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UNDERSTANDING TEACHERS' CONCEPTUALISATION OF ANGLE

John Lawton
Deakin University

Concept image, and its interaction with formal mathematics, is central to the learning of geometry. This paper discusses findings from a survey of teachers which seeks to better understand their concept image for angle, and how they relate that to their definitions for it. The survey is part of a study of the way that angle is conceptualised in geometry teaching in Australia. Teachers in the survey in this study were able to provide clear formal definitions for angle in most cases. However, there also appears to be some fragmentation of concept image and definitions employed by teachers in the survey. The potential for employing a more integrated, complex, and subtle understanding of angle is discussed.

BACKGROUND

The importance of mental imagery in geometry learning was highlighted by Vinner (1991) when he argued that students learn primarily through the development of a *concept image*. When students are asked to consider a concept, its name evokes "something non-verbal" (p. 68). This *cognitive structure* is defined as the set of "mental pictures, associated properties and processes as well as strings of words and symbols" that we associate in our minds with a concept. Formally structured mathematics is described as *concept definition*. Although it is important for students to interact with definitions, it is argued that they are not central to the learning of geometry.

Mitchelmore and White (1998) found three distinct conceptualisations for angle in use in schools. These are angle as a *geometric structure* formed by two lines meeting at a vertex, as a *region* formed by two intersecting half- planes and as *turn*, about a point between two lines. These are thought to originate from different sets of physical experience. It is argued that no single form of words can completely define angle, and that no individual conceptualisation for angle is sufficient for use in all angle contexts. For instance, when measuring the angle formed by the corner of a room, the region concept for angle is appropriate. However, this concept is not adequate when needing to measure an amount of rotation about a point that exceeds 360º. To understand angle, students need to be able to operate different concepts for it simultaneously. We learn angle by experiencing many different angle situations. The *standard angle concept* of "two lines meeting at a point with some relation between them" (p .5) is eventually discovered to be the one abstraction that is common to all angle situations. This concept image for angle can be taught by learning to identify structural elements of the standard angle concept in a variety of different real-world contexts. It is often necessary to imagine or remember structural angle components which are otherwise invisible, such as the opening position of the arm in an angle formed by an open door. This complex understanding of angle is operated mentally. We learn to measure angle by operating this concept image as a *rotating radius*, transforming one arm of the angle until it is directly above the other arm.

METHODOLOGY

The findings in this paper are from a PhD study. The research question is "How do teachers conceptualise angle, and how do enhancements to that understanding inform geometry teaching practice?" This is explored initially with a survey assessing teachers' use of angle in their classrooms

and their opinions of problems associated with angle as described in extant research. The study further involves experimental classroom lessons with the standard angle concept as a central focus, implemented with teacher case studies.

The survey uses Qualtrics (https://www.qualtrics.com). An average of 12 questions for each teaching level and 20 general questions are asked. The questions are mostly multiple choice with respondents encouraged to provide written explanations for specific choices. A sketching tool is used by participants to show how they typically draw angle in class. The survey takes about 30 minutes. Responses have been received during 2022 and 2023 from 43 teachers. Fourteen are primary teachers, and 29 are secondary, at years three to eight. They have on average been teaching for 17 years each. Twenty-one respondents teach in government schools, 13 in Catholic schools and 9 at independent schools. Thirty-one teachers in the survey work in the Australian state of Victoria and 12 work in the state of New South Wales (NSW). A different curriculum applies to teachers in each state. The results reported in this paper are from three questions in the survey that map teachers' concept image and defines their concept definitions for angle. They are,

Question 1, "do you employ a formal definition of angle in your teaching?" Where a yes answer is given respondents are asked to provide the definition in writing.

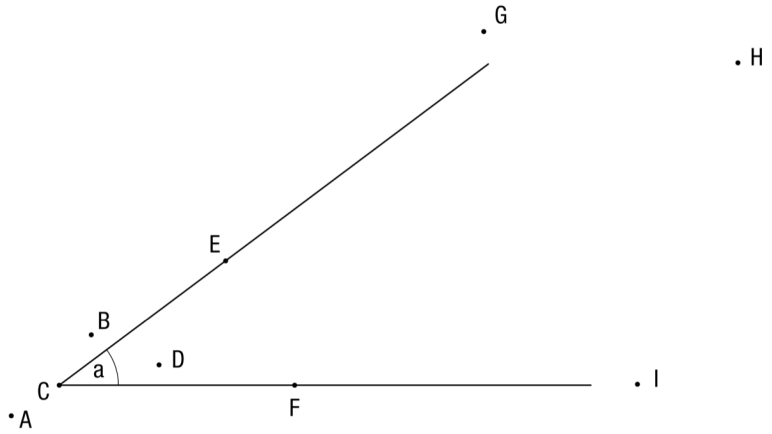

Figure 1. Angle illustration used in survey question. Based on Silfverberg, H., & Joutsenlahti, J. (2014). Prospective teacher's conceptions about a plane angle and the context dependency of the conceptions. (p. 5). In P. Liljedhal, C. Nicol, S. Oesterle, & D. Allen (Eds.), *Proc.38th Conf. of the Int. Group for the Psychology of Mathematics Education Vol. 5* (pp. 185-192). Vancouver, Canada: PME.

