Fair Share

A Rational Approach to Division!

Topic Involved: multiples, division, number sense, algebraic thinking.

Grade Level: Certain aspects of this lesson can be applied from as early as kindergarten and up.

Materials: Pencil and paper, base-ten blocks (students). Overhead projector, transparencies, base-ten blocks (teacher).

Type of Activity: Group

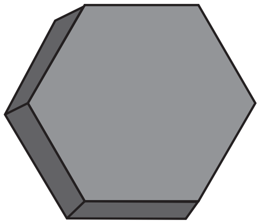
Relation to NCTM Standards: Understanding the meanings of operations and how they relate to one another, developing fluency in dividing whole numbers, understanding fractions as a division of numbers.

Fair Share

"Division means everyone gets a fair share—no one gets any more or any less than anyone else." It is a simple statement that is delightfully characteristic of WI's resident mathematician, Rachel McAnallen. An educator for over 40 years, Rachel is also a life-long learner—her lessons reflect a continuous ability to learn from colleagues and students alike. The end result is Rachel's straightforward, enthusiastic approach to numbers and their operations, and this issue's lesson on division is no exception.

"Years ago a friend of mine who was a kindergarten teacher told me that her students could divide better than my seventh and eighth graders," Rachel recalls. "Of course, I was highly offended by that." Despite her initial reaction, Rachel went to her friend's class to observe. As Rachel looked on, the kindergarten teacher put a large quantity of pattern blocks out on the floor and gathered her students around the pile. She told them, "You can't play with the pattern blocks until we have fair share. What do you want to fair share first?" "Yellow," the students told her. "How many yellow do you think everyone will get?" she asked them. Most students guessed two or three. "Everyone go in and get three yellow hexagons," she instructed.

"The kids all dove in," Rachel remembers, "and they all picked out three yellow pattern blocks and there were still a lot left in the pile."



"What should we do?" asked the teacher. "Go in and get more," the students told her. "How many more should you get?" she asked.

"They looked, and they estimated, and they decided they should get two more each," says Rachel. "They kept going in and getting more yellow until there weren't enough to go around. They never counted anything."

With eight hexagons left in the pile, the teacher asked her students, "What should we do?" "Those are the leftovers," the students told her.

"They talked about what they should do with the remainder," says Rachel. "They didn't want to cut the hexagons up, so they decided to put the eight remaining blocks off to the side. No one could play with those blocks, because there weren't enough to fair share."

Their teacher gave the students the choice of playing with the hexagons or continuing to divvy up the pattern blocks. The students chose to fair share the red trapezoids.

Rachel watched with interest as one learner suggested that everyone take twenty trapezoids. "Everyone go in and get 20 trapezoids," the teacher instructed.

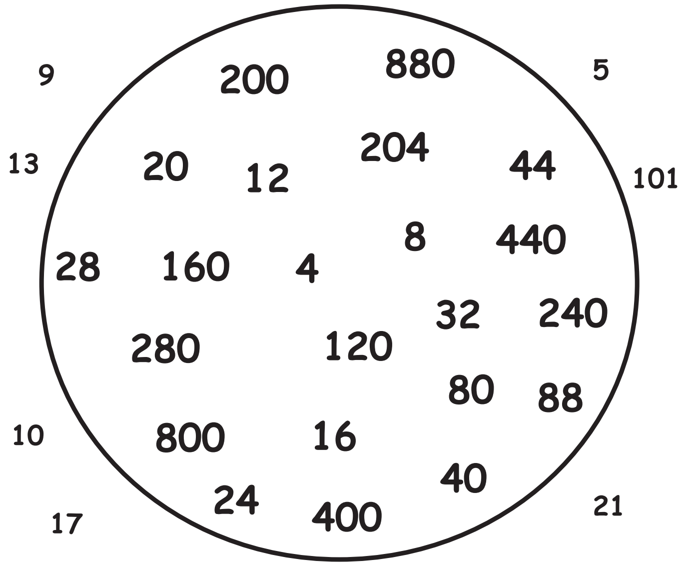
"Of course there weren't enough to go around," recalls Rachel, "but the teacher just let it occur. It was a beautiful thing to watch. Some kids got twenty, and other students didn't have enough." The class was quick to recognize that this was not fair share. The teacher asked her class what they should do to make it fair share. "There were loads of suggestions," says Rachel. "I don't remember exactly how they solved it, but it was quite wonderful. They fair shared every one of those pattern blocks and then they were allowed to play with them."

