

LESSON

7

Straight Line Curves

Years 5 to 9

Constructing Envelopes of Curves

This lesson involves students using the number line on their MATHOMAT to construct envelopes for the parabola and, in the higher year levels, the ellipse and hyperbola. Students specify the 'rules' for drawing the straight lines in terms of either numeric or algebraic relationships.

In this lesson students will:

- use geometric instruments to make accurate constructions;
- construct envelopes of curves;
- identify various curves and investigate their properties; and
- devise rules to specify relationships in either numeric or algebraic form.

Materials Required

For each student:

- a MATHOMAT;
- unlined paper (scrap paper will do); and
- well sharpened pencils.

Lesson Summary

- Constructing the envelope of a parabola by joining points on perpendicular straight lines;
- investigating the effect of changing the angle of intersection of the lines;
- devising a numeric or algebraic 'rule' for joining the points;
- constructing envelopes of circles by joining evenly spaced points on a circle;
- investigating envelopes of other curves produced by joining points circles, intersecting and parallel lines; and
- (optional) creating patterns by joining points on a circle using arcs instead of straight lines.

For the Teacher

We usually think of a curve as being the *locus* of its set of points—i.e. the set of points which lie on the curve.

However, we can also think of a curve as being produced by its *envelope*—i.e. the set of lines which are tangent to the curve at each point. Of course we can't draw all the lines in the envelope—if we could we would get a solid black region representing the area 'outside' the curve.

In this lesson, students use the number lines on their MATHOMAT to join points on pairs of intersecting lines according to different rules to produce envelopes for the parabola and, in higher year levels, the ellipse and hyperbola. The rules for these constructions can be specified using either numeric or algebraic relationships.

The MATHOMAT template contains several circles with evenly spaced markings on their circumferences. For example, circle 29 has 100 evenly spaced marks on its circumference. This circle can therefore also be used to produce 50, 25, 20, 10, 5, 4 or 2 evenly spaced points. Similarly, circle 3, which has 60 evenly spaced marks on its circumference, can also be marked with 30, 20, 15, 12, 10, 6, 5, 4, 3 or 2 evenly spaced points.

This lesson also asks students to join points on their MATHOMAT circles using different rules in order to produce envelopes of the circle and the cardioid.

As an optional conclusion to the lesson, students can create patterns by joining points on the circumference of a circle using arcs of the same circle instead of straight lines.

Encourage students to use sharp pencils in order to construct figures as accurately as possible. (Pencils are preferable to pens in this lesson as incorrect lines can be more easily erased if necessary.)

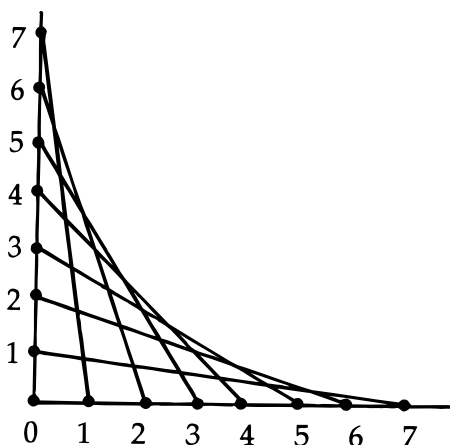
Lesson Outline

1. Constructing the envelope of a parabola using perpendicular lines

Ask students to draw perpendicular lines as shown below and use their MATHOMAT number line to mark off unit intervals from 0 to 7. (For quick ways to construct perpendicular lines, see page 2 of the MATHOMAT Instruction Book.)

Students can then use straight lines to join points on the horizontal and vertical lines, joining point 7 to point 1, point 6 to point 2, etc, as shown here.

Discuss with students the 'shape' formed by the straight lines—the 'white part' looks like a curve. Explain that this session is mainly about producing curves in this way—i.e. by producing what is called an envelope of the curve. With older students, you may wish to introduce the term tangent, at least intuitively, by pointing out that the straight lines 'touch' the curve but never cut it.

