

OTHER DERIVATIVE APPLICATIONS

In this chapter we will consider several other topics concerning the derivative of a function. First we will look at the Mean Value Theorem and several ideas that often get confused with the Mean Value Theorem. Then we will discuss Related Rate Problems, L'Hôpital's Rule, and optimization problems.

The Mean Value Theorem

Previously, we looked at the *average rate of change* of a function over an interval. For a continuous function, $f(x)$, on the interval $[a, b]$ this is simply the slope of the segment between the endpoints:

$$\frac{f(b) - f(a)}{b - a}$$

If the function is differentiable on the interval the Mean Value Theorem tells us that there will be someplace in the open interval where the derivative is equal to the average rate of change.

The Mean Value Theorem: If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) then there exist a number c in the open interval (a, b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

I have mixed feelings about proof in high school math and high school calculus. I am not one for proving everything. For one thing, it cannot be done and, if it could be done, proof would become the whole focus of high school math. Proofs are not the focus of first-year calculus or AP Calculus. The place for proving “everything” is a real analysis course in college.

However, students should know about proof and there are places where you can demonstrate some of the power of proof and show how proof works in calculus. The MVT is one of those places. If you do not want to prove the MVT, there are two ways that you can use to convince your class of the MVT without proofs.

To prove the MVT we will start with two preliminary, but none the less important theorems.

Fermat’s Theorem on Extreme Values

Perhaps not as well-known as another of Fermat’s Theorems, this theorem is much more practical. The theorem establishes that if a function is continuous on a closed interval and has a maximum (or minimum) value on that interval at $x = c$, then the derivative at $x = c$ is either zero or does not exist.

There are two cases. In each case we will look at the limit of the difference quotient that defines the derivative at $x = c$, namely, $\frac{f(c+h)-f(c)}{h}$ and look at what happens as h approaches 0 from the left and from the right. These two limits are the same and equal to the derivative if, and only if, the derivative at c exists.

Assuming that $f(c)$ is a maximum, then $f(c) \geq f(c+h)$ regardless of whether h is positive or negative. Therefore, the numerator of the difference quotient is always zero or negative. Then if $h < 0$, the quotient is non-negative; likewise, if $h > 0$, the quotient is non-positive.

Case I: The two limits are not equal. In this case the derivative does not exist. This could occur with a piecewise function, where two pieces with different derivatives meet at $x = c$.

Case II: The limits are equal. In this case the limit from the left ($h < 0$) must be greater than or equal to zero (since the function is increasing there) and the limit from the right ($h > 0$) must be less than or equal to zero. Then, the only way the limits can be equal is if both limits are zero; therefore the derivative is zero.

The proof for a minimum is essentially the same.

Recall that any place where the derivative of a continuous function is zero or undefined is called a critical point and the number c is called a critical number. I think this proof is interesting because while there are lots of symbols flying around the key is interpreting what kind of number (positive, zero or negative) the symbols represent. Another thing I like is having to “read” the symbols and see that $f(c) \geq f(c+h)$

Rolle's Theorem

Rolle's Theorem says that if a function is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b) and if $f(a) = f(b)$, then there exists a number c in the open interval (a, b) such that $f'(c) = 0$. ("There exists a number" means that there is at least one such number; there may be more than one.)

The proof has two cases:

Case I: The function is constant (all of the values of the function are the same as $f(a)$ and $f(b)$). The derivative of a constant is zero so any (every, all) value in the open interval qualifies as c .

Case II: If the function is not constant then it must have a (local) maximum or minimum in the open interval (a, b) by the Extreme Value Theorem). So by Fermat's Theorem the derivative at that point must be zero.

So, Fermat's Theorem makes Rolle's Theorem a piece of cake.

A *lemma* is a theorem whose result is used in the next theorem and makes it easier to prove. So Fermat's Theorem is a lemma for Rolle's Theorem. On the other hand a *corollary* is a theorem is a result (theorem) that follows easily from the previous theorem. So Rolle's Theorem could also be called a corollary of Fermat's Theorem.

Rolle's Theorem makes its appearance in the MVT and then more or less disappears from the stage. Nevertheless, I like this proof because it's so simple. It comes immediately from Fermat's Theorem.