"After watching that, I realized my friend was absolutely right," says Rachel. "Her kindergartners could divide better than my seventh graders. MY students were just doing an algorithm on paper, but her students really understood what division was all about. As a result, I went back to my own students and I began to teach division quite differently."

Multiples—a Circle of Friends

In order to learn division students must know the multiples of numbers. Rachel draws a large circle on the overhead projector.

Multiples of 4
(aka as Friends of 4)



"We want to know what the multiples of 4 are," she tells students. "I also call them friends of four." While she was teaching this lesson in South Africa at a school for the deaf and hearing impaired, Rachel was told that there was no sign word for the term multiple. "I wrote the multiples of four on the board and asked the students what they thought we ought to call them. They decided to call them the friends of four." Rachel liked their new terminology and has permanently incorporated the phrase into her lesson. Above the large circle she now writes "Multiples or Friends of 4."

"Knowing the multiples of a number isn't the same as knowing the multiplication tables," notes Rachel. "If students only write down the multiplication tables of four, they will only write them as far as they have memorized." As the class suggests multiples of four, Rachel writes them within the circle. "Eight is a friend of four," says one student. "Yes," responds Rachel, "So is eighty, and so is eight hundred." "How about eighty-eight?" asks a student. Rachel writes this down and then asks, "Who is not a friend of four?" "Five," a student tells her. Outside of the circle, Rachel writes 5. "Sixteen is a friend of four, but seventeen isn't," she says, continuing to write. "One hundred is, but one hundred one is not."

Rachel suggests that teachers display a large sheet of paper in the classroom with circles for the multiples of 2 through 12. Students can then add multiples to the sheet as they think of them. "If they think of a multiple of four, they can just go up and write it in the appropriate circle," she explains. "They can also develop divisibility rules as they look at the paper. How do you know when a number is divisible by four? How do you know if a number is divisible by 3?"

Once the class has filled the circle with a good amount of friends of four, Rachel decides they are ready to move on. On a new transparency, she writes the number 28. "Let's write a math sentence for twenty-eight in terms of multiples of four," she says. "We know that twenty-eight is already a multiple of four, so we could write it as twenty eight equals twenty-eight."

28=28

"But let's look at other multiples of four," Rachel suggests. "Twenty-eight equals twenty-four plus four."

28=24+4

Students catch on quickly, coming up with their won sentences:

28=20+4+4

28=20+8

28+12+12+4

28=4+4+4+4+4+4+4

Rachel reminds students that they are not restricted to using multiples that are less than twenty-eight:

28=32-4

"There is a new term called chunking that some teachers use in reading," comments Rachel. "Chunking is when you break a word down. A teacher who was watching me teach students to break down a number as we've done here with twenty-eight, said to me, 'That's like chunking with numbers."

In the beginning the students are rewriting the number in terms of multiples of a single digit number. She cautions teachers from going to bigger numbers too soon. "It's important that the teacher does not finish the lesson way before the kids do."

Next Rachel chooses a number that is not evenly divisible by four:

35

"Let's write 35 in terms of numbers that are friends of four," she says. "What are all the ways we can think of to write thirty-five using multiples of four?" She writes an example:

35=32+3

"The number you have left over should be less than the multiple you are working with," Rachel explains. "If I'm writing in terms of multiples of four, this number," she points to the three, "has to be less than four, or you need to continue chunking."

35=8+8+8+8+3

and

35=28+4+4-1

Students come up with several sentences, for instance:

Rachel selects a bigger number:

41

Let's write forty-one in terms of multiples of three," she instructs. After the class creates three sentences, Rachel announces, "You have a homework assignment tonight, and the answer is forty-one! Give me three ways to write forty-one in terms of multiples of four." She asks them to write three sentences for 41 using multiples of five, of six, of seven, of eight, and of nine. "That's twenty-one sentences," she clarifies. "Look how much we can get from just forty-one!"

