LESSON

10

Stars, Circles and Mystic Signs

Years 4 to 9

Creating Patterns with Circles

This lesson requires students to investigate various figures, including the Mystic Rose and star polygons, which can be formed using equally spaced points on the MATHOMAT circles.

In this lesson students will:

- discover properties of polygons inscribed in a circle;
- select, use and communicate strategies for solving a non-routine problem; and
- use geometrical tools to make accurate drawings.

Materials Required

For each student:

- a MATHOMAT;
- unlined paper (scrap paper will do);
- extra-fine-point pens or very sharp pencils; and
- an eraser.

Additional materials:

- one overhead projector version or one photocopy for each group of students of Handout 10.1, Stars, Circles and Mystic Signs (a photocopy master is included at the end of this lesson); and
- (optional) one overhead projector version or one photocopy for each group of students of Handout 10.2, Other Designs Using Points on Circles (a photocopy master of this is also included).

Lesson Summary

- Identifying the number of equally spaced points on various MATHOMAT circles;
- investigating the number of lines in Mystic Rose patterns for different numbers of points;
- producing star polygons and investigating their properties; and
- (optional) creating other attractive designs using points on a circle.

For the Teacher

In this lesson, students use the equally spaced marks on various MATHOMAT circles to construct and investigate figures such as the Mystic Rose pattern (see Handout 10.1, Stars, Circles and Mystic Signs) and star polygons. In order to choose appropriate circles to create their designs, students will need to begin by finding the number of points marked on the various MATHOMAT circles (refer also to Lesson 7).

The Mystic Rose pattern is produced by marking points at equal intervals on the circumference of a circle and joining each point to every other point. Examples for 5 and 18 points are provided on Handout 10.1, Stars, Circles and Mystic Signs, at the end of this lesson. In this lesson, all students produce a 3 and 4 point Mystic Rose pattern. Depending on their year level and construction skills, each student also produces another Mystic Rose pattern with 5, 6, 8, 10 or 12 points. Students can then investigate the number of lines in a Mystic Rose pattern and attempt to find a way of calculating this number without needing to count. Older students can generalise their results for 100 points, or even for n points, where n is any natural number.

All regular polygons can be inscribed in a circle, with their vertices producing equally spaced points on the circumference. So, for example, the vertices of a regular pentagon form 5 equally spaced points on the circumference of a circle. This means that one way of drawing a regular pentagon is to take 5 equally spaced points on the circumference of a circle and join each point to the 'next' point.

Star polygons are produced by joining every second (or third, or fourth, etc) point. So, for example, the pentagram on Handout 10.1, Stars, Circles and Mystic Signs can be produced by starting with 5 equally spaced points on the circumference of a circle and joining every second point. Similarly, the hexagram on the same sheet can be produced by starting with 6 equally spaced points and joining every second point. In this case (because 2 is a factor of 6) the hexagram will need to be produced using two separate lines, each joining 3 points. Star polygons can easily be identified by using the symbol (n,m) to indicate that they are formed by starting with n equally spaced points and joining every m-th point. So the pentagram has the symbol (5, 2) and the hexagram the symbol (6, 2). Older students can investigate questions such as 'When do two star polygons look the same?' and 'How can we tell in advance how many separate lines will be needed to draw a star polygon?'

Other attractive patterns can also be formed using equally spaced points on a circle. A few of these are illustrated on Handout 10.2, Other Designs Using Points on Circles. Students can copy these or create their own.

An extra-fine-point pen or very sharp pencil is essential for patterns based on a large numbers of points.

Lesson Outline

1. Identifying the number of equally spaced points on MATHOMAT circles

This lesson requires students to use equally spaced points on the various MATHOMAT circles. In order to produce neat designs, it is preferable to use the largest circle possible in each case—therefore of the circles with 4 equally spaced marks (circles 13, 14, 28, 30 and 35) only circle 28 will be used. However, a balance needs to be struck between using larger circles and ones where it is easy to mark the required number of equally spaced points.

Ask students to work in groups or individually to investigate the possible numbers of equally spaced points which can be marked on the various MATHOMAT circles. Each student should complete a table like the one shown below.