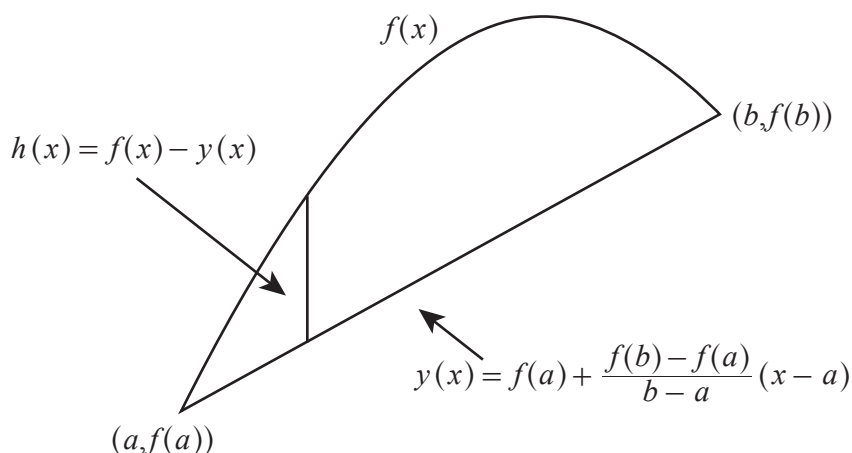
The Mean Value Theorem's Proof

The Mean Value Theorem says that if a function, f , is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) then there is a number c in the open interval (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

The proof, which once you know where to start, is straight forward and rests on Rolle's Theorem.

AP Questions

- (FR) **2007** AB 3b
- (MC) **2008** AB 89
- (FR) **2011** AB 4d
- (FR) **2013** AB 3b



In the figure we see the graph of f and the graph of the (secant) line, $y(x)$, between the endpoints of f . We define a new function $h(x) = f(x) - y(x)$, this is the vertical distance from $f(x)$ to $y(x)$. The equation of the line is in the figure and so

$$h(x) = f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a} (x - a) \right)$$

The function h meets all the conditions of Rolle's Theorem. In particular, $h(a) = h(b) = 0$ since at the endpoint the two graphs intersect and the distance between them is zero. You can also verify this by substituting first $x = a$ and then $x = b$ into h . Therefore, by Rolle's Theorem there is a number $x = c$ between a and b such that $h'(c) = 0$. So we'll find the derivative and substitute in $x = c$.

$$h'(x) = f'(x) - 0 - \frac{f(b) - f(a)}{b - a}$$

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a}$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

This last equation is what we were trying to prove, the MVT.

The arc from the definition of derivative, through Fermat's Theorem and Rolle's Theorem to the MVT is, I think, a good way to demonstrate how theorems and their proofs work together. Since I would not like my students not to have any familiarity with proof and definition I think this is a good place to show them just a little of what it's all about.

On the other hand we have ended up with a strange equation, which apparently has something to do with mean value, whatever that is.

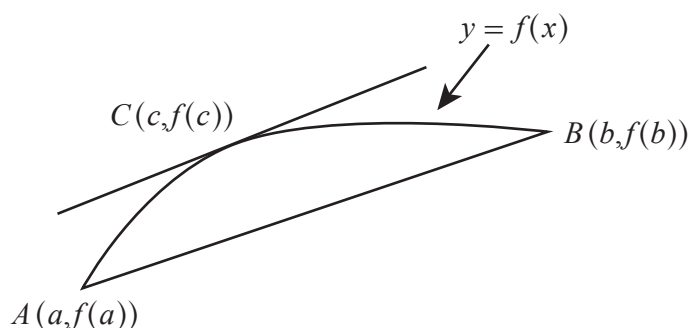
What I don't like about this proof is that you have to know to set up the function h at the beginning. It is "legal" to do that, but how do you know to do it? On the other hand doing things like that is something that has to be done sometimes and students need to know this too. Here is an easier approach, not a proof, to convince your students that the MVT is true.

AP Questions

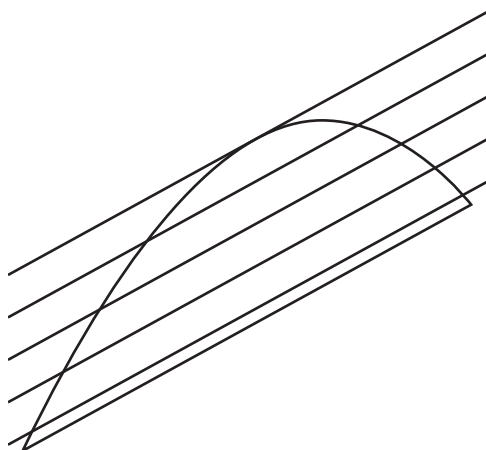
(FR) 1999 AB 3b
(FR) 2007 (form B)
AB 6a

The MVT Graphically

Graphically, the MVT says that there is at least one point in the interval where the tangent line is parallel to the line between the end points of the interval.