When teachers start division very early with children, the beginning learners don't have to write anything down. Primary teachers can teach fair share with kids using pattern blocks, or unifix cubes—they can use anything they want in their classroom. The creative teacher can find all sorts of things to fair share in their classrooms.

Putting the Remainder in Context

When young learners to fair share using pattern blocks or other manipulatives as Rachel observed in her friend's kindergarten classroom, they make decisions about the remainder without writing anything out. For instance, if the item can be cut up, the students can divvy the pieces amongst themselves as a fraction. If it cannot be divided, they set the remainder aside. Teaching division in this kind of context—asking students to think about what it is they are dividing—is a lesson that Rachel emphasizes at every grade level.

"I am absolutely amazed at the way we deal with the remainder in education," she says. "Traditionally we teach students that what they do with the remainder depends on what grade they are in. Fourth-grade students write the remainder as "R", then fifth-grade students are taught they can write a fraction, and in sixth grade they learn to write it out as a decimal."

During teacher workshops, Rachel will often create a question that typically appears in standardized tests. "Eleven people are going on a field trip. A car holds four people. How many cars will they need?" She writes out the multiple choice answers:

a) 2

b) 2 $\frac{3}{4}$

c) 3

"This is a wonderful question," says Rachel. "The answer, of course, is three cars. But our kids constantly put down 2 ¾ cars." Students who make this mistake have grown accustomed to looking only at the numbers in a problem, she explains. They don't stop to realize that four people aren't going to get very far in ¾ of a car.

"Remember, in math, numbers are adjectives," Rachel points out. "Students need to know what noun they are dividing. If I have twenty-six balloons and I want to fair share them among five people, each person gets five balloons. Now, let's think—are we going to cut up the remaining balloon and write it as a fraction? Of course not. It's a remainder."

When Rachel works on this concept in the classroom, she asks students, "What are some things that we would fair share but wouldn't want to cut up into pieces?" The students create a long list: Books, trading cards, bikes, and so on. "What are things we would fair share that we could break into pieces to make a fraction?" Rachel asks. "Cookies!" suggests one student. "Sure," says Rachel with a smile. "If we have twenty-six cookies divided among five people, everybody gets five cookies with one cookie left over. Do you think that one left over cookie is going to last very long?" "No!" the class laughs. "No, right away we're going to divide it into five pieces!" she says. "Everyone would get 5 1/5 cookies."

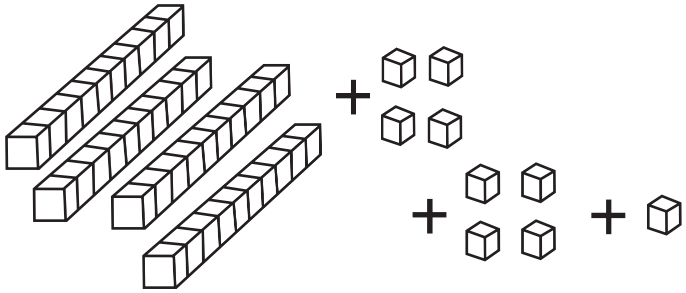
"The remainder has nothing to do with what grade a student is in, or how old they are," reasons Rachel. "It has to do with the noun. What is it they are divvying up? When we teach division we need to put the nouns into the problems, and tell our students to treat the remainder according to the noun they are working with."

From Base-ten to Pen and Paper

Money is one of the most familiar nouns that we describe using numbers. When the class has finished their work writing a number in terms of the multiples of another number, Rachel brings out the base-ten blocks to teach them how to fair share money. The small cube represents one dollar, and the long ten is worth $10. She places four tens and nine ones on the overhead projector. Next she selects four students to come up to the front of the room. "We've got forty-nine dollars, and we need to fair share this money between you four," she explains. "Everyone gets the same amount—nobody gets any more or any less." Each student comes up and takes $10. Next, they each take $1, then they go around and take $1 again. "They do not automatically take $2," notes Rachel.

