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performing on paper

Putting the Operations in Their Place

All year long we have "danced with decimals" here at WI, guided by our own Rachel McAnallen. A woman of many talents, she skillfully led us through decimal place values and taught us how to truly express ourselves numerically. Rachel has long-insisted that mathematics is a language to be spoken, an art to be seen, a music to be heard, and a dance to be performed. It seems only fitting that the conclusion of the series is a performance of all the steps she's taught us this year. In this issue, our resident mathematician shows readers how to perform the mathematical operations—addition, subtraction, multiplication and division—on paper.

Topics Involved: number sense, justification of operations using different forms of numerical expression
Materials: Each student will need notepaper and pencil. Small groups of students can share five dice marked with the place value for hundreds, tens, ones, tenths, and hundredths, respectively.
Type of Activity: large group, small group and individual instruction
Note to teachers: Most students will only require a notebook and pencil for this activity, provided they have done enough lead-up work with the money and digits in previous lessons. (See WI Vol. 16, 1-4) If a student or a group of students has difficulty working in the abstract, it is possible to teach this lesson with the aid of the place value wallet and money. We also recommend labeling a sheet of paper to mark the location of the "Bank" when playing the money game.

Addition—Expanded Form

Beginning the lesson with addition, Rachel selects a learner to roll three place-value dice marked with tens, ones and tenths. "Roll 'em and read 'em," urges Rachel.

Giving the dice a shake, the student rolls, then announces, "Twenty-seven and two-tenths."

"We are going to write that number down in standard form," Rachel instructs, modeling the each step of the writing process on the overhead for the class.

10s 1s 1/10s

27.2

"Now, we're going to write an equal sign to the right of the standard form and write it in expanded form:

10s 1s 1/10s

27.2 = 20. + 7. + 0.2

"Standard form equals expanded form," she reminds the class, writing SF = EF above the numbers.

Another student rolls the three dice to generate a second number.

"We aren't rolling the operation cube today because we know that we are adding," notes Rachel. "What are we adding to twenty-seven and two-tenths?" she asks.

"Forty-five and five-tenths."

"Good," says Rachel. "We're going to write that down in standard and expanded form, right below our first number. Following the example on the overhead, the class writes + 45.5 = + 40. + 5. + .5 in their notes.

"Now we can draw a line underneath our problem and put an addition sign beside forty-five and five-tenths," Rachel directs.

$$\begin{array}{l} \quad 10s\ 1s\ \tfrac{1}{10}s \\ +27.2 = +20.\ +7.\ +0.2 \\ +\underline{45.5} = +40.\ +5.\ +0.5 \\ \hline \end{array}$$

"We're going to add these numbers using the expanded form." She asks a learner, "What would you like to add first?"

"I'd like to add twenty and forty."

"What do we get when we add those two numbers?"

"Sixty."

"Everyone write sixty underneath twenty and forty," instructs Rachel.

$$\begin{array}{l} \quad 10s\ 1s\ \tfrac{1}{10}s \\ +27.2 = +20.\ +7.\ +0.2 \\ +\underline{45.5} = \underline{+40.\ +5.\ +0.5} \\ \qquad\qquad\ +60. \end{array}$$

"What next?" she asks.

"Seven plus five equals twelve," answers the student.

$$\begin{array}{l} \quad 10s\ 1s\ \tfrac{1}{10}s \\ +27.2 = +20.\ +7.\ +0.2 \\ +\underline{45.5} = \underline{+40.\ +5.\ +0.5} \\ \qquad\qquad\ +60.\ +12. \end{array}$$

"And then what do we get when we add two-tenths and five-tenths?" asks Rachel.

"Seven-tenths," the student replies.

$$\begin{array}{l} \quad 10s\ 1s\ \tfrac{1}{10}s \\ +27.2 = +20.\ +7.\ +0.2 \\ +\underline{45.5} = \underline{+40.\ +5.\ +0.5} \\ \qquad\qquad\ +60.\ +12.+0.7 \end{array}$$

Next the students combine similar terms, which Rachel writes out on the overhead:

$$\begin{array}{l} \quad 10s\ 1s\ \tfrac{1}{10}s \\ +27.2 = +20.\ +7.\ +0.2 \\ +\underline{45.5} = \underline{+40.\ +5.\ +0.5} \\ \qquad\qquad\ \underbrace{+60.\ +12.}_{+72.}+0.7 \end{array}$$

"In expanded form, we have seventy-two plus seven-tenths," Rachel observes as she writes out this step. "How do we say that in standard form?"

"Seventy-two and seven-tenths."

$$\begin{array}{l} \quad 10s\ 1s\ \tfrac{1}{10}s \\ +27.2 = +20.\ +7.\ +0.2 \\ +\underline{45.5} = \underline{+40.\ +5.\ +0.5} \\ \qquad\qquad\ \underbrace{+60.\ +12.}_{+72.}+0.7 \\ 72.7 = \qquad +72.\ +0.7 \end{array}$$

"Seventy-two and seven-tenths equals seventy-two plus seven-tenths," she reads. "Standard form equals expanded form.

Addition—Partial Sum Form

"Expanded form is one way we can add numbers," says Rachel. "Now we are going to add the same numbers in a different way. We're going to add them using the partial sum form.

She rewrites the same problem on the overhead:

$$\begin{array}{l} \quad 10s\ 1s\ \tfrac{1}{10}s \\ +27.2 \\ +\underline{45.5} \end{array}$$

"When we add in partial sum form, we look at the numbers and we decide which column we want to add first—tens, ones, or tenths," Rachel explains.

