 =$$
$$3 \cdot 64 + 7 \cdot 5 - 121 =$$
$$192 + 35 - 121 =$$
$$227 - 121 =$$
$$106$$

EXAMPLE 3

Find the value of $\frac{28}{4} - \frac{6^2}{12} - \frac{32}{2^3}$.

SOLUTION

$$\frac{28}{4} - \frac{6^2}{12} - \frac{32}{2^3} =$$
$$\frac{28}{4} - \frac{36}{12} - \frac{32}{8} =$$
$$7 - 3 - 4 =$$
$$4 - 4 =$$
$$0$$

Exercises

Set I

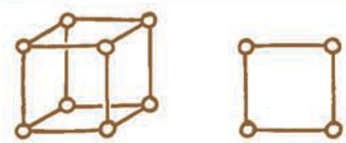
1. If possible, express each of the following numbers as a power of the number given.
   a) 125 as a power of 5.
   b) 10 as a power of 0.
   c) 64 as a power of 2.

2. A parking meter will take nickels or dimes.
   a) If it contains x coins and someone puts in a dime, how many coins does it contain in all?
   b) If it contains 17 coins of which x are nickels, how many dimes does it contain?
   c) If it contains x nickels and 24 dimes and someone puts in 2 more nickels, how many coins does it contain in all?

3. Mr. Webster is trying to improve his vocabulary.
   a) If he learns x new words each day, how many words will he learn in a week?
   b) If he learns x new words each day, how long will it take him to learn 1,000 new words?
   c) If he knows 15,000 words now and learns 10 new words each day, how many words will he know in x days?

Set II

4. The figure shown here illustrates the expression $2^3 + 2^2$.



Which figure below illustrates each of the expressions in parts a through g? (The crossed-out circles indicate subtraction.)

Figure 1

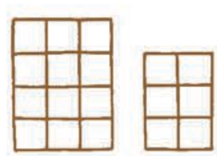


Figure 2

Figure 3

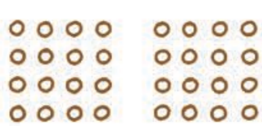


Figure 4

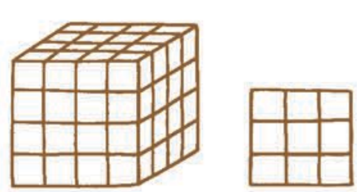


Figure 5

Figure 6

a) $4^2 + 4^2$
b) $4^3 + 3^2$
c) $4 \cdot 3 + 3 \cdot 2$
d) $6^2 - 3 \cdot 2$
e) $6 \cdot 2 - 3 \cdot 2$
f) $2 \cdot 4^2$
g) $4 \cdot 2^2$

5. Find the value of each of the following expressions.
   a) $2 \cdot 5 + 4 \cdot 10$
   b) $2 + 5 \cdot 4 + 10$
   c) $3 \cdot 2^4$
   d) $3 + 2^4$
   e) $5^2 - 4^2$
   f) $5^2 \cdot 4^2$
   g) $6 \cdot 7 - 12 + 3^3$
   h) $6 \cdot 7 + 3^3 - 12$
   i) $6 \cdot 7 - 3^3 + 12$
   j) $6 \cdot 7 + 12 - 3^3$
   k) $\frac{4^2}{8} + \frac{8^2}{4}$
   l) $\frac{4^2}{4} + \frac{8^2}{8}$
   m) $\frac{8^2}{4} + \frac{4^2}{8}$
   n) $11 - 2 \cdot 3 + 7 \cdot 2$
   o) $11 - 2^3 + 7^2$
   p) $11 \cdot 2^3 - 7^2$
   q) $11 \cdot 7^2 - 2^3$

6. Write an expression for each of the following.
   a) The sum of the squares of x and y.
   b) Ten decreased by the product of x and 5.
   c) The quotient of x and 5, decreased by 10.
   d) The product of 8 and the cube of x.
   e) The difference between the fourth power of y and y.
   f) Two more than the quotient of 12 and x.
   g) The sum of x and the product of x and y.

7. The value of the expression $x^2 + 3x - 2$ depends on the number with which we replace x. For example, if x is 5,

$$\begin{aligned} x^2 + 3x - 2 &= \\ 5^2 + 3 \cdot 5 - 2 &= \\ 25 + 15 - 2 &= \\ 40 - 2 &= \\ 38 \end{aligned}$$

   Find the value of $x^2 + 3x - 2$ if
   a) x is 1.
   b) x is 4.
   c) x is 10.
   d) x is 20.

