LESSON

9

Round and Round the Circle

Years 4 to 8

Rotational Symmetry

This lesson involves students in investigating rotational symmetry of various MATHOMAT and other shapes and using MATHOMAT shapes to create two-dimensional shapes and patterns with rotational symmetry.

In this lesson students will:

- investigate rotational symmetry in two-dimensional shapes.

Materials Required

For each student:

- a MATHOMAT;
- unlined paper (scrap paper will do);
- a copy of the Worksheet 9.1, Rotational Symmetry in Complex Shapes (a photocopy master is provided at the end of this lesson);
- fine-point pens or pencils; and
- scissors.

Additional materials:

- one overhead projector version or photocopies for each group of students of Transparency 5.1, Lines of Symmetry which was used in Lesson 5.

Lesson Summary

- Identifying rotational symmetry in MATHOMAT shapes;
- classifying shapes according to the number of rotational symmetries;
- completing and constructing two-dimensional shapes and patterns with rotational symmetry;
- identifying rotational symmetry in more complex shapes; and
- (optional) having a brief discussion on the finite symmetry groups of two-dimensional figures.

For the Teacher

This lesson is an extension of Lesson 5, for older students, who may be interested in knowing more about *symmetries*. Students should have completed Lesson 5 before doing this lesson.

A *symmetry* of an object permutes its points while reproducing the same shape exactly. As well as using lines of symmetry to produce reflections, the only other transformations which can be used to transform a *finite two-dimensional* figure into itself are rotations.

Some of the figures on Transparency 5.1, Lines of Symmetry, which was used in Lesson 5 have one or more rotational symmetries. This sheet can either be reproduced on an overhead projector transparency for class discussion or photo-copied for groups of students to look at and discuss.

A regular pentagon, for example, has 10 possible symmetries—five reflections and the five rotations about its centre through 72°, 144° (= 2 x 72°), 216° (= 3 x 72°), 288° (= 4 x 72°), and 0° (which, although it leaves the pentagon 'unchanged' is still counted as a rotation).

In a similar way, any regular polygon with n sides has 2n symmetries—n reflections and n rotations. So, for example, a square has 2 x 4 = 8 symmetries—4 reflections and 4 rotations, while a regular dodecagon (12 sides) has 2 x 12 = 24 symmetries—12 rotations and 12 reflections.

A circle, of course, has infinitely many reflections and rotations.

The full set of symmetries of an object form what is called its *symmetry group*. It turns out that every *finite two-dimensional* figure, apart from the circle—which should perhaps not be considered as finite here—has a symmetry group which consists of either n rotations only or n reflections together with n rotations, where n is some natural number. (Motivated maths students in the upper years may wish to know that the symmetry group consisting of n rotations is *called the cyclic group of order n*, written C_n—because it 'goes round in a cycle' and comes back to the start—while the symmetry group consisting of n reflections and n rotations is called the dihedral group of order 2n, written D_n—because it includes the reflections and so is 'two-faced').

This lesson follows a similar format to Lesson 5.

The lesson provides opportunities for students to explore the rotational symmetries of the MATHOMAT shapes and classify them according to the number of rotational symmetries. For some shapes, students can use the markings on the perimeter to locate the 'centre' and use the protractor to determine the angle of rotation.

The MATHOMAT shapes and curves include those which only have one rotational symmetry, the 'trivial' rotation through 0° (these are the non-equilateral triangles, the trapezium, the sin/cos curve, the parabola and the half dodecagon); those which have two rotational symmetries, the rotations through 0° and 180° (the ellipses and the non-square parallelograms); those which have three, four, five, six or eight rotational symmetries (the regular polygons with the corresponding number of sides); and the circles, which have infinitely many rotational symmetries.

Depending on the year level, students should be encouraged to attempt to describe their results from the classification activity in general terms like those given above. This can again, as in Lesson 5, form the basis for a valuable class discussion. In the case of the regular polygons, older students should be encouraged to explore the relationship between the number of sides, the number of rotations and the angles of rotation—i.e. the fact that the rotations for an n-sided regular polygon are always through angles which are multiples of $(360/n)°$.

Following the classification activity, students can use their MATHOMAT to construct and create attractive shapes and patterns with rotational symmetry, as well as identify rotational symmetries in more complex patterns.

For older students who are interested, the lesson can conclude with a discussion of the fact that the only finite symmetry groups of two-dimensional figures are C_n and D_n (see above). They can then go back and label each shape with its symmetry group, which then allows them to 'see' and recognise at a glance the types of symmetries.