Circle number	Number of equally spaced points	Numbers of equally spaced points possible
2	10	2, 5, 10
3	60	2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60
15	32	2, 4, 8, 16, 32
28	4	2, 4,
29	100	2, 4, 8, 16, 32

2. Drawing the Mystic Rose pattern

The Mystic Rose pattern is produced by marking points at equal intervals on the circumference of a circle and joining each point to every other point. Examples for 5 and 18 points are provided on Handout 10.1, Stars, Circles and Mystic Signs at the end of this lesson. Show the class these on an overhead projector or a photocopy can be provided for each group.

Ask students to produce their own 3 and 4 point Mystic Rose patterns using circle 3. Depending on their year level and construction skills, students should also produce one or more Mystic Rose pattern using 5, 6, 8, 10 or 12 points (using circles 2, 3, 15, 29 or 3 respectively).

It is a good idea to check that students have the correct number of equally spaced points before they start drawing the lines.

3. The Mystic Rose challenge

The challenge for students is to investigate the number of lines in a Mystic Rose pattern—their own and perhaps the ones on Handout 10.1, Stars, Circles and Mystic Signs.

Ask older students to generalise their results for 100 points, or even for n points, where n is any natural number.

For the 3, 4 and 5 point Mystic Rose patterns, it is easy to count the number of lines, but for larger numbers of points students will need to find a way of calculating without counting.

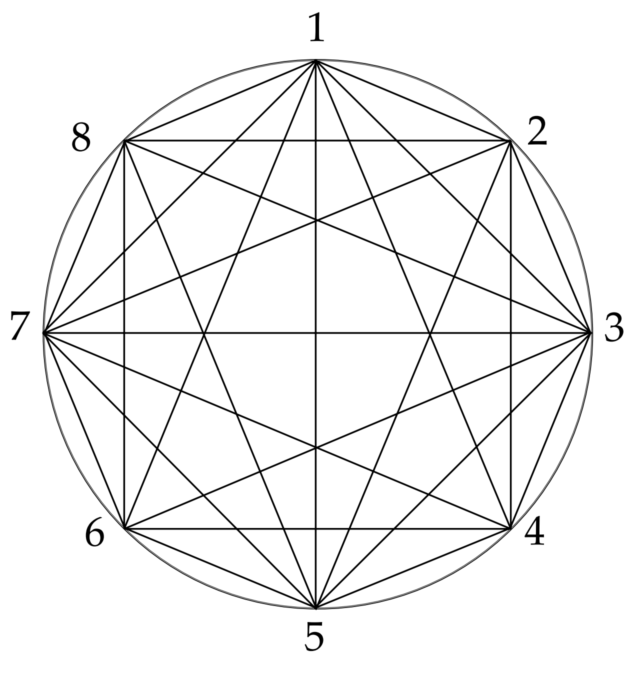
Allow students to find and use their own solution strategies. This is likely to take some time—perhaps the best part of a lesson—but it can lead to a very valuable class discussion afterwards.

At the end of this part of the lesson, ask students to share their solution strategies with the class. Those students who were able to generalise their results to large numbers can explain how they did this. Different strategies can be compared in terms of how easy they are to generalise and how easy they are to explain.

Possible strategies

A number of possible strategies which students might use are illustrated below, using the 8 point Mystic Rose as an example.

Mystic Rose pattern for 8 points



Strategy 1: Counting the number of lines—version 1

Point 1 is joined to 7 other points
Point 2 is joined to 6 other points (not counting point 1)
Point 3 is joined to 5 other points (not counting points 1 & 2)
...
Point 7 is joined to 1 other point
Point 8 is joined to 0 new points
So there are 7+ 6 + 5 + 4 +3 + 2 + 1 = 28 lines.

While this is a good method, it is rather difficult to generalise to larger numbers of points. However, a positive aspect of using this solution strategy is that it can lead to students finding their own creative ways of adding lists of consecutive numbers. For older students, this can be used as an introduction to arithmetic progressions.

Strategy 2: Counting the number of lines—version 2

Each point is joined to 7 other points, so there are (8 x 7) ÷ 2 = 28 lines. We need to divide by 2 since each line joins two points and, if we just used 8 x 7 = 56, we would count each line twice.