The class decides to begin with the tens column, arriving at a partial sum of sixty. "Underneath the line we've drawn we're going to write sixty as sixty and zero-tenths," instructs Rachel, writing this out on the overhead.

$$\begin{array}{l} \quad 10s\ 1s\ \tfrac{1}{10}s \\ +\ 27.2 \\ +\ \underline{45.5} \\ +\ 60.0 \end{array}$$

"Now, we need to show how we got the sixty," she continues. "Over to the right of sixty, we are going put an equals sign and write twenty plus forty." She waits for students to complete this step in their notebooks. "Just to make sure we are being clear, we are going to label that line as the tens column."

$$\begin{array}{lll} \quad 10s\ 1s\ \tfrac{1}{10}s & & \\ +\ 27.2 & & \\ +\ \underline{45.5} & & \\ +\ 60.0 & = 20.\ +\ 40. & \text{(tens column)} \end{array}$$

Next, the students decide to add the ones column.

"Go ahead," says Rachel. "What is seven plus five?"

"Twelve," the class tells her.

"We are going to write that as twelve and zero-tenths. Look how nicely it lines up underneath that sixty and zero-tenths," she says, guiding the students through the process as she writes the numbers out on the overhead. "Next we draw our equals sign and write seven plus five, and put ones column in parentheses."

$$\begin{array}{lll} \quad 10s\ 1s\ \tfrac{1}{10}s & & \\ +\ 27.2 & & \\ +\ \underline{45.5} & & \\ +\ 60.0 & = 20.\ +\ 40. & \text{(tens column)} \\ +\ 12.0 & = 7.\ +\ 5. & \text{(ones column)} \end{array}$$

The only column left to add is the tenths column. "What is two-tenths plus five-tenths?" Rachel asks.

"Seven-tenths."

"Let's line up the seven-tenths in the tenths-column." She draws an equals sign. "How did we get seven-tenths?"

"Two-tenths plus five-tenths."

"And what column are we adding?" asks Rachel.

"Tenths."

$$\begin{array}{lll} \quad 10s\ 1s\ \tfrac{1}{10}s & & \\ +\ 27.2 & & \\ +\ \underline{45.5} & & \\ +\ 60.0 & = 20.\ +\ 40. & \text{(tens column)} \\ +\ 12.0 & = 7.\ +\ 5. & \text{(ones column)} \\ +\ \ \underline{0.7} & = 0.2\ +\ 0.5 & \text{(tenths column)} \end{array}$$

"These are our partial sums," Rachel informs the class. She writes this term to the left of the three sums and draws a bracket.

The students add the three numbers—60.0, 12.0, and 0.7—beginning with the tens column. "Sixty plus ten is seventy. Zero plus two is two. Zero plus zero, plus seven-tenths is seven-tenths."

"Seventy-two and seven-tenths," reads Rachel. "Out to the side we draw our equals sign and write down sixty plus twelve plus seven-tenths. Now we can label this as our final sum."

$$\begin{array}{llll} & \quad 10s\ 1s\ \tfrac{1}{10}s & & \\ & +\ 27.2 & & \\ & +\ \underline{45.5} & & \\ \text{Partial Sum} \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. & \begin{array}{l} +\ 60.0 \\ +\ 12.0 \\ +\ \ \underline{0.7} \end{array} & \begin{array}{l} = 20.\ +\ 40. \\ = 7.\ +\ 5. \\ = 0.2\ +\ 0.5 \end{array} & \begin{array}{l} \text{(tens column)} \\ \text{(ones column)} \\ \text{(tenths column)} \end{array} \\ & +\ 72.7 & = 60.\ +\ 12.\ +\ 0.7 & \text{Final Sum} \end{array}$$

Practice, Practice, Practice!

Rachel selects another pair of learners to roll the dice.

After each roll, the class writes out the number in standard form = expanded form in their notebooks.

$$\begin{array}{ll} \quad 10s\ 1s\ \tfrac{1}{10}s & \\ +\ 75.7 & = +\ 70.\ +\ 5.\ +\ 0.7 \\ +\ \underline{29.8} & = +\ \underline{20.\ +\ 9.\ +\ 0.8} \end{array}$$

Narrating her thought process, Rachel works out the problem on the overhead. "Seventy plus twenty is ninety. Five plus nine is fourteen." For a moment she pauses with her pen in the air, and then declares, "I'm going to write fourteen as ten plus four!"

Next she adds the tenths. "Seven-tenths plus eight-tenths is what?" Rachel asks.

"Fifteen-tenths," volunteers one student.

"What is another way to say fifteen tenths? Remember the trading rules for the place value wallet," she reminds the learner, "ten for one and one for ten."

"One and five-tenths," replies the student.

"I'm going to write one plus five-tenths," decides Rachel. "Now I can add. Ninety plus ten is one hundred, four plus five is five, and I have five-tenths. My answer is one hundred five and five-tenths."

10s 1s 1/10s

$$
\begin{aligned}
+75.7 &= +70. + 5. + 0.7\\
+\underline{29.8} &= +\underline{20. + 9. + 0.8}\\
&\qquad +\underbrace{90. + 10.} + \underbrace{4. + 1.} + 0.5\\
+105.5 &= \quad +100. \quad + \quad 5. \quad + 0.5
\end{aligned}
$$

"Why did you split up some of the numbers?" a sudent wants to know.

"It is an algebraic process mathematicians use to make things easier," she explains. "If I can break the numbers down into tens and ones, it ismuch easier to add the similar terms."

"Does anyone have another way to add this in expanded form?" Rachel asks. One student is chosen to go up the overhead projector and share the way they added the problem in expanded form:

10s 1s 1/10s

$$
\begin{aligned}
+75.7 &= +70. + 5. + 0.7\\
+\underline{29.8} &= +\underline{20. + 9. + 0.8}\\
&\qquad +\underbrace{90. + 14.} + 1.5\\
+105.5 &= \quad +104. \quad + 1.5
\end{aligned}
$$

Finally, they add the problem in partial sum form:

10s 1s 1/10s

$$
\begin{array}{ll}
 & +75.7\\
 & +\underline{29.8}\\
\text{Partial Sum} \left\{ \begin{array}{l} +90.0 = +70. + 20.\\ +14.0 = +5. + 9.\\ +\underline{\ 1.5} = +0.7 + 0.8 \end{array} \right. & \begin{array}{l}\text{(tens column)}\\ \text{(ones column)}\\ \text{(tenths column)}\end{array}\\
 & +105.5 = +90. + 14. + 1.5 \quad \text{Final Sum}
\end{array}
$$

"I want students to justify their work using the process that works best for them," notes Rachel. "Whichever way makes sense to the learner." The key is to allow students enough practice. "I do several examples with the group adding tens, ones and tenths in both forms," she stresses. "Teachers sometimes rush through a lesson because they are worried about getting to the test, but students will perform much better on the test if they have had enough experience showing their work."