8. Find the value of each of the following expressions for the numbers given.
   a) $2x + 7$ if x is 6.
   b) $15 - 3x$ if x is 2.
   c) $1 + 4x^2$ if x is 5.
   d) $x^3 - x^2$ if x is 10.
   e) $x^4 + x$ if x is 3.
   f) $5x^2 - x + 6$ if x is 4.

9. At Frankenfurter's Delicatessen, salami costs 80 cents a pound and liverwurst costs 95 cents a pound.
   a) How much would an order of 7 pounds of salami and 3 pounds of liverwurst cost?
   b) How much would an order of x pounds of salami and y pounds of liverwurst cost?

Set IV

Because very few people enjoy doing arithmetic, pocket calculators have become very popular. Although they are easy to use, getting the correct answer to a problem that requires more than one operation is not as simple as it might seem.

Consider the problem

$$12 \cdot 5 - \frac{8}{4} + 7 \cdot 2$$

for example. If you push the keys for these numbers and operations in the order shown here,

[1] [2] [x] [5] [−] [8] [÷] [4] [+] [7] [x] [2] [=]

a calculator will give the wrong answer.

1. What is the correct answer to the problem?
2. What answer do you think the calculator might give instead?
3. Why would it give that answer?
4. What would you do if you wanted to use such a calculator to get the correct answer to the problem?

EJERCICIO 80

Resolver las siguientes ecuaciones:

1. $x+3(x-1)=6-4(2x+3)$.
2. $5(x-1)+16(2x+3)=3(2x-7)-x$.
3. $2(3x+3)-4(5x-3)=x(x-3)-x(x+5)$.
4. $184-7(2x+5)=301+6(x-1)-6$.
5. $7(18-x)-6(3-5x)=-(7x+9)-3(2x+5)-12$.
6. $3x(x-3)+5(x+7)-x(x+1)-2(x^2+7)+4=0$.
7. $-3(2x+7)+(-5x+6)-8(1-2x)-(x-3)=0$.
8. $(3x-4)(4x-3)=(6x-4)(2x-5)$.
9. $(4-5x)(4x-5)=(10x-3)(7-2x)$.
10. $(x+1)(2x+5)=(2x+3)(x-4)+5$.
11. $(x-2)^2-(3-x)^2=1$.
12. $14-(5x-1)(2x+3)=17-(10x+1)(x-6)$.
13. $(x-2)^2+x(x-3)=3(x+4)(x-3)-(x+2)(x-1)+2$.
14. $(3x-1)^2-5(x-2)-(2x+3)^2-(5x+2)(x-1)=0$.
15. $2(x-3)^2-3(x+1)^2+(x-5)(x-3)+4(x^2-5x+1)=4x^2-12$.

Problems from a Spanish algebra book

LESSON 8 Parentheses

Parentheses are among the most frequently used symbols in algebra. One way in which parentheses are used is to change the usual order of operations. For example, suppose that the sum of 3 and 5 is to be subtracted from 10. Because $3 + 5 = 8$ and $10 - 8 = 2$, the answer to this problem is 2. If we tried writing the problem as

$$10 - 3 + 5$$

however, we would get the wrong answer because, according to our rules of operation, additions and subtractions are done from left to right:

$$\begin{aligned} 10 - 3 + 5 &= \\ 7 \quad + 5 &= \\ 12 & \end{aligned}$$

In order to show that we want to add 3 and 5 before subtracting the result from 10, we write

$$10 - (3 + 5)$$

The parentheses indicate that the operation inside them is to be done first:

$$\begin{aligned} &10 - (3 + 5) = \\ &10 - \quad 8 \quad = \\ &\quad 2 \end{aligned}$$

► In an expression containing parentheses, the parentheses indicate that the operations enclosed within them are to be done before anything else.

Division is usually indicated in algebra by a fraction bar. To show, for example, that the sum of 9 and 3 is to be divided by the difference of 5 and 1, we write

$$\frac{9 + 3}{5 - 1}$$

The fraction bar here means not only to divide, but also to *add and subtract before dividing.*

$$\frac{9 + 3}{5 - 1} = \frac{12}{4} = 3$$

Because the usual procedure is to divide (and multiply) before adding and subtracting, the fraction bar acts here as a parentheses symbol.

Here are more examples of how the value of an expression containing parentheses is found.

EXAMPLE 1
Find the value of $(7 + 4)(7 - 4)$.

SOLUTION

$$\begin{aligned} &(7 + 4)(7 - 4) = \\ &\quad 11 \cdot 3 \quad = \\ &\quad 33 \end{aligned}$$

EXAMPLE 2
Find the value of $4 + (11 - 2)^2$.

