

Chapter 2 - Statics: Principles of Equilibrium

2.1 Introduction

Structures are supposed to resist the loads that are placed upon them by developing counteracting forces that keep the structure in “equilibrium”. These counteracting forces developed in a structure are typically referred to as reactions. Equilibrium of a structure will exist when the resultant of all the forces and moments on an object equals zero. Equilibrium is the foundational principle that underlies all techniques used in structural design.

2.2 Types of Equilibrium

Forces have the tendency to move bodies in one of two ways. The first is referred to as translational movement which is movement that occurs in a straight-line path. Such movement occurs in many everyday applications, and can easily be viewed right now by pushing your book across your desk. Notice the book tends to move in the line of the push.

If your push was not “centered” along one side of the book, however, another type of movement may have observed while pushing the book across the desk. The book may have started to spin or twist as it moved across. This twisting is evidence of the other movement that may take place. This type of movement is rotational movement is quantified as a “moment”. In two dimensions (which is what this book focuses on) the magnitude of a “rotational force” or moment is dependent on two items—the magnitude of the force and the perpendicular distance between the point or axis of rotation and the force involved. This can be formalized in the equation shown as follows:

$$\text{Moment} = \text{Force} \times \text{Distance} \qquad (\text{Eq. 2-1})$$

Remember the distance from the point (or axis) must be perpendicular to the force. An easy mistake for a beginner to make is to use the wrong distance when calculating the moment of a force. Units of moment are typically inch-pound (lb-in), foot-pound (lb-ft), or newton-meter (N-m)

In order to be in equilibrium a structure must:



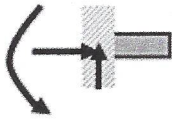

- have no net forces on it, and
- have no net moments on it

If this occurs there will be no translational or rotational movement and the structure will “stay at rest”. Typically, the equation of force equilibrium is broken down into rectangular components for use in planar systems. It is simpler to think about forces in terms of their x-and y-components because these can be readily added. Therefore, the concept of two-dimensional equilibrium is most often expressed by the three basic equations as follows:

$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma M = 0$$

2.3 Free Body Diagrams

A fundamental skill needed is the ability to graphically represent all the forces that act on a structure or an element within a structure. This representation of all the forces acting on an object is what is known as a **free-body diagram**. To draw a free-body diagram correctly, it is necessary to isolate the object from its support conditions. Once the body is isolated, it is necessary to replace the actual supports with the potential forces that can “react” at those supports. Typical support conditions are shown below which represent the possible forces that could be generated for these support conditions. These forces at the supports are often called reactions.

Support Type	Idealized View	Potential Reactions
Roller		1 Force - Perpendicular to Surface
Pin		2 Forces – Perpendicular and Parallel to Surface
Fixed		2 Forces – Perpendicular and Parallel to Surface 1 Moment – About the support
Cable (Link)		1 Force – Along the line of action of the cable (link)

$$\Sigma F_y = 0$$

$R_{ay} - 600 \text{ lbs} = 0$; Simplifying, this equation can be written as:
 $R_{ay} = 600 \text{ lbs}$

$$\Sigma M_a = 0$$

$$+600 \text{ lbs}(3 \text{ ft}) - M_R = 0$$

Rearranging, this equation becomes:
 $1800 \text{ lb-ft} = M_R$

The moment reaction at the building is $M_R = 1800 \text{ lb-ft}$

The support reactions are $R_{ax} = 0$, $R_{ay} = 600 \text{ lbs}$, and $M_R = 1800 \text{ lb-ft}$

