Wonderful Ideas

...for teaching, learning, & enjoying Mathematics!

Express Yourself

Exploring Multiplicative and Exponential Forms

Welcome to part two of this year's five-issue focus series, "Dancing with Decimals." In our first installment (V16 N1), WI's Rachel McAnallen introduced learners to a number game featuring dollars, digits, dice, and decimal points. Students became well-versed in the place value system on the left side of the decimal point, manipulating numbers and digits in their wallet, and speaking the language of the numbers in both standard and expanded form. The money game continues in this issue, serving as the backdrop for a lesson designed to teach students how to express numbers in written standard form all the way through to algebraic form.

Topics Involved: speaking and writing numbers in different forms of expression, prime factorization, exponents, number sense, place value (number versus digit), decimal placement in whole numbers, addition and subtraction of whole numbers.

Materials: notepaper and pencil, as well as materials to play the money game--pairs of students share a set of base-ten blocks or $1, $10; $100 bills * and yellow chips; an egg carton or tray containing digits & symbols 0 through 9, +, -, . , and < ; an addition/subtraction hexahedron; and four decahedrons marked with the place value for ones, tens, hundreds, and thousands, respectively.* Each student also needs a white sheet of paper to represent their wallet.

Type of Activity: large group instruction and small group activity

Grade Level: 2nd and up. *We recommend using base-ten blocks with students the lower grades.

Mastering Multiplicative Form

Students begin by playing the number game they learned in the last lesson. Rolling the place value dice, they add and subtract money from their wallets and adjust their digits accordingly. "Don't forget to talk!" Rachel reminds them. "You must tell your partner how much money you have in your wallet in standard and expanded form!" After a few minutes, she stops the activity and calls their attention to the overhead projector.

"I have five hundred seventy-nine dollars in my wallet," she tells the class. "In standard form, I have five hundred seventy-nine dollars," she says, writing $579 on the overhead. "In expanded form, I have five hundred, plus seventy, plus nine." She writes:

$500 + 70 + 9

"Now, there is another way I can express this amount of money—it's known as the multiplicative or distributive form. A good way to remember multiplicative form is to remember we are going to do some multiplying," explains Rachel. On the overhead, she writes:

5 ($100) + 7 ($10) + 9 ($1)

"As we get further down the road in mathematics," she tells students, "mathematicians don't use a little x for a multiplication sign because we may get it mixed up with letters in the alphabet. Instead, mathematicians put things in parenthesis—it means we have a group or groups of something. So reading our money in multiplicative form, we have five groups of a one hundred dollar bill, plus seven groups of a ten dollar bill, plus nine groups of a one dollar bill. Or we could also say, five times the quantity one hundred, plus seven times the quantity ten, plus nine times the quantity one."

If this way of expressing the number seems familiar to the students, it is because they've used it before. By repeating the chant learned in the first activity—"this is the digit five, when I put it in it's place it becomes a number. It tells a story. It says I have five one hundreds"—they were speaking in multiplicative form.

Returning to the money game, students are instructed that they are no longer required to read their money in standard form. Instead, the partners take turns using expanded form and multiplicative form. Traveling around the room, Rachel listens to make sure each group is speaking the language. "Remember—if I catch someone who isn't talking, I'll come take money from that group's wallets!" she warns them good-naturedly.

After several minutes, she stops the game and announces, "You're all getting so good! You might want to start writing your answer down. Algebra students write things down in their notebooks. I think you all are ready to write out your answers out in all three forms."