Hundreds & Hundredths

Once students are comfortable with adding tens, ones, and tenths, they are ready to move on. "Who wants to roll more dice?" Rachel asks.

"Hundreds!" implore several voices.

"If we roll hundreds, then we must also roll hundredths," she decides.

Just as they did working with tens, ones, and tenths, after each roll the students write out the five digits in standard form, then expand the number.

"Eight hundred twenty-five and thirty-seven hundredths."

"Two hundred eighty-five and ninety-eight hundredths."

First they write out the problem in expanded form:

100s 10s 1s 1/10s 1/100s

$$
\begin{aligned}
+825.37 &= +800. + 20. + 5. + 0.3 + 0.07\\
+\underline{285.98} &= +200. + 80. + 5. + 0.9 + 0.08\\
&\quad +\underbrace{1000. + 100. + 10.} + \underbrace{1.2 + 0.15}\\
+1111.35 &= \qquad +1110. \qquad + \quad 1.35
\end{aligned}
$$

As they begin writing out the problem in partial sum form, the students are allowed to use abbreviations for hundreds, tens, ones, tenths and hundredths if they prefer. Together, the class works out a system, using capital letters for hundreds and tens, and lower-case letters for tenths and hundredths. "I give the kids a choice on whether they want to abbreviate," says Rachel. "Some learners want to save time, while others love the elegance of writing out the words."

100s 10s 1s 1/10s 1/100s

	+ 825.37		
	+ 285.98		
Partial Sum	+1000.00	= + 800. + 200.	HC
	+100.00	= + 20. + 80.	TC
	+10.00	= + 5. + 5.	OC
	+1.20	= + 0.3 + 0.9	$t^{ths}C$
	+0.15	= + 0.07 + 0.08	$h^{ths}C$
	+1111.35	Final Sum	

The class works out a few more five-digit addition problems before they are given their homework assignment. Each student is given a set of five polyhedra dice to bring home for the evening. "Go home and make up five addition problems by rolling the dice," Rachel tells them. "You must roll at least one of the decimal dice—you cannot just roll whole numbers. I want you to show your work. If you love adding in expanded form, you can write out four of your problems in that form and do one in partial sum form. If you love adding in partial sum form, work out four that way, and write out one problem in expanded form."

Teachers should be prepared for students to do more than the required five problems. "Students want to know if they can write out each problem in both forms, and they ask if they are allowed add more than two numbers together," says Rachel. "When learners are allowed to be more creative with their homework, they want to do more work than they were originally assigned.

Subtraction—Expanded & Partial Difference Form

Students are introduced to subtraction of decimal fractions on paper using the same format they followed in their work with addition. In the first example, however, the numbers are not chosen by rolling the decimal dice. Rather than leaving things to chance, Rachel selects the numbers she would like students to work with in the beginning.

She starts the class off with two three-digit numbers, using tens, ones and tenths.

10s 1s 1/10s

$$\begin{array}{r} +\ 35.4 \\ -\ \underline{26.8} \end{array}$$

"We're going to write this out in expanded form," instructs Rachel. "The top number is addition—it's the amount of money we've added to our wallet—so we need to put addition signs in front of the thirty, the five, and the four-tenths." Next, she expands 26.8, drawing subtraction signs in front of each number. "All these positive and negative symbols are very important later on in algebra." Rachel notes, carefully modeling this for the students in her examples.

10s 1s 1/10s

$$\begin{array}{rcl} +\ 35.4 & = & +\ 30.\ +\ 5.\ +\ 0.4 \\ -\ \underline{26.8} & = & -\ \underline{20.\ -\ 6.\ -\ 0.8} \end{array}$$

"What is thirty subtract twenty?" she asks one learner.

"Ten."

"What is five subtract six?" she asks another.

"Negative one."

"What is four-tenths subtract eight-tenths?"

"Negative four-tenths."

Rachel records the numbers on the overhead projector. "Be sure to you are writing down all the addition and subtraction signs," she reminds the students, "and don't forget the equals sign!"

10s 1s 1/10s

$$\begin{array}{rcl} +\ 35.4 & = & +\ 30.\ +\ 5.\ +\ 0.4 \\ -\ \underline{26.8} & = & -\ \underline{20.\ -\ 6.\ -\ 0.8} \\ & & +\ 10.\ -\ 1.\ -\ 0.4 \end{array}$$

A volunteer helps to work out the problem.

"Ten subtract one is what?"

"Nine."

"And nine subtract four-tenths is what?"

The student thinks for a moment. "Eight and six-tenths."

10s 1s 1/10s

$$\begin{array}{rcl} +\ 35.4 & = & +\ 30.\ +\ 5.\ +\ 0.4 \\ -\ \underline{26.8} & = & -\ \underline{20.\ -\ 6.\ -\ 0.8} \\ & & +\ \underbrace{10.\ -\ 1.}\ -\ 0.4 \\ +\ 8.6 & = & +\ 9.\ -\ 0.4 \end{array}$$

Rewriting the same problem, Rachel shows the class another way think about the numbers in expanded form. "We're going to talk to ourselves," she tells the class. "We're going to ask ourselves, 'What is the easiest number to subtract twenty-six from?'"

"Thirty," suggests one learner." Rachel writes this down:

Step 1.

10s 1s 1/10s

$$\begin{aligned} +\ 35.4 &= +\ 30. \\ -\ \underline{26.8} &= -\ 26. \end{aligned}$$

"And what is the easiest number to subtract the eight-tenths from?"

"One." She writes these numbers down on the overhead:

Step 2.