2.5 Other Applications of Equilibrium in Construction

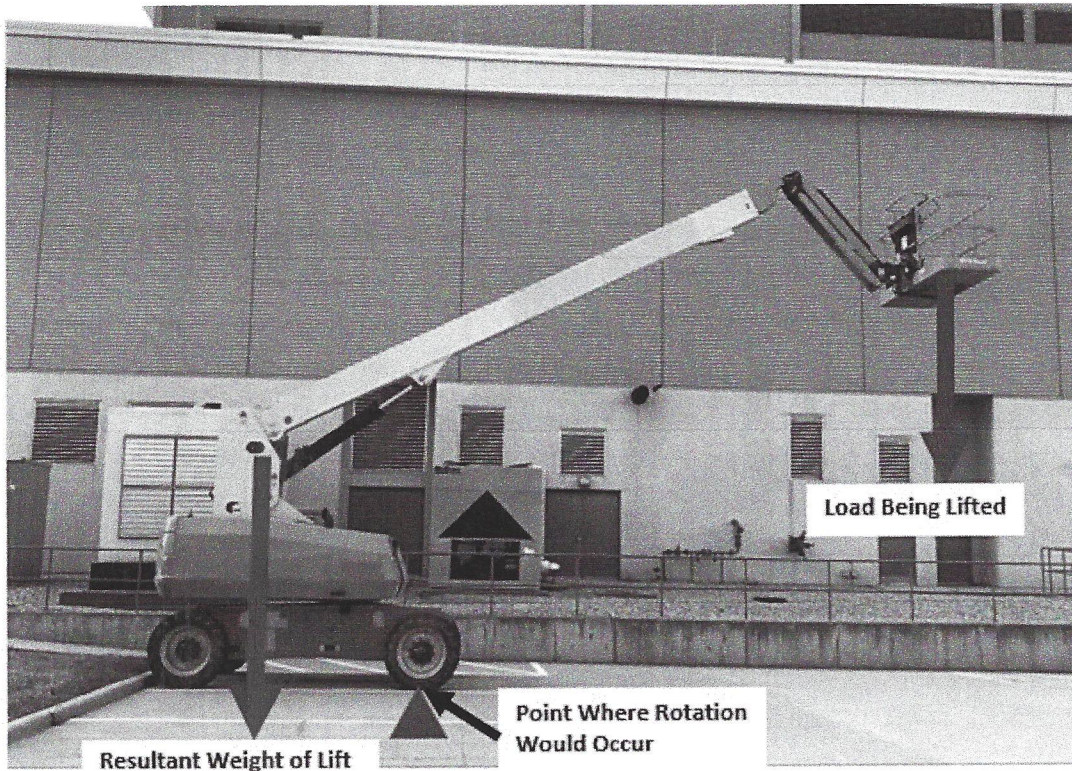
The previous examples demonstrate the solution process for determining the magnitude and direction of the forces generated at the supports. This is integral in our understanding of structures because these support reactions HAVE to develop to keep the object from moving, i.e. to keep the object in equilibrium.

There is, however, another use of equilibrium that involves construction equipment that is used to lift objects. These pieces of equipment include a variety of cranes, lifts, and backhoes. The safety issue that arises with such pieces of equipment is based upon their anticipated “reach”, what is the maximum load that they can theoretically lift? Of course, we will consider that the theoretical maximum as the actual maximum should be much lower for safety.

Consider the lift shown below. At some point, as the load being lifted get larger and larger, the lift would begin to “tip” or “rotate” about the front wheels. If tipping occurs, then equilibrium of the lift does not exist and a hazardous condition presents itself for those working with and around the crane. In such cases, construction personnel must know how much the crane can safely lift.

Rotation about the front wheels depends on the moment caused by the load on the one side of the wheels and the moment caused by the self-weight of the lift on the other side. If the moment caused by the self-weight of the lift is larger than the moment from the load being lifted, then the lift remains stable. But as the moment on one side equal the moment on the other side then the lift is on the verge of tipping. So a

tipping problem will be solved by considering the moment that act around the location of tipping (in this case, the front wheels).



For simple stress, both axial cases as well as shear cases, the formula for calculating stress is as follows:

$$\sigma = \frac{P}{A} \quad (\text{Eq. 3-1})$$

Where

σ = stress (units of psi, ksi, Pa, etc.)

