

LESSON

11

Unwrapping the Circle

Years 9 to 12

Constructing the Sine Graph

This lesson introduces students to the number π and its location on a number line. Students then use the unit circle definition of the sine of an angle to construct a sine graph.

In this lesson students will:

- identify π as a number;
- locate multiples of π on a number line;
- define the sine of an angle; and
- construct the graph of the sine function.

Materials Required

For each student:

- a MATHOMAT;
- a photocopy of Worksheet 11.2, Measuring the Sines of Angles, (a photocopy master is included at the end of this lesson);
- unlined paper (scrap paper will do);
- a fine-point pen or pencil;
- access to a scientific calculator or spreadsheet; and
- graph paper.

Additional materials:

- an overhead projector and coloured pens;
- a photocopy for each group of students of Worksheet 11.1, Estimating π (a photocopy master is included at the end of this lesson);
- a collection of circular objects of various radii (plates, jars, bike wheel, etc); and
- a transparency of each of the diagrams in Transparencies 11.1, 11.2, and 11.3 (a photocopy master is included at the end of this lesson).

Lesson Summary

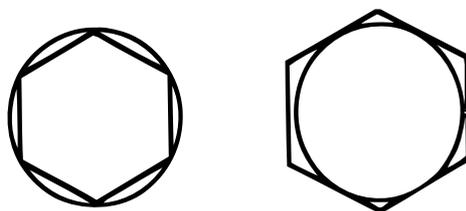
- Finding approximate value of π using circular objects;
- measuring distance around the unit circle;
- using radians to measure angles
- using the MATHOMAT to produce an 'unwrapped' circle;
- defining the sine of an angle;
- drawing the sine graph; and
- estimating the sines of angles.

For the Teacher

Students need a thorough understanding of the number π before they are introduced to the sine function and its graph. Included among students' common misconceptions of the nature of π are that it is 'part of a circle' and 'another way of measuring angles'. It is essential that students understand that π is a number that can be represented on a number line.

The number π has a fascinating history. In the Bible, in the Book of Kings and Chronicles, the value of π is given as 3. Ancient Egyptians used the more accurate value of 3.16—their geometrical investigations were triggered by the regular flooding of agricultural land by the Nile. By 150 AD the value of π was assumed to be 3.1416.

Archimedes, the Greek philosopher and mathematician, was fascinated by π . He devised a method of calculating its value by inscribing and circumscribing polygons about a circle; the area of the circle lies between that of the inscribed and circumscribed hexagons—its perimeter lies between that of the two hexagons.



The calculation of the value of π achieved almost cult status and was the goal of many prominent mathematicians. In the late 16th century a German mathematician, van Ceulen, calculated π to 36 decimal places and requested that this value be inscribed on his tombstone. In 1824 its value was calculated by Gauss to 200 places and in 1854 by Shanks to 707 places, after 10 000 hours of endeavour.

This lesson begins with a revision of the concept of π and an introduction to the concept of radian measure. From this foundation MATHOMAT shapes are used to construct diagrams that demonstrate the relationship between the unit circle and the number line on which the circle is 'unwrapped'. From here the definition of the sine of an angle follows and the graph is drawn as a natural consequence.

Before the lesson begins make overhead transparencies of the seven diagrams provided in Transparencies 11.1, 11.2 and 11.3. Diagram 1 on Transparency 11.1 includes a face. Cut this from the transparency for use as an independent entity—it will be used to illustrate key aspects of a ride on a ferris wheel.

A toy ferris wheel is a useful discussion aid—ask interested students to make one from a construction kit before the lesson.

A number of sessions will be required to complete all of the activities in this lesson.

Lesson Outline

1. Estimating the value of π

Although most students can recite the deceptively simple formula $C=2\pi r$, a thorough understanding of what it and its components really mean is necessary before using the unit circle to introduce trigonometric functions.

Initial discussions and demonstrations with real circles (bike wheels, cans) should establish that π is a number 'a bit bigger than 3'—the activities to achieve this are described in the following paragraphs.

This part of the lesson works best if students are organised into groups of five or six.

Ask each group to measure the radius and circumference of each of a number of circular objects, enter their measurements into Worksheet 11.1, Estimating π , and calculate the ratio $C/2r$, using a calculator or spreadsheet.