SOLUTION

$$\begin{aligned} 4 + (11 - 2)^2 &= \\ 4 + \quad 9^2 \quad &= \\ 4 + \quad 81 \quad &= \\ 85 \end{aligned}$$

EXAMPLE 3
Find the value of $\frac{10}{6 - 5} + \frac{6 \cdot 5}{10}$.

SOLUTION

$$\begin{aligned} \frac{10}{6-5} + \frac{6 \cdot 5}{10} &= \\ \frac{10}{1} + \frac{30}{10} &= \\ 10 + 3 &= \\ 13 \end{aligned}$$

Exercises

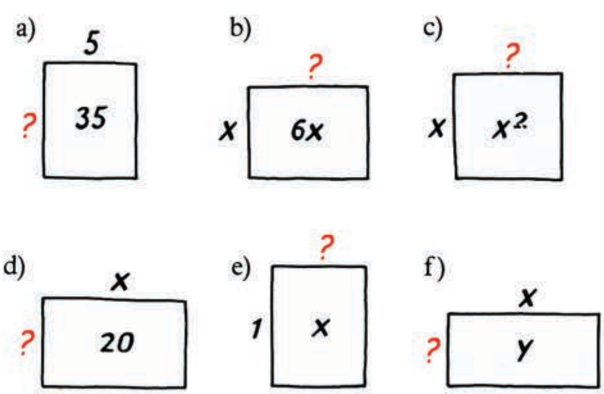
Set I

1. If possible, find the value of each of the following.

   a) $0 \cdot 100$ c) $\frac{0}{100}$ e) $\frac{1}{100}$

   b) $1 \cdot 100$ d) $\frac{100}{0}$ f) $\frac{100}{1}$

2. Find the missing dimension for each of these rectangles. (Some of your answers will be in terms of the letters.)



3. The tenrec, a native animal of Madagascar, is capable of giving birth only ten weeks after it itself is born.

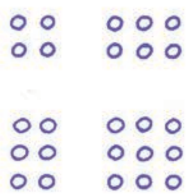
   a) How many generations of descendants of one of these animals could be born in 50 weeks?
   b) How many generations of descendants could be born in x weeks if x is a multiple of 10?

Set II

4. Tell whether or not the expressions in each of the following pairs are equal.
   - a) $(11 + 5) + 2$ and $11 + (5 + 2)$
   - b) $(11 - 5) - 2$ and $11 - (5 - 2)$
   - c) $(11 + 5) - 2$ and $11 + (5 - 2)$
   - d) $(11 - 5) + 2$ and $11 - (5 + 2)$
   - e) $12 \cdot 6 \cdot 3$ and $12 \cdot (6 \cdot 3)$
   - f) $12 + 6 \cdot 3$ and $12 + (6 \cdot 3)$
   - g) $12 + 6 \cdot 3$ and $(12 + 6) \cdot 3$
   - h) $\frac{12 + 6}{3}$ and $\frac{(12 + 6)}{3}$

5. Find the value of each of these expressions.
   - a) $7 \cdot 3^2$
   - b) $(7 \cdot 3)^2$
   - c) $4 + 2 \cdot 3 + 5$
   - d) $(4 + 2) \cdot 3 + 5$
   - e) $4 + 2 \cdot (3 + 5)$
   - f) $(4 + 2) \cdot (3 + 5)$
   - g) $15 - 3 \cdot 4 - 2$
   - h) $(15 - 3) \cdot (4 - 2)$
   - i) $15 - (3 \cdot 4 - 2)$
   - j) $15 - 3 \cdot (4 - 2)$
   - k) $\frac{30}{10} + \frac{6}{2}$
   - l) $\frac{30 + 6}{10 + 2}$
   - m) $\frac{30}{10} \cdot \frac{6}{2}$
   - n) $\frac{30 \cdot 6}{10 \cdot 2}$
   - o) $5^2 - 5 \cdot 2^2$
   - p) $(5^2 - 5) \cdot 2^2$
   - q) $(5^2 - 5 \cdot 2)^2$

6. The figure shown here illustrates the expression $(2 + 3)^2$.



Which figure below illustrates each of the expressions in parts a through h?