This is a neater method and generalises easily to finding that there are (100 x 99) ÷ 2 = 4950 lines for 100 points and (n x [n–1]) ÷ 2 lines for n points.

Many students, however, start by using this method, but abandon it after finding it gives the wrong answer when they forget to divide by two.

This provides an excellent opportunity for students to find a way to 'check their answers'—it can be quite difficult to persuade students that true checking involves finding some other way to confirm their answer than by 'asking the teacher' or just repeating the arithmetic! This problem can be used to introduce students to the valuable strategy of checking their solution method by testing it on a simpler case.

Strategy 3: Looking at simpler cases and finding a pattern

Students can find the number of lines for 2 points, 3 points, etc, and make a table like the one below in order to look for a pattern.

Number of Points	2	3	4	5	6	7	...
Number of Lines	1	3	6	10	15	21	...

+ 2   + 3   + 4   + 5   + 6   ...

Filling in a few entries of the table usually leads to students finding a pattern. However, it is important that students should not be satisfied with just finding a pattern but should understand the need to find a 'reason why it works'. In this case, each time you add a point it will need to be joined to all the existing points. So, for example, adding the eighth point will add 7 new lines, since it will need to be joined to each of the previous 7 points, and, in general, adding the n-th point will create an extra (n–1) lines.

While looking at simpler cases is a valuable problem solving strategy for students to use, this strategy is again difficult to apply to larger Mystic Rose patterns or use to provide a general answer for the number of lines.

4. Investigating star polygons

The pentagram

All regular polygons can be inscribed in a circle, with their vertices producing equally spaced points on the circumference—for example, the vertices of a regular pentagon form 5 equally spaced points on the circumference of a circle. So one way of drawing a regular pentagon is to take 5 equally spaced points on the circumference of a circle and join each point to the 'next' point.

Ask students to make three copies of circle 29 and mark 5 equally spaced points on each, starting at 'the top'.

Now ask students to draw a pentagon on the first of their circles by joining adjacent points.

Star polygons are produced by joining every second (or third, or fourth, etc) point.

Ask students to use their second copy of circle 29 to produce a star polygon by joining every second point—i.e. by joining the points in the following order: 1 – 3 – 5 – 2 – 4 – 1.

This star polygon is called the *pentagram* and has the symbol (5, 2)—meaning that it was formed by using 5 equally spaced points and joining every second one.

Ask students to use their third circle to draw the star polygon (5, 3)—i.e. the one obtained by joining every third point. Why does it look the same as the pentagram? What would happen if they drew (5, 4)? What would be a good symbol to use for the pentagon?

Students may be interested to know that, during the time of Pythagoras, a secret society or brotherhood called the Pythagoreans took the pentagram as their badge or symbol by which any member was able to immediately recognise a fellow member. The pentagram was also regarded as a symbol of health.

(Optional) ask students to make their own copy of the nested pentagram shown on Handout 10.1, Stars, Circles and Mystic Signs, using their drawing of the star polygon (5, 3)—each new pentagram can easily be produced by joining every second point of the pentagon formed at the 'centre' of the previous one. Of course, in theory, this process can be repeated infinitely!

An amazing property of the infinite set of nested regular pentagrams is that the length of every line segment is in the Golden Ratio to the length of the next smallest segment. This fact was well known to the Pythagoreans.

The hexagram

Ask students to produce the star polygon (6, 2) by using 6 equally spaced points on circle 3. This six-pointed star, which is called the *hexagram*, will probably be familiar to students as the Star of David. Like the pentagram, it was widely used in the past to ward off evil.

Two other ways of producing the hexagram using the MATHOMAT are shown on Handout 10.1, Stars, Circles and Mystic Signs. The first of these methods uses shape 23 to produce two interlocking triangles. Because the lines drawn are relatively thick, it is probably best to draw the hexagram by tracing one of the triangles and positioning the other 'by eye'—using the marks for the mid-points to position the second triangle results in an unbalanced design. The second quick way to draw the hexagram is to draw the hexagon (shape 9) and, working clockwise around it, extend each of the sides sufficiently to form the six-pointed star. Demonstrate these ways to the class.

