Achievement Standard 91262

Apply calculus methods in solving problems

MATHEMATICS AND STATISTICS

2.7

Externally assessed 5 credits

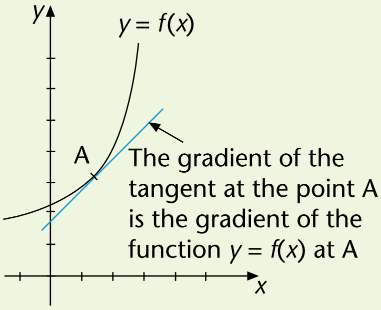
Gradient functions and differentiation

In problems with a *constant* **rate of change** between two variables, the graph of one variable against the other is a straight line whose **gradient** or **slope** equals the rate of change. In most situations, the graph of one variable against the other is not a straight line because the rate of change varies (and is not constant).

Gradient function

The function which gives the gradient (slope or rate of change) at each point of a function's graph is called the **gradient function** or **derivative** for that particular graph.

At points on the graph where there is a tangent, the gradient function gives the slope of the tangent to the curve at that point.

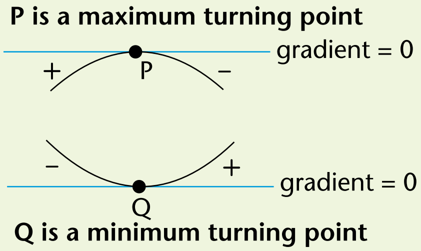


Sketching the gradient function

The graph of a function f can be used to draw the graph of its gradient function, f'.

A function $y = f(x)$ has a **maximum turning point** when its graph changes from **increasing** (positive gradient) to **decreasing** (negative gradient). At this turning point the gradient is zero ($f'(x) = 0$).

Similarly a function has a **minimum turning point** when its graph changes from decreasing to increasing.



Example

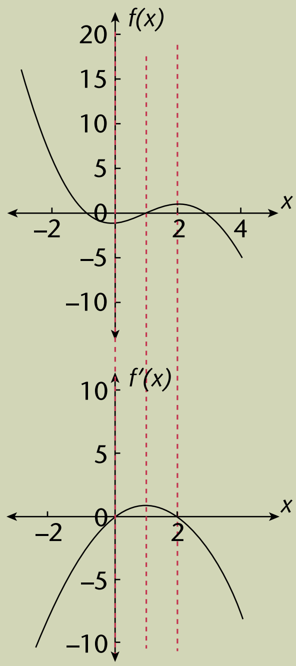
The graph of a function f is drawn alongside (upper graph). The graph of its gradient function f' is drawn below.

The first and third dotted vertical lines pass through the maximum and minimum turning points of f. At these two points, the gradient of f is zero ($f' = 0$), so the x-values of the maximum and minimum turning points correspond to the x-intercepts of f'.

Left of the first dotted vertical line and right of the third dotted vertical line, the gradient of f is negative, so for these x-values, the graph of f' lies below the x-axis.

Between the first and third dotted vertical lines, the gradient of f is positive, so for these x-values, the graph of f' lies above the x-axis.

The middle dotted vertical line passes through the point where f has maximum gradient, so this corresponds to the vertex of f' (a maximum turning point).



Finding derivatives

The gradient function or derivative of $f(x) = x^2$ is easily shown to be the function $y = 2x$.
By using identical methods, it can be shown that the derivative of x^3 is $3x^2$ and of x^4 is $4x^3$. In general:

The derivative of x^n is nx^{n-1}

Example

Q. Find the derivatives of x^4 and x^{43}

A. The derivative of x^4 is $4x^3$ and the derivative of x^{43} is $43x^{42}$

Note:
- The derivative of x is 1 — **Proof:** x can be written as x^1. The derivative of x^1 is $1x^0 = 1$
- The derivative of 1 is 0 — **Proof:** 1 can be written as x^0. The derivative of x^0 is $0x^{-1} = 0$

Differentiation

The process of *finding a derivative* is called **differentiation**. Functions which *can be differentiated* are called **differentiable**. Substitute to find the derivative for a particular value of x.

Example

Q. Differentiate x^8 and find the value of the derivative when $x = 2$

A. Differentiating x^8 gives the derivative $8x^7$ — derivative of x^n is nx^{n-1}

When $x = 2$, the value of the derivative is $8 \times 2^7 = 1\,024$ — substituting $x = 2$

There are several notations for showing derivatives.