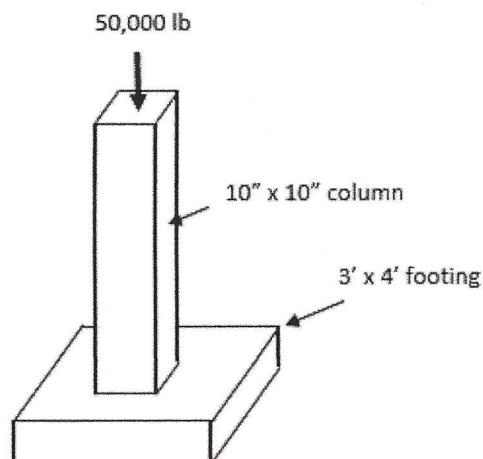
P = externally applied force (units of lbs, kips, N, etc.)

A = resisting area (units of in², ft², m², etc.)

Note that the units of stress are pounds per square inch (psi), kips per square inch (ksi), or newton per meter (which is termed a pascal in SI units). The units of stress underscore the difference between stress and force — stress is force divided by area (i.e. distributed over the area) while force is the total force on the object. Recognize that stress and force are different! The examples below will focus on stress calculations.

Example 1

A column measuring 10 in. x 10 in. is supported by a concrete footing measuring 3 ft. x 4 ft. Calculate the stress in the column and the stress on the soil beneath the footing.



The stress in the column and the stress under the footing are both compressive stresses, although technically we refer to the stress in the soil under the footing as a “bearing” stress. Both of these stresses will be calculated by the following formula:

$$\sigma = \frac{P}{A}$$

The stress in the column is calculated as follows:

$$\sigma = \frac{50,000 \text{ lb}}{100 \text{ sq. in.}}$$

$$\sigma = 500 \text{ psi (500 pounds per square inch)}$$

The stress in under the footing is calculated as follows:

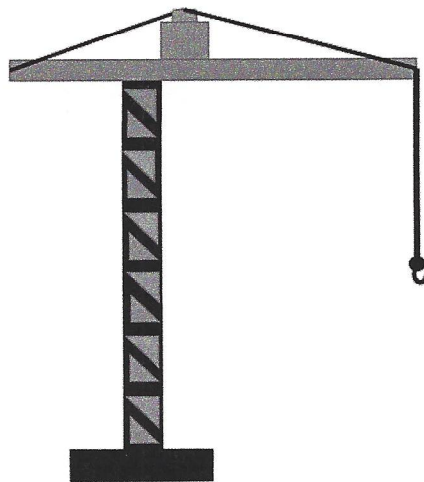
$$\sigma = \frac{50,000 \text{ lb}}{12 \text{ sq. ft.}}$$

$$\sigma = 4167 \text{ psf (four thousand one hundred sixty seven pounds per square foot)}$$

Which stress is larger? Hopefully you said 500 psi! (Remember units—there are 144 square inches in a square foot. So 4167 psf = 29 psi approximately)

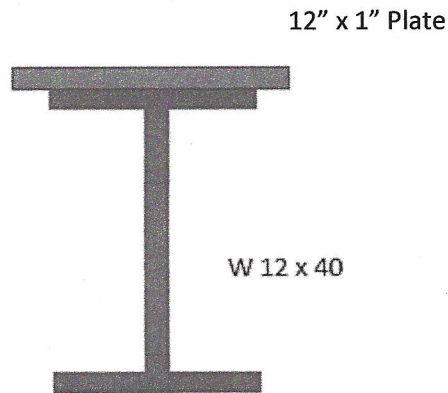
Example 2

A tower crane lifts a load weighing 8000 pounds. The cable used in the hoisting mechanism is 1 inch in diameter. What is the stress in the cable?