Rachel is careful to stress that she does not introduce writing into the activity until she is sure that the students absolutely understand the concept and have had lots of practice. "As they play the game, students are rolling the dice, they are using the money, they are using the digits, and they are still speaking the language, and now they are writing the language down in mathematical terms."
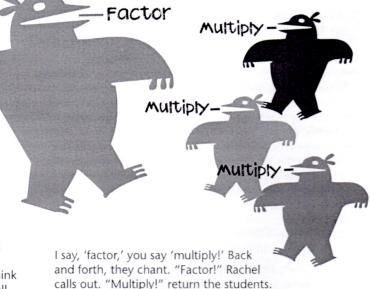
Just the Factors, Ma'am

In order to teach students all the ways to write a number from standard form to algebraic form, they must understand exponents. "Traditionally, learners are

not taught about exponents—other than squared and cubed—until they are in algebra class." notes Rachel. "What I have found is that we can teach exponents to students at a much earlier age, while they are learning about factoring and prime numbers."

Once the students are confident speaking and writing numbers out in standard, expanded and multiplicative forms, they are ready to move on to the next section of the lesson. With their math notebooks in front of them, the class is instructed to take notes as Rachel teaches from the overhead projector. "If I write something down, you should be writing it down, too," she emphasizes. At the top of the page, she writes:

Factor x Factor = Product

"When you hear the word factor," Rachel tells students, "you should immediately think, multiply. So, when

I say, 'factor,' you say 'multiply!' Back and forth, they chant. "Factor!" Rachel calls out. "Multiply!" return the students.

"Factor!"

"Multiply!"

"Factor!"

"Multiply!"

"Good!" says Rachel. "Now, I'm going to give you two factors." Directly beneath the words "factor x factor," Rachel writes:

4 x 2

"What is the product?" she asks.

"Eight," the class tells her. In their notebooks, they write 8 underneath the word "product."

"We can also say four groups of two equals eight," Rachel reminds them, writing:

4 (2) = 8

She gives them another set of factors:

5 x 10

"Fifty," the students respond.

"Our product is 50," agrees Rachel.

5 x 10 = 50

"Or, we can write it as five groups of ten equals fifty."

5 (10) = 50

"How about thirteen times one?" Rachel suggests.

13 x 1 = 13 or 13 (1) = 13

"Notice that I've given you the two factors, and then you give me the product," she observes. "Now we are going to do the opposite—I am going to give you the product and I want you to give me two factors."

She writes: 12 = {

"Someone give me two factors of twelve."

"Three times four." "Two times six." "One times twelve."

Occasionally, a student will suggest ½ x 24, or – 2 x – 6 as a pair of factors. "A creative thinker will remind me when I've forgotten to explain that we are only looking for factors that are positive whole numbers," laughs Rachel. "Otherwise, we could go on listing factors infinitely!"

On the overhead, Rachel lists the six factors of twelve, beginning with 1 x 12.

$$12 = \begin{cases} 1 \times 12 \\ 2 \times 6 \\ 3 \times 4 \end{cases}$$

"One, two, three, four, six and twelve are all factors of twelve," she reads. "Why isn't seven a factor of twelve? Seven plus five equals twelve."

"Seven plus five is adding," answers one student. "A factor means multiplying."

"Oh, great remembering!" says Rachel. She gives them another product. "Let's try 25. What are the factors?"

$$25 = \begin{cases} 1\ (25) \\ 5\ (5) \end{cases}$$

"How about 32?"

$$32 = \begin{cases} 1\ (32) \\ 2\ (16) \\ 4\ (8) \end{cases}$$

"Let's look at something," Rachel says, pausing to examine the list of products and factors they have created. "Twelve has six factors. Twenty-five only has three factors. Thirty-two has six factors. Twelve and thirty-two have an even number of factors, and twenty-five has an odd number of factors. I wonder if we can make a conjecture here…"

"Oh!" exclaims one student. "Maybe even numbers

have an even number of factors, and odd numbers have an odd number?"

"Well, let's try another odd number," reasons Rachel. "What are the factors for fifteen?"

$$15 = \begin{cases} 1\ (15) \\ 3\ (5) \end{cases}$$

Counting, they discover that fifteen has an even number of factors. "How about sixteen?" asks Rachel.

$$6 = \begin{cases} 1\ (16) \\ 2\ (8) \\ 4\ (4) \end{cases}$$

"One, two, four, eight, sixteen," reads Rachel. "So sixteen has five factors. I guess that blows a hole in our conjecture." If students were to factor out more numbers, they would eventually discover the pattern. "As it turns out," Rachel notes, "only the square numbers have an odd number of factors. All the rest of the numbers have an even amount of factors. Isn't that cool?"