The 'magic hexagram' on Handout 10.1, Stars, Circles and Mystic Signs was formed by joining the opposite points of a hexagram and numbering all of the points and intersections in a 'magic way'. Challenge students to find its magical properties.

Other star polygons

Ask students to use circle 13 or 24 to investigate star polygons with 10 points. How many different shapes can be produced using star polygons with 10 points? Which star polygons with 10 points look the same? Which star polygons with 10 points need more than one continuous line to produce? How many lines do they need? Why? What is 'different' about the star polygon (10,5)?

Ask older students to make conjectures about star polygons with different numbers of points (e.g. 15, 16, 100, n) without drawing them and attempt to answer general questions such as 'When do two star polygons look the same?', 'How can we tell in advance how many lines will be needed to draw a star polygon?' and 'What other star polygons would look similar to the star polygon (10, 5)?'

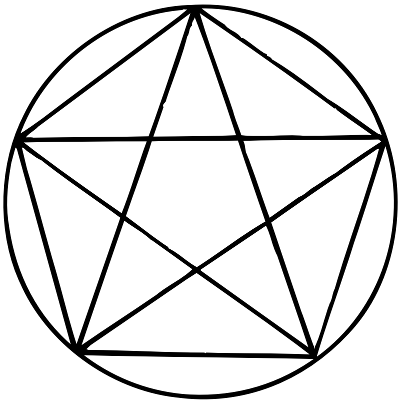
5. (Optional) creating designs using polygons in circles

Handout 10.2, Other Designs Using Points on Circles, illustrates a number of attractive designs which can be produced using equally spaced points on a circle.

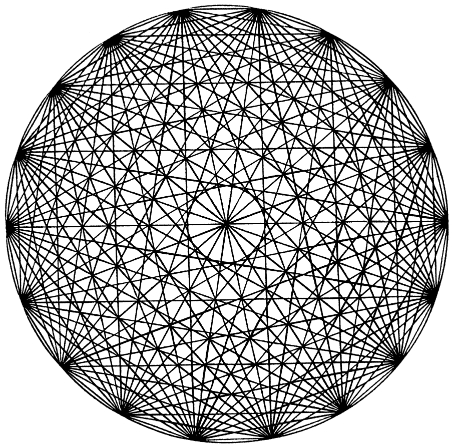
To complete this lesson, ask students to copy these or create their own designs. A good quality presentation can be achieved by drawing the construction lines with a soft pencil, completing the design with a waterproof pen and finally erasing unwanted construction lines.