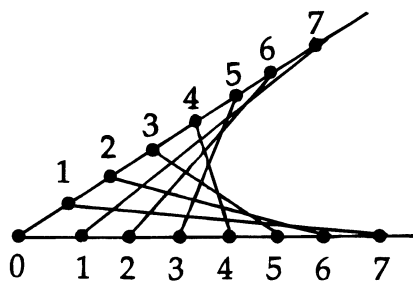


2. Investigating the effect of changing the angle of intersection

Ask students to work individually or in groups to investigate the effect of changing the angle of intersection of the lines on the curve produced—continue to use the same rule of joining points 7 and 1, etc, at this stage.

Students should be given the opportunity to discuss their drawings in order to establish that, no matter what angle is used, the curve obtained is still similar in many ways to the one produced in part 1 above—for example, the diagram below illustrates the case when the lines intersect at 30°.

In each case, the curve produced is part of a parabola. Among similarities students should notice, is the fact that each curve is symmetrical about a line which bisects the angle formed by the pair of intersecting lines. The term parabola can be introduced here—perhaps with a few other examples drawn on the board—as it will be useful to have names for the various curves drawn.



3. Devising a rule for joining the points

Ask students to devise a rule to use when joining the points. Explain that they may need such a rule, if, for example, they wanted to tell a friend over the telephone how to reproduce their curve.

After students have spent a few minutes devising and writing down their rule, hold a brief class discussion where students share the different rules obtained.

Depending on the year level, possible answers might include:

‘The two numbers used always add to 8.’

‘Join the point x to the point $(8 - x)$.’

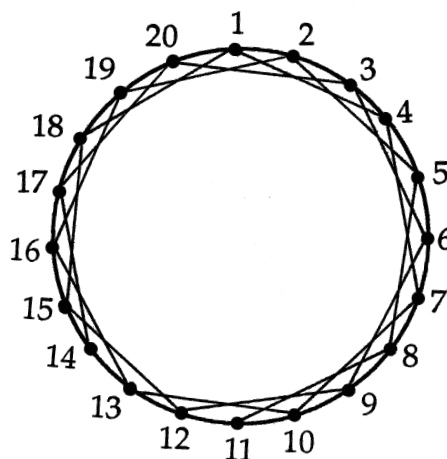
4. Circles within circles

Ask students to work individually or in groups to investigate the effect of joining evenly spaced points on the circumference of a circle.

Suggest that students begin by drawing either circle 3 or circle 29 from their MATHOMAT and marking a chosen number of equally spaced points on the circumference. This involves careful counting, marking and numbering of the appropriate MATHOMAT points. (Initially younger students could be encouraged to use circle 29 with 10 or 20 points.)

Students can then join designated points on the circumference of their circle, as illustrated here.

In this example, the figure was produced using circle 29, with 20 evenly spaced points as marked. Each point x was then joined to the point $x + 3$ (i.e. point 1 was joined to point 4, point 2 joined to point 5, etc). Note that point 18 is joined to ‘point 21’, which doesn’t exist but is in fact understood to be point 1.



Depending on the year level, instructions for joining points using the MATHOMAT straight edge can be given in many different ways. For example, students might give instructions such as those listed below:

‘Label the points from 1 to 20, in order. Now join one point to the point three away from it, and the next point to the one three from it, and so on.’

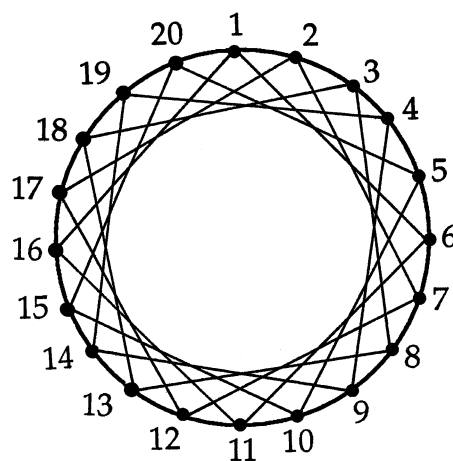
‘Label the points from 1 to 20, in order. Join point 1 to point 4, point 2 to point 5, point 3 to point 6, etc.’

