Algebraic Addition!

Taking Students Beyond the Answer

When students have a strong foundation in place value, they can better understand the different operations applied to those numbers. "The next thing I do with students is add," says Rachel. "Keep in mind, we are adding money," she reiterates. "We are adding nouns, not adjectives."

Rachel is very deliberate in planning the numbers she uses in this lesson. She selects numbers that will best illustrate the concept without confusing students to begin with. "It's important to put compatible numbers in the ones place," she emphasizes. In this case, the compatible numbers are numbers that are multiples of ten.

She writes the following problem on the overhead projector:

$15
+ $25
+ $37

Then, at the top of each column, she writes the place value in dollars:

$10 column | $ 1 column
$15
+$25
+$37

Now the students are asked to expand the numbers:

$10 | $ 1
$15 = $10 + $5
+$25 = $20 + $5
+$37 = $30 + $7

"We are using an algebraic process," Rachel points out. "The left hand side equals the right hand side."

The questions Rachel poses to students are sequential and well planned. "Who would like to do this problem?" she asks. Looking at the right hand side of the problem, students are given the choice of adding the $10 bills or the $1 dollar bills first. "It doesn't make any difference which one they choose, because we will add it both ways," says Rachel. For the most part, kids want to add the $10 bill column first.

Because she wants students to say the number, Rachel asks, "Add up the tens column—what is your answer?" When a student responds with the number's digit name (six, rather than sixty), Rachel simply points to the twenty in the $10 column and asks, "What is the name of that number?" Once the student is given this reminder, they quickly correct their answer. "Sixty!"

"What numbers did you add to get sixty?" asks Rachel.

$10 | $ 1
$15 = $10 + $5
+$25 = $20 + $5
+$37 = $30 + $7
$60

"I added $10 + $20 + $30."

Now the class adds the $1 bills. "What do you get?" asks Rachel. "Seventeen," students tell her. "Who would like to write out seventeen as 10 + 7?" she asks. Many of the students raise their hands. "Lots of students choose this, because it's easier to add tens," she notes. "We teach learners to count by tens and then we never let them use it, because we always make them add the ones column first."

$10 | $ 1
$15 = $10 + $5
+$25 = $20 + $5
+$37 = $30 + $7
$60 + 10 + $7

Adding the $1 column:

Next, students add together the right hand side of the problem for their total:

```
            $10 $1
  $15 = $10 + $5
+ $25 = $20 + $5
+ $37 = $30 + $7
------------------------
  $77 = $60 + $10 + $7
```

"We have 60 + 10 + 7, so our answer is seventy-seven," confirms Rachel.

The Math Process

The NCTM Standards ask that students learn to justify their work. Teaching students to "carry" numbers merely clouds the math rather than clarifying it, maintains Rachel. "Putting down a number and then writing a little number at the top of the next column is not justifying your work," she says. "I carry my groceries from the car. But how do you carry an adjective? You can't carry a red."

"In the writing process, writers have a first draft, a second draft and a final draft of their stories," explains Rachel. "In this lesson, we are teaching the math process."

By learning to add algebraically, students create a draft in which they justify their work:

```
               $10 $1
                $15
              + $25
              + $37
              -------
First Draft:    $60 = $10 + $20 + $30
                $17 = $5 + $5 + $7
                ---------------------
Final Draft:    $77 = $60 + $10 + $7
```

"If you want the kids to understand that the ones add up to seventeen, then write seventeen," she says logically. "Putting a little number at the top does not work. Teachers who are totally honest with themselves know that when a student puts down the number and carries, the child does not know what number they are writing."

Addition is advanced counting, Rachel often reminds students and teachers alike. "It doesn't matter where you start or end, just get 'em all." She illustrates the idea this way:

```
 $15=111111111111111
+$25=1111111111111111111111111
+$37=1111111111111111111111111111111111111
```

"If I put 15 little ones out here, and 25 ones here and 37 here...does it make any difference where I start or end? No. Just get it all."

In keeping with this concept, Rachel models adding those same numbers in different ways. "I model adding in descending order—adding from the left to the right," she says. "And I show students how to add the ones first and still justify their work."