   - a) $4^2 + 1^2$
   - b) $(4 + 1)^2$
   - c) $4(4 + 1)$
   - d) $(4 - 1)^2$
   - e) $4^2 - 1^2$
   - f) $4(4 - 1)$
   - g) $4^2 + 4$
   - h) $4^2 - 4$

Figure 1

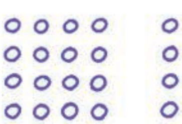


Figure 2

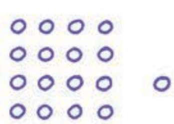


Figure 3

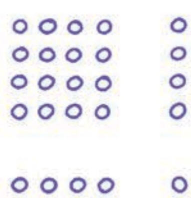


Figure 4

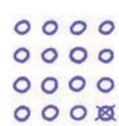


Figure 5

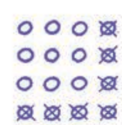
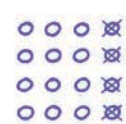


Figure 6



7. To show that someone is to add x and 3 and then square the result, we write $(x + 3)^2$. Write an expression for each of the following sets of operations.
   a) Subtract 5 from x and then cube the result.
   b) Multiply x by 6 and then add y.
   c) Add y to 6 and then multiply by x.
   d) Divide 10 by x and then subtract y.
   e) Subtract y from 10 and then divide by x.
   f) Multiply the sum of x and 2 by the sum of x and 7.
   g) Divide the difference of x and y by twice x.
   h) Square the product of 3 and x and subtract the result from 11.
   i) Subtract the product of 3 and x from 11 and square the result.
   j) Add the cubes of x and y and multiply the result by 8.

8. Find the values of the following expressions for the numbers given.

   $x^2 + 2x - 15$
   a) if x is 3
   b) if x is 4
   c) if x is 10
   d) if x is 50

   $(x - 3)(x + 5)$
   e) if x is 3
   f) if x is 4
   g) if x is 10
   h) if x is 50

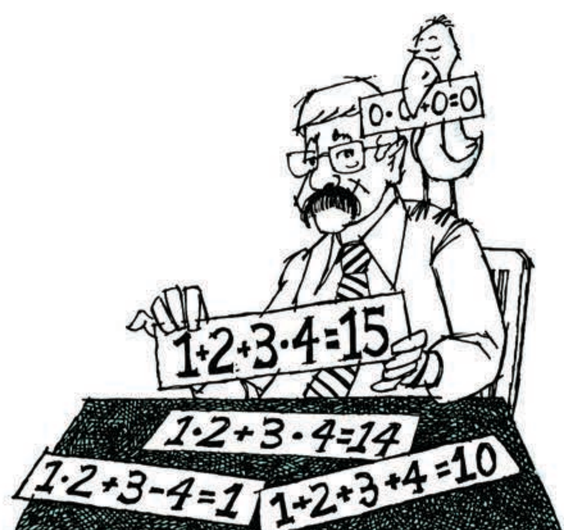
Set IV

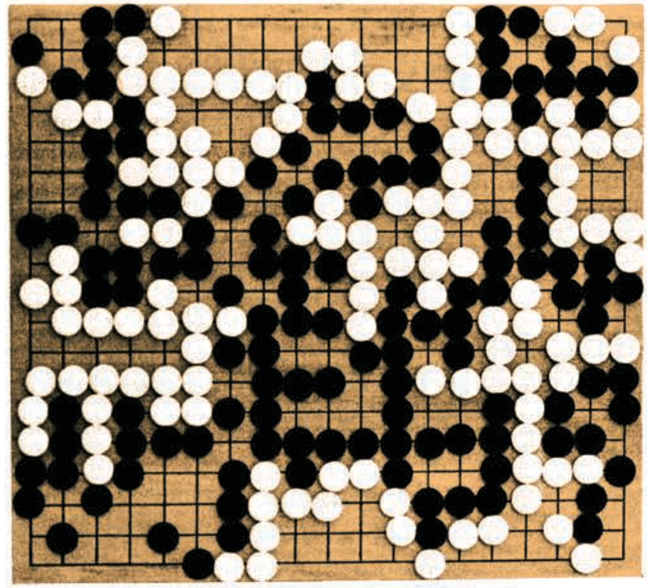
The value of the expression 1 ___ 2 ___ 3 ___ 4 depends on the symbols of operation with which we replace the blanks. Examples are shown in the picture at the right.

1. Can you figure out which of the following symbols of operation, +, −, ·, and ÷, should be used to replace the blanks in the expression

   1 ___ 2 ___ 3 ___ 4 ___ 5 ___ 6 ___ 7 ___ 8 ___ 9 ___ 10

   in order to make it as large a number as possible?
2. What is the value of the number?
3. Suppose that, in addition to replacing the blanks with symbols of operation, you may add parentheses wherever you wish. What would you do to make the expression as large a number as possible?
4. What is the value of the number?





PHOTOGRAPH BY PETER RENZ.

