CHAPTER 3

Points and Curves

Felix Hausdorff

Felix Hausdorff (November 8, 1868 – January 26, 1942) was a German mathematician. He is known for his essential contributions to topology and set theory. He played a key role in developing the concepts of Hausdorff dimension and Hausdorff space. The Hausdorff measure is an important notion in geometric measure theory that laid the foundation for modern topology. Hausdorff, who was Jewish, faced difficulties during the Nazi era, ultimately leading him to take his own life to avoid being sent to the Endenich camp. Source: https://en.wikipedia.org/wiki/Felix_Hausdorff

In this chapter, we introduce fundamental concepts of topology, including curves, neighborhoods, types of points, open and closed sets, boundedness, connectedness, and compactness, all in a simple-to-understand way. A common joke among mathematicians is that a

topologist is someone who can't tell the difference between a mug and a donut: Pour coffee into a donut and eat a mug.

A **region** refers to a set of points in the plane, which is a two-dimensional space (see the Section "Four-dimensional space" for notions of the usual two-dimensional and three-dimensional spaces).

A **body** means a set of points in our three-dimensional space. Consequently, a region can also be regarded as a body, as shown in Figure 3.1.

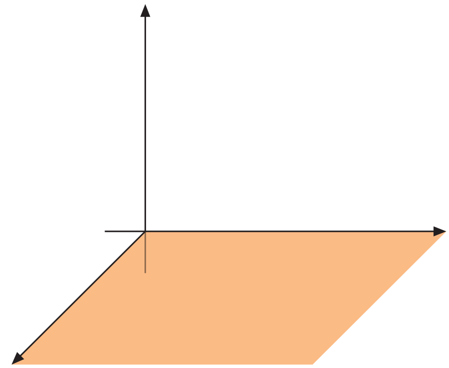


Figure 3.1. A region in the usual three-dimensional space

3.1 Curve

A tiny dot '·' is used in mathematics as a visual representation of a **point**, which indicates an exact position in the plane or space. It has no size or dimensions such as length, width, or height.

Figure 3.2 Point A

To highlight a point, we often represent it as the intersection of two small lines, symbolized as X. We usually designate points with capital letters; see Figure 3.2.

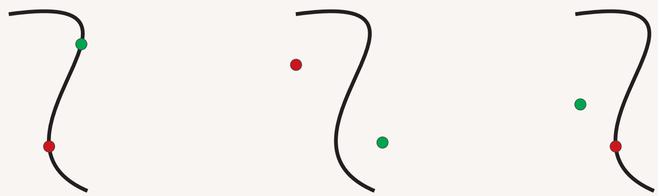
A **curve** is a path that connects one point (the starting point) to another (the end point) without interruption. When you draw a curve with a pencil, the curve should be created without lifting the pencil off the paper and continuing to draw from another point. A real model of a curve is the path we take through a forest from our hut to a river to catch fish. Another example is a map showing overland roads in a country or even a treasure map.

A curve, as a set in the plane, can also be considered a region.

Curves can take the form of either a straight line or a curved line.

Example 3.1

1. Figure 3.3 (i) illustrates two points both of which are on a curve.
2. Figure 3.3 (ii) shows two points none of which are on a curve.
3. Figure 3.3 (iii) presents two points one of which is on a curve and the other is not.



(i) (ii) (iii)

Figure 3.3 Two points and a curve

Activity 3.1

In Figure 3.4, three points are given. Draw

1. a curve through them;
2. a curve such that a point is on the curve and the other two points are not;
3. a curve such that none of the points are on the curve.

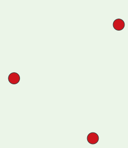


Figure 3.4 3 points

Activity 3.2

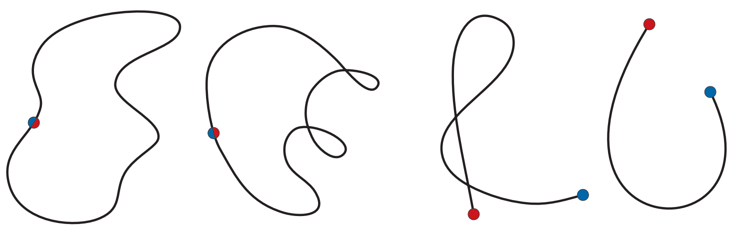
Find a maze on the internet and try to solve it.

3.2 Closed and open curves

A **closed curve** is a curve in which the start and end points coincide. Otherwise, it is said to be an **open curve**; see Figure 3.5. It becomes apparent that a curve can intersect itself at more than one point.

A curve that does not intersect itself (in other words, is non-self-intersecting) is known as a **simple curve**.

A curve can be directed by arrows from its starting point to its end point. For example, curves (a) and (d) in Figure 3.5 are simple.