10s 1s 1/10s

$$\begin{aligned} +\ 35.4 &= +\ 30.\ +\ 1. \\ -\ \underline{26.8} &= -\ 26.\ -\ 0.8 \end{aligned}$$

"So far we've expanded thirty plus one out of our thirty-five and four-tenths," observes Rachel. "How much do we have left to expand?"

"Four and four tenths."

"We don't need subtract anything from four and four-tenths, so we can just subtract zero," she says, writing this out.

Step 3.

10s 1s 1/10s

$$\begin{aligned} +\ 35.4 &= +\ 30.\ +\ 1.\ +\ 4.\ +\ 0.4 \\ -\ \underline{26.8} &= -\ \underline{26.\ -\ 0.8\ -\ 0\ -\ 0} \end{aligned}$$

"Now, see how nicely this works," Rachel tells the class. "Subtract twenty-six from thirty."

"Four," several students reply.

"Subtract eight-tenths from one."

"Two-tenths."

"Subtract zero from four."
"Four."

"Subtract zero from four-tenths."

"Four-tenths."

Step 4.

10s 1s 1/10s

$$\begin{aligned} +\ 35.4 &= +\ 30.\ +\ 1.\ +\ 4.\ +\ 0.4 \\ -\ \underline{26.8} &= -\ \underline{26.\ -\ 0.8\ -\ 0\ -\ 0} \\ &\quad +\ 4.\ +\ 0.2 +\ 4.\ +\ 0.4 \end{aligned}$$

"Let's combine similar terms," says Rachel. "What is four plus four?"

"Eight."

"Now we add the tenths—two-tenths plus four-tenths?"

"Six-tenths."

"There's our answer—eight and six-tenths."

Step 5.

10s 1s 1/10s

$$\begin{aligned} +\ 35.4 &= +\ 30.\ +\ 1.\ +\ 4.\ +\ 0.4 \\ -\ \underline{26.8} &= -\ \underline{26.\ -\ 0.8\ -\ 0\ -\ 0} \\ &\quad +\ 4.\ +\ 0.2 +\ 4.\ +\ 0.4 \\ +\ 8.6 &= \quad +\ 8.\ +\ 0.6 \end{aligned}$$

Finally, they work out the same problem in partial difference form.

10s 1s 1/10s

$$\begin{array}{lllll} & +\ 35.4 & & & \\ & -\ \underline{26.8} & & & \\ \text{Partial} & +\ 10.0 & = & +\ 30. - 20. & \text{TC} \\ \text{Difference} & -1.0 & = & +\ 5. - 6. & \text{OC} \\ & \underline{-0.4} & = & +\ 0.4 - 0.8 & \text{t}^{\text{ths}}\text{C} \\ & +8.6 & \text{Final Difference} & & \end{array}$$

For the next problem Rachel chooses two students to roll the three decimal dice. "We're going to subtract the smallest number from the biggest number," she announces.

"So we're going to subtract seventy-six and nine-tenths from ninety-one and two-tenths," Rachel says, writing. She shows them a number of ways to work out the problem, talking them through the process step by step.

Expanded Form:

$$
\begin{array}{rcl}
 & & 10s\ 1s\ \tfrac{1}{10}s \\
+\ 91.2 & = & +\ 90. + 1. + 0.2 \\
-\ \underline{76.9} & = & -\ \underline{70. - 6. - 0.9} \\
 & & +\ \underbrace{20. - 5.} - 0.7 \\
+\ 14.3 & = & +\ 15. - 0.7
\end{array}
$$

OR

$$
\begin{array}{rcl}
 & & 10s\ 1s\ \tfrac{1}{10}s \\
+\ 91.2 & = & +\ 80. + 10. + 1. + 0.2 \\
-\ \underline{76.9} & = & -\ \underline{70. - 6. - 0.9 - 0} \\
 & & +\ \underbrace{10. + 4.} + \underbrace{0.1 + 0.2} \\
+\ 14.3 & = & +\ 14. + 0.3
\end{array}
$$

OR

$$
\begin{array}{rcl}
 & & 10s\ 1s\ \tfrac{1}{10}s \\
+\ 91.2 & = & +\ 80. + 10. + 1. + 0.2 \\
-\ \underline{76.9} & = & -\ \underline{76. - 0. - 0.9 - 0} \\
 & & +\ \underbrace{4. + 10.} + \underbrace{0.1 + 0.2} \\
+\ 14.3 & = & +\ 14. + 0.3
\end{array}
$$

Partial Difference Form:

$$
\begin{array}{rrl}
 & 10s\ 1s\ \tfrac{1}{10}s & \\
 & +\ 91.2 & \\
 & -\ \underline{76.9} & \\
\text{Partial} & +\ 20.0 & = +\ 90. - 70.\quad \text{TC} \\
\text{Difference} & -\ 5.0 & = +\ 1. - 6.\quad \text{OC} \\
 & -\ \underline{0.7} & = +\ 0.2 - 0.9\quad \text{t}^{\text{ths}}\text{C} \\
 & +\ 14.3 & \text{Final Difference}
\end{array}
$$

5-digit Subtraction

After they have worked out several subtraction problems using tens, ones and tenths, the students are eager to add the hundreds and hundredths dice into the mix.

$$
\begin{array}{r}
100s\ 10s\ 1s\ \tfrac{1}{10}s\ \tfrac{1}{100}s \\
+\ 621.58 \\
-\ \underline{393.53}
\end{array}
$$

"If we think about our money, what is the easiest thing to subtract three-hundred ninety-three from?" asks Rachel.

"Four hundred." On the overhead, Rachel writes:

Step 1.

$$
\begin{array}{rcl}
100s\ 10s\ 1s\ \tfrac{1}{10}s\ \tfrac{1}{100}s & & \\
+\ 621.58 & = & +\ 400. \\
-\ \underline{393.53} & = & -\ 393.
\end{array}
$$

"Now, what is the easiest way to subtract fifty-three hundredths?"