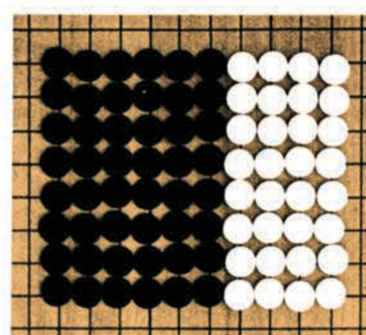
LESSON 9 The Distributive Rule

The oldest game in the world may be the game of Go. It originated in China and is thought to have been played as long ago as the twenty-fourth century B.C.

Go is played with black and white stones on a square board. The object is to capture more territory than the other player while losing as few stones as possible in doing so. The photograph above shows how the board might look at the end of a game.

Although the way in which the stones are arranged on the board makes them difficult to count, the stones in the pattern below are easy to count. Two ways to count them illustrate a simple but very useful pattern called the *distributive rule*. One way is to multiply the sum of the numbers of black and white stones in one row, 6 + 4, by the number of rows, 8:

$$8(6 + 4) = 8(10) = 80$$



PHOTOGRAPH BY PETER RENZ.

The other way is to multiply each of these numbers, 6 and 4, by 8 and add the results:

$$8(6) + 8(4) = 48 + 32 = 80$$

Comparing the first way with the second, we see that

$$8(6 + 4) = 8(6) + 8(4)$$

This pattern is true for *any* set of three numbers.

The Distributive Rule (Addition)
For any three numbers a, b, and c, $a(b + c) = ab + ac$.

Notice that this rule is about a relationship between multiplication and addition. Sometimes it is stated by simply saying that "multiplication distributes over addition."

There is a similar rule relating multiplication and subtraction.

The Distributive Rule (Subtraction)
For any three numbers a, b, and c, $a(b - c) = ab - ac$.

Because the product of two numbers does not depend on the order of the numbers, the two distributive rules can also be written with the numbers in each product interchanged; that is, because

$$a(b + c) = ab + ac \quad \text{and} \quad a(b - c) = ab - ac$$

it is also true that

$$(b + c)a = ba + ca \quad \text{and} \quad (b - c)a = ba - ca$$

The distributive rules are among the most fundamental patterns of algebra. Here are examples of how they are used.

EXAMPLE 1
Use the distributive rule to write the product $10(x + 2)$ as a sum.

SOLUTION
$10(x + 2) = 10(x) + 10(2) = 10x + 20$

EXAMPLE 2
Use the distributive rule to write the product $(5 + x)y$ as a sum.

SOLUTION
$(5 + x)y = 5y + xy$

EXAMPLE 3
Use the distributive rule to write the product $x(x - 1)$ as a difference.

SOLUTION
$x(x - 1) = x(x) - x(1) = x^2 - x$

Exercises

Set I

1. Simplify each of the following expressions.
   a) $a + a + a + a + a$
   b) bbb
   c) $\frac{a}{1} + \frac{b}{1}$
   d) $0(a + b)$

2. Mr. Hunt can type 20 words per minute and Miss Peck can type x words per minute.
   a) If they type at the same time, how many words can they type in a minute?
   b) How many words can Miss Peck type in 5 minutes?
   c) How long would it take Mr. Hunt to type y words?

3. The largest pizza ever baked weighed 1,000 pounds.
   a) If it contained x pounds of cheese, how much did the other ingredients weigh?
   b) If the pizza were cut into y equal pieces, how much would each piece weigh?
   c) If 10 people ate z pounds each, how much would be left?

Set II

4. The figure below illustrates the pattern $2(4 + 3) = 2(4) + 2(3)$.

Write a pattern illustrated by each of the following figures.

a)

b)

c)

d)

5. The multiplication problem $5x^2$ and the addition problem $x^2 + x^2 + x^2 + x^2 + x^2$ are equivalent. Write a multiplication problem equivalent to each of the following addition problems.
   a) $x^3 + x^3 + x^3 + x^3$
   b) $2x + 2x + 2x + 2x + 2x + 2x + 2x$
   c) $(x + 1) + (x + 1) + (x + 1)$
   d) $\underbrace{(x + y) + (x + y) + \cdots + (x + y)}_{\text{9 of them}}$

   Write an addition problem equivalent to each of the following multiplication problems.
   e) $2x^4$
   f) $5(3x)$
   g) $4(x + 7)$

6. According to the distributive rule, $4(x+2) = 4x + 8$. One way to prove this is by writing $4(x+2)$ as a repeated addition problem and rearranging the numbers being added:

$$\begin{aligned} 4(x+2) &= (x+2)+(x+2)+(x+2)+(x+2) \\ &= x+x+x+x+2+2+2+2 \\ &= 4x+8 \end{aligned}$$