Circular object	Circumference, C	Radius, r	$C/2\pi$
Plate			
Jam jar			
Bike wheel			
Trundle wheel			

An efficient method of measuring circumferences is by wrapping string around the circular objects; another is by rolling the objects for exactly one revolution on a flat surface. Since each of these methods relates directly to concepts developed later in the lesson, students should experience them both.

Use class discussion to compare different groups' methods and their results. A final value of π should be agreed upon—this might be the mean of all calculated values.

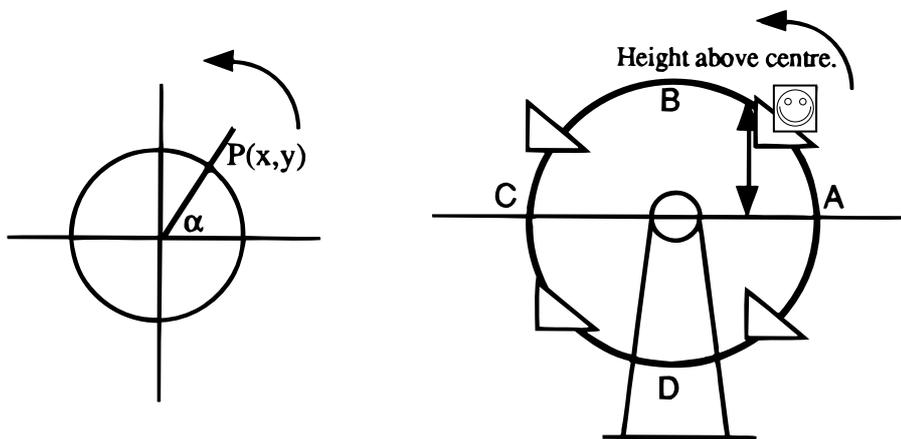
As a closing activity for this part of the lesson, ask students to graphically summarise the information in their tables by plotting C (vertical axis) against r (horizontal axis) and discuss the outcome.

Present a brief account of the history of π and its value (3.14159265) to several decimal places.

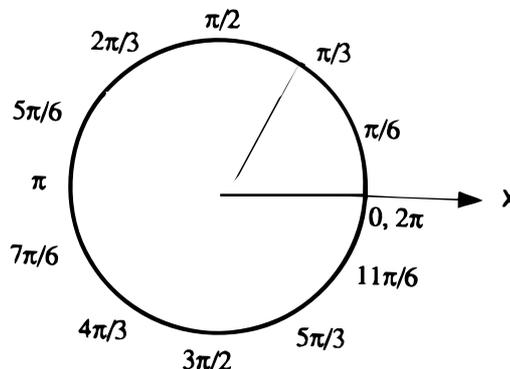
2. Measuring distances around the unit circle

Introduce the idea of a point moving around the circumference of a circle using the model of a person's ride on a ferris wheel. Use the transparencies of the ferris wheel and the face, together with coloured pens, to illustrate key aspects of the person's ride as mentioned in the next paragraph. A toy ferris wheel would also aid discussion.

At any instant the person's height above or below the axis can be estimated and the person's journey should be discussed in terms of height, the distance travelled along the circumference and its repetitive aspects. For example: how far has the person travelled in moving from A to B; from A to C; from A to D; from A to A?



Revise the concept of the angle of rotation, conventionally measured anti-clockwise from the positive x axis, in terms of the person's ride on the ferris wheel. For example: what is the angle from A to D; what is the angle of each of the four seats on the wheel in the diagram? Now guide discussion from the ferris wheel to a similar discussion of a point moving around a unit circle. Emphasise that if the radius of a circle is 1 then the length of the circumference is 2π , the length of the semi-circle is π , the length of the arc of a quarter of the circle is $\pi/2$ etc. The diagram below (Diagram 4, Transparency 11.2) summarises significant distances around the circumference of the unit circle. In the next part of the lesson students produce copies of this diagram.