"From fifty-eight hundredths." Rachel writes this out:

Step 2.

$$
\begin{array}{rcl}
100s\ 10s\ 1s\ \tfrac{1}{10}s\ \tfrac{1}{100}s & & \\
+\ 621.58 & = & +\ 400. + 0.58 \\
-\ \underline{393.53} & = & -\ 393. - 0.53
\end{array}
$$

"We've expanded four hundred and fifty-eight-tenths out of six hundred twenty-one and fifty-eight-tenths," says Rachel. "How much more do we have to add to the right-hand side to make it equal with the left-hand side?"

"Two hundred twenty-one." She writes this down.

"We don't have anything left to subtract from two hundred twenty-one," Rachel observes, "so we can just put a zero." She writes:

Step 3.

$100s\ 10s\ 1s\ \frac{1}{10}s\ \frac{1}{100}s$

$$\begin{array}{rcl} +\ 621.58 & = & +\ 400.\ +\ 0.58\ +\ 221. \\ -\ \underline{393.53} & = & \underline{-\ 393.\ \ -\ 0.53\ \ -\ \ 0\ \ } \end{array}$$

"Four hundred subtract three-hundred ninety-three is what?" Rachel asks.

"Seven."

"Fifty-eight hundredths subtract fifty-three hundredths?"

"Five-hundredths."

"And two hundred twenty-one subtract zero?"

"Two hundred twenty-one."

Step 4.

$100s\ 10s\ 1s\ \frac{1}{10}s\ \frac{1}{100}s$

$$\begin{array}{rcl} +\ 621.58 & = & +\ 400.\ +\ 0.58\ +\ 221. \\ -\ \underline{393.53} & = & -\ \underline{393.\ \ -\ 0.53\ \ -\ \ 0} \\ & & +\ \ 7.\ +\ 0.05\ \ +\ 221. \end{array}$$

"Let's combine similar terms," instructs Rachel. "When we add seven and two-hundred together, what do we get?"

"Two hundred twenty-eight."

"Add five-hundredths and we have an answer of how much?"

"Two-hundred twenty-eight and five-hundredths."

Step 5.

$100s\ 10s\ 1s\ \frac{1}{10}s\ \frac{1}{100}s$

$$\begin{array}{rcl} +\ 621.58 & = & +\ 400.\ +\ 0.58\ +\ 221. \\ -\ \underline{393.53} & = & -\ \underline{393.\ \ -\ 0.53\ \ -\ \ 0} \\ & & +\ \ 7.\ +\ 0.05\ \ +\ 221. \\ +\ 228.05 & = & +\ 228.\ \ +\ 0.05 \end{array}$$

"Now let's try it in partial difference form."

$100s\ 10s\ 1s\ \frac{1}{10}s\ \frac{1}{100}s$

$$\begin{array}{lrl} & +\ 621.58 & \\ & -\ \underline{393.53} & \\ \text{Partial Difference} \left\{ \begin{array}{l} \\ \\ \\ \\ \\ \end{array} \right. & \begin{array}{r} +\ 300.00 \\ -\ 70.00 \\ -\ 2.00 \\ -\ 0.00 \\ \underline{+\ 0.05} \end{array} & \begin{array}{ll} =\ +\ 600.\ -\ 300. & \text{HC} \\ =\ +\ 20.\ -\ 90. & \text{TC} \\ =\ +\ 1.\ -\ 3. & \text{OC} \\ =\ +\ 5.\ -\ 5. & \text{t}^{\text{ths}}\text{C} \\ =\ +\ 0.08\ -\ 0.03 & \text{h}^{\text{ths}}\text{C} \end{array} \\ & +\ 228.05 & \text{Final Difference} \end{array}$$

One of the next rolls produces a problem with some zeros in the larger number.

$100s\ 10s\ 1s\ \frac{1}{10}s\ \frac{1}{100}s$

$$\begin{array}{r} +\ 304.05 \\ -\ \underline{165.97} \end{array}$$

"Let's think," says Rachel. "What is the easiest thing to subtract one hundred from?"

"One hundred?" a learner suggests.

"Sure," she agrees, writing this down on the overhead in expanded form.

Step 1.

$100s\ 10s\ 1s\ \frac{1}{10}s\ \frac{1}{100}s$

$$\begin{array}{rcl} +\ 304.05 & = & +\ 100. \\ -\ \underline{165.97} & = & -\ 100. \end{array}$$

"What is the easiest thing to subtract sixty-five from?

"Seventy." She writes this next step out:

Step 2.

$100s\ 10s\ 1s\ \frac{1}{10}s\ \frac{1}{100}s$

$$\begin{array}{rcl} +\ 304.05 & = & +\ 100.\ +\ 70. \\ -\ \underline{165.97} & = & -\ 100.\ \ -\ 65. \end{array}$$

"And what is the easiest way to subtract ninety-seven hundredths?" Rachel presses.

"One."

She writes this down:

Step 3.

$$
\begin{array}{l}
\quad 100s\ 10s\ 1s\ \tfrac{1}{10}s\ \tfrac{1}{100}s \\
+\ 304.05 = +\ 100. + 70. + 1. \\
-\ \underline{165.97} = -\ 100.\ -\ 65.\ -\ 0.97
\end{array}
$$

"We've used one used one hundred seventy one," says Rachel, "But we want to get up to three hundred four and five-tenths. Let's clean up the hundreds first. How much do I need to add to get to three hundred?"

"One hundred thirty."

"Okay, now we're up to three hundred one," Rachel calculates, "and I want to get to three hundred four and five-hundredths."

"Add three and five-hundredths," a student advises.

"Since we are going to be subtracting zero, let's combine those numbers." She writes the number as 133.05.

Step 4.

$$
\begin{array}{l}
\quad 100s\ 10s\ 1s\ \tfrac{1}{10}s\ \tfrac{1}{100}s \\
+\ 304.05 = +\ 100. + 70. + 1.\ +\ \ 133.05 \\
-\ \underline{165.97} = -\ \underline{100.\ -\ 65.\ -\ 0.97\ \ -\ \ 0.00}
\end{array}
$$

"Now, let's see how this works out," says Rachel. "One hundred subtract one hundred is what?"

