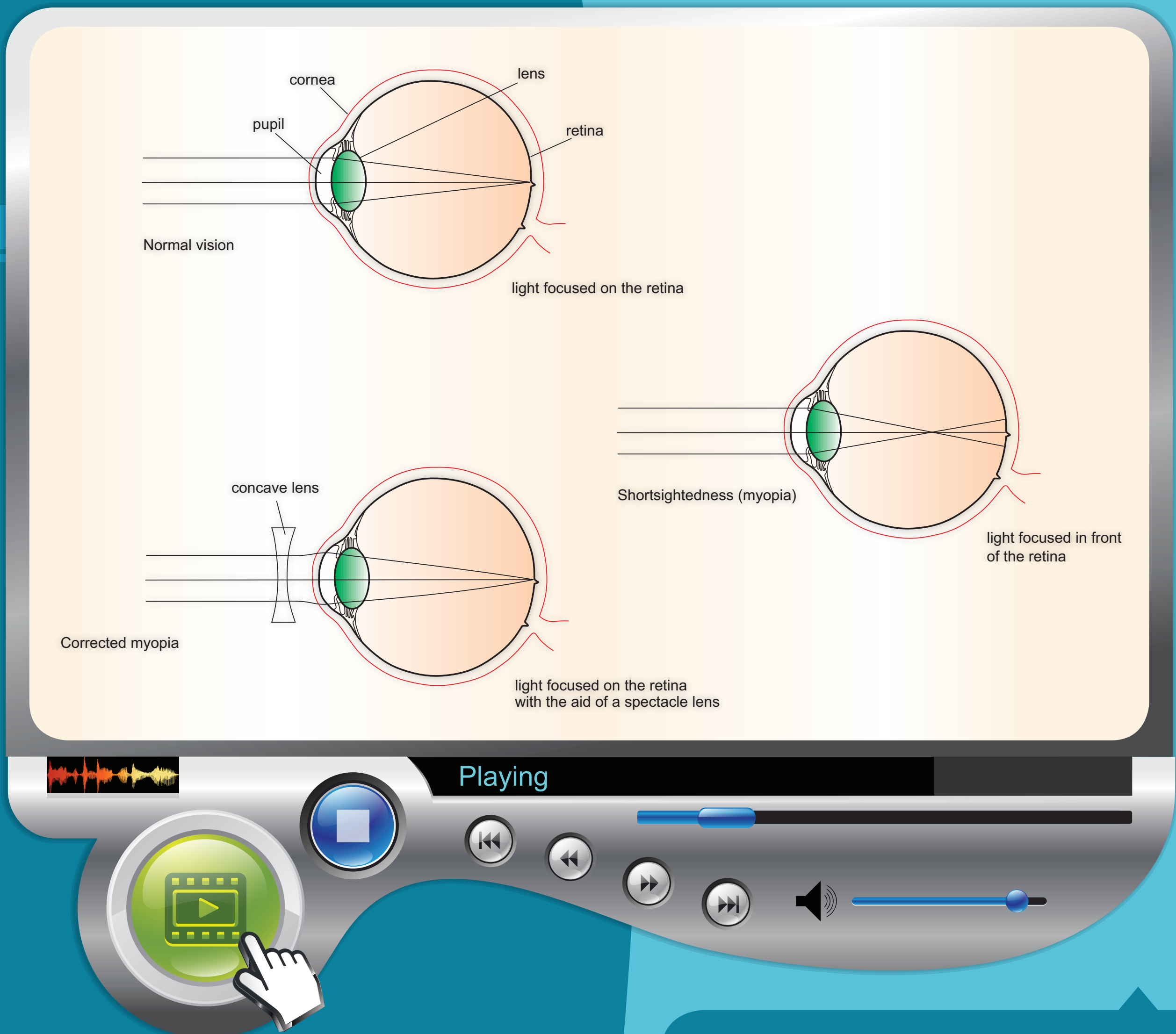
5 SIMPLE ALGEBRAIC FRACTIONS



LET'S LEARN TO...