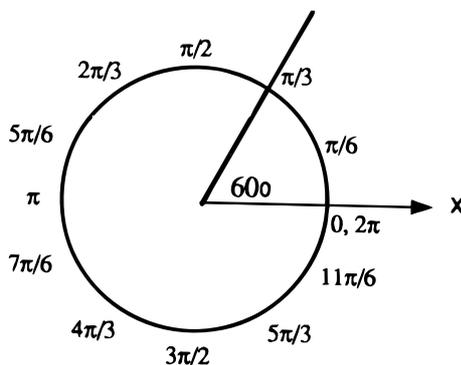
3. Using radians to measure angles

The idea of the radian being another way of measuring angles is introduced in this section.

Use Diagram 3, Transparency 11.2, to show that a string wrapped around part of the circumference of a unit circle subtends an angle at the centre of the circle.

Explain that if the length of the string is equal to the radius of the circle then the angle is, by definition, one radian. Extend this to 'other pieces of string': if the string has length $\pi/2$ (a quarter of the circumference) then the angle is $\pi/2$ radian and similarly for each of the 'string lengths' marked on the circle below (Diagram 5, Transparency 11.2).

Ask students to use MATHOMAT shape 3 to draw a unit circle and accurately reproduce the diagram below: emphasise that each of the 12 evenly spaced points on the circumference has associated with it an arc whose length is measured from the x-axis, and an angle subtended at the centre of the circle. (Each gradation on MATHOMAT shape 3 is 6° . (See Lesson 7, part 4, for further discussion of dividing circles into equal parts.)

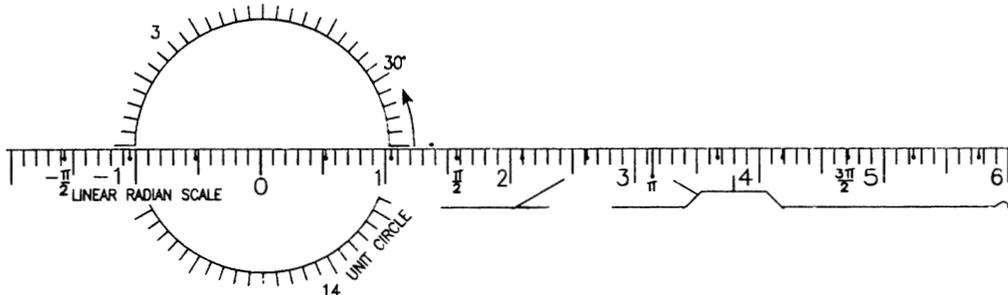


Ask students to calculate the angles associated with each point marked on the circumference of the circle and use these to complete the second and third columns of the table in Worksheet 11.2, Measuring the Sines of Angles.

4. Using the MATHOMAT to produce an 'unwrapped' circle

MATHOMAT shape 3 is designed to be used as a unit circle in association with the linear radian scale on the right hand side of the MATHOMAT. The diagram below shows how one unit on this scale is equal to the radius of the unit circle, shape 3.

The radius of the circle is the unit of the linear scale.



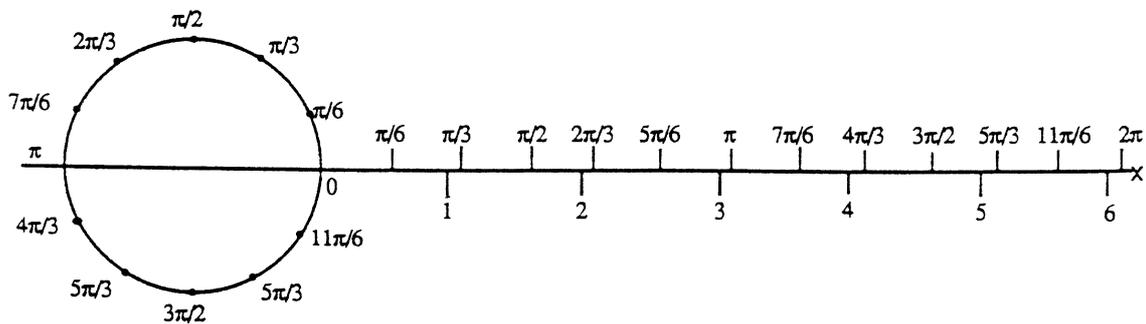
Students must understand the relationship between these components of the MATHOMAT if they are to use them in the remainder of the lesson.