Prime Numbers & Composite Numbers

"What are the factors for eleven?" asks Rachel.

"One times eleven."

She writes:

11 = 1 (11)

"There aren't any others. How about seven?"

"One times seven."

7 = 1 (7)

"So some numbers have just two factors—one and itself," Rachel explains. "Mathematicians have a name for numbers that have only two factors. We call them prime."

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
											1 x 12						1 x 18		1 x 20
			1 x 4		1 x 6		1 x 8	1 x 9	1 x 10		2 x 6		2 x 7	1 x 15	2 x 8		2 x 9		2 x 10
	1 x 2	1 x 3	2 x 2	1 x 5	2 x 3	1 x 7	2 x 4	3 x 3	2 x 5	1 x 11	3 x 4	1 x 13		3 x 5	4 x 4	1 x 17	3 x 6	1 x 19	4 x 5
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
	P	P	C	P	C	P	C	C	C	P	C	P	C	C	C	P	C	P	C

P = Prime   **C = Composite**

On the overhead, she draws a horizontal line and lists the numbers 1 through 20, and then allows the students time to create this line of numbers in their notebooks. "Above each number we are going to write out the factors, and we are going to start with the number two—we will come back to one later. The factors for two are one and itself," says Rachel, writing. "So that means two is a prime number." Below the 2 she draws a P. "P for prime."

After listing the factors of four, Rachel introduces another vocabulary word. "Four is not prime, because it has more than two factors. Mathematicians call numbers with more than two factors a composite--it is composed of other numbers." Underneath the 4, she draws a C for composite. "Is five prime or composite?" she asks.

"Prime."

When they reach the number nine, one learner mistakenly identifies it as prime. "No!" retorts another student.

"If a person makes a mistake, telling them 'no' doesn't help them," says Rachel. "If someone says nine is prime, you should say, 'three times three.' Give them the factors that mean nine is a composite."

In almost every class, when the group identifies the numbers 2 through 20 as prime or composite, one student will become fascinated with prime numbers. "Whenever I teach this activity, I will discover a student who has written the numbers all the way up to fifty or one hundred and are figuring out all the primes," laughs Rachel. "It looks like they are not paying attention to me at all as I direct the activity—they are off in their own little world of prime numbers."

Looking over the number line they have created, one student asks, "What about one?"

"The strict definition of a prime number is a number that has only two distinct factors—one and itself," explains Rachel. "Distinct means that the two factors are different." Above the number 1 on the number line, she writes 1 (1). "One is not prime because it only has one distinct factor. It is known a unique number—uni meaning one."
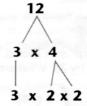
Prime Factors

"Now that we know who the prime numbers are, we are going to break down or decompose a composite number into it's prime factors," announces Rachel. On the overhead, she writes: 12

"Underneath the twelve, we draw a little upside down V," she says, continuing to write as she talks students through the process, "then we ask ourselves some questions." With a hand on her hip she says, "Self? What are the factors of twelve, other than twelve and one?" She decides to use 3 x 4, writing this into her factor tree. Rachel poses the next question to herself, encouraging the students to join in. "Self? Is three prime?" "Yes," she answers. Drawing a vertical line under the 3, she writes another 3 below it. Moving to the next number, she asks herself, "Self? Is four prime?" "No" she answers. "Self," Rachel instructs, "go factor." Working on the factor tree, she is left with 3 x 2 x 2.

$$
\begin{array}{c}
\mathbf{12} \\
\diagup \diagdown \\
\mathbf{3 \ x \ 4} \\
| \quad \diagup \diagdown \\
\mathbf{3 \ x \ 2 \ x \ 2}
\end{array}
$$

"Now I check—is three prime? Yes. Is two prime? Yes." On the side of the factor tree, she writes: prime factors (p. f.) of 12

"Let's try another number—how about twenty-eight?" suggests Rachel. "Self? Is twenty-eight prime?"
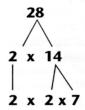
"No," both she and the students answer.