Adding the $1 column first:

```
 $10 $1
  $15
+ $25
+ $37
-------
  $17 = $5 + $5 + $7
  $60 = $10 + $20 + $20
  ---------------------
  $77 = $60 + $10 + $7
```

Often a student will raise their hand and announce, "I found another way to do it." For instance, one learner added the problem this way:

```
 $10 $1
  $15
+ $25
+ $37
-------
  $40 = $20 + $10 + $5 + $5
  $37 = $30 + $7
  -------------------------
  $77 = $60 + $10 + $7
```

"Once teachers open up the wonderful world of mathematics, beautiful things will happen with the kids," says Rachel with a smile.

An Awesome Axiom!

Rachel prepares students for their homework assignment with a fun activity. "In mathematics, we have things called axioms," she tells the class. "An axiom is a proposition that we can't prove is true but we've never found a case that hasn't worked. Here is an axiom: If we take a set of equals and add them to a set of equals, the sums will be equals."

To demonstrate this proposition she asks students to volunteer two sets of addition facts. "Four plus two equals six," calls out one student. "Five plus five equals ten," suggests another. Rachel writes these on the overhead:

$$\begin{array}{r} 4 + 2 = 6 \\ + \quad 5 + 5 = 10 \\ \hline \end{array}$$

"Let's add them together and see what happens," she says with a gleam in her eye. (Add 4 + 5, then add 2 + 5).

$$\begin{array}{r} 4 + 2 = 6 \\ + \quad 5 + 5 = 10 \\ \hline 9 + 7 = \end{array}$$

Add the result (Add 9 + 7):

$$\begin{array}{r} 4 + 2 = 6 \\ + \quad 5 + 5 = 10 \\ \hline 9 + 7 = 16 \end{array}$$

Finally, notice what happens when you add the right-hand side of the original sets (Add 6 + 10)!

$$\begin{array}{r} 4 + 2 = 6 \\ + \quad 5 + 5 = 10 \\ \hline 9 + 7 = 16 \end{array}$$

"What they are learning is the algebraic assumption that left-hand side must always equal right-hand side," explains Rachel.

$$\begin{array}{r} 4 + 2 = 6 \\ + \quad 5 + 5 = 10 \\ \hline 9 + 7 = 16 \end{array}$$

"Kids have a lot of fun with this," she says. "It's also a great way to practice math facts."

On occasion, Rachel has offered students an incentive to try and disprove the axiom. For example, "I taped a fifty dollar bill to the board in an eighth grade classroom in Arizona," she recalls. She announced to the students, "If anyone can find a case where this doesn't work, I'll give you this money."

Word of Ms. Math's challenge spread quickly, until students in other classrooms were working on the problems. "Everybody was trying to find one—kids were having their parents work on them." One student told Rachel that he and his father had sat working on the problems for 3 hours!

Despite vigorous efforts from students in schools across the country, Rachel has been able to hold on to her money. "No one can find a case," she laughs. "Oh, they get creative. They want to know if they can use more than two sets of equals, and the students will experiment with the axiom at their own developmental level, using fractions, decimals, and integers.

$$\begin{array}{r} 1 + 2 + 3 = 6 \\ + \quad 2 + 2 + 4 = 8 \\ + \quad 10 + 10 + 10 = 30 \\ \hline 13 + 14 + 17 = 44 \end{array}$$

One Problem Added 10 Different Ways

After the students have played with the axiom in class, Rachel brings them back to the original problem. "I keep using 15 + 25 + 37," she explains. "If you want kis to learn the process, give them the answer. In every class there is a kid who wants to get the answer first—they want to get it first and get it down fast. I give them the answer, and ask them to show me 10 different ways to get there."

Indeed, Rachel only assigns that one problem as homework, and requests that students find 10 different ways to add it. They must justify their work in the way that she has demonstrated. Sometimes a student will tell Rachel that they were taught a certain way to add.

"That can be one of your ways," she'll tell them.