Discuss with students the fact that integer units on the linear scale are labelled, as are the positions of $-\pi$, $-\pi/2$, $\pi/2$, π , $3\pi/2$ and 2π . Indicate that if the circumference were to be ‘unwrapped’ and laid along the linear scale then correspondingly labelled points would exactly match.

Also marked on the linear scale, but not explicitly labelled, are points that are multiples of $\pi/6$. Identify these as they allow a direct mapping of the circle onto the scale.

Ask students to use MATHOMAT shape 3 and the linear radian scale to produce their own accurate copy of the following diagram—they will use their copies of this diagram in part 7 of the lesson.

Multiples of π on the unit circle and on the corresponding linear scale



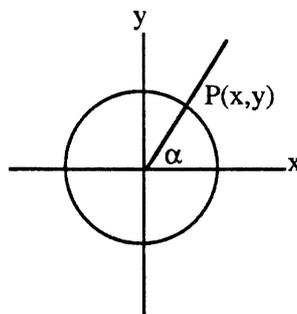
Two important points to emphasise in discussion are that:

- the linear radian scale is a number line with labels for fractions of π as well as the integers; and
- the unit length on the linear scale is equal to the radius of the circle, and therefore the scale can be considered an ‘unwrapping’ of the circle.

5. Defining the sine of an angle

Corresponding to every point $P(x,y)$ on the unit circle there is an angle α .

Point $P(x,y)$ on the unit circle and the corresponding angle α



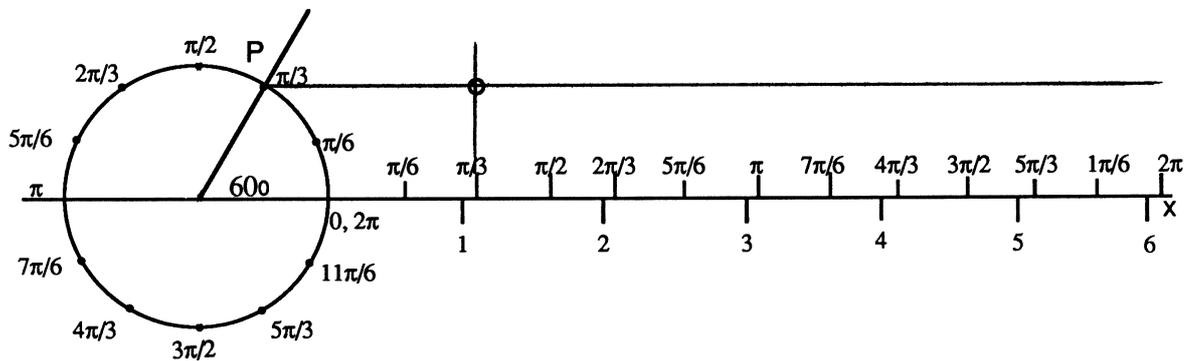
The sine of the angle α is defined as y , the height of P above the x -axis.
So, by definition:

$$\sin \alpha = y.$$

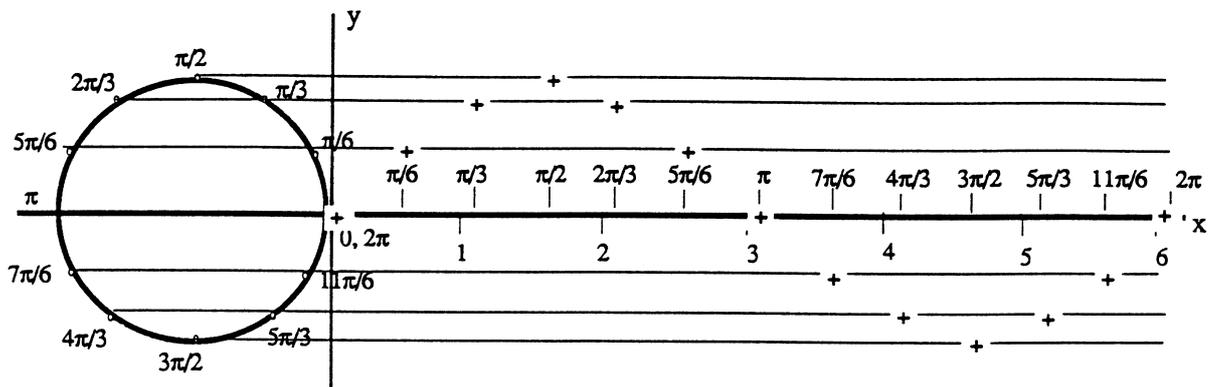
Illustrate how 'y' varies by selecting, on the transparency of the ferris wheel or the unit circle, a variety of angles in the range 0° to 360° . Do not give precise numerical values at this stage; however, the increasing and decreasing nature of y and its minimum and maximum values and periodicity should be discussed.