(a) (b) (c) (d)

Closed Curves Open Curves

Figure 3.5 Closed and open curves

Activity 3.3

1. Is a triangle a simple open curve?
2. Draw a closed curve and an open curve that intersect each other.
3. Draw two closed curves that intersects each other.

The **interior of a closed curve** is the area fully encircled by the curve, while the **exterior of a closed curve** refers to the region beyond the curve, as shown in Figure 3.6.

Activity 3.5

The points $A, B, C, D, E, F, G, M, N$, and P are given in Figure 3.8. Draw

1. an open curve such that the point A is on the curve;
2. an open curve such that the point A is not on the curve;
3. an open curve through both B and C;
4. a simple closed curve oriented counterclockwise through both D and E;
5. a closed curve such that the point G is on the curve;
6. a simple closed curve such that the point G is in its interior;
7. a closed curve such that the point G is in its exterior;
8. a closed curve featuring the point M in its interior, the point N on the curve, and the point P in its exterior.

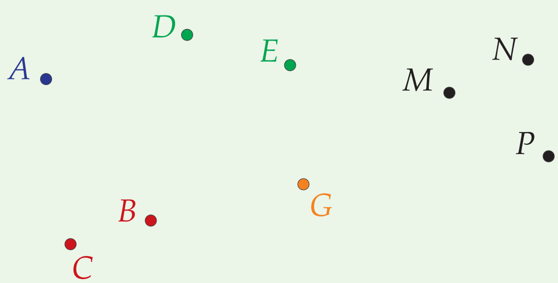


Figure 3.8 Points and drawing curves

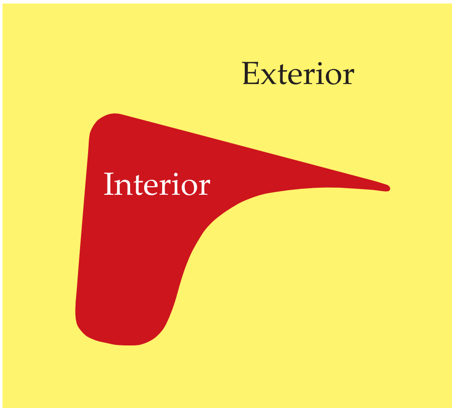


Figure 3.6. Interior and exterior of a closed curve

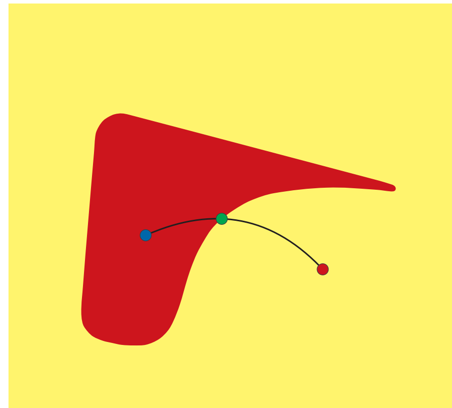


Figure 3.7 Jordan theorem

An interesting result, known as the Jordan curve theorem, which goes back to Camille Jordan (1838–1922), states that every simple closed curve on a plane divides the plane into two distinct regions: an inner and an outer region. Moreover, any curve connecting a point in the interior to a point in the exterior necessarily intersects the curve at some point, as depicted in Figure 3.7.

A simple closed curve is called **oriented clockwise** when we walk on it through its direction, we have its interior on the right side, otherwise, we say that the curve is **oriented counterclockwise**. For example, curve (a) in Figure 3.5 is oriented clockwise.

Activity 3.4

1. Draw a simple closed curve oriented clockwise inside a given circle.
2. Draw a simple closed curve oriented counterclockwise outside a given circle.

Activity 3.6

In Figure 3.9, hatch

1. the interior of the curve with the color red;
2. the exterior of the curve with the color yellow.

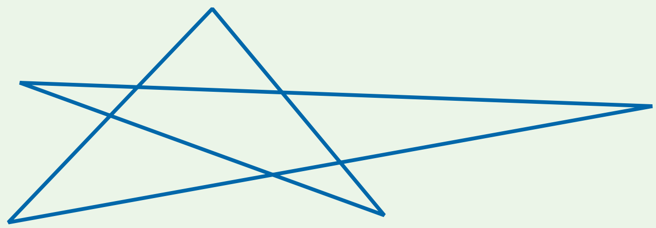


Figure 3.9 Interior and exterior of a curve

Activity 3.7

Figure 3.10. illustrate a closed curve, a point A located in its interior, and a point B situated in its exterior. Draw a curve starting from the point A and ending at the point B. Check to see if this curve intersects the given curve.