Rachel emphasizes that she gives the students lots of practice with the equation. She does not rush the process. "What is your favorite number?" she asks a student. "Seven," he replies. "Okay, let's write out 15 in terms of seven!" Rachel tells him. "And let's write 25 in terms of eight, and write out 30 in terms of 10."

$$\begin{array}{r} \$15 = 7 + 7 + 1 \\ + \ \$25 = 8 + 8 + 8 + 1 \\ + \ \$37 = 10 + 10 + 10 + 7 \\ \hline \$77 = 25 + 25 + 20 + 7 \\ \hline \end{array}$$

$$\$77 = 50 + 27$$

Students can use any numbers they want, as long as the left-hand side is equal to the right-hand side. "Who is in charge of the problem?" Rachel asks the class. "We are. It's up to us how we add this, because we've got the pencils."

An Infinite Number of Ways...

Another student wants to break down the numbers in fives:

$$\begin{array}{l} \ \ \$15 = 5 + 5 + 5 \\ + \ \$25 = 5 + 5 + 5 + 5 + 5 \\ + \ \$37 = 5 + 5 + 5 + 5 + 5 + 5 + 5 + 2 \\ \hline \end{array}$$

The way a student chooses to write the problem out depends on the developmental level of the learner. Some students may be ready for a more algebraic approach. "If we are using all fives, we don't want to have to write them all out," says Rachel. "We know we have fifteen fives, so let's write it this way:

$$\begin{array}{r} \$15 = 3\ (\$5) \\ + \ \$25 = 5\ (\$5) \\ + \ \$37 = 7\ (\$5) + 2 \\ \hline 10\ (\$5) + 5\ (\$5) + \$2 \\ \hline \$77 = \$50 + \$25 + \$2 \end{array}$$

In order for students to use this algebraic method, it is essential that they understand that they can only add similar terms: 3 (5) + 5 (5) + 7 (5) + 2, but not 3 (5) + 5 5) + 6 (6) + 1.

"How old are you?" Rachel asks one student. "Nine," she tells her. "Okay, let's do the problem using nine!" Rachel tells her. "But we're going to be writing down a lot of nines aren't we? Well look how we can do this:

$$\begin{array}{r} \$15 = 1\ (9) + 6 \\ + \ \$25 = 2\ (9) + 7 \\ + \ \$37 = 4\ (9) + 1 \\ \hline \end{array}$$

Rachel talks students through the problem. "How many nines do we have? We can write 7 (9), and then look, there is another nine in fourteen if we want to take it. Wo we could write that fourteen our of 1 (9) + 5:

$$\begin{array}{r} \$15 = 1\ (9) + 6 \\ + \ \$25 = 2\ (9) + 7 \\ + \ \$37 = 4\ (9) + 1 \\ \hline 7\ (9) + 1\ (9) + 5 \\ \hline 8\ (9) + 5 \\ \hline \$77 = 72 + 5 \end{array}$$

Another creative approach, write the problem in terms of 25!

$$\begin{array}{r} \$15 = 25 - 10 \\ + \ \$25 = 25 \\ + \ \$37 = 25 + 10 + 2 \\ \hline \$77 = 75 + 0 + 2 \end{array}$$

"The tens zero out," explains Rachel. "Ten dollars I owe, plus ten dollars I have, equals zero."

Adding three-digit numbers

Eventually, students will be ready to move on to adding three-digit numbers, but Rachel cautions against moving to this step too quickly. "Sometimes teachers jump to adding three-digit numbers before kids have had enough practice with two-digits."

Rachel writes:

$$\begin{array}{r} \$358 \\ + \ \$256 \\ + \ \$174 \\ \hline \end{array}$$

(Notice the placement of compatible numbers in both the $1 and $10 columns.)

"What is the name of this number?" she asks, pointing to the three hundred. She does this with each number, always making sure that students are using the correct choice of starting to add in whichever column they wish. "Most of the time kids want to add the $100 bill column first," she says. "Then they may want to add the $1 column next, and then the $10 column":

```
   $358
 + $256
 + $174
   $600 = 300 + 200 + 100
   $ 18 = 6 + 4 + 8
   $170 = 50 + 50 + 70
   $788 = 600 + 100 + 70 + 10 + 8
```

"There are an infinite number of ways to add this or any problem," says Rachel. "There is a quotation I heard somewhere that I just love: 'Arithmetic is answering the question, Mathematics is questioning the answer."

That is why I give one addition problem as homework, and ask students to add it ten different ways. I want kids to grow up to be thinking human beings."