With that in mind, it is very easy to see why the MVT is true. Picture the line between point A and B moving upwards parallel to its original position. Eventually, it must become tangent to the function, and at that point $f'(c) = \frac{f(b) - f(a)}{b - a}$.



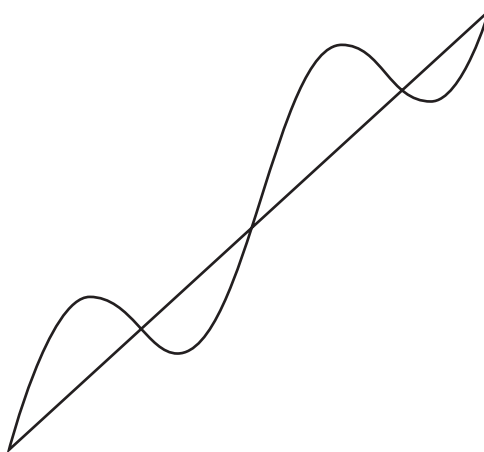
Yet another way to show the MVT is this. Near the left end of the first graph above the slope of the tangent to the graph (the derivative) is greater than the slope of the secant line; near the right end the slope of the tangent is less than the slope of the secant. So somewhere in between, by the Intermediate Value Theorem, the slope of the tangent must equal the slope of the secant. (For the purists out there, this is from Darboux's theorem, and actually requires a slightly stronger hypothesis, namely that the one-sided derivatives at a and b exist.)

A good counterexample to show your students is the absolute value function. On the interval $[-2, 5]$ there is no place where the slope is $3/7$, which is easy to see because there is no place where the tangent line could be parallel to the line between the endpoints. Why? Because the hypothesis about differentiability is not true.

AP Question

(FR) 2005
AB 5/BC 5d

Finally, note that there may be more than one place where the tangent line is parallel to the line between the endpoints, more than one value of c , as the next figure illustrates.



AP Questions

(FR) **2004** AB 1 c, d
(FR) **2007** AB 3c

While neither of these is quite enough for a proof, they are easier to see. Your textbook probably has a series of exercises where a function and an interval are given with the directions to “find the value of c guaranteed by the MVT (or Rolle’s Theorem).” This rote type questions has not appeared on the AP Calculus exams, at least not lately. The MVT and Rolle’s Theorem (which is really just a special case of the MVT) are tested quite often in other ways. Students should know the hypotheses and conclusion of both theorems and know what they mean graphically. This is what is tested. See the examples cited in the margin to understand how the MVT is tested.

Avoiding Confusion

Students often confuse the several concepts that have the word “average” or “mean” in their title. This may be partly because the formulas associated with each are also very similar, but I think the main reason may be that they are keying in on the word “average” rather than the full name. See “Other Averages” in chapter 15, page 168.

Optimization

An optimization problem is one in which we are trying to find the best way to make or do something. That may mean the cheapest way, the way that uses less material, the way that gives us the most for a given amount of material, and so forth. Here is a typical problem similar to one on an AP exam many years ago:

AP Question

(FR) **1982** AB 6
BC 3

A tank with a rectangular base and rectangular sides is to be open at the top. It is to be constructed so that its width is 4 meters and its volume is 36 cubic meters. If building the box costs \$12 per square meter for the base and \$6 per square meter for the sides, what is the cost of the least expensive tank?

That's the entire question. Notice that there are no parts a , b , and c . The students had to write the equation to be optimized, then differentiate, and find the minimum cost. The problem is that if a student cannot write the equation, they cannot earn any of the other points. Questions on today's exams are written with parts that are almost always do not require anything from other parts. This is done so that students who miss one part of the question have a chance of earning points on the other parts. They are not out the whole 9 points because they cannot get started. For this reason optimization problems of this type have not appeared on the exam since 1982.