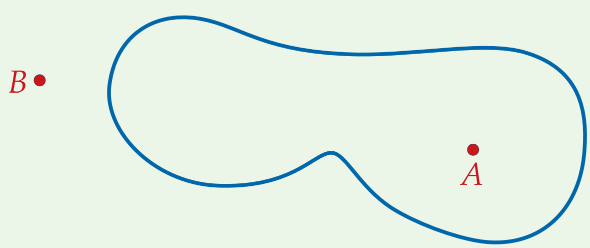


Figure 3.10 Illustrating the Jordan curve theorem

3.3 Interior, exterior, and boundary of regions

A **neighborhood** of a point refers to the interior of a closed curve that surrounds the point. It is important to note that the points on the curve are not considered part of the neighborhood. To clarify this fact, we draw the corresponding curve in the form of a dashed line in the presentation of a neighborhood. It is worth noting that the point itself is included in each of its neighborhoods; see Figure 3.11.

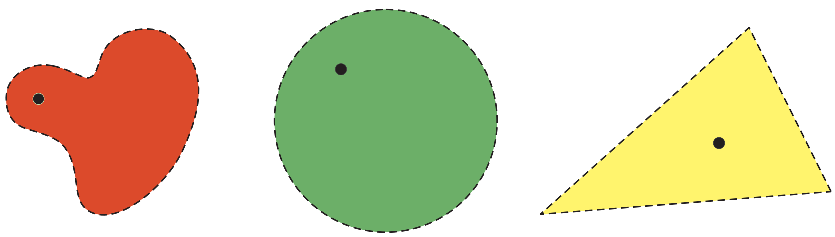


Figure 3.11 Three neighborhoods of three points

Activity 3.8

The points A, B, C, and D are given in Figure 3.12. Draw

1. two arbitrary neighborhoods for A and B;
2. two neighborhoods for C and D such that do not intersect.

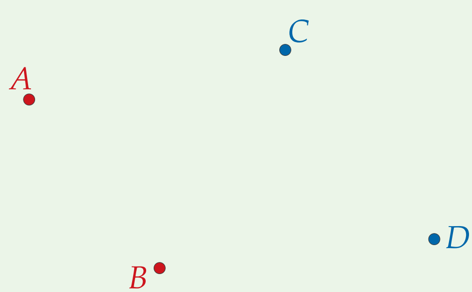


Figure 3.12 Drawing neighborhoods of given points

A point is said to be an **interior point** of a region if there exists a neighborhood that is entirely contained in the region as illustrated in Figure 3.13. An interior point is a part of the region itself. The interior of a region consists of all its interior points.

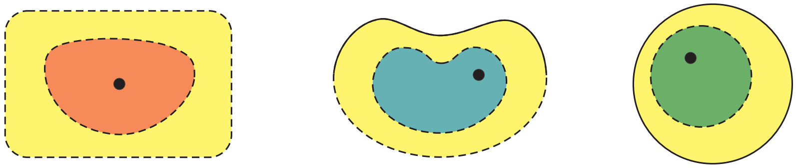


Figure 3.13 Interior points for yellow regions

A point is called an **exterior point** of a region if there exists a neighborhood that is entirely outside of the region, as shown in Figure 3.14. An exterior point of a region is not any part of the region itself. The exterior of a region consists of all its exterior points.

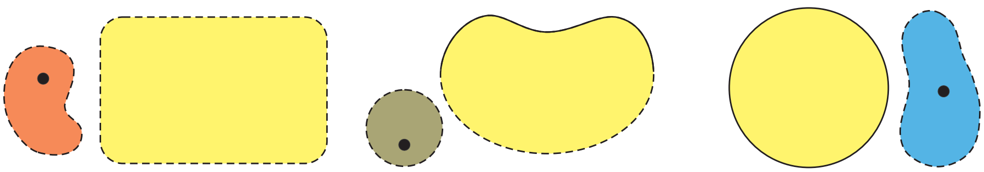


Figure 3.14 Exterior points for yellow regions

A point is considered a topological boundary point, or simply a **boundary point**, of a region if it is neither an interior point nor an exterior point, as shown in Figure 3.15. Therefore, a point is a boundary point if every neighborhood of the point intersects both the region and its complement. A boundary point of a region may belong to the region or not. The boundary of a region consists of all its boundary points.

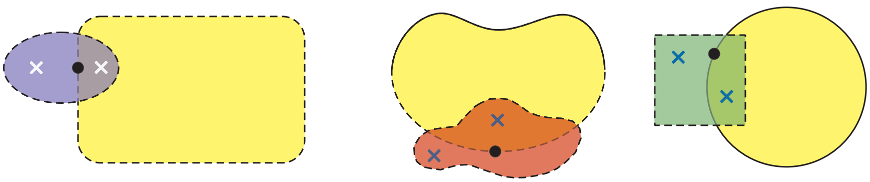


Figure 3.15 Boundary points for yellow regions