Now there is just the $1 cube remaining. "What do you want to do with this one dollar?" asks Rachel. "Do you want me to keep it?" "No!" the four students tell her quickly. "We want to cut it into parts." "You want me to cut it into five parts?" Rachel asks them. They give her looks of disbelief. "There are only four of us," explains one girl in the group. Rachel mimics cutting the small cube into four parts, and uses her fingers to cover all but ¼ of the $1. "We call that one fourth, or one quarter," she explains. "Everyone has twelve dollars and then you each get a quarter, so everyone has twelve dollars and twenty-five cents."

The four students return to their chairs. "Now we're going to write down what they did," says Rachel. "We had forty-nine dollars fair shared among 4 people. How did they chunk this forty-nine? The first thing they did was fair share the forty dollars," she says, writing. "Then they fair shared the four dollars. Then they shared another four dollars, and there was one dollar left over."



$$4p\,\overline{\big)\,\$40+4+4+1}$$

Rachel is careful to write out the process exactly as the students performed it with the blocks. J "I'm not going to write that they fair shared eight dollars-that's not how the kids did it. They fair shared four twice," says Rachel.

She asks the class, "When they fair shared the forty dollars, how much did they each get?" "Ten dollars." She writes this down. "What did they get when they fair shared the first four dollars?" "One dollar." "Then what did they get when they fair shared the next four dollars? "One Dollar." 'Okay, and finally, they each got a quarter, right?"

$$\begin{array}{rl} & \$10+1+1+\frac{1}{4}\quad=\$12.25 \\ 4p & \overline{\big)\,\$40+4+4+1} \end{array}$$

Teaching students to write out or "chunk" a problem this way came from watching them work with the base-ten blocks. "It isn't anything brilliant on my part," she says matter-of-factly. "I would give them the base-ten blocks. I would give them the situation, and tell them to solve it. After watching students use the base-ten blocks, I realized that what they do makes a lot more sense than the way I was taught in the 1950s," she laughs. "I am just transferring the efficient way that students do the problems into the symbolic."

Still using the base-ten blocks on the overhead, Rachel shows students that there are other ways to break up $49. "Since we're dividing by four people,

$$4p\,\overline{\big)\,\$20+20+8+1}$$

and fives are easy, we could chunk it into groups of twenty," she says. "So we'd have":

$$\begin{array}{rl} & \$5+5+2+\frac{1}{4} \\ 4p & \overline{\big)\,\$20+20+8+1} \end{array}$$

Once students have practiced with the base-ten blocks, Rachel sets them aside. J "Let's suppose we want to divide thirty-seven cookies by three people," she says.

$$3p\,\overline{\big)\,37\text{cookies}\quad}$$

"We're going to write thirty-seven using friends of three," she explains. "Say that number," she urges the class. "Thirty. Seven. We have thirty," she says, writing, "and then seven...let's use six as a friend of 3, plus one."

$$3p\,\overline{\big)\,30c+6c+1c}$$

"What's thirty divided by 3?" asks Rachel. "Ten." "And six divided by three?" "Two." "What do we have left?" "One." "If we're fair sharing cookies, we'll cut that into thirds. Everyone gets twelve and one third cookies."

$$\begin{array}{rl} & 10c+2c+\frac{1}{3}c\quad=12\tfrac{1}{3}c \\ 3p & \overline{\big)\,30c+6c+1c} \end{array}$$

"What's another way we could have done that?" Rachel asks. One student wants to write 37 in terms of 9.

$$\begin{array}{rl} & 3c+3c+3c+3c+\frac{1}{3}c \\ 3p & \overline{\big)\,9c+9c+9c+9c+1c} \end{array}$$

"What's our answer?" "Twelve and one third cookies," he tells her. "Oh my gosh, we got the same thing!" she exclaims.