1. express an algebraic fraction in its simplest form
2. perform the four operations (+, –, ×, ÷) on algebraic fractions
3. solve fractional equations that can be reduced to quandratic equations
4. solve everyday problems involving algebraic fractions
5. find the value of an unknown quantity in a given formula
6. change the subject of a formula

A human eye uses a lens to form the image of an object on the retina. A short-sighted person has to use a concave lens to correct his or her vision. We apply the lens formula $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$ to find the required focal length f of a lens to correct short-sightedness, where u and v denote the object distance and image distance from the lens respectively.

5.1 *Simplifying Simple Algebraic Fractions*

We know that a number of the form $\frac{p}{q}$, where p and q are integers, and $q \neq 0$, is called a fraction. For example, $\frac{2}{3}$, $\frac{17}{5}$, and $-\frac{9}{26}$ are fractions.

An algebraic expression of the form $\frac{P}{Q}$, where P and Q are expressions involving addition, subtraction, and/or multiplication and $Q \neq 0$, is called an **algebraic fraction**. For example,

$$\frac{ab}{c}, \frac{m}{n-2}, \frac{x^2}{(x+1)(x-3)}, \text{ and } \frac{4pq-5rs}{y+z} \text{ are algebraic fractions.}$$

It must be emphasized that the value of the denominator of an algebraic fraction **cannot** be zero. This is because division by zero is undefined. Therefore, in the above examples of algebraic fractions, $c \neq 0$, $n \neq 2$, $x \neq -1$ or $x \neq 3$, and $y \neq -z$. Therefore, in this chapter, we will take the value of all denominators to be non-zero.

> If $R \neq 0$, we have $\frac{R}{R} = 1$, and hence,
>
> $$\frac{P \times R}{Q \times R} = \frac{P}{Q} \times \frac{R}{R} = \frac{P}{Q} \quad \text{and} \quad \frac{P \div R}{Q \div R} = \frac{P}{Q} \div \frac{R}{R} = \frac{P}{Q}.$$

We see that the value of an algebraic fraction is unchanged when both its numerator and denominator are multiplied or divided by the same non-zero number or expression.

MATH WEB

You can see more examples of and practice with algebraic fractions at the website http://regentsprep.org/Regents/math/ALGEBRA/AV5/indexAV5.htm.

DISCUSS

Is $\frac{a}{b} = \frac{a+c}{b+c}$ for $c \neq 0$?

Example 1

Express each of the following in its simplest form.

(a) $\frac{12a^2b}{15ab^3}$

(b) $\frac{8x^2+6x}{18tx}$

Solution

(a) $\frac{12a^2b}{15ab^3} = \frac{(3ab)(4a)}{(3ab)(5b^2)}$ — Factorize the numerator and the denominator.

$= \frac{4a}{5b^2}$ — $\frac{3ab}{3ab} = 1$

(b) $\frac{8x^2+6x}{18tx} = \frac{(2x)(4x+3)}{(2x)(9t)}$

$= \frac{4x+3}{9t}$ — $\frac{2x}{2x} = 1$

RECALL

The simplest form of a fraction (also called a lowest term) is one where the numerator and denominator have no factors in common, other than 1.

Try It! 1

Simplify each of the following algebraic fractions.

(a) $\dfrac{28x^3y^2}{20x^2y^5}$

(b) $\dfrac{25mn}{5m^2 + 15mn}$

Example 2

Simplify each of the following algebraic fractions.

(a) $\dfrac{2ax - 6ay + bx - 3by}{10ax + 8ay + 5bx + 4by}$

(b) $\dfrac{2x^2 - x - 15}{9 - x^2}$

Solution

(a) $\dfrac{2ax - 6ay + bx - 3by}{10ax + 8ay + 5bx + 4by}$

$= \dfrac{2a(x - 3y) + b(x - 3y)}{2a(5x + 4y) + b(5x + 4y)}$ Factorize by grouping terms.