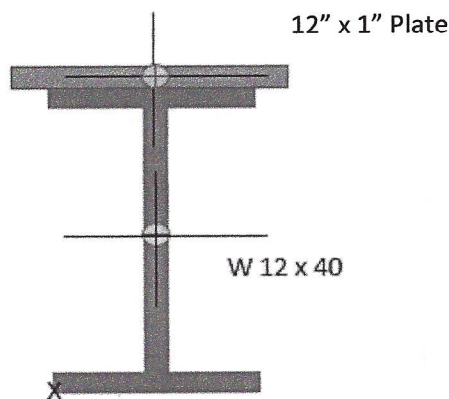


Example 3

Find the centroid for the shape shown below. Assume the plate is centered over the beam.



The shape is a composite shape and the origin has not been explicitly established. It is common in situation like this to choose the origin at the bottom or bottom corner so we will place it at the lower left hand edge of the W 12 x 40 shape. So the next step is to split the shape into several standard shapes that can easily be handled. In this case we should split it into the W 12 x 40 and a rectangular plate as such. Both the W 12 x 40 and the rectangle have a centroid which is a specific distance away from the origin. Remember parameters such as area, depth, flange width can all be located in the property tables in Appendix A.



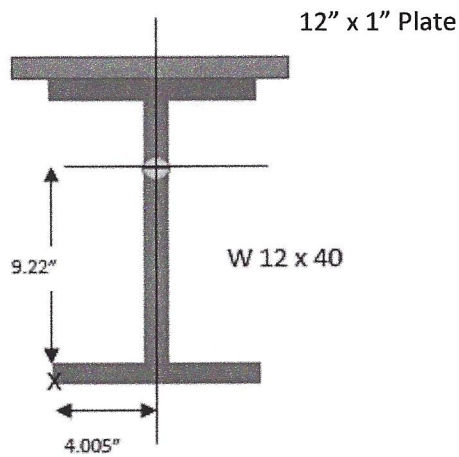
Filling out the table with a , x , and y values we can then sum the " a ", " ax " and " ay " columns and calculate the \bar{x} and \bar{y} formulas as shown below.

Shape #	a	x	ax	y	ay
W12x40	11.7 in ²	4.005 in	46.86 in ³	5.95 in	69.62 in ³
12x1	12 in ²	4.005 in	48.06 in ³	12.4 in	148.8 in ³
	23.7 in ²		581.3 in ³		218.42 in ³

$$\bar{x} = \frac{\sum ax}{\sum a} = \frac{94.92}{23.7} = 4.005 \text{ in.}$$

$$\bar{y} = \frac{\sum ay}{\sum a} = \frac{218.42}{23.7} = 9.22 \text{ in.}$$

Notice how the \bar{x} distance is exactly one-half the width of the flange. That is due to the fact that the cover plate was centered over the beam. Notice also that the composite shape is symmetrical about the centroidal axis. Any axis of symmetry will also be a centroidal axes. The centroid is located as shown below.



Where

M = applied bending moment to the cross-section

c = distance from the neutral axis to location of calculated bending stress

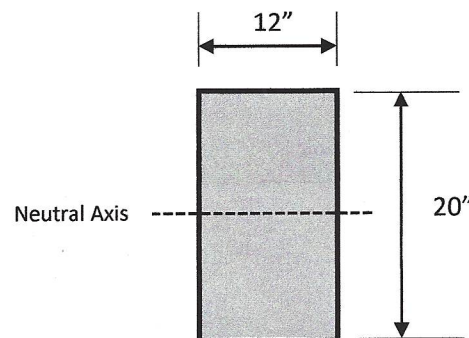
I = cross-section's moment of inertia

The moment of inertia, I , in this equation is a property of the beam's cross-section and, as discussed in a previous chapter, is the parameter describing the shape's resistance to bending (as I becomes larger, the more bending resistance). Determining the moment of inertia for composite shapes as well as those using standardized shapes is an important competency. For standardized shapes (steel, concrete, etc.) the moment of inertia can be found in property tables such as those found in the back of this textbook.

This bending stress equation can be used to either calculate bending stresses on an existing beam or to design the necessary moment of inertia, I , which is required to resist a certain amount of applied moment. It is important to keep consistent units, remembering that the resulting units should be in units such as psi, ksi, etc.