6. Drawing the sine graph

The following diagram illustrates how to plot points for the sine graph. The point shown is obtained as the intersection of the line drawn vertically from $\pi/3$ on the x -axis with the line drawn horizontally from $\pi/3$ on the unit circle.



Ask students to use the MATHOMAT to draw additional horizontal lines on the diagrams of the 'unwrapped' circles they produced in part 4 of this lesson, and then plot points corresponding to the sines of each of the angles marked on the rim of the circle.



Ask the students to carefully join the points with a smooth curve to obtain a sine graph.

7. Estimating the sines of angles

To obtain a numerical estimate of the sine of an angle, the angle can be drawn onto the unit circle and 'y' measured directly with the MATHOMAT linear scale.

As a final activity, ask students to obtain numerical estimates for the sines of each of the angles in the table in Worksheet 11.2, Measuring the Sines of Angles, and record these in column 4 of the table. They can compare these values with those obtained from their calculators, which they enter into the fifth column of the table.

Estimating π

Circular object	Circumference, C	Radius, r	$C/2\pi$
Plate Jam jar Bike wheel Trundle wheel			

Measuring the Sines of Angles

Distance round circle	Angle		Sine of angle	
	degrees	radians	measured	calculator
0				
$\pi/6$				
$\pi/3$				
$\pi/2$				
$2\pi/3$				
$5\pi/6$				
π				
$7\pi/6$				
$4\pi/3$				
$3\pi/2$				
$5\pi/3$				
$11\pi/6$				
2π				

Unwrapping the Circle Diagrams 1 and 2

Diagram 1

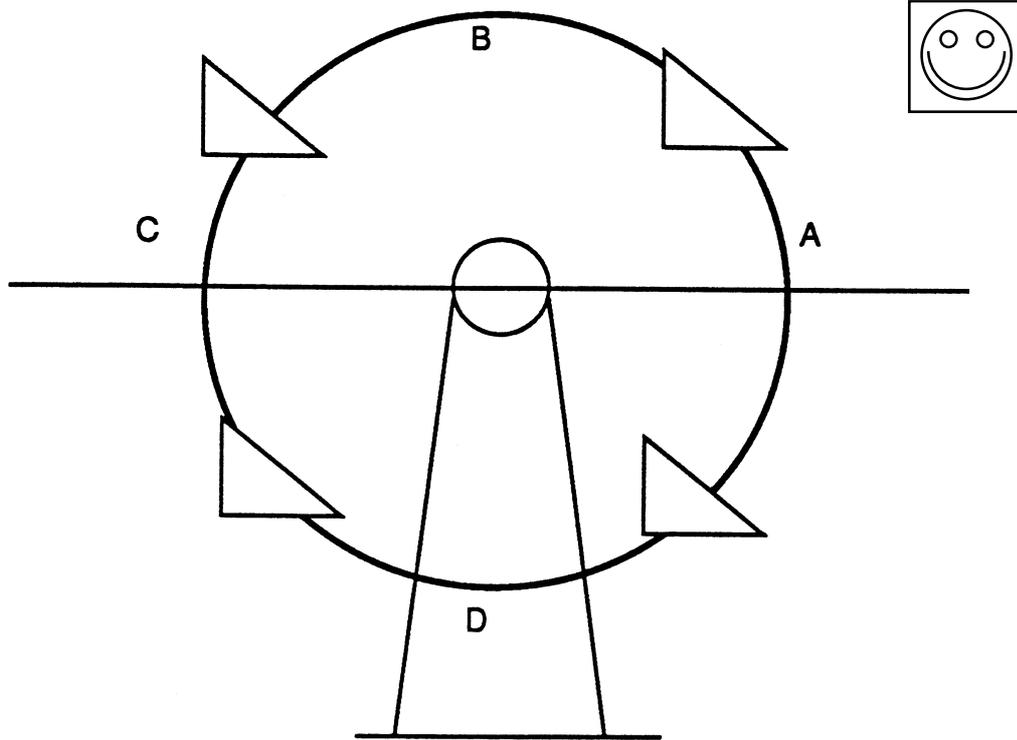
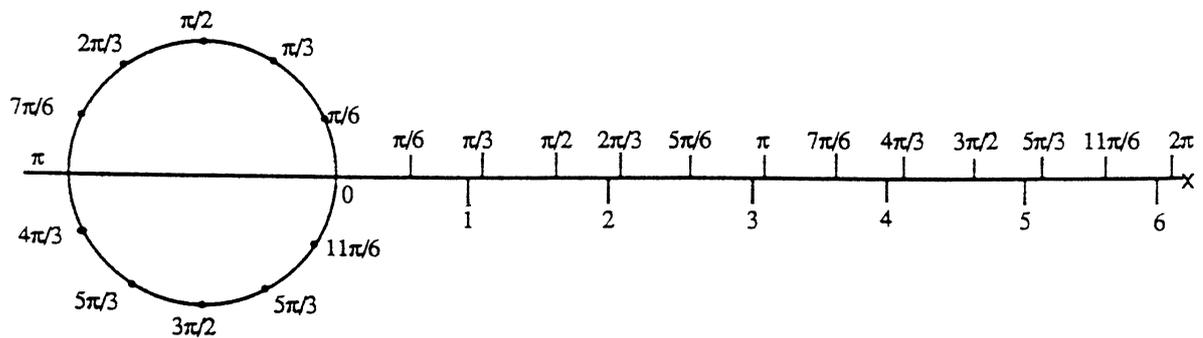