$= \dfrac{\cancel{(2a + b)}(x - 3y)}{\cancel{(2a + b)}(5x + 4y)}$

$= \dfrac{x - 3y}{5x + 4y}$

(b) $\dfrac{2x^2 - x - 15}{9 - x^2}$

$= \dfrac{(2x + 5)(x - 3)}{(3 + x)(3 - x)}$ Using $a^2 - b^2 = (a + b)(a - b)$, $9 - x^2 = (3 + x)(3 - x)$.

$= \dfrac{(2x + 5)\cancel{(x - 3)}}{-(3 + x)\cancel{(x - 3)}}$ $3 - x = -(x - 3)$

$= -\dfrac{2x + 5}{x + 3}$ $\dfrac{a}{-b} = -\dfrac{a}{b}$

REMARKS

In general,

$y - x = -(x - y)$.

Try It! 2

Simplify each of the following.

(a) $\dfrac{3px + 15py - 2qx - 10qy}{21px - 3py - 14qx + 2qy}$

(b) $\dfrac{3x^2 - 10x - 8}{16 - x^2}$

5.3 *Addition and Subtraction of Algebraic Fractions*

In Grade 7, we learned to simplify algebraic expressions such as $\frac{x-3}{4} + \frac{x+1}{5}$, where the denominator of each algebraic fraction is an integer. Recall that to add these algebraic fractions, we can find the LCM of the denominators so that both fractions have a common denominator as shown below.

$$\begin{aligned}\frac{x-3}{4} + \frac{x+1}{5} &= \frac{5(x-3)}{20} + \frac{4(x+1)}{20} \quad \text{LCM of 4 and 5 is 20.}\\ &= \frac{5x-15+4x+4}{20}\\ &= \frac{9x-11}{20}\end{aligned}$$

RECALL

The least common multiple (LCM) of two or more numbers is the smallest number that is a multiple of the given numbers.

Similarly, when we add or subtract algebraic fractions with denominators that involve variables, we can find the LCM of the denominators first. The LCM of two algebraic expressions is an expression that is the common multiple of the given expressions with the least number of factors.

Consider the expressions $8a^2b$ and $12ab^3$.

$$\begin{aligned}8a^2b &= 2^3 \times a^2 \times b\\ 12ab^3 &= 2^2 \times 3 \times a \times b^3\\ \therefore \text{ their LCM} &= 2^3 \times 3 \times a^2 \times b^3 \quad \text{the smallest product that contains every factor of the given expressions}\\ &= 24a^2b^3\end{aligned}$$

For $2(x+1)^3(x-3)$, $(x+1)^2(x+2)$, and $3(x+1)(x+2)(x-3)$, their LCM is $6(x+1)^3(x+2)(x-3)$.

Example 5

Express each of the following as a single fraction in its simplest form.

(a) $\frac{7}{8n} + \frac{1}{n^2}$ **(b)** $\frac{x}{3y} - \frac{y}{4x}$

Solution

(a) LCM of $8n$ and n^2 is $8n^2$.

$$\begin{aligned}\frac{7}{8n} + \frac{1}{n^2} &= \frac{7n}{8n^2} + \frac{8}{8n^2}\\ &= \frac{7n+8}{8n^2}\end{aligned}$$

(b) LCM of $3y$ and $4x$ is $12xy$.

$$\begin{aligned}\frac{x}{3y} - \frac{y}{4x} &= \frac{4x^2}{12xy} - \frac{3y^2}{12xy}\\ &= \frac{4x^2-3y^2}{12xy}\end{aligned}$$

MATH@WORK

11. A group of people hired a boat for $1,200 and each person paid an equal amount for it. Five people did not turn up, so each person on the trip had to pay $8 more. Find the number of people in the group who went on the trip.

12. The numerator of a positive fraction is 2 less than its denominator. When both the numerator and the denominator are increased by 3, the new fraction is greater than the original one by $\frac{1}{18}$. Find the original fraction.