Example 1

The beam cross-section shown below has 100 kip-ft of bending moment applied to it. Calculate the bending stress at the outside surfaces (it is assumed that the neutral axis is the centroidal x-axis).



Rectangular shapes have a moment of inertia that can be calculated from the formula:

$$I = \frac{1}{12} b h^3$$

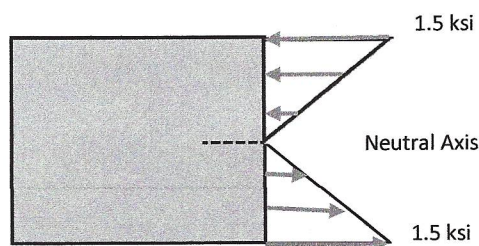
Which in this case gives a moment of inertia of:

$$I = \frac{1}{12} (12)(20)^3 = 8000 \text{ in}^4$$

Since the problem wants the stress at the outside edge, the “c” distance is from the neutral axis to the outside which in this case is 10 inches. So the bending stress at the outside edge of the beam becomes:

$$\sigma_b = \frac{(100 \text{ k-ft})(12 \frac{\text{in}}{\text{ft}})(10 \text{ in})}{8000 \text{ in}^4} = 1.5 \text{ ksi}$$

Notice the unit conversion from kip-ft to kip-in for the moment. That was done to keep the units consistent. This bending stress of 1.5 ksi occurs at both the top and bottom of the beam since it is symmetrical (i.e. “c” is the same 10 inches whether you are measuring to the top or the bottom of the beam). However, one side is 1.5 ksi in tension and the other side is 1.5 ksi compression. In an upcoming chapter, we will discuss how you determine tension and compression.



If a beam is not symmetrical, then its neutral axis will not lie “in the middle” of the shape and therefore this will result in a “c” value for the top of the beam and another “c” value for the bottom of the beam. Of course, the different “c” values will result in different bending stresses at the top outside edge and the bottom outside edge. The following example will demonstrate this concept.

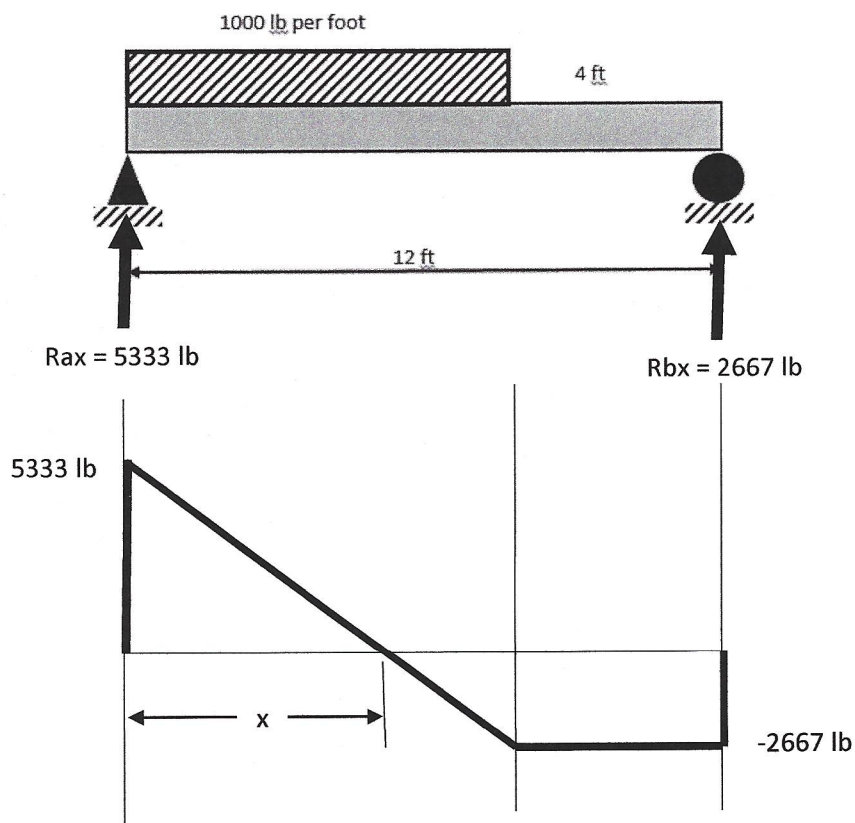


Example 2

The beam cross-section shown below has 100 kip-ft of bending moment applied to it. Calculate the bending stress at the outside surfaces (it is assumed that the neutral axis is the centroidal x-axis). Consider $I = 260.8 \text{ in}^4$