Diagram 2



Unwrapping the Circle

Diagrams 3, 4 and 5

Diagram 3

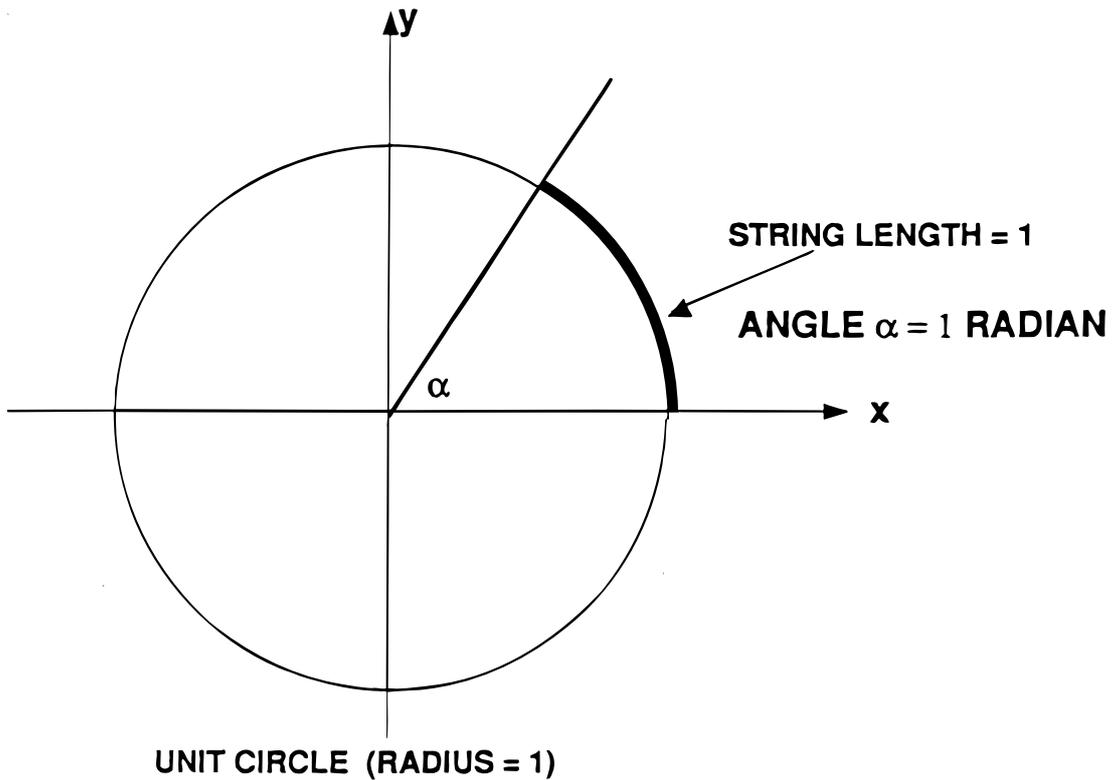


Diagram 4

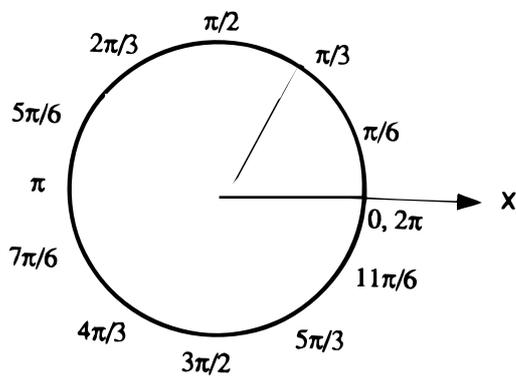