For the next problem Rachel makes the dividend a bit bigger. "Let's divide seventy-nine of something," she says, "Seventy-nine what?" "Pokéman cards," says one student. "Okay, let's divide seventy-nine Pokéman cards by four kids."

$$4k\,\overline{)\,79p}$$

"What is a friend of four?" Rachel asks. "Forty." She writes this down. "What's another friend of four?" "Twenty." Rachel continues asking for multiples of four until they have broken it up into the following:

$$\begin{array}{rl} & 10p+5p+3p+1p+R3p \\ 4k & \overline{)\,40p+20p+12p+4p+3p} \end{array}$$

$$=19p+R3p$$

"Everyone gets nineteen Pokéman cards, with a three left over, or R3," says Rachel. "If it were cookies, everyone would still get nineteen cookies, but then we'd cut each of the three cookies into four pieces. So everyone would get nineteen plus three fourths of a cookie.

Rational Numbers

"In upper level math, we write everything as a rational number," says Rachel. "A rational number is a number that can be expressed as a/b but b cannot equal zero. Rational refers to a ratio—a comparison of two numbers by division," she explains. "If we were to write all our division as a rational number, we would never have to use the word gozinta," she laughs. "We'd never have to say, "Four gozinta seventy-nine." She writes out:

$$\frac{79}{4}$$

"Let's take 79/4 and write out all the ways we could do it."

$$\frac{79}{4}=\frac{40}{4}+\frac{32}{4}+\frac{4}{4}+\frac{3}{4}$$

$$\frac{79}{4}=10+8+1+\frac{3}{4}$$

$$\frac{79}{4}=19\frac{3}{4}$$

Next she writes out:

$$\frac{79}{4}=\frac{40}{4}+\frac{20}{4}+\frac{12}{4}+\frac{4}{4}+\frac{3}{4}$$

$$\frac{79}{4}=10+5+3+1+\frac{3}{4}$$

$$\frac{79}{4}=19\frac{3}{4}$$

Rachel asks the students, "What's the nearest multiple of four to seventy-nine?" After a moment to think it over, one student says, "Eighty." "This is super, super cool," Rachel says excitedly. "Watch this, are you ready?" She writes:

$$\frac{79}{4}\quad\frac{80}{4}-\frac{1}{4}$$

$$\frac{79}{4}=20-\frac{1}{4}$$

"What's twenty subtract one fourth?" she asks. "Nineteen and three fourths!" a student replies.

$$\frac{79}{4}\quad\frac{80}{4}-\frac{1}{4}$$

$$\frac{79}{4}=19-\frac{3}{4}$$

"We usually teach students to come up with the nearest multiple of four under the number, but that's not true," Rachel says. "Put the numbers in context and the students can do this."

"It's important that kids have lots of practice dividing a two digit by a one digit," maintains Rachel. "I can't emphasize that enough." She suggests putting students with a partner. "I give them a book problem, and tell them each to do the problem a different way. I also give them the answer," she adds, "to take the emphasis off doing it fast."

Three Cheers for Dividing Three-digits!

When students have had sufficient practice dividing two-digit numbers by one digit numbers, Rachel moves on to three-digit numbers. "We do easy ones in the beginning," she says. "In the beginning, I'm the one controlling the numbers. After they have the idea, kids can make up their own problems.

$$4p\,\overline{)\,\$148}$$

"If learners have done a lot of work with two digit divided by one digit, they don't have any trouble with this." Again, students try more than one way to solve the problem:

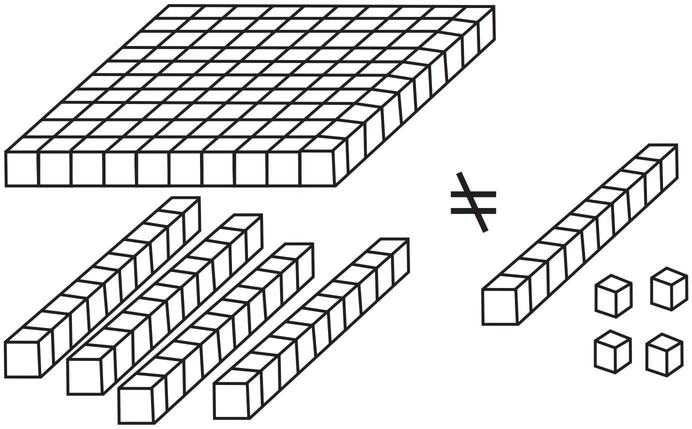
$$\begin{array}{rl} & \$25+10+2 \\ 4p & \overline{)\,\$100+40+8} \end{array} =\$37$$

and also!