Use the same method to prove that
a) $3(x+5) = 3x + 15$
b) $2(x+y) = 2x + 2y$
c) $4(x^2+1) = 4x^2 + 4$

7. Use the distributive rule to write each of the following as a sum or difference.

a) $8(x+3)$	f) $(4+x)y$
b) $5(y-2)$	g) $(y-x)7$
c) $x(x+1)$	h) $(x-6)x$
d) $y(x-y)$	i) $10(x^2+4)$
e) $(x+9)2$	j) $x(x^3-1)$

8. The way in which you learned to multiply numbers in arithmetic has as its basis the distributive rule. For example, to multiply 51 by 32 we write

$$\begin{array}{r} 51 \\ \times\ \ 32 \\ \hline 102 \\ +1530 \\ \hline 1632 \end{array}$$

To see how the distributive rule applies, consider the fact that $32 = 2 + 30$ so that

$$\begin{aligned} 32 \cdot 51 &= (2+30)51 \\ &= 2 \cdot 51 + 30 \cdot 51 \\ &= 102 + 1530 \\ &= 1632 \end{aligned}$$

a) Do the following multiplication problem.

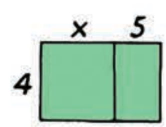
$$\begin{array}{r} 72 \\ \times 43 \\ \hline \end{array}$$

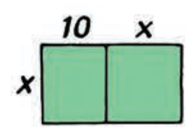
b) Show, by using the distributive rule, why what you have done is correct.
c) Now do this mutiplication problem.

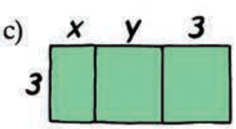
$$\begin{array}{r} 43 \\ \times 72 \\ \hline \end{array}$$

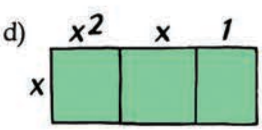
d) Explain your method by using the distributive rule.

9. Write the total area of each of these rectangles in two different ways.

a) 

b) 

c) 

d) 

10. Buster Brown owns x pairs of tennis shoes and y pairs of loafers.
a) If he has no shoes other than these, how many pairs of shoes does he own in all?
b) Write the number of shoes that he owns altogether as a product.
c) How many individual tennis shoes does he own?
d) How many individual loafers does he own?
e) Write the number of shoes that he owns altogether as a sum.

Set IV

You know from the distributive rule of multiplication over addition that, for all values of x and y,

$$2(x + y) = 2x + 2y$$

Is it also true that

$$(x + y)^2 = x^2 + y^2?$$

1. To find out, find the values of the following expressions for the numbers given.

   $(x + y)^2$
   a) if x is 2 and y is 0.
   b) if x is 0 and y is 6.
   c) if x is 3 and y is 4.
   d) if x is 9 and y is 1.

   $x^2 + y^2$
   e) if x is 2 and y is 0.
   f) if x is 0 and y is 6.
   g) if x is 3 and y is 4.
   h) if x is 9 and y is 1.

2. What do you conclude about $(x + y)^2$ and $x^2 + y^2$ on the basis of your results?

Summary and Review

In this chapter, we have reviewed the fundamental operations and their relationships.

Addition (*Lesson 1*) The result of adding two numbers, say a and b, is called their *sum* and is written as $a + b$. It does not depend on the order of the numbers, and so $a + b = b + a$.

Subtraction (*Lesson 2*) The result of subtracting one number from another, say b from a, is called their *difference* and is written as $a - b$. It may be understood to mean either "b taken away from a" or "the number that must be added to b to give a."

Multiplication (*Lesson 3*) The result of multiplying two numbers, say a and b, is called their *product* and is written as ab. As in addition, it does not depend on the order of the numbers, and so $ab = ba$. Multiplication can be understood as repeated addition; for example, $3a$ means $a + a + a$.

Division (*Lesson 4*) The result of dividing one number by another, say a by b, is called their *quotient* and is written as $\frac{a}{b}$. It is the number that must be multiplied by b to give a.

Raising to a Power (*Lesson 5*) To raise a number to a power means to multiply the number by itself one or more times; for example, a^4 is read as "a to the fourth power" and means $a \cdot a \cdot a \cdot a$. The 4 is called an *exponent*. The second and third powers of a number such as a are called "a squared" and "a cubed."

Zero and One (*Lesson 6*) Zero is the only number that can be added to or subtracted from another number without changing it. For every number a, $a + 0 = a$ and $a - 0 = a$.

Whenever any number is multiplied by zero, the result is zero. For every number a, $a \cdot 0 = 0$.