Stars, Circles and Mystic Signs

THE MYSTIC ROSE

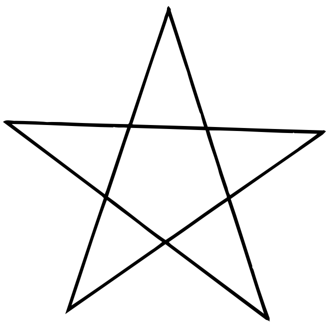


The 5 point Mystic Rose pattern

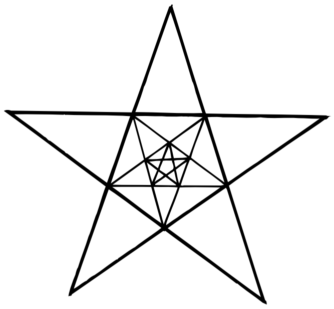


The 18 point Mystic Rose pattern

PENTAGRAMS

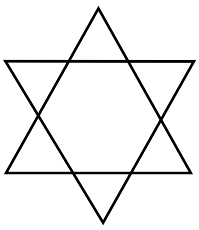


The pentagram

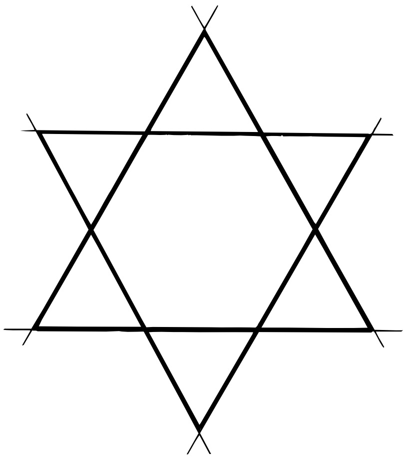


Nested pentagrams

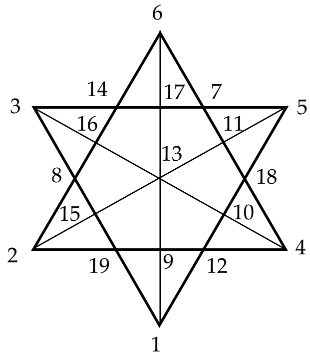
HEXAGRAMS



The hexagram



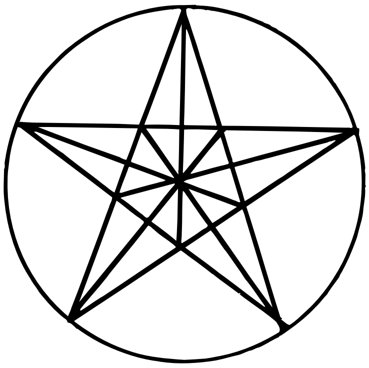
Another way of drawing
a hexagram

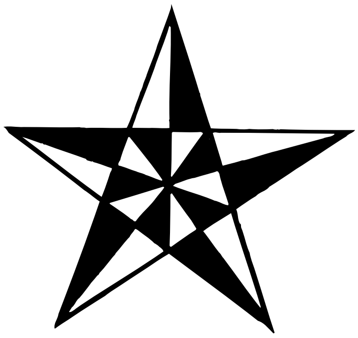


The magic hexagram

Other Designs Using Points on Circles

A DESIGN BASED ON THE PENTAGRAM

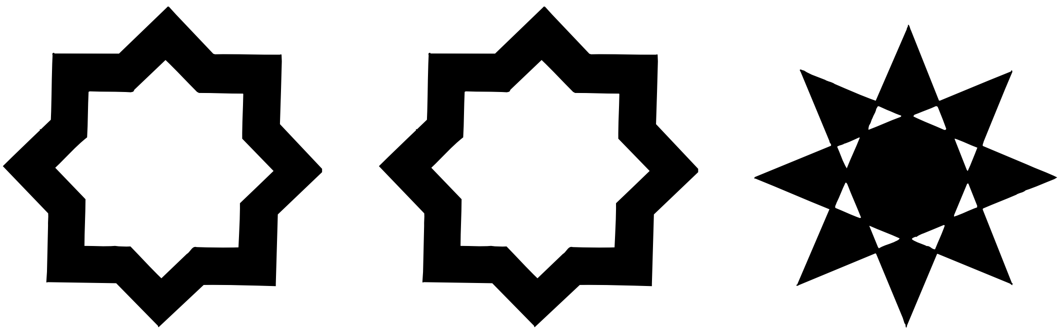




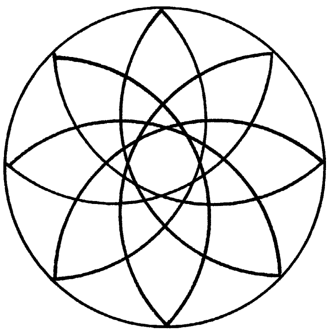
Use a circle with 5 equally spaced points to construct a pentagram and join the points as shown.

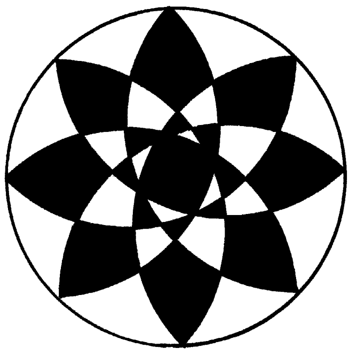
Carefully erase the circle and shade the regions as shown.

SOME OTHER DESIGNS BASED ON REGULAR POLYGONS



A DESIGN BASED ON CIRCLES





Use a circle with 8 equally spaced points and join every third point with an arc of the circle as shown.

Carefully shade the regions as shown.