BRAIN WORKS

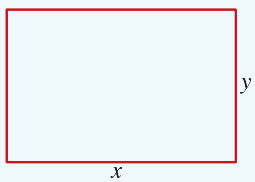
13. A store owner has 80 copies of a computer game. Based on past experience, if the price is set at $60 per copy, all of them will be sold. For each $5 increase in price, an additional 4 copies of the game will be unsold.
    (a) If the price is set at $70 per copy,
    (i) find the number of copies that will be sold,
    (ii) find the total sales amount.
    (b) If the total sales amount is $5,100, find the possible price per copy of the game.
    (c) What should the price per copy be in order to get the maximum total sales amount?
    [Assume that the price per copy of the game is a multiple of 5 in **(b)** and **(c)**.]

5.5 *More about Formulas*

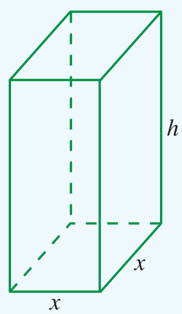
A Finding the Value of an Unknown Quantity in a Formula

In Grade 7, we learned that a formula is an equation that relates two or more quantities. Here are two examples of formulas.

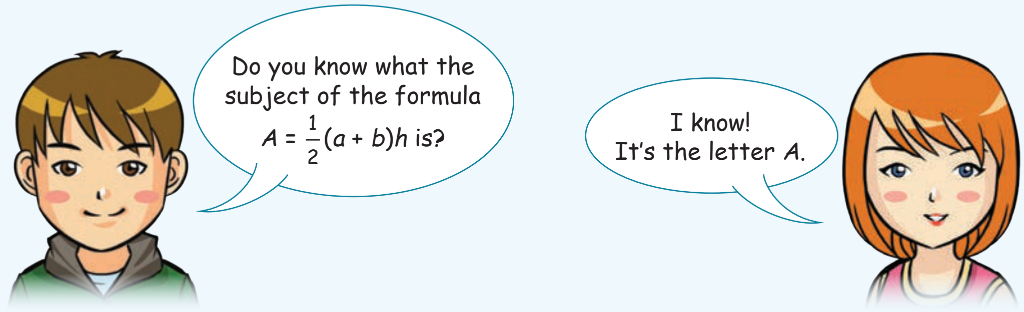
1. The perimeter P cm of a rectangle of dimensions x cm by y cm is given by the formula, $P = 2(x + y)$.



2. The volume V cm^3 of a rectangular prism with square base of side x cm and height h cm is given by the formula, $V = x^2h$.



B Changing the Subject of a Formula



The **subject** of a formula is a **single term** that appears only on the left-hand side of a formula. We can find the value of a subject, in the above case, A, by substituting the values of a, b, and h in the formula. However, if we want to find the value of b, given the other values in the formula, then we have to manipulate the terms. Alternatively, we can rewrite the formula by expressing b as the subject which is expressing b in terms of A, a, and h. This process is called **changing the subject of a formula**.

Example 14

Given the formula $d = \frac{1}{2}(u + v)t$,

(a) express v as the subject of the formula,
(b) find the value of v when $t = 4$, $u = 54$, and $d = 230$.

Solution

(a)

$$d = \frac{1}{2}(u + v)t$$

$$2d = (u + v)t \quad \text{Multiply both sides by 2.}$$

$$\frac{2d}{t} = u + v \quad \text{Divide both sides by } t.$$

$$\therefore \ v = \frac{2d}{t} - u \quad \text{Subtract } u \text{ from both sides.}$$

(b) When $t = 4$, $u = 54$, and $d = 230$, we have

$$v = \frac{2 \times 230}{4} - 54$$
$$= 115 - 54$$
$$= 61$$

DISCUSS

Compare the solution for Example 14(b) with that of Example 12(b). What do you notice?

Try It! 14

Given the formula $s = ut + \frac{1}{2}at^2$,

(a) express u as the subject of the formula,
(b) find the value of u when $a = 6$, $s = 203$, and $t = 7$.