'Label the points from 1 to 20, in order. Join the point x to the point $x + 3$, except that when you get a number greater than 20 you will need to subtract 20 from the answer.'

The diagram below was produced using the same circle with the same spacing, but this time joining point x to point $x + 5$.

After each student has produced one drawing, discuss as a class the effects of:

- varying the number of points on the circumference; and
- varying the value of 'c' when we join point x to point $x + c$ (as happened above where in the first example we used $c = 3$ while in the second example we used $c = 5$).

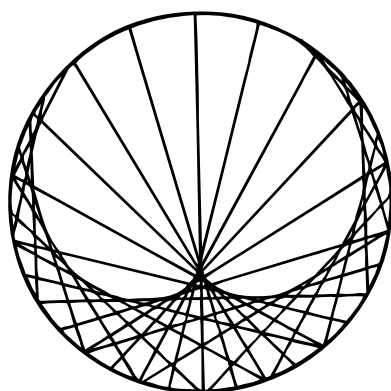
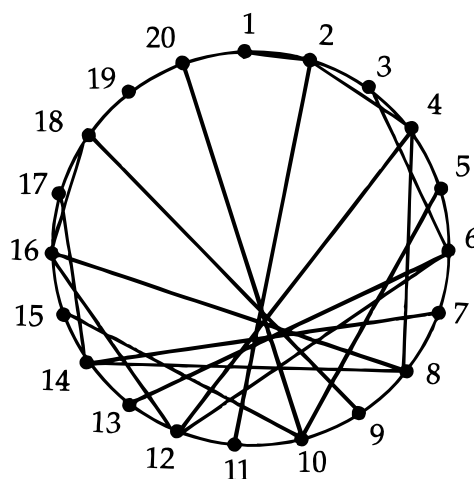


(Of course you would not necessarily use this type of language with students!)

5. Other curves

Several other well-known families of curves can be obtained by joining points on the circumference of a circle or on parallel or intersecting straight lines.

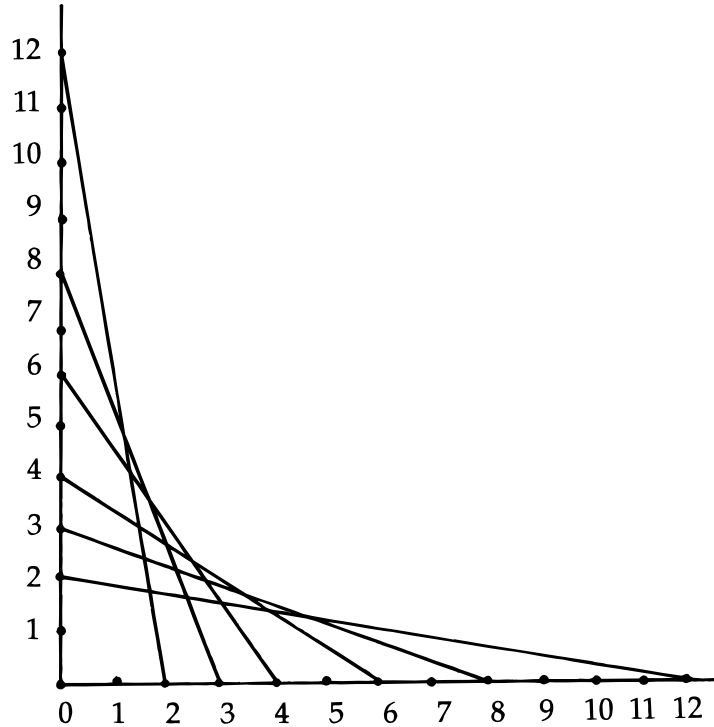
For example, the heart-shaped *cardioid* can be obtained by joining the point x to the point $2x$ on the circumference of a circle—the example here was produced using circle 29 with 20 evenly spaced points.