$$\begin{array}{l} \qquad \$10+10+5+10+2 \\ 4p\,\overline{)\,\$40+40+20+40+8} \end{array}$$

Rachel takes out the base-ten blocks, and places a one hundred, four tens, and eight ones on the overhead.

“Let’s look at how we usually teach students this problem,” she says. “We spend a lot of time explaining place value to students, and then we look at one hundred forty-eight divided by four and right off the bat we say, ‘Four doesn’t go into one. We’ve made one hundred,” she points to the big square one hundred, “into this little one,” and points the little one cube. “Next we say, ‘take four into fourteen.” Rachel shakes her head in confusion. “Magically one hundred forty has become fourteen!” she says, again pointing out the difference in the blocks in disbelief.



“I don’t know about you, but I don’t like working with money that way!” she jokes. “What we’ve done is totally misrepresent the number. Then we have the audacity to say that students don’t have number sense! How can they possibly have number sense when we call this big square a one hundred, and then go and turn it into a tiny little one for division? That is where we lose the language-based learners.”

Rachel chooses another set of numbers to divide:

$$5\,\overline{)\,397}$$

She takes the class through the problem in several different ways, talking them through it slowly. “I model the thinking process for kids,” she says. “If we don’t model what we teach, then we’re teaching something else. If I don’t model slowing down and using the thinking process, then students think I have the answer really fast all the time.” She also tells students about her math thesaurus. “Lots of people use a thesaurus for words,” says Rachel. “If I’m writing a story, and I don’t like a certain word, I can open a thesaurus and find a different word that means the same thing,” she explains. “The cool thing about having a math thesaurus is that it’s in your brain! If I don’t know how many times five goes into three hundred fifty, I can go into my brain, chunk it! I can chunk it into three hundred plus fifty. I use my math thesaurus all the time!”

She writes out the problem as:

$$\begin{array}{l} \quad 60+10+8+1+\frac{2}{5} \\ 5\,\overline{)\,300+50+40+5+2} \end{array} = 79\tfrac{2}{5}$$

A student asks, “Why didn’t you use three hundred fifty?”

“Because I didn’t think of it,” says Rachel simply.

Now she writes it out:

$$\begin{array}{l} \quad 70+9+\frac{2}{5} \text{ or } R2 \\ 5\,\overline{)\,350+45+2} \end{array}$$

“Now, are you ready for big time?” she asks the class. “If forty is a friend of five, then four hundred is also a friend of five.” She decides to write this problem out in rational form:

$$\frac{397}{5} = \frac{400}{5} - \frac{3}{5} = 39\tfrac{2}{5}$$

Students are given lots of practice dividing three-digit numbers by one digit numbers. “They ought to know how to do any two-digit or three-digit number divided by a one digit pretty easily,” says Rachel. “We are still not giving out worksheets with 30 long division problems on it—that causes more tears for students and their parents. You don’t have to give a sheet of 30 division problem. You only have to give two or three problems and have students do each one 5 different ways.

Dividing Three-digit by Two-digit

"In order to divide three-digit numbers by two-digit numbers, students should know how to double numbers and they should know how to multiply any number by ten," says Rachel. She writes:

$$5\overline{)397}$$

"We want to break two hundred ninety-six into reasonable chunks," says Rachel. "What do you get when you double thirteen?" she asks. "Twenty-six," a student answers. "Now multiply it by ten" she tells the student. "Two-hundred sixty," he replies. "So right off the bat, we've taken two hundred sixty out of two hundred ninety-six," says Rachel, writing. "That leaves us with thirty-six left to chunk." She says again, "Double thirteen," "Twenty-six," the class answers. "What's thirty-six subtract twenty-six?" she asks. "Ten." Rachel writes this down saying, "Ten is less than thirteen, so we're done chunking."