How would this change if the beam carried a uniformly distributed load?

With a uniformly distributed load, the shear diagram decreases at a rate that would be equal to the magnitude of the load. For instance, if a uniform load is 3 kips per foot the shear diagram would decrease by 3 kips for every foot of the load's length. This would result in a diagonal line on the shear force diagram. Still, the shear at any point is equal to the sum of the forces. This is demonstrated on the beam below.



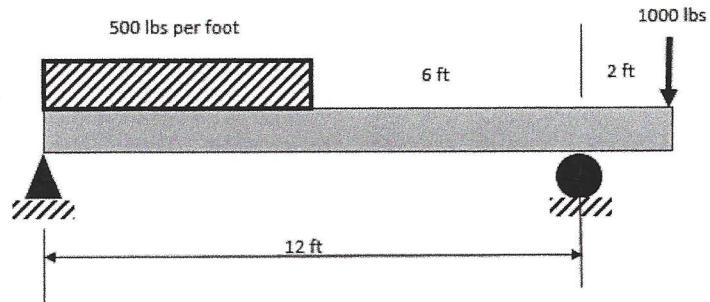
One item to determine is where the diagonal line of the shear diagram crosses zero. The diagonal line is a straight line with a slope of -1000 lbs per foot so the “x” distance is simply calculated as:

$$x = 5333 \text{ lb} / 1000 \text{ lb} / \text{ft} = 5.33 \text{ ft}$$

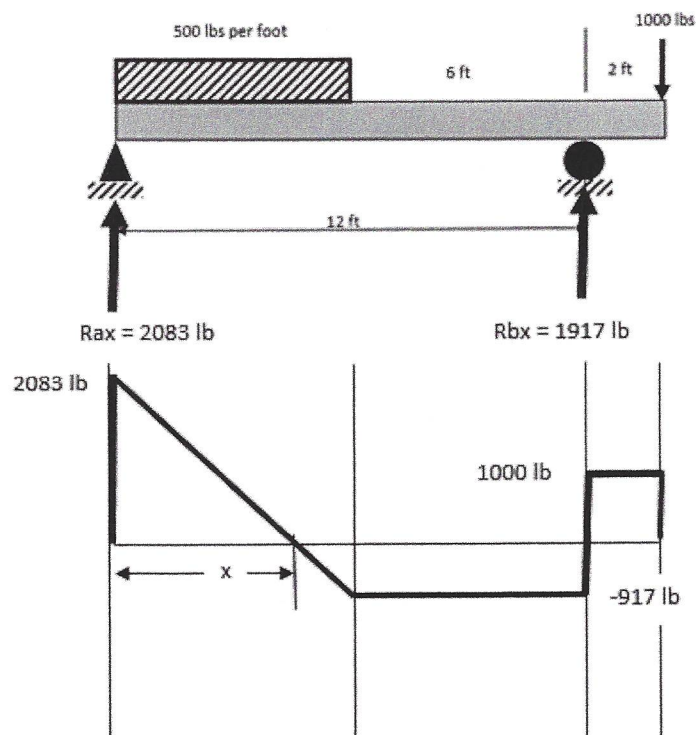
As before, the shear diagram plainly shows the maximum shear force on the beam (5333 lbs) and where that occurs (at the left support). The following example will reinforce the development of the shear force diagram.

Example 7

Draw the shear force diagram for the following beam.