$f'(x)$, $\frac{dy}{dx}$ and y' *all* mean the derivative of the function $y = f(x)$

$f'(a)$ means the derivative of the function $y = f(x)$ when $x = a$

Examples

Q. 1. $f(x) = x^{20}$; what is $f'(x)$? 2. $u = r^{12}$; what is $\frac{du}{dr}$? 3. If $y = x^6$; find y'.

A. 1. $f'(x) = 20x^{19}$ 2. $\frac{du}{dr} = 12r^{11}$ 3. $y' = 6x^5$

When a term has a coefficient a, the term is differentiated using:

$$\frac{d}{dx}(ax^n) = anx^{n-1}$$

Examples

1. If $f(x) = 6x^3$ then $f'(x) = 6 \times 3x^2 = 18x^2$

2. If $g(x) = 5x^4$ then $g'(x) = 5 \times 4x^3 = 20x^3$

3. If $h(x) = \frac{3x^6}{2} = \frac{3}{2}x^6$ then $h'(x) = \frac{3}{2} \times 6x^5 = 9x^5$

If an expression contains several terms added together, the expression is differentiated term by term.

Examples

Q. 1. Differentiate: $y = x^{20} + x^9 - x^7 + 2x + 3$ 2. Differentiate: $f(x) = \frac{x^3}{3} - \frac{5x}{6} + \frac{1}{2}$

A. 1. $\frac{dy}{dx} = 20x^{19} + 9x^8 - 7x^6 + 2x^0 + 0$
$= 20x^{19} + 9x^8 - 7x^6 + 2$ — $x^0 = 1$

2. $f'(x) = \frac{3x^2}{3} - \frac{5}{6} + 0$
$= x^2 - \frac{5}{6}$

It is important to recognise that π is a constant; also that some constants are represented by a letter such as k. Thus if $f(x) = \pi x^2$ then $f'(x) = 2\pi x$ and if $g(x) = k^2x^3$, then $g'(x) = 3k^2x^2$.

The second derivative

If the derivative of a function f' (or $\frac{dy}{dx}$) is differentiated again, then this function is called the **second derivative**, and is denoted $f''(x)$ (or $\frac{d^2y}{dx^2}$).

For example, if $f(x) = x^4$ then $f'(x) = 4x^3$ and $f''(x) = 12x^2$.

Gradient of a curve

The **gradient** of a curve $y = f(x)$ at a point whose x-coordinate is a is found by evaluating $f'(a)$. This involves differentiating to find $f'(x)$ (or $\frac{dy}{dx}$ or y') then substituting $x = a$ into the expression for the derivative.

Example

Q. Find the gradient of the curve $y = x^4 - 3x^2 + 1$ at the point where $x = 1$.

A. $y' = 4x^3 - 6x$ — differentiating

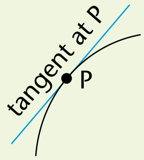
At $x = 1$, $y' = 4(1)^3 - 6(1)$ — substituting

$= -2$

So the gradient of the curve at $x = 1$ is -2

Equations of tangents

A **tangent** to a curve at a point is a straight line which touches the curve at that point.



The gradient of the tangent to the curve $y = f(x)$ at $(a, f(a))$ is $f'(a)$

Example

Q. Find the equation of the tangent to $y = x^2$ when $x = 3$.

A. When $x = 3$, $y = 9$, — since $y = 3^2$

so the tangent goes through the point (3,9)

The gradient of the tangent at x is $\frac{dy}{dx} = 2x$ — differentiating

So the gradient when $x = 3$ is $2 \times 3 = 6$ — substituting $x = 3$

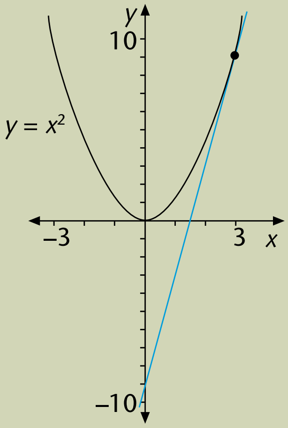
The equation of the straight line is:

$y - 9 = 6(x - 3)$ — using $y - y_1 = m(x - x_1)$

$y - 9 = 6x - 18$ — expanding

$y = 6x - 9$ — simplifying

Note: The graph illustrates the situation.



Increasing and decreasing functions

Suppose a tangent is drawn at a point on the graph of a function. For this point

- if the y-values are increasing as the x-values increase then the tangent will be tilted up and have a positive gradient.
- if the y-values are decreasing as the x-values increase then the tangent will be tilted down and have a negative gradient.

Hence if $y = f(x)$ is a function then $f'(x) > 0$ when the function is increasing and $f'(x) < 0$ when the function is decreasing.