IN A NUTSHELL

Simple Algebraic Fractions

- Algebraic expressions such as

  $$\frac{3}{x},\ \frac{ab+c}{pq-r},\ \text{and}\ \frac{x^2-9}{x^2+2x-3}$$

  are algebraic fractions, where the values of the denominators cannot be zero.

- An algebraic fraction can be simplified if there is a common factor in its numerator and its denominator.

  e.g., $$\frac{x^2-9}{x^2+2x-3} = \frac{(x+3)(x-3)}{(x+3)(x-1)} = \frac{x-3}{x-1},\ \text{where } x \neq 1$$

Multiplication and Division of Algebraic Fractions

- $$\frac{P}{Q} \times \frac{R}{S} = \frac{P \times R}{Q \times S}$$

- $$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \times \frac{S}{R} = \frac{P \times S}{Q \times R}$$

Formulas

- An equation connecting or relating variables is a formula.
- The value of a variable in a formula can be found by substitution.
- The variable that appears as a single term only on the left-hand side of a formula is called the **subject** of the formula.
- The subject of a formula can be changed using algebraic manipulations.

Addition and Subtraction of Algebraic Fractions

- Find the LCM of the denominators of the algebraic fractions in an expression.
- Convert each fraction in the given expression to an equivalent fraction with the LCM as the denominator.

  e.g., $$\frac{a}{4x} + \frac{b}{6y} = \frac{3ay}{12xy} + \frac{2bx}{12xy} = \frac{3ay+2bx}{12xy},$$

  where $x \neq 0$, $y \neq 0$.

  $$\frac{1}{x-2} - \frac{1}{x+3} = \frac{(x+3)-(x-2)}{(x-2)(x+3)} = \frac{5}{(x-2)(x+3)},$$

  where $x \neq 2$, $x \neq -3$.

Fractional Equations

$$\frac{a}{x-p} + \frac{b}{x-q} = c$$

$$(x-p)(x-q)\left(\frac{a}{x-p} + \frac{b}{x-q}\right) = c(x-p)(x-q)$$

$$a(x-q) + b(x-p) = c(x-p)(x-q)$$

- Simplify a fractional equation by multiplying both of the equation by their common denominator.
- Check and reject any derived roots that cause the “division-by-zero” error in the original equation.

REVIEW EXERCISE 5

1. Simplify each of the following algebraic fractions.

(a) $\dfrac{18u^2 - 30u}{35 - 21u}$

(b) $\dfrac{9ac + 12bc}{9a^2 - 16b^2}$

(c) $\dfrac{k^2 + 6k - 7}{k^2 - k}$

(d) $\dfrac{x^2 - 9}{x^2 - 7x + 12}$

2. Simplify each of the following.

(a) $\dfrac{x^2 - 36}{x^2 - y^2} \times \dfrac{x + y}{x - 6}$

(b) $\dfrac{a^2 + a - 12}{8a + 12b} \times \dfrac{14a + 21b}{a^2 + 2a - 8}$

(c) $\dfrac{t^2 + 9t + 14}{4t^2 - 1} \div \dfrac{t + 7}{1 - 2t}$

(d) $\dfrac{p^2 - 2p + 1}{4q^2 + q - 3} \div \dfrac{p - 1}{q + 1}$

3. Simplify each of the following as a single fraction in its simplest form.

(a) $\dfrac{3x + 5}{4} + \dfrac{5}{3x - 5}$

(b) $\dfrac{1}{2x - 7} - \dfrac{1}{2x + 9}$

(c) $\dfrac{2}{x + 6} - \dfrac{3}{(x + 6)^2}$

(d) $\dfrac{10}{x^2 - 8x + 16} + \dfrac{3}{4 - x}$

4. Simplify each of the following as a single fraction in its simplest form.

(a) $\dfrac{2}{x^2 - 1} - \dfrac{1}{x + 1} + \dfrac{1}{1 - x}$

(b) $\dfrac{2}{3y - 2} + \dfrac{4}{4 - 9y^2} - \dfrac{1}{2 + 3y}$

5. **(a)** Simplify $\dfrac{1}{t - 3} - \dfrac{6}{t^2 - 9}$.