To begin with, the support reactions must be solved correctly (refer back to the earlier section if you need guidance on equilibrium equations – it is imperative that you get the reactions correct!). The reaction are as shown below.



The “x” distance where shear would be zero is?

$$x = 2083 \text{ lb} / 500 \text{ lb} / \text{ft} = 4.17 \text{ ft}$$

On the compression side of the beam, the stress is approximated by a rectangular block having a stress magnitude of $0.85f_c$. The area of the block is the product of the width of the beam, b , and the depth of the block, a . The compression resultant is as follows:

$$C = 0.85 f_c (a)(b)$$

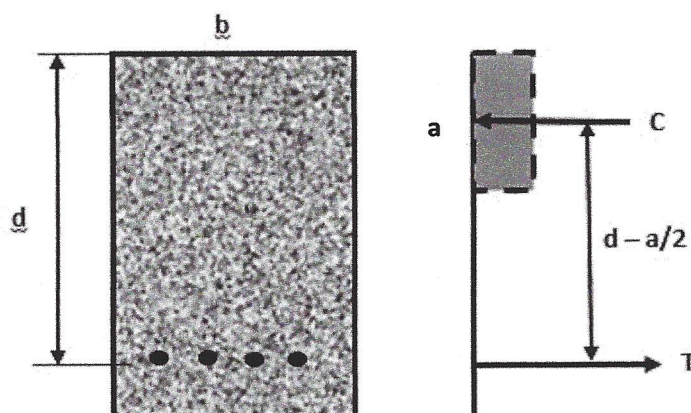
where

f_c = compressive strength of concrete, psi or ksi

a = depth of the rectangular stress block, inches

b = width of the beam, inches

The tension, T , and compression, C , forces on each side of the neutral axis must be equal for purposes of internal equilibrium. The T and C forces are separated by a distance which forms an internal moment couple that counteracts the externally applied moment. This is shown as follows:



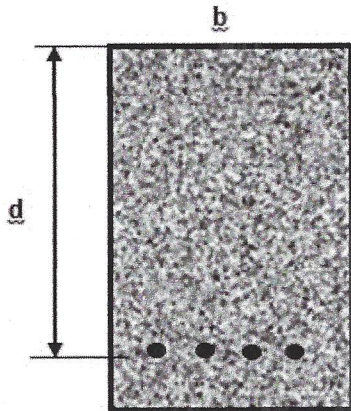
The “ d ” distance shown stretched from the compression side of the beam to the centroid of the steel (where the T force acts). The “ d ” distance is sometimes referred to as the design depth or the effective depth. Therefore, the nominal moment capacity of a beam is given by the following equation:

$$M_n = T(d - a/2) \quad (\text{Eq. 6-5})$$

The distance “ a ” is the depth of the stress block as previously mentioned. The following example will demonstrate how the nominal moment capacity of a beam can be calculated.

Example 5

Calculate the nominal moment capacity, M_n , for the beam shown below. Consider $b = 10$ inches and $d = 16$ inches. Reinforcing is 4 - #6 bars. $f'_c = 3$ ksi and $f_y = 60$ ksi



Starting out, calculate the resultant tensile force, T .

$$T = A_s f_y$$

where

$$A_s = 4 \text{ bars}(0.44 \text{ in}^2) = 1.76 \text{ in}^2$$

$$f_y = 60 \text{ ksi}$$

So $T = A_s f_y$ and for this example:

$$T = 1.76 \text{ in}^2(60 \text{ ksi}) = 105.6 \text{ kips}$$

The compression resultant must be equal to is as follows:

$$C = 0.85f'_c(a)(b)$$

$$105.6 \text{ kips} = 0.85(3 \text{ ksi})(a)(10 \text{ in})$$

Solving for "a" we find

$$a = 4.14 \text{ inches}$$

The nominal moment capacity is then:

$$M_n = T(d - a/2) = 105.6 \text{ kips}[16'' - (4.14\text{in}/2)] = 1470 \text{ kip-in or } 122.6 \text{ kip-ft}$$

It should be noted that there are other formulas that could be used to calculate the nominal capacity of a reinforced concrete beam. The student is encouraged to explore other methods on their own.