Symmetry groups of three-dimensional objects have proved a powerful tool in crystallography, as well as playing a key role in modern theoretical physics. For example, a current first year university physics text (Ohanian, 1989) includes a section discussing not only three-dimensional symmetry but also an analysis of the repetitions in the (two-dimensional) work of M. C. Escher—see also Lesson 2 here. It states that

> the repetitive pattern of behavior of the atoms listed in the Periodic Table … can be traced to an underlying symmetry of the equations governing the motion of the electrons in the atoms. And the regular pattern of the behavior of the elementary particles in 'families' of particles … can be traced to a symmetry of the parameters describing the internal structure of these particles. Such abstract mathematical symmetries play a key role in physics. In fact, much of modern theoretical physics can be described as a search for symmetry. (p. I-1)

Lesson Outline

1. Introducing rotational symmetry

Use an overhead projector transparency or photocopy of Transparency 5.1, Lines of Symmetry to introduce a class discussion about rotational symmetry.

The four examples on the sheet include two figures, the mask and the Balmain Bug, which can only be reproduced by rotations through 0° (or other multiples of 360°)—i.e. they have only one rotational symmetry, the 'trivial' rotation.

The top right hand figure, however, has three rotational symmetries—the rotations about its 'centre' through 0°, 120° and 240°. This can be demonstrated on the overhead projector by reproducing the figure on an overhead transparency, cutting it out roughly, placing a mark of some kind at the top, and rotating it while asking students to say stop when it 'looks the same as at the start', ignoring the mark for the moment. This can be repeated until it 'returns to its original position' after three such rotations—the fact that it has returned to its original position is shown by the mark being again at the top. Alternatively, groups of students could cut out the figure and do this at their desks.

It is important to realise that the rotation must be about the centre of the figure. When working at their desks, students might wish to 'pin down' their figures at the centre with the point of a ball point pen.

Note that we usually disregard the fact that rotations of 480° (= 120° + 360°) or 840° (= 120° + 2 x 360°), etc, would have the same effect as a rotation through 120°.

Through class discussion or by working in groups, students can now discover that the bottom left hand figure has twelve rotational symmetries —the rotations through 0°, 30°, 60°, 90° …, 330°, since 30 = 360/12.

A class discussion can also include a brief investigation of the rotational symmetries of upper case letters of the alphabet—e.g. The letter A has only the trivial rotation through 0°, while N and Z have two rotational symmetries, the rotations through 0° and 180°.

2. Identifying rotational symmetries of the MATHOMAT shapes

Ask students to work individually or in groups to find the rotational symmetries for most of the MATHOMAT shapes and curves—shapes such as 13, 14, 26, 28, 30, 32 and 35, which are small and which are repeated in a larger size should be omitted.

If students work individually, they will need to be allocated shapes to test as testing all the shapes would take far too long. If students work in groups, they can decide how to share the shapes to be tested—a good way which results in a variety for each student would be to take every fifth shape in a group of five (e.g. the first student gets shapes 1, 6, 11, 18, etc to test, since 13 and 14 are omitted).

Each student can cut out the shapes to be tested and find the number or rotational symmetries for each one as was done in part 1 of this lesson. A good way to do this is to get students to draw the shape they are testing again, superimpose the cut-out shape, mark one vertex and the centre, 'pin down the centre' and rotate the shape until it fits the drawing again. Students can repeat the rotation until the shape returns to its original position, in order to find the number of rotational symmetries.

Students should also be encouraged to find the angles of rotation, either by measurement or by calculation. With experience, some students will be able to work out the number of rotations by reasoning such as: A regular pentagon has 5 equal sides and 5 equal angles, so it will have 5 rotational symmetries. In order to return to its original position it must rotate through 360°. So each rotation must be through an angle of (360/5)° = 72°.

3. Classifying MATHOMAT shapes according to the number of rotational symmetries

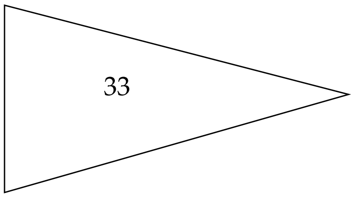
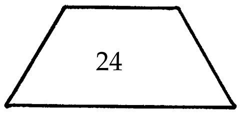
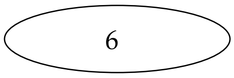
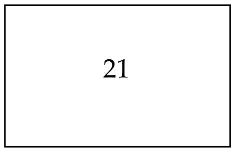
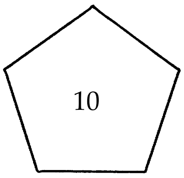
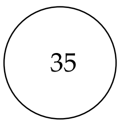
Ask students to work in groups to classify the MATHOMAT shapes according to the number of rotational symmetries.

As in Lesson 5, each group should be encouraged to record and display their results in preparation for a class discussion. Students can complete a table like the one shown on the following page.

Also as in Lesson 5, there are many different ways to produce such a table and students should be encouraged to plan how to display their results.