Besides writing the model, the calculus concepts involved in solving an optimization problem are finding extreme values. This is tested with other kinds of questions. Occasionally there are fairly simple questions that could be called optimization questions on the multiple-choice.

Optimization is a good application of the derivative to use with your class, perhaps for a day or two. Writing the equation to be optimized is the part that gives students the most trouble. Do not get bogged down doing that. The important thing to know how to find extreme values, given the equation.

Related Rate Problems

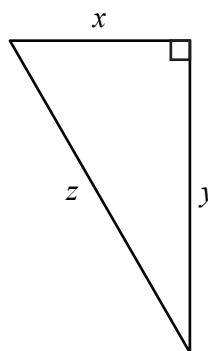
Since calculus is the study of change it is hardly surprising that Related Rate problems are included in the course. In fact, Related Rate problems are one of the few "type" problems mentioned by name in the Course Description. Almost all of the textbooks have examples of Related Rates problems.

Related Rate questions often include geometric considerations that usually end up being the more difficult part of the problem. I suggest you start with questions like those in Activity 2. These two questions do not involve geometry and let students start by looking for the relationships between the change in the variables. Then move onto geometry based questions.

Another way to get started is by doing a problem like the one below two ways and contrasting the answers.

Two cars travel on perpendicular roads towards the intersection of the roads. The first car starts 100 miles from the intersection and travels at a constant speed of 55 mph. The second car starts at the same time, 250 miles from the intersection and travels at a constant speed of 60 mph. How fast is the distance between them changing 1.5 hours later?

Method 1: Let t = the time the cars travel. Let x and y be their distance from the intersection, and z be the distance between the cars. Then $x = 100 - 55t$ and $y = 250 - 60t$ and $z(t) = \sqrt{(100 - 55t)^2 + (250 - 60t)^2}$.



Activity 2

Differentiating gives:

$$\frac{dz}{dt} = \frac{2(100 - 55t)(-55) + 2(250 - 60t)(-60)}{2\sqrt{(100 - 55t)^2 + (250 - 60t)^2}}$$

Substituting $t = 1.5$ will give us the answer.

$$\frac{dz}{dt} = \frac{2(17.5)(-55) + 2(160)(-60)}{2\sqrt{17.5^2 + 160^2}}$$

Don't simplify. Stop here and move on to

Method 2: $z^2 = x^2 + y^2$, $\frac{dx}{dt} = -55$ mph, and $\frac{dy}{dt} = -60$ mph. Using

implicit differentiation gives $\frac{dz}{dt} = \frac{2x \frac{dx}{dt} + 2y \frac{dy}{dt}}{2z}$. Substituting $t = 1.5$,

$x = 17.5$ miles, $y = 160$ miles, and $z = \sqrt{17.5^2 + 160^2}$ miles. Then

$$\frac{dz}{dt} = \frac{2(17.5)(-55) + 2(160)(-60)}{2\sqrt{17.5^2 + 160^2}}.$$

AP Questions

(MC) **2003** AB/BC 78

(FR) AB 3a

(FR) **2009** (Form B)

AB 1b

(FR) **2010** (Form B)

AB 3/BC 3d

(MC) **2012** AB 88

Compare the answers from the two methods and discuss the factors of the derivative. The surprise is not that the answers are the same, but that the un-simplified expressions are the same. Each of the numbers (except the 2's) represents one of the physical aspects of the problem. The second method seems more in the spirit of a Related Rate problem. That is, the idea of expressing the change in one variable in terms the other variables and the rates of change of the other variables (their derivatives) is what Related Rate problems are all about.

A unit analysis is also in order here: The units of the numerator are *miles²/hour*; the units of the denominator are miles, and so the rate is miles/hour, as expected.

The majority of the Related Rate problems involve geometry. The variables are related by the Pythagorean Theorem as in the example above, by similar triangles, or right-triangle trigonometry. The formula is often given in the stem of the question.

Related Rate question do appear on the exam, but they are not a major item. Often they account for a single multiple-choice question or one or two parts of a free-response question.

L'Hôpital's Rule

L'Hôpital's Rule is a BC topic that will be included in AB starting with the 2017 exams. It is an efficient way of finding limits of indeterminate forms and the "proof" is an amazing demonstration of the power of the local linearity concept.

Limits that are tested on the AB Exams are often capable of being done by algebraic methods, other than L'Hôpital's Rule (direct substitution, factoring, reducing, using trig identities, etc.). L'Hôpital's Rule is often easier and more direct and students may use it whenever it is applicable.