So the design capacities would be:

$$\text{In ASD, } \frac{P_n}{1.67} = 394 \text{ kips} / 1.67 = 236 \text{ kips}$$

$$\text{In LRFD, } \phi P_n = (.9)394 \text{ kips} = 355 \text{ kips}$$

In the next example we will explore the effect that bracing has on a column.

Example 2

A W 12 x 50 column is 15 feet long and has pinned end conditions ($K = 1.0$). If it is braced at its mid-height in the weak direction, determine its design capacity using the ASD and LRFD methods. Use $F_y = 50$ ksi and $E = 29,000$ ksi.

Start by determining the slenderness ratio of this column. Since the column is braced in the weak direction we will need to calculate the slenderness ratio about both the x and y axes.

The slenderness ratio about the weak axis (i.e. the weak direction, r_y) will control the column's behavior.

Note that the "length" in the weak direction is only 7.5 feet.

$$KL/r_x = (1.0)(15' \times 12''/') / 5.18 = 34.8$$

$$KL/r_y = (1.0)(7.5' \times 12''/') / 1.96 = 45.9 \Rightarrow \Rightarrow \Rightarrow \text{the weak axis controls since its slenderness is larger}$$

$$4.71 \sqrt{\frac{E}{F_y}} = 4.71 \sqrt{\frac{29000}{50}} = 113.4$$

Since the column's controlling slenderness ratio (45.9) is less than 113.4, the critical stress is calculated as follows:

$$F_{cr} = \left(.658 \sqrt{\frac{F_y}{E}} \right) F_y$$

$$F_y = 50 \text{ ksi and}$$

$$F_e = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2} = F_e = \frac{\pi^2 29000}{(45.9)^2} = 135.9 \text{ ksi}$$

Therefore

$$F_{cr} = \left(.658 \sqrt{\frac{50}{135.9}} \right) 50 = 42.9 \text{ ksi}$$

So then the nominal compressive capacity is equal to:

$$P_n = F_{cr} A_g = (42.9 \text{ ksi})(14.6 \text{ in}^2) = 626 \text{ kips}$$

So the design capacities would be:

$$\text{In ASD, } \frac{P_n}{1.67} = 626 \text{ kips}/1.67 = 375 \text{ kips}$$

$$\text{In LRFD, } \phi P_n = (.9)626 \text{ kips} = 563 \text{ kips}$$

Note how the bracing reduced the effective length of the column which caused an increase in its critical stress, nominal capacity and ultimately an increase in its design capacity.

7.6 Steel Column Design

As with beam design and the design of many other structural elements, column design may be a trial and error process. A designer could simply choose a column section, and then calculate its design compressive strength to compare it to the loads that it needs to carry. Of course, this procedure could take some time as finding the “correct” section is much easier if the first trial is fairly close to the final answer. To begin with, remember the two “categories” of column behavior—inelastic buckling, and elastic buckling which were presented previously. The real unknown in design is the value of critical buckling stress, F_{cr} . And, although this stress is unknown, we know that as the slenderness ratio decreases then the critical stress increases.

The AISC’s Steel Construction Manual (Part 4) contains some design tables for a section’s “Available Strength in Axial Compression”. It should be noted that these tables apply only to members subjected to axial compression and that columns subjected to both compression and bending would have to be designed as a beam-column. Nonetheless, the use of these tables is presented here to present another option for the design of steel columns subjected only to axial compression. Several of these column tables are found in Appendix A of this text.

A sample of such a table is shown in the subsequent example. To determine a column’s design capacity we select the column size and read off the design capacity at a particular effective length, KL . Each member shape has two capacities shown – the left (shaded) side is the design capacity for the ASD method while the right side is the design capacity for the LRFD method. Also, notice that the table itself is based on the initial assumption that the controlling slenderness for the column is about the weak axis. For the design of an appropriate column size given a particular required load, you would assume that the