Questions 2 & 3 asked respondents to consider the angle representation in Figure 1, then to indicate which of the points A to I are part of angle a, and which are not. Question 2 requires each point A-I to be dragged and dropped into one of three possible boxes, being "belongs to Figure 1", "does not belong to Figure 1" and "not sure". In question 3 respondents are asked to drag all the points in Figure 1 into "does" or "does not" boxes, reallocating any of the points they were unsure of in question 2 using a best guess. These questions are derived from a study by Silfverberg and Joutsenlahti (2014) of 191 Finnish university students studying to be teachers.

FINDINGS

Responses to the concept definition question are shown in Table 1. Seventy two percent of teachers gave a written definition which includes one or more of the three angle concepts described by Mitchelmore and White (1998). This students in the Silfverberg and Joutsenlahti (2014) study had difficulty defining angle, the study did not report statistically on concept definitions.

Concept definitions for angle			Number	%
Classification		Example		
Static	Geometric	"A figure which is formed by two rays or lines that share a common end point is called an angle."	4	9
	Geometric & region	"An angle is the space made between two rays that meet at a point."	7	16
Dynamic	Turn only	"(angle is) measure of turn."	6	14
	Turn & (static) geometric	"Measurement of the turn between two lines that meet at a point."	14	33
No formal angle definition used in teaching			5	12
Question not answered			7	16
Total			43	100

Table 1: Teacher definitions for angle

The most common definition (33% of responses) involves the dynamic notion of turn combined with the static geometric concept of angle. Example definitions for each of the three angle categories are shown in Table 1. Twenty five percent of respondents define angle solely in static terms and 47% use a dynamic definition. Six (14%) of responses were dynamic definitions that excluded any reference to static angle concepts. Nine (21%) of respondents included a specific unit (degrees) in their definition for angle.

The findings from questions 2 & 3 for concept image in relation to Figure 1 are shown in Table 2. To answer these questions, it is necessary to mentally construct a concept image for a plane angle and to decide which of the points in the illustration lie within the boundaries of that image. The collection of points chosen by each respondent are then used to categorise their *enacted concept image* when answering the question. Enacted concept image is the portion of their total concept image that respondents engaged while answering. Answers are categorised into four concept images for angle shown in Table 2. If points C, E, F only were chosen, it is argued that a concept image involving the two arms of the angle as line segments was enacted and the answer was classified as geometric, finite. If points C, E, F, I only was chosen the answer was classified as geometric infinite, as the angle arms were argued to be seen as rays. The set of points C, D, E, F was classed as regional finite because point D is part of the region between the lines of the angle. C, D, E, F, I indicate an infinite region.

Nine respondents (21%) gave a set of points which could not be classified, and 8 (19%) did not attempt the question. Twenty (46%) respondents enacted a geometric concept image for angle involving two arms and a vertex, 14% enacted a regional concept image. Ten of the 20 respondents with a geometric

enacted concept image included that concept in their definition for angle. The remaining 10 gave a solely turn based definition (in 3 cases), had no definition for angle (in 4 cases), or gave a regional concept definition (in 3 cases).

Teachers' concept image for angle			Finnish study		Australian study	
			No.	%	No.	%
No answer attempted					8	19
Answer not classified			1	6	9	21
Geometric image	Finite	C, E, F	56	32	9	21
	Infinite	C, E, F, I	66	38	11	25
Regional image	Finite	C, D, E, F	4	2	1	2
	Infinite	C, D, E, F, H, I	47	27	5	12
Total			174	100	43	100

Table 2: Teacher concept image for angle

There were 10 cases in which a finite concept image was enacted, these used the infinite terms "line", or "ray" to describe the arm of an angle when providing concept definitions.

DISCUSSION

Most teachers in this survey (72%) gave a clear written definition for angle. However, the teaching of angle using the concept image centered approach proposed in this study involves a clear and explicit concept image for angle, and a multifaceted definition for it. This is more complex and integrated than the angle conceptualisations expressed by many participants in this survey. For example, only 33% of respondents defined angle in both static and dynamic terms. Mitchelmore and White (1998) argue that both concepts for angle are needed, such as when developing an ability to measure angle as a rotating radius in students. There is an apparent disconnect between concept image and definition for angle in many cases. For instance, the 50% of respondents who enacted a geometric concept image for angle without including that concept in their angle definitions. Mitchelmore and White describe "some relationship" between the lines of an angle in their definition for it, and they encourage the exploration of the underlying nature of angle in teaching. The use of a specific unit (degrees) by 21% of respondents in their angle definitions is problematic in that it may discourage the use of different formal and informal measurement units when exploring the relationship between the lines of an angle.

References

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