"Zero!"

"Seventy subtract sixty-five?"

"Five."

"One subtract ninety-seven hundredths?"

"Three-hundredths."

"One hundred thirty-three and five hundredths subtract zero?"

"One hundred thirty-three and five hundredths."

Step 5.

$$
\begin{array}{l}
\quad 100s\ 10s\ 1s\ \tfrac{1}{10}s\ \tfrac{1}{100}s \\
+\ 304.05 = +\ 100. + 70. + 1.\ +\ \ 133.05 \\
-\ \underline{165.97} = -\ \underline{100.\ -\ 65.\ -\ 0.97\ \ -\ \ 0.00} \\
\qquad\qquad\qquad\quad 0. +\ 5. + 0.03 + 133.05
\end{array}
$$

Adding the numbers together, the students arrive at an answer of one hundred thirty-eight and eight hundredths.

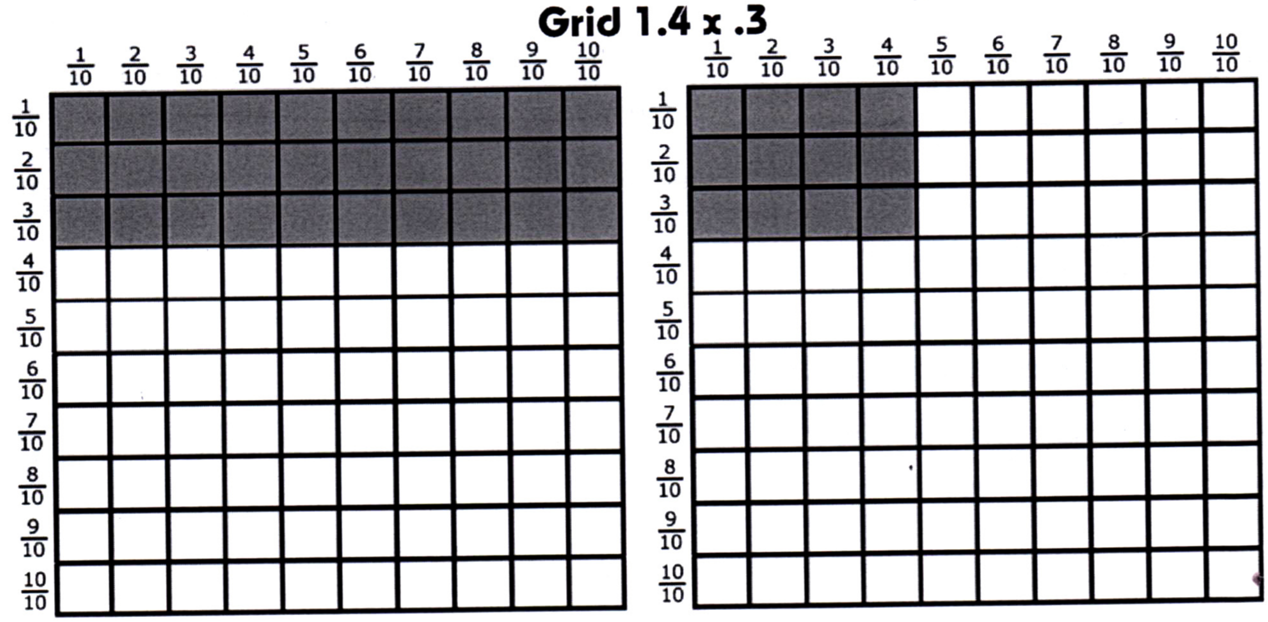
$$
\begin{array}{l}
\quad 100s\ 10s\ 1s\ \tfrac{1}{10}s\ \tfrac{1}{100}s \\
+\ 304.05 = +\ 100. + 70. + 1.\ +\ \ 133.05 \\
-\ \underline{165.97} = -\ \underline{100.\ -\ 65.\ -\ 0.97\ \ -\ \ 0.00} \\
\qquad\qquad\qquad\quad 0. + \underbrace{5. + 0.03} + \underbrace{133.05} \\
+\ 138.08 = \qquad\qquad\quad 5. \quad + \quad 133.08
\end{array}
$$

"Isn't that beautiful?" Rachel says proudly. "And we didn't have to cross out any numbers!

Multiplication—Beyond "B-CUZ"

When it comes to the multiplication of decimals, many learners perform the operation with no understanding of the mathematics involved. Moving a decimal point without

Grid 1.4 x .3



knowing the reason is a practice used in what Rachel derisively calls the "B-CUZ" theory of mathematics. "Why do you do it? B-CUZ." The unwitting action falls into the same senseless category as "crossing out," "carrying," and "borrowing. "There is a mathematical reason we move the decimal point," Rachel assures students and teachers alike, "and it is not B-CUZ."

On the overhead, she writes a simple example:

$$\begin{array}{r} .1 \\ \times\ \underline{.1} \\ .01 \end{array}$$

"Most people looking at this problem would write down a one and blindly move the decimal over two places to the left, giving them an answer of one one-hundredth," observes Rachel. "They have absolutely no idea why they've moved it—it's just a rule."

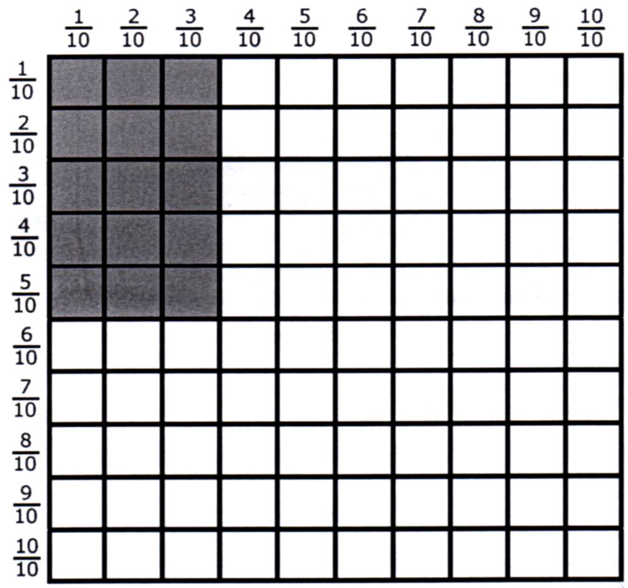
Next she writes:

$$\frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$$

"If we look at the same problem using fractions instead of decimals—multiplying the numerator by the numerator and the denominator by the denominator—we can see that one times one is one, and ten times ten is one hundred, giving us an answer of one one-hundredth," Rachel explains.