Graphics calculators

Finding the gradient of a curve at a point is very easily done using a graphics calculator (there are special options for finding the values of derivatives). However, when finding gradients, you must show the formula for the derivative in your answer, so use this calculator method only as a check.

Using the derivative to locate turning points

At turning points of a curve, the gradient of the curve is zero.

To locate the x-coordinates of the turning points of the curve $y = f(x)$, solve $f'(x) = 0$

Substitute the x-coordinate of the turning point into the equation of the curve to find the y-coordinate of the turning point.

Example

Q. Find the turning points of the curve $f(x) = x^3 + 3x^2 - 24x + 1$

A. Solve $f'(x) = 0$ to find the turning points.

$3x^2 + 6x - 24 = 0$ — differentiating

$x^2 + 2x - 8 = 0$ — dividing by 3

$(x + 4)(x - 2) = 0$ — factorising

$x = -4$ or 2

If $x = -4$, $y = (-4)^3 + 3(-4)^2 - 24(-4) + 1 = 81$ — substituting in $y = f(x)$

If $x = 2$, $y = (2)^3 + 3(2)^2 - 24(2) + 1 = -27$ — substituting in $y = f(x)$

So the turning points are $(-4,81)$ and $(2,-27)$.

To determine the **nature of the turning points** a sign table for $f'(x)$ can be used.

Example

Continuing the previous example ...

By calculating values of $f'(x)$ for x values in the neighbourhood of the turning points $(-4,81)$ and $(2,-27)$ it can be seen that

$f'(x)$ is positive (i.e. f is increasing) for $x < -4$

$f'(x)$ is negative (i.e. f is decreasing) for $-4 < x < 2$

$f'(x)$ is positive (i.e. f is increasing) for $x > 2$

Thus $(-4,81)$ is a maximum turning point and $(2,-27)$ is a minimum turning point.

Sign table

x	-6	-4	0	2	3
$f'(x)$	+	0	–	0	+
slope	↗	—	↘	—	↗

Alternatively, the second derivative can be used to find the nature of a turning point.

Suppose a function f has a stationary value at $x = a$ (i.e. $f'(a) = 0$)

- if $f''(a) < 0$ then f has a maximum turning point at $x = a$
- if $f''(a) > 0$ then f has a minimum turning point at $x = a$

For example, in the worked example above, the curve $f(x) = x^3 + 3x^2 - 24x + 1$ has turning points at $x = -4$ and $x = 2$.

Differentiating $f'(x) = 3x^2 + 6x - 24$ gives $f''(x) = 6x + 6$

$f''(-4) = 6 \times -4 + 6 = -18$ (negative), so $x = -4$ is a maximum turning point.

$f''(2) = 6 \times 2 + 6 = 18$ (positive), so $x = 2$ is a minimum turning point.

Questions: Gradient functions and differentiation

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1. A function f is given by $f(x) = x^5 + 3x^2 - 7x + 2$.
   Find the gradient of the graph of the function at the point where $x = 1$.

2. A function f is given by $f(x) = 2 - 4x + 5x^2 + ax^3$ Year 2016 Ans. p. 103

   The gradient of the graph of the function at the point where $x = 1$ is 3.

   Find the value of a.

3. Use calculus to show that the line $y = 15x - 12$ Year 2017 Ans. p. 103

   is a tangent to the graph of the function $f(x) = 4x^2 - x + 4$.

4. Find the equation of the tangent to the graph of the function Year 2017 Ans. p. 103

   $f(x) = 6 + 14x - 2x^3$

   at the point (2,18) on the graph.

5. Use calculus to find the value of k if the line $y = 6x + k$ is a tangent to the graph of the function $f(x) = x^2 + 2x - 1$. Year 2017 Ans. p. 103

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6. A tangent to the graph of the function $f(x) = 3x^2 - 4x$ has a gradient of 2, and passes through the point $(5,a)$, where a is a constant.
Find the value of a.

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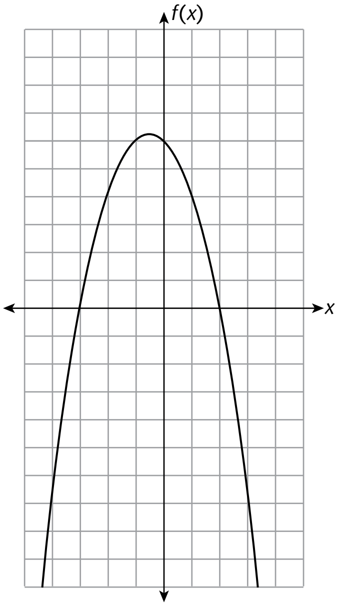