"What are two factors of twenty-eight besides one and itself?"

"Two times fourteen." volunteers a student. "Four times seven," offers another.

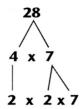"Let's start with two times fourteen," decides Rachel.

Going through the same process, she asks herself questions as she writes out the factor tree.

"Self? Is two prime? Yes." She draws a line and writes 2 below it. "Is fourteen prime? No. What are the factors of fourteen? Two and seven." Below 14 she writes 2 x 7. "Is two prime? Yes. Is seven prime? Yes."

$$28$$
$$2 \text{ x } 14$$
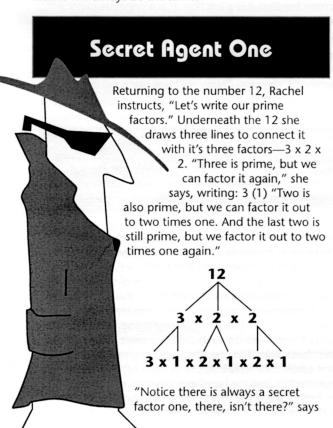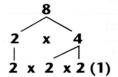$$2 \text{ x } 2 \text{ x } 7$$

Over to the side of the completed factor tree, she writes: p. f. of 28

"Just for fun, let's factor twenty-eight again," says Rachel, "but this time we'll start with four times seven."

$$28$$
$$4 \text{ x } 7$$
$$2 \text{ x } 2 \text{ x } 7$$

"We still have twenty-eight," she says. "It doesn't make any difference how we do the factor tree, the prime factors will always be the same."

Secret Agent One

Returning to the number 12, Rachel instructs, "Let's write our prime factors." Underneath the 12 she draws three lines to connect it with it's three factors—3 x 2 x 2. "Three is prime, but we can factor it again," she says, writing: 3 (1) "Two is also prime, but we can factor it out to two times one. And the last two is still prime, but we factor it out to two times one again."

$$12$$
$$3 \text{ x } 2 \text{ x } 2$$
$$3 \text{ x } 1 \text{ x } 2 \text{ x } 1 \text{ x } 2 \text{ x } 1$$

"Notice there is always a secret factor one, there, isn't there?" says Rachel. "No one ever writes it down—but we can write it in parenthesis." On the overhead projector she writes out a factor three for the number 8. "The prime factors for eight are two times two times two," she says. "Either to the left or right of that, I can write a one in parenthesis. It is a secret factor.

$$8$$
$$2 \text{ x } 4$$
$$2 \text{ x } 2 \text{ x } 2 \text{ (1)}$$

The Amazing Power of Exponents!

"Mathematicians don't like to write out a lot of numbers," explains Rachel. "Instead they have invented ways to keep things simple." On the overhead, she writes:

$$8 = 2 \text{ x } 2 \text{ x } 2$$

"We know the prime factors of eight are two times two times two," she says. "Mathematicians have invented an easier way to say the same thing. She writes: 2^3

"The little three is a more simple way of writing that two is used as a factor three times. Keep in mind, there is also our secret factor one hiding there."

$$8 = 2 \text{ x } 2 \text{ x } 2 = 2^3$$

Emphasizing the language, Rachel has students repeat the sentence with her. "Eight is equal to two used as a factor three times."

"It's important to remember that two used as a factor three times is not the same as two times three," advises Rachel, illustrating this idea on the overhead:

$$2 \text{ x } 2 \text{ x } 2 \neq 2 \text{ x } 3$$

For the next example, Rachel writes:

$$5 \text{ x } 5 \text{ x } 5 \text{ x } 5 \text{ (1)} = 5^4 \text{ (1)}$$

"Instead of writing out all those fives, mathematicians write five with that little number four. We call that little number an exponent. The four makes the number more powerful—five is raised to the fourth power. It means five is used as a factor four times."