AP Questions

(FR) **2010** BC 5b

(FR) **2013** BC 5a

For example this limit may be found by reducing:

$$\lim_{x \rightarrow \pi} \frac{\tan(x)}{\sin(x)} = \lim_{x \rightarrow \pi} \frac{1}{\cos(x)} = -1$$

or by L'Hôpital's Rule

$$\lim_{x \rightarrow \pi} \frac{\tan(x)}{\sin(x)} = \lim_{x \rightarrow \pi} \frac{\sec^2(x)}{\cos(x)} = \frac{(-1)^2}{-1} = -1$$

Keep in mind that to use L'Hôpital's Rule, the limit must be one of the indeterminate forms discussed later in this chapter; this must be established before applying L'Hôpital's Rule.

On the BC exam L'Hôpital's Rule may be the only practical way to find the limit. It may also be used in connection with evaluating improper integrals and infinite series. For example:

$$\begin{aligned} \int_1^{\infty} (3-x) e^{-x} dx &= \lim_{b \rightarrow \infty} \int_1^b (3-x) e^{-x} dx \\ &= \lim_{b \rightarrow \infty} (x-2) e^{-x} \Big|_1^b \\ &= \lim_{b \rightarrow \infty} \left(\frac{b-2}{e^b} \right) - (-e^{-1}) \\ &= \lim_{b \rightarrow \infty} \frac{1}{e^b} + e^{-1} \\ &= 0 + e^{-1} \\ &= e^{-1} \end{aligned}$$

The antiderivative was found using the technique of Integration by Parts (chapter 13). The limit in the third line is an indeterminate form of the type $\frac{\infty}{\infty}$. L'Hôpital's Rule was used to express this limit as one whose value could be found (fourth line).

Note that when evaluating improper integrals on the free-response section the limit forms must be shown as in the example above. Skipping right to the second line could lose points.

L'Hôpital's Rule and Local Linearity¹³

The proof of L'Hôpital's Rule begins with a lemma – a simple theorem that will be used in proving the main theorem:

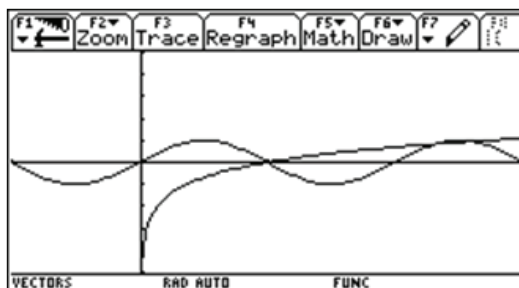
If two lines intersect at a point on the x -axis, then the ratio of their y -coordinates is equal to the ratio of their slopes. That is, if $y_1 = m_1(x - a)$ and $y_2 = m_2(x - a)$, then for all $x \neq a$, $\frac{y_1}{y_2} = \frac{m_1}{m_2}$.

L'Hôpital's Rule begins with a similar assumption: two differentiable functions that intersect at a point on the x -axis:

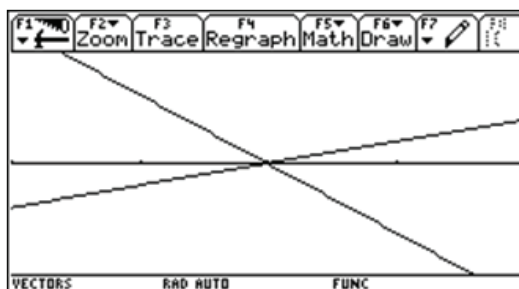
Let f and g be differentiable functions and let $f(a) = g(a) = 0$, then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$, if the limit exists.

As an example, find $\lim_{x \rightarrow 1} \frac{\ln(x)}{\sin(\pi x)}$. Let $f(x) = \ln(x)$, $g(x) = \sin(\pi x)$, and $a = 1$.

The figure below shows the calculator graph of the functions in the window $x[-1,3]$ by $y[-5, 5]$.



After zooming in on the point $(1,0)$ several times the local linearity property shows the graphs looking like lines.



Activity 3

And this is where the lemma comes into play. The local linearity property of differentiable functions has produced two “lines” intersecting on the x -axis. Since the hypothesis requires differentiable functions, it will always be the case that the graphs will become “lines” due to local linearity.