The cardioid can be seen more clearly on the second example shown here, which is based on 32 evenly spaced points.

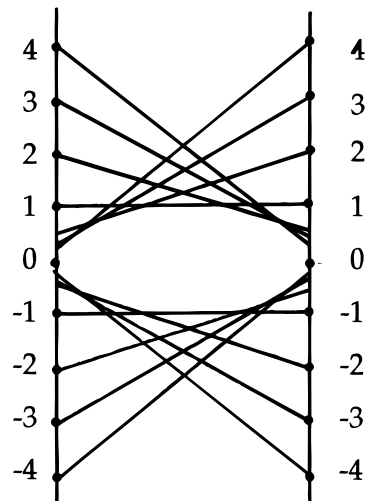
A *hyperbola* can be obtained by joining a point x to the point c/x on perpendicular lines. The example below shows some of the lines obtained when $c = 24$.

The lines shown were obtained by choosing pairs of points so that their product is 24 (e.g. the point 12 is joined to the point 2, while the point 3 is joined to the point 8). Other lines can be added by estimating the fractional points on the number lines.



The envelope of an *ellipse* can be obtained by using parallel lines and joining the point x to the point $1/x$, as shown here. For example, the point 4 is joined to the point $1/4$, while the point -2 is joined to the point $-1/2$.

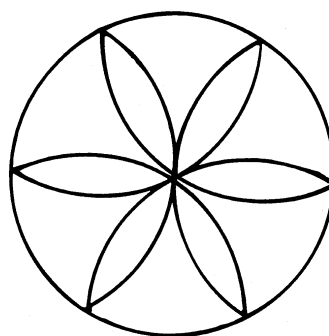
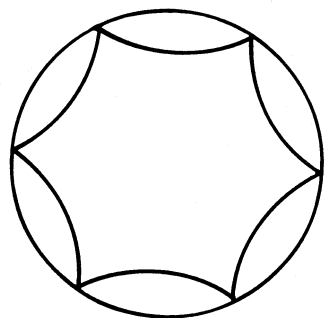
Ask students to produce the envelopes of one or more of these curves, using the instructions given above.



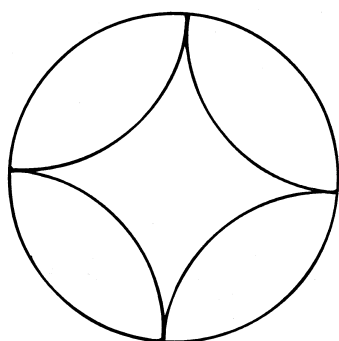
6. (Optional) creating patterns using arcs of circles

Ask students to draw additional circles and mark up to 10 evenly spaced points on their circumference. They can then join selected points using circular arcs (from the same circle) by positioning the MATHOMAT circle template over the chosen points. Some possible results are shown below:

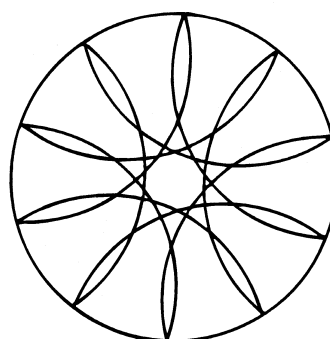
Circle 3, using six points



Circle 29, using four points



Circle 29, using ten points



Students can vary the number of points and the rules for joining them in order to produce other designs.

Other ideas for designs can be found in books such as Phillips (1996) and Seymour, Silvey and Snider (1974).

References

Phillips, G. (1996). *GeoMat Investigations*. Croydon North, Vic: Geoff Phillips Publications.

Seymour, D., Silvey, L. & Snider, J. (1994). *Line Designs* Palo Alto, CA: Creative Publications.