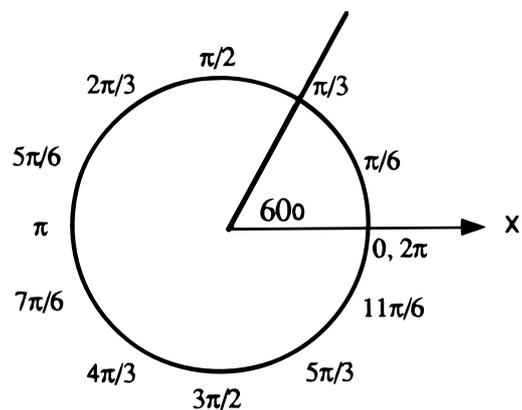


Diagram 5



Unwrapping the Circle Diagrams 6 and 7

Diagram 6

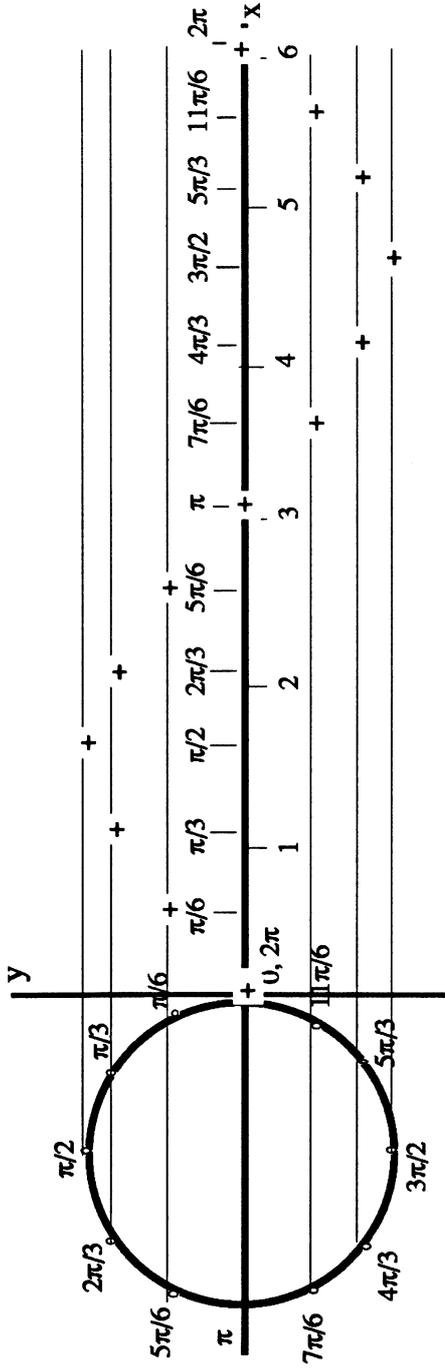


Diagram 7