Although zero may be divided by another number, giving zero as the result, we never divide a number by zero. For every number a (except 0), $\frac{0}{a} = 0$;. $\frac{a}{0}$ and $\frac{0}{0}$ are meaningless.

One is the only number that can be multiplied by or divided into another number without changing it. For every number a, $a \cdot 1 = a$ and $\frac{a}{1} = a$.

Several Operations (*Lesson 7*) In performing a series of operations, we work from left to right, first raising to powers, then multiplying and dividing, and finally adding and subtracting.

Parentheses (*Lesson 8*) Parentheses are often used to change the usual order of operations by indicating that the operation inside them is to be done first. The fraction bar used to indicate division acts as a parentheses symbol.

The Distributive Rule (*Lesson 9*) The distributive rule relates multiplication and addition. It says that for any three numbers a, b, and c,

$$a(b + c) = ab + ac \quad \text{and} \quad (b + c)a = ba + ca$$

A similar rule relates multiplication and subtraction. For any three numbers a, b, and c,

$$a(b - c) = ab - ac \quad \text{and} \quad (b - c)a = ba - ca$$

Exercises

Set I

1. Write another expression equivalent to each of the following.

   a) $7 + 7 + 7 + 7$
   b) $7 \cdot 7 \cdot 7 \cdot 7$
   c) $2x$
   d) y^6

2. Write a number for each of the following:
   a) The number w squared.
   b) The product of 3 and x.
   c) The number y taken away from 17.
   d) The fifth power of z.

3. Here are directions for a number trick.

   Step 1. Think of a number.
   Step 2. Multiply by five.
   Step 3. Add eight.
   Step 4. Subtract three.
   Step 5. Divide by five.
   Step 6. Subtract the number that you first thought of.

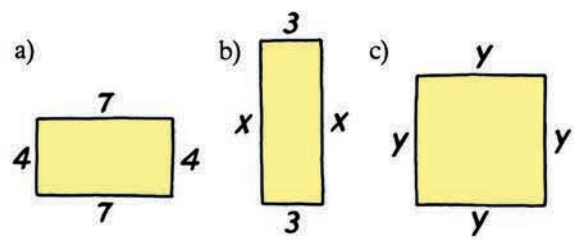
   a) Show how the trick works by drawing boxes and circles to illustrate the steps.
   b) What is the result at the end of the trick?
   c) Two steps in the trick could be combined into one without changing the end result. Which are they?
   d) What would the step replacing them be?

4. Which figure below and at the right illustrates each of the following expressions?
   a) $4 + 3^2$ b) $4 \cdot 3^2$ c) $(4 + 3)^2$

5. This problem is about the powers of 4.
   a) Make a table showing the values of the second through sixth powers of 4.
   b) Can you guess what any of the digits of 4^{100} might be?

6. A chessboard contains 2^6 small squares.
   a) How many squares is that?
   b) Can you write 2^6 as a number squared?

7. The perimeter of a rectangle is the sum of the lengths of its sides. The area of a rectangle is the product of its length and width. What are the perimeter and area of each of these rectangles?



8. Find the value of each of these expressions.
   a) $30 - 9 - 7$
   b) $30 - (9 - 7)$
   c) $1 + 4^3$
   d) $(1 + 4)^3$

Figure 1

Figure 2

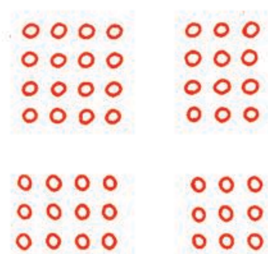


Figure 3

9. A can of Goober's Mixed Nuts contains almonds, cashews, and peanuts.
   a) If one can contains 9 almonds, x cashews, and 142 peanuts, how many nuts does it contain in all?
   b) If another can contains 160 nuts of which x are almonds and y are cashews, how many peanuts does it contain?

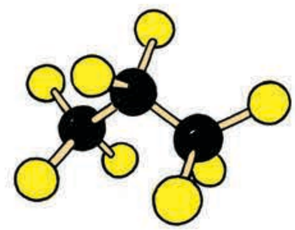
10. Division by zero makes no sense.
   a) Explain why there is no number equal to $\frac{2}{0}$.
   b) Is there any number equal to $\frac{0}{0}$?

11. Mr. Bunyan is a lumberjack.
   a) If he can cut down 600 trees in an hour, how many trees can he cut down in x hours?
   b) If he can saw up x logs in a day, how many days would it take him to saw up 10,000 logs?

12. Write an expression for each of the following sets of operations.
   a) Multiply x by 5 and then add 1.
   b) Add 3 to x and then square the result.
   c) Raise x to the sixth power and then subtract 7.