(b) Hence simplify

$$\left(\frac{1}{t - 3} - \frac{6}{t^2 - 9}\right) \times (t^2 + 10t + 21).$$

6. Solve each of the following equations.

(a) $\dfrac{x - 2}{x + 4} = x$

(b) $\dfrac{x + 1}{2} = \dfrac{8}{x + 1}$

(c) $5x - \dfrac{5}{x} = 24$

(d) $\dfrac{7 - x}{x + 2} - \dfrac{1}{x - 1} = \dfrac{1}{3}$

7. Make the letter in the brackets the subject for each of the given formula.

(a) $y = \dfrac{x}{5}(2h + k)$ [k]

(b) $z = \dfrac{3u + 2v}{2u - 5v}$ [v]

(c) $\dfrac{1}{2p^2} - \dfrac{2}{x^2} = 1$ [p]

(d) $H = 4\pi\sqrt{m^2 + n^2}$ [m]

8. The cost $\$C$ of making a circular brooch of radius r cm is given by the formula,

$$C = 5 + 8r^2 + \frac{200}{n},$$

where n is the number of brooches made.

(a) If 100 brooches of radius 2 cm are made, find the cost of making each brooch.

(b) Express r as the subject of the formula.

(c) If 400 brooches are made and the cost of making each brooch is \$55.50, find the radius of each brooch.

9. A formula is given by $y = 3\sqrt{\dfrac{c^2 + x^2}{c^2 - x^2}}$, where $0 < x < c$.

(a) Make x the subject of the formula.

(b) When $c = 5$ and $y = 4$, find the value of x, rounded to 3 significant figures.

10. Jim and Peter each earn \$176 a week from their part-time jobs. Jim is paid \$$x$ per hour and Peter gets \$3 per hour more than Jim.
   (a) Write down, in terms of x, the number of hours that Jim and Peter each work in a week.
   (b) Given also that Jim works 6 hours more per week than Peter, form an equation in x and find the value of x.
   (c) Find the number of hours that Jim works in a week.

11. The denominator of a fraction is 2 more than 2 times of its numerator. If 3 is subtracted from the numerator and 5 is subtracted from the denominator, the new fraction is equivalent to $\frac{2}{5}$. Find the original fraction.

12. Two good friends, Betty and Susan, each ran the 42 km of a marathon race. Susan ran the race at an average speed which was 1 km/hr less than Betty's average speed. Given that the difference between the times taken by the two girls to complete the race was 12 minutes,
   (a) form an equation to represent this situation,
   (b) find, in hours and minutes, the time Betty took to complete the race.

EXTEND YOUR LEARNING CURVE

Complex Fractions

A complex fraction is a fraction in which the numerator or the denominator or both are fractions.

The following are three examples of complex fractions. Only one of them is simplified correctly. Identify the correct one and point out the mistakes made in the other two.

1.
$$\frac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{c}+\frac{b}{d}} = \frac{\frac{ad+bc}{bd}}{\frac{ad+bc}{cd}} = \frac{\frac{1}{bd}}{\frac{1}{cd}} = \frac{b}{c}$$

2.
$$\frac{x^2-\frac{1}{x^2}}{x+\frac{1}{x}} = \frac{\left(x-\frac{1}{x}\right)\left(x+\frac{1}{x}\right)}{x+\frac{1}{x}} = x-\frac{1}{x} = \frac{x^2-1}{x}$$

3.
$$\frac{\frac{2y}{x}-\frac{x}{2y}}{\frac{1}{2y}-\frac{1}{x}} = \frac{\frac{4y^2-x^2}{2xy}}{\frac{x-2y}{2xy}} = \frac{4y^2-x^2}{x-2y} = \frac{(2y+x)(2y-x)}{x-2y} = -2y+x$$

WRITE IN YOUR JOURNAL

1. Explain what you would do to add or subtract algebraic fractions with quadratic expressions in the numerator and/or denominator.

2. Discuss why it is necessary to change the subject of a formula.