As soon as most groups have completed their recordings or displays, hold a class discussion to compare results and attempt to come to agreement on the types of shapes which have different numbers of rotational symmetries. This discussion can be valuable in developing students' ability to use conventional geometric language to describe classes of shapes at all of the year levels suggested for this lesson.

Sample table classifying MATHOMAT shapes according to numbers of rotational symmetries

Number of rotational symmetries	Shapes
1	 
2	 
3	
4	
5	
6	
8	
∞	

4. Identifying rotational symmetries in more complex shapes

Ask students to work individually or in groups to complete the Worksheet 9.1, Rotational Symmetry in Complex Shapes.

The first part asks students to find the number of rotational symmetries, while question two asks fo the number of lines of reflection (see Lesson 5) .

As before, students need to realise that when they are provided with photos of objects occurring in real life, it is the object not the photo which is being considered and also that real life objects won't be perfectly symmetrical!

5. (Optional) brief discussion of finite symmetry groups of two-dimensional figures

For older students who are interested, this portion of the lesson can conclude with a discussion of the fact that the only finite symmetry groups of two-dimensional figures are C_n and D_n (see For the Teacher). They can then go back and label each of the figures on the Worksheet 9.1, Rotational Symmetry in Complex Shapes with its symmetry group, which then allows them to 'see' and recognise at a glance the types of symmetries.

6. Creating symmetrical patterns

Ask students to create their own patterns with rotational symmetry using their MATHOMAT. Older students can be asked to produce patterns with specified numbers of rotational symmetries and/or lines of reflection. (Note that the number of reflections is always either zero or the same as the number of rotations.)

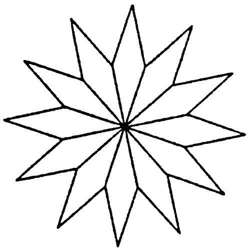
Reference

Ohanian, H. C. (1989). *Physics* (2nd edn). New York: W. W. Norton.

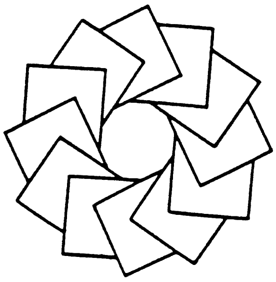
Rotational Symmetry in Complex Shapes

1. For each of the diagrams below, find the number of rotational symmetries and record your answer in the first space provided (R for rotations).

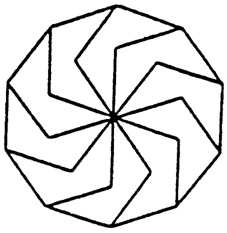
   a) MATHOMAT DESIGNS



R ☐ L ☐

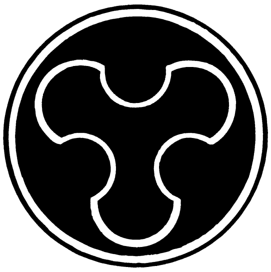


R ☐ L ☐

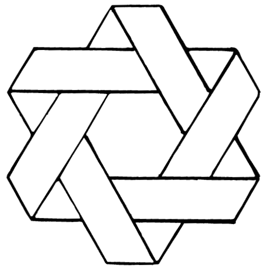


R ☐ L ☐

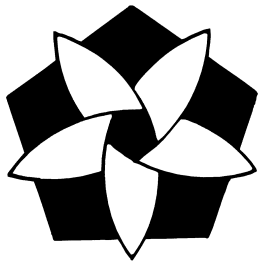
   b) OTHER DESIGNS



R ☐ L ☐

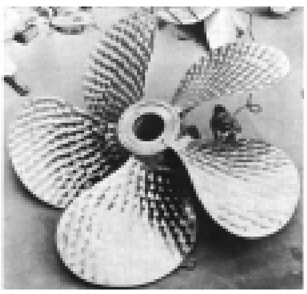


R ☐ L ☐

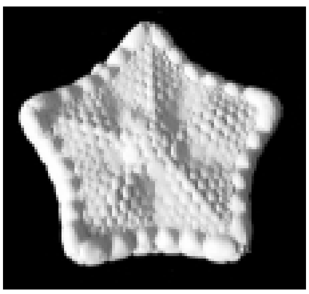


R ☐ L ☐

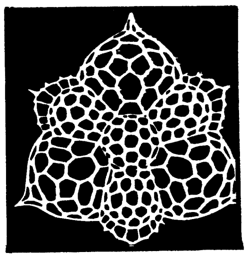
   b) REAL-LIFE OBJECTS—of course these will not be perfect!



R ☐ L ☐



R ☐ L ☐



R ☐ L ☐

2. Using the diagrams in question 1, find the number of lines of symmetry (reflectional symmetries) each has and record your answer in the second space provided (L for lines of reflection).