She selects a new example. "Let's look at three-tenths times five-tenths." On the overhead, she draws a 10 x 10 grid and marks it in 1/10 increments. Using her pen, Rachel demonstrates, "If I go over three-tenths, and down five-tenths and I shade in that area, it should be made up of fifteen squares. Fifteen out of how many little squares in this big square? Fifteen out of one hundred. Fifteen-hundredths."

Grid .3 x .5



She writes:

$$\frac{15}{100} = .15$$

"Let's look at another one." Rachel writes out a new problem on the overhead.

$$\begin{array}{r} 1.4 \\ \times\ \underline{.3} \end{array}$$

"If we look at this problem using the grid, it's telling me that I need to go over one and four-tenths, and then come down three tenths," she says. "Oh goodness, I can only go across ten-tenths. I need to draw another grid so I can go over another four-tenths." Using two grids, she counts over and down, then shades in the rectangles. See Grid 1.4 x .3 on page 10.

Returning to the numbers, she works out the problem using partial product form.

$$\begin{array}{rll} & 1.4 & \\ & \times\ \underline{.3} & \\ \text{Partial Product} \left\{ \begin{array}{l} \\ \\ \end{array} \right. & \begin{array}{r} .3 \\ \underline{.12} \end{array} & \begin{array}{l} = .3 \times 1. \\ = .3 \times .4 \end{array} \\ & .42 & \text{Final Product} \end{array}$$

"Forty-two hundredths," she announces. "Going back to our shaded grid, how many squares did we cover? Forty-two."

"Notice that the decimal point always moves two to the left when we multiply tenths times tenths," says Rachel after they have worked on a few more examples. "What do you think will happen if we multiply tenths times hundredths?"

She writes:

$$\begin{array}{r} .32 \\ \times\ \underline{.4} \end{array}$$

Working from the left, she multiplies four-tenths by three-tenths. Rachel writes this out in fractional form. "Four over ten times three over ten is what?" she asks.

"Twelve hundredths."

$$\begin{array}{r} .32 \\ \times\ \underline{.4} \\ .12 \end{array} = \frac{3}{10} \times \frac{4}{10}$$

"Now let's multiply four-tenths times two-hundredths." She writes out the fractions so students can multiply the numerators and denominators. "What is four times two?"

"Eight."

"What is ten times one hundred?"

"One thousand."

Rachel writes the result in decimal form. "Eight one-thousandths."

$$\begin{array}{rrcl} & .32 & & \\ & \times\ \underline{.4} & & \\ \text{Partial} & .12 & = & \frac{3}{10} \times \frac{4}{10} \\ \text{Product} & \underline{.008} & = & \frac{4}{10} \times \frac{2}{100} \end{array}$$

"One tenth plus zero tenths is one tenth," she reads, adding the partial products together. "Two hundredths plus zero hundredths is two-hundredths. Zero plus eight-thousandths is eight-thousandths. One hundred twenty-eight thousandths.

$$\begin{array}{rrcl} & .32 & & \\ & \times\ \underline{.4} & & \\ \text{Partial} & .12 & = & \frac{3}{10} \times \frac{4}{10} \\ \text{Product} & \underline{.008} & = & \frac{4}{10} \times \frac{2}{100} \\ & .128 & & \text{Final Product} \end{array}$$

"Remember when we multiplied tenths times tenths, we noticed that the decimal point was always moved two places to the left," says Rachel. "When we multiply tenths times hundredths, the decimal moves three places to the left. When you take something that is less than one, and multiply it times something less than one, you're answer will be waaaaaaaaay less than one," she reasons.

Using the same numbers, Rachel writes out the problem in expanded form.

$$\begin{array}{rcl} \text{SF} & = & \text{EF} \\ .32 & = & .3 + .02 \\ \times\ \underline{.4} & = & \underline{\quad \times .4 \quad} \\ .128 & = & .12 + .008 \end{array}$$

Next, she decides to try expanding the problem in fractional form.

$$\begin{array}{rcl} .32 & = & \frac{3}{10} + \frac{2}{100} \\ \times\ \underline{.4} & = & \underline{\times \frac{4}{10} \qquad} \end{array}$$

"Three-tenths times four tenths is twelve over one hundred. Two hundredths times four tenths is eight over one thousand," she says, writing as she talks.

$$\begin{array}{rcl} .32 & = & \frac{3}{10} + \frac{2}{100} \\ \times\ \underline{.4} & = & \underline{\times \frac{4}{10} \qquad} \\ & & \frac{12}{100} + \frac{8}{1000} \end{array}$$

Rachel considers the two fractions. "Before we can add our fractions together, we must find a common denominator," she says. "If we multiply twelve-tenths by Secret Agent One in the form of ten over ten, we will have a common denominator of one thousand."

$$\begin{array}{rcl} .32 & = & \frac{3}{10} + \frac{2}{100} \\ \times\ \underline{.4} & = & \underline{\times \frac{4}{10} \qquad} \\ & & \frac{12}{100} \times \frac{10}{10} + \frac{8}{1000} \end{array}$$

"We multiply the numerators—twelve times ten is one hundred twenty," explains Rachel, writing. "Then we multiply the denominators—ten times ten is one thousand. Now we can add our fractions together. One hundred twenty-thousandths plus eight-thousands is one hundred twenty-eight thousands.