13. Write each of these products as a sum or difference.
   a) $7(a + 2)$ b) $b(1 - b)$ c) $(c + 9)5$

14. A molecule of propane gas consists of three carbon atoms and eight hydrogen atoms, as shown in the model below.

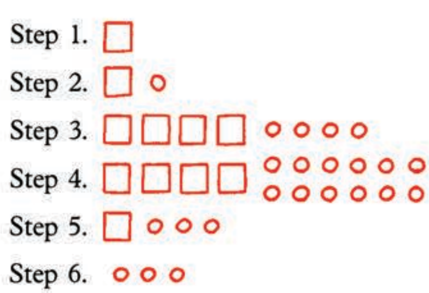


   a) How many of each atom do x molecules of propane contain?
   b) Write the total number of atoms in x propane molecules as a sum.
   c) How many atoms does one propane molecule contain?
   d) Write the total number of atoms in x propane molecules as a product.

Set II

1. Write another problem equivalent to each of the following.
   a) $11 + 11 + 11$ c) $5x$
   b) $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ d) y^4

2. The pictures below illustrate the steps of a number trick. Tell what is happening in each step.



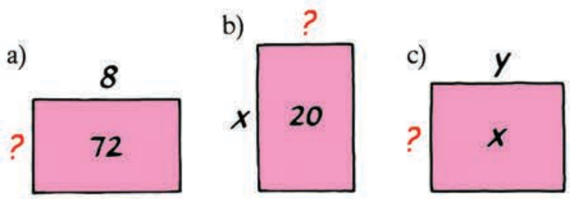
3. Write a number for each of the following:
   a) The difference between a and 5.
   b) The number b cubed.
   c) The sum of 2 and c.
   d) The quotient of 1 and d.

4. If possible, express each of the following as a power of the number given.
   a) 32 as a power of 2.
   b) 3 as a power of 1.
   c) 1,000,000 as a power of 10.

5. During the month of July, there were x shark attacks off the shore of Amity Beach.
   a) If 3 of the attacks were within 50 feet of the shore, how many were farther away?

b) If 5 attacks occurred in August, how many were there in all?

6. Find the missing dimension for each of these rectangles.



7. Par on the Shady Acres Golf Course is 72.
   a) If Colonel Bogey's score is x strokes above par, what is his score?
   b) If Miss Birdie's score is y strokes below par, what is her score?
   c) Mr. Bunker's score on the first nine holes is 75 (he has a terrible time with sand traps) and his score on the second nine is x. How many strokes above par is his total score?

8. Find the value of each of these expressions.
   a) $6 \cdot 10^2$
   b) $(6 \cdot 10)^2$
   c) $(2 + 7)(8 - 3)$
   d) $2 + 7 \cdot 8 - 3$

9. The cube numbers are related to the differences of square numbers in an interesting way.
   a) Copy and complete the following table.

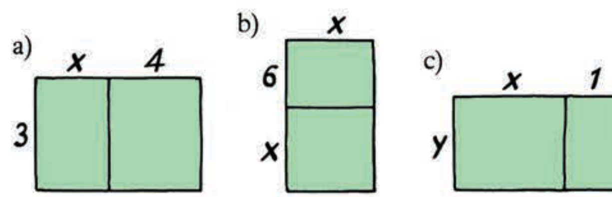
$$3^2 - 1^2 = 8 = 2^3$$
$$6^2 - 3^2 = \_\_\_ = \_\_\_^3$$
$$10^2 - 6^2 = \_\_\_ = \_\_\_^3$$
$$15^2 - 10^2 = \_\_\_ = \_\_\_^3$$

   b) Can you guess what the next line of this table is?

10. Write an expression for each of the following sets of operations.
    a) Subtract 6 from x and then multiply by 2.
    b) Divide x by 8 and then add 4.
    c) Cube x and then subtract the result from 150.

11. Write each of these products as a sum or difference.
    a) $8(v + 11)$
    b) $(w - 6)3$
    c) $x(y + z)$

12. Show how each of these figures illustrates the distributive rule by writing its area as both a product and a sum.



13. Since going on a diet, Mrs. Uppington has lost 3 kilograms each week.
    a) At this rate, how many kilograms would she lose in x weeks?
    b) If she weighed 200 kilograms before beginning the diet, how much would she weigh after x weeks of it?
    c) If she wants to lose x kilograms, how many weeks will it take her?

14. Find the values of the following expressions for the numbers given.

$x^2 + 5x - 14$
    a) if x is 2
    b) if x is 3
    c) if x is 10

$(x + 7)(x - 2)$
    d) if x is 2
    e) if x is 3
    f) if x is 10