On the overhead, Rachel writes: 10 x 10 (1) "Ten times ten times the secret factor one," she reads. "Ten is used as a factor two times." She writes: 10^2

To help students see the pattern of exponents, Rachel draws a table. In the first column of the first row, she writes one million in word form. In the next column she writes = 1,000,000,000.0 in standard form, making sure to include the decimal point. Next she writes

5

one million	= 1,000,000.0	= 10 x 10 x 10 x 10 x 10 x 10 x 1	= 10^6 (1)
one hundred thousand	= 100,000.0	= 10 x 10 x 10 x 10 x 10 x 1	= 10^5 (1)
ten thousand	= 10,000.0	= 10 x 10 x 10 x 10 x 1	= 10^4 (1)
one thousand	= 1,000.0	= 10 x 10 x 10 x 1	= 10^3 (1)
one hundred	= 100.0	= 10 x 10 x 1	= 10^2 (1)
ten	= 10.0	= 10 x 1	= 10^1 (1)
one	= 1.0	= 1	= 10^0 (1)
one tenth	= 0.1	= $\frac{1}{10}$ x 1	= 10^{-1} (1)
one hundredth	= 0.01	= $\frac{1}{10}$ x $\frac{1}{10}$ x 1	= 10^{-2} (1)
one thousandth	= 0.001	= $\frac{1}{10}$ x $\frac{1}{10}$ x $\frac{1}{10}$ x 1	= 10^{-3} (1)
ten thousandth	= 0.0001	= $\frac{1}{10}$ x $\frac{1}{10}$ x $\frac{1}{10}$ x $\frac{1}{10}$ 1	= 10^{-4} (1)
one hundred thousandth	= 0.00001	= $\frac{1}{10}$ x $\frac{1}{10}$ x $\frac{1}{10}$ x $\frac{1}{10}$ x $\frac{1}{10}$ x 1	= 10^{-5} (1)
one millionth	= 0.000001	= $\frac{1}{10}$ x $\frac{1}{10}$ x $\frac{1}{10}$ x $\frac{1}{10}$ x $\frac{1}{10}$ x $\frac{1}{10}$ x 1	= 10^{-6} (1)

= 10 x 10 x 10 x 10 x 10 x 10 x 1.

"Now we count how many times we use ten as a factor," she tells the students. Together, they count six tens. In the fourth column, Rachel writes 10^6 (1).

"Ten used as a factor six times," they read, "times the secret one."

Row by row, they fill in the chart. When they come to the number 1, Rachel asks, "One is ten used as a factor how many times?" "None," a student tells her. "Correct—ten is used as a factor zero times. Ten to the zero power times one—remember, there is always secret agent one!"

Drawing a line from top to bottom along the left side of the chart, Rachel explains, "What we are doing is dividing by ten each time. If we look at the next number, we are dividing our secret one by ten—which gives us one tenth, or one over ten used as a factor once." She writes this out in the chart.

"Mathematicians really don't care to have anything in the denominator, other than one," Rachel continues, "so they follow the pattern of the exponents—they write it as ten raised to the negative one power. She writes this in the chart. "We are following the mathematical pattern of the exponents. We can't really say that ten is used as a factor negative one times— instead we say ten used in the denominator as a factor once. When we say ten to the negative one, we have to

remember that it is a fraction. It is a negative exponent, not a negative number."

When they have completed the chart from one million to one millionth, Rachel instructs, "Rotate your chart 90 degrees clockwise." Rotating her own chart on the overhead, she says, "Now we have place value!"

Place Value Symmetry

Next, Rachel writes the number 574.289, marking the place values above each digit.