$$
\begin{aligned}
.32 &= \frac{3}{10} + \frac{2}{100} \\
\times\ \underline{.4} &= \times \frac{4}{10} \\
& \underbrace{\frac{12}{100} \times \frac{10}{10}} + \frac{8}{1000} \\
.128 &= \frac{120}{1000} + \frac{8}{1000}
\end{aligned}
$$

Distributive Form

Rachel thinks of still another way to write the problem, using the distributive property:

$$\frac{4}{10}\left(\frac{3}{10} + \frac{2}{100}\right)$$

"We need to rewrite this so that everything in our parentheses has the same denominator," she tells students. "I know that three-tenths is equal to thirty-hundredths, so I am going to multiply by Secret Agent One in the form of ten over ten." The problem now reads:

$$
\begin{aligned}
&\frac{4}{10}\left(\underbrace{\frac{3}{10} \times \frac{10}{10}} + \frac{2}{100}\right) \\
&\frac{4}{10}\left(\frac{30}{100} + \frac{2}{100}\right)
\end{aligned}
$$

"Now I can distribute. Four-tenths times thirty one-hundredths is one hundred twenty-thousandths, plus four-tenths times two-hundredths is eight-thousandths. When I add those two fractions together, I still have one hundred twenty-eight-thousandths!" she exclaims, writing the answer in both rational and decimal form.

$$\frac{4}{10}\left(\frac{30}{100} + \frac{2}{100}\right) = \underbrace{\frac{120}{1000} + \frac{8}{1000}}_{\frac{128}{1000} \text{ or } .128}$$

Students enjoy rolling the decimal dice to generate numbers for multiplication problems in class and at home. For the first number, they may decide to roll the hundredths and tenths dice together, and then roll only the tenths die to get their second number.

Learners are directed to justify their homework problems in whichever form they prefer. "The only rule is that you must show your work," says Rachel. "Putting down and carrying is not showing your work."

Dividing by Decimals

When dividing decimal fractions, learners are often taught to employ the B-CUZ method used in multiplication. Once again, decimal points are moved around without much explanation or understanding. Rachel offers the following division problem for consideration:

$$2.5 \overline{)\,1.75}$$

"The way that this kind of problem is traditionally taught, students are told to move the decimal point in two and five-tenths all the way to the right, making it read twenty-five," she says. "Next, learners are told that they must also move the decimal point in one and seventy-five hundredths, so that it now reads seventeen and five-tenths. They go through all this movement of the decimal point, and then are instructed to follow the "Dead Mice Smell Bad" method of division. Why? B-CUZ."

In order to clarify the reasoning behind moving the decimal point, Rachel rewrites the problem as a rational number:

$$\frac{1.75}{2.5}$$

"Mathematicians work very hard to try and make numbers rational*," she explains, "but they do not like to have decimals or fractions in the denominator. Looking at the rational number one and seventy-five hundredths over two and five tenths, we know that mathematicians would like the denominator to be a whole number—they would like it to be twenty-five."

Rachel appears to puzzle over this idea for a moment. "What we need to do is multiply that fraction by some secret number," she muses. "We need a number you can multiply or divide any number by and not change it's value."

"One?" a student ventures.

"Of course," says Rachel. "Secret Agent One." She draws the large outline of the character on the overhead projector. "What can we multiply two and five-tenths by so that it will be a whole number?

"Ten," answer several students.

"Yes, but we can't just multiply by ten, can we? We have to multiply by ten over ten—Secret Agent One."

$$\frac{1.75}{2.5} \times \frac{10}{10} = \frac{17.5}{25.}$$

"When we move the decimal point, what we are really doing is multiplying by Secret Agent One," Rachel says simply.

"Couldn't you multiply by one in the form of one hundred over one hundred?" a student asks Rachel. " You could have all whole numbers."

"Sure you can," she tells him.

$$\frac{1.75}{2.5} \times \frac{100}{100} = \frac{175}{250}$$

"Usually when you are working from a text book, the authors want your answer in decimal form, not fraction form. We can simplify the number by dividing by Secret Agent One in the form twenty-five over twenty-five. If we divide one hundred seventy-five by twenty-five, what do we get?"

"Seven."

She writes draws a line and writes 7 above it. "What is two hundred fifty divided by twenty-five?"

"Ten."

Rachel writes ten below the line, giving her a new rational number.

$$\frac{175}{250} \div \frac{25}{25} = \frac{7}{10}$$

"Seven-tenths!" she exclaims. "How do we write that in decimal form?"

$$2.5\overline{\smash{)}1.75}^{\;0.7}$$

"Let's do another one," says Rachel, choosing numbers for a new example.

$$.09\overline{\smash{)}3.6}$$

She rewrites this as a rational number:

$$\frac{3.6}{.09}$$

"All the mathematicians are shouting boo-hiss, boo-hiss!" says Rachel. "They don't want a decimal fraction in the denominator. So what is the only thing we can do?

"Multiply by one," says one learner.

"In what form?" Rachel asks.

"One hundred over one hundred."

$$\frac{3.6}{.09} \times \frac{100}{100} = \frac{360}{9}$$

"Three hundred sixty divided by nine is what?"

"Forty."

"Exactly right," confirms Rachel. "Our answer is forty."

$$.09\overline{\smash{)}3.6}^{\;40.}$$

"What the division problem is really asking," Rachel explains, "is how many nine one-hundredths are there in three and six-tenths? Nine one-hundredths is a teensy fraction. It takes a lot of nine-hundredths to fit into three and six-tenths." She draws dollar signs in front of the numbers.

$$\$.09\overline{\smash{)}\$3.6}^{\;40.}$$

"If we look at this in terms of money, we are asking how many groups of nine cents are in three dollars and sixty cents," continues Rachel. "Does our answer make sense? Are there a bunch of groups of nine pennies in three dollars and sixty cents? Yes—forty of them."

She selects another pair of numbers and writes them on the overhead.

$$.5\overline{\smash{)}4.5}$$

"Four and five-tenths divided by five-tenths." Rachel writes out the numbers again in rational form.

$$\frac{4.5}{.5}$$