$$13\overline{)260+26+10\qquad\qquad}$$

"So, what do we have?" says Rachel as she talks through the problem. "We have twenty plus two, plus ten thirteenths. So our answer is 22 and 10/13."

$$\begin{array}{r}70+\ 9+\frac{2}{5}\ \text{or R2}\\ 5\overline{)350+45+2\qquad}\end{array}$$

By this time, the class is not longer putting nouns with the numbers. "There comes a point where we don't bother," explains Rachel, "because the students know how it works. At this point, we have chosen to write everything as a fraction, unless the situation tells us otherwise."

Now Rachel writes:

$$17\overline{)779}$$

"We know if we double a number, we've multiplied by two," she says. "If we double again, we've multiplied by four. You can divide anything if you know about doubles."

She asks a student. "Double seventeen, what do you get?" "Thirty-four." "Now double thirty-four," Rachel tells her. "Sixty-eight," says the student. "Multiply by ten," says Rachel. "Six hundred eighty."

"Okay!" exclaims Rachel. "So six hundred eighty is a friend of seventeen!" She writes this down. "If we subtract six hundred eighty from seven hundred seventy-nine, we have ninety-nine left. Now, what's a friend of seventeen that is close to ninety-nine?" The students double seventeen, to get thirty-four, and then double thirty-four, to get sixty-eight. "Okay, so let's take sixty-eight out of ninety-nine," says Rachel. "Now we have thirty-one left to chunk." The students tell her to take 17 from 31. "What's left?" she asks. "Fourteen."

$$\begin{array}{r}40+4\ +1\ +\frac{14}{17}\\ 17\overline{)680+68+17+14}\end{array}$$

Dead Mice Smell Bad – Getting Creative With Algorithm

"When we teach division to students," says Rachel, "we usually teach them the algorithm: Divide, Multiply, Subtract, Bring Down. It's the DMSB method." She writes on the overhead:

DMSB

Divide
Multiply
Subtract
Bring down

"One student I met called it the Dead Mice Smell Bad" method she laughs. "I've heard it called all sorts of things – Does McDonald's Sell Burgers, Del Monte String Beans, Daddy Mommy Sister Brother. In South Africa it was Daddy Makes Soccer Balls. I've heard them all."

Even within the old algorithm, Rachel finds plenty of wiggle-room for creativity. "Mathematics is finding the pattern in nature and then putting symbols to it," says Rachel. "Let's think about the patterns we teach kids. We teach kids that they have to add the ones first. We tell them that they have to start with the numbers on the right when they add, subtract, and multiply," she says skeptically. “In division, notice that we tell them that they have to start from the left! None of this is necessarily so. There is nothing in the Dead Mice Smell Bad Method that tells us that we must start from the left. If just says Divide.” She writes a familiar problem on the board:

$$4\overline{)148}$$

“Many teachers have had a student look at a problem like this and ask if they can take the four into the eight first,” says Rachel. “And we teachers just automatically say no. But the algorithm doesn’t say start from the left, so just for the fun of it let’s break tradition and start from the right! Divide eight by four and what do you get?” “Two.” “Now we multiply – what is two times four?” “Eight,” students tell her. “Now we subtract: eight minus eight?” “Zero.” “Finally we bring down the next number, forty,” says Rachel. “Now we start the process again.”

$$\begin{array}{r} 2+10+25 \quad =37 \\ 4\overline{)148} \\ \underline{-8} \\ 40 \\ \underline{-40} \\ 100 \\ -\underline{100} \\ 0 \end{array}$$

“Look at that,” laughs Rachel when they complete the problem. “Our answer is still 37! Have I used the algorithm? Of course!”

Teachers Take Note

“Teachers should not just jump right in with this lesson without the proper lead up,” cautions Rachel. “In order to do division this way, it’s really important that they have worked with their students using the algebraic methods for addition, subtraction, and multiplication. If students aren’t thinking algebraically, if they are not calling numbers by their correct place value names, this lesson will not work.”