Addressing the teachers in the room, she explains, "Every year a student will ask me why the place value to the right of the decimal point isn't one-oneth. They think that the decimal point is the line of symmetry. If you have ones, tens and hundreds on the left side, they think you should have one-oneth, one-tenth, and so on. In reality, the line of symmetry is one." She draws a dotted line down through the ones place. "If we folded this sheet along the line, the tens column and the one-tenth column fall on top of each other, the one-hundredths and the hundreds columns match up, and so forth. One is the most unique number there is, and it is the mathematical line of symmetry.

Expanded Multiplicative Form

"Now that you understand about prime factors and Secret Agent One, there is another form I want you to learn to write as you play the game," Rachel tells the class. "I call it the expanded multiplicative form. She writes this out on the overhead. "Let's take five hundred seventy-nine again," she says, writing:

5 (10 x 10 x 1) + 7 (10 x 1) + 9 (1)

"What we have really done is decompose 100 into its tens factors," says Rachel. "Now let's suppose that we have a really big amount of money in your wallet." She writes: $2,586 "So we have standard form," she says, labeling the place value of each digit. "Then we have expanded form": $2000 + $500 + $80 + $6

"Then we have multiplicative form":

2 ($1000) + 5 ($100) + 8 ($10) + 6 ($1)

"And now we have expanded multiplicative. She writes:

2 (10 x 10 x 10 x 1) + 5 (10 x 10 x 1) + 8 (10 x 1) + 6 (1)

After a working out a few examples as a group, students resume the money game and write each answer in all four forms of expression.

At the end of class, Rachel assigns homework. "Go home tonight and pick five numbers," she directs. "Each number must be three digits or greater, and each digit must be different. I want you to write out each of your five numbers in standard, expanded, multiplicative, and expanded multiplicative form."

Some students may want to create large numbers using 10 or more digits. In that case, Rachel allows them to repeat a digit more than once.

Exponential Form

At the start of math class, students go over their homework, and then get with a partner to play the money game for a few minutes. "It is review," explains Rachel. "The game is a mental warm up." She instructs students on the rules. "While you are playing, I want you to say your answers in the multiplicative form and the expanded multiplicative form." After ten minutes, Rachel calls their attention up to the overhead projector. "I need someone to give me a four digit number." Hands go in the air. "Each digit must be different," she continues, "It has to be a whole number. Each digit must be odd. I want the smallest four digit number you can make under those conditions."

There is a thoughtful pause, and then a student calls

out, "One three five seven,"

"I asked for a number," Rachel stresses. "You just gave me four digits."

"One thousand, three hundred fifty-seven."

"Good—now, let's write it out in all the forms."

1357

1000 + 300 + 50 + 7

1 (1000) + 3 (300) + 5 (10) + 7

1 (10 x 10 x 10 x 1) + 3 (10 x 10 x 1) + 5 (10 x 1) + 7 (1)

"Okay," says Rachel, "Now we're going to write this out in what is known as exponential form. On the overhead she writes:

$1 (10^3) + 3 (10^2) + 5 (10^1) + 7 (10^0)$

Together, the class reads the expression out loud, "One times the quantity ten used as a factor three times, plus three times the quantity 10 used as a factor two times, plus five times the quantity ten used as a factor one time, plus seven times the quantity ten used as a factor zero times."

"We are expressing the number in terms of it's base, which is ten," Rachel explains. "We are looking at the pattern of the exponents." She works out several examples with the class before they return to the money game.

"Now, while you are playing the game, I want you to give your answers in multiplicative or exponential form," Rachel informs students. After a few rolls, she announces, "While you are playing I am going to yell out the different forms and you must say your answer in the form that I tell you."

"Multiplicative form!" she calls out. Walking around, she lets a couple of minutes pass and then yells, "Exponential form!" "Standard form!" "Expanded multiplicative form!" "Expanded form!" "Just listen!" exclaims Rachel in amazement. "Listen to how many languages of math you can speak!" Ω

Exponential Form!

Expanded multiplicative form!

Standard Form!