The rest is easy: in the limit the ratio of the y -coordinates $\frac{f(x)}{g(x)}$ is equal to the ratio of their slopes $\frac{f'(x)}{g'(x)}$ and

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$$

$$\lim_{x \rightarrow 1} \frac{\ln(x)}{\sin(\pi x)} = \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{\pi \cos(\pi x)} = \frac{1}{-\pi}$$

In addition to making clear why L'Hôpital's Rule is true in a way that is much easier to see than the analytic proofs in most textbooks, this approach also nicely ties together the graphical and analytic aspects of the problem.

It may be that the limit of the derivatives is also an indeterminate form. If so, L'Hôpital's Rule may be used again until a limit is found or the expression is no longer indeterminate.

Note that this is just the quotient of the derivatives, not a Quotient Rule differentiation situation.

Demonstration using the Tangent Line Approximations:

Another way of demonstrating L'Hôpital's Rule uses the tangent line approximations of the functions. Lines that intersect on the x -axis approximate the functions.

$$\begin{aligned}\lim_{x \rightarrow a} \frac{f(x)}{g(x)} &\approx \lim_{x \rightarrow a} \frac{f(a) + f'(a)(x - a)}{g(a) + g'(a)(x - a)} \\ &\approx \lim_{x \rightarrow a} \frac{0 + f'(a)(x - a)}{0 + g'(a)(x - a)} \\ &\approx \lim_{x \rightarrow a} \frac{f'(a)}{g'(a)} \\ &= \frac{f'(a)}{g'(a)}\end{aligned}$$

Both this method and the previous method show the power of the concept of local linearity. Here, since the functions are differentiable, they can be approximated by their tangent lines. Since $f(a) = g(a) = 0$, the quotient reduces to an equation like that in the lemma. Previously, zooming in showed that the functions were locally linear and suggested the use of the lemma.

Indeterminate and Determinate Forms:

In addition to $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$ (with any combination of signs), $\infty - \infty$, $\infty \cdot 0$, 0^0 , 1^∞ , and ∞^0 are also considered indeterminate forms. These are called indeterminate forms because their limits cannot be determined directly and the same form may have different values depending on the particular functions and the point at which the limit is found. These other forms are handled by manipulating them into $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$, or reducing them to something whose limit can be found. This may involve algebra, trigonometric substitutions or using logarithms. Textbooks have many examples of these; the AP Calculus exam questions stick to the two basic forms.

It is helpful to realize that there are determinate forms that resemble the indeterminate, but always have the same value. The main determinate forms and their limit values are:

- $\infty + \infty \rightarrow \infty$
- $-\infty - \infty \rightarrow -\infty$
- $0^\infty \rightarrow 0$
- $\frac{1}{0^\infty} = 0^{-\infty} \rightarrow \infty$

AP Questions

- (MC) **2003** BC 2
 (MC) **2008** BC 3
 (MC) **2012** BC 28

AP Question(MC) **2012 AC 18**

Finally, realize that the limit definition of derivative is always an indeterminate form. A limit like $\lim_{h \rightarrow 0} \frac{\sin(\frac{\pi}{3} + h) - \sin(\frac{\pi}{3})}{h}$ is an indeterminate form of the type $\frac{0}{0}$. It can be evaluated using the definition of the derivative of $\sin(x)$ evaluated at $\frac{\pi}{3}$, or by using L'Hôpital's Rule. In either case the result is the same: $\cos(\frac{\pi}{3}) = \frac{1}{2}$.

Activity 1: MVT discussion question:

The procedure used in the proof of the MVT to find the value of c for the function $h(x)$ is quite similar to the procedure for finding extreme values of $h(x)$. Are the extreme values of $h(x)$ at the value(s) of c ? Why or why not? Think graphically.

Activity 2: Related Rate Questions without Geometry.

Physics Formulas: This is a different type Related Rate problem that is not found in many textbooks. They are good beginning questions because there is no geometry to worry about.

- a. The kinetic energy, K in joules, of a moving object is given by the equation $K = \frac{1}{2}mv^2$ where m is the mass of the object and v is its velocity. The mass of a rocket decreases at a constant rate of 25 kg/sec due to the burning of its fuel. When the mass of the rocket is 6000 kg, the velocity is 12 m/sec and increasing at the rate of 2 m/sec/sec. At this instant how fast is the kinetic energy in joules / sec., changing?
- (A) 142,200
(B) 144,000
(C) 145,800
(D) 217,600
(E) 432,000
- b. The force, F in Newtons, of a moving object is given by the equation $F = ma$ where m is the mass of the object and a is its acceleration. A rocket sled is propelled along a track with an acceleration given by $a(t) = 5t^2 + 6t$ for $t \geq 0$. When $t = 6$ sec. its mass is 10 kg and is decreasing at the rate of 0.2 kg/sec due to the burning of its fuel. At this instant how fast is the force changing in Newtons / second?
- (A) 228
(B) 327.4
(C) 519.2
(D) 616.8
(E) 703.2

Answers: a. A, b. D

Activity 3: L'Hôpital's Rule by graphing.

For each of the limits graph the numerators and denominators separately on the same graph. Zoom-in several times at the point where they intersect on the x -axis until the graphs look like lines. Compare their slopes at the point where they meet on the x -axis. What is the *ratio of the slopes*? Explain how you can determine this by looking at the graph and not doing any computation.

a. $\lim_{x \rightarrow \pi} \frac{\tan(x)}{\sin(x)}$

b. $\lim_{x \rightarrow 0} \frac{\sin(x)}{x}$

c. $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x}$

Answers: a. -1 ; b. 1 ; c. 0

Activity 4: Cutting a wire.

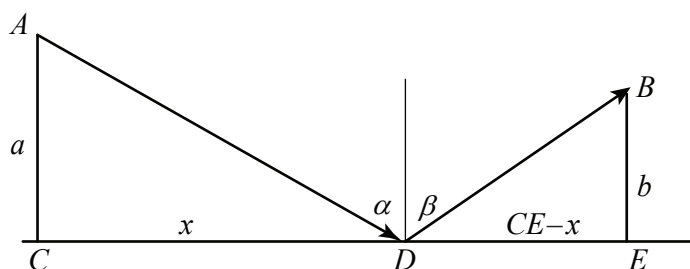
Often when student finish this problem they have found the minimum; unfortunately the maximum is required.

A wire 3 feet long is cut and formed into a square and a circle. Where should the wire be cut so that the total area of the square and circle is a maximum?

Answer: The maximum occurs when the wire is formed entirely into a circle. This is an endpoint maximum.

Activity 5: Project – The Law of Reflection. An optimization exercise

1. Light travels at a constant speed from a point a units above a flat surface, to the surface, and then is reflected to a second point b units above the surface, so that the total time is a minimum. Show that this implies that the angle between the incoming ray and the normal to the surface (*angle of incidence*, α) equals the angle between the normal and the reflected ray (*angle of reflection*, β).



2. Using the Law of Reflection and the formula for the angle between two lines (see hints), prove the reflection properties of the conic sections:
- (A) A light ray originating from the focus of a parabola is reflected parallel to the axis of the parabola; a ray traveling parallel to the axis of a parabola is reflected to the focus.
 - (B) A light ray originating from one focus of an ellipse is reflected to the other focus.
 - (C) A light ray originating from one focus of a hyperbola is reflected (from either branch of the hyperbola) as through it originated from the other focus.

Hints:

For the Law of Reflection

1. Since the speed is constant, minimizing the distance is equivalent to minimizing the time.
2. This involves a very messy computation; there is, however, a way to bail out long before the end.
3. The *acute* angle rotated *counterclockwise* from a line with slope of m_1 to a line with a slope of m_2 is $\tan^{-1} \left(\frac{m_2 - m_1}{1 + m_1 m_2} \right)$.

Solution: The total distance $T = \sqrt{a^2 + x^2} + \sqrt{(CE - x)^2 + b^2}$

and $\frac{dT}{dx} = \frac{x}{\sqrt{a^2 + x^2}} + \frac{-(CE - x)}{\sqrt{b^2 + (CE - x)^2}}$. This equals zero when

$$\frac{x}{\sqrt{a^2 + x^2}} = \frac{CE - x}{\sqrt{b^2 - (CE - x)^2}}.$$

But $\sin \alpha = \frac{x}{\sqrt{a^2 + x^2}}$, and $\sin \beta = \frac{CE - x}{\sqrt{b^2 - (CE - x)^2}}$. So

$\sin \alpha = \sin \beta$, and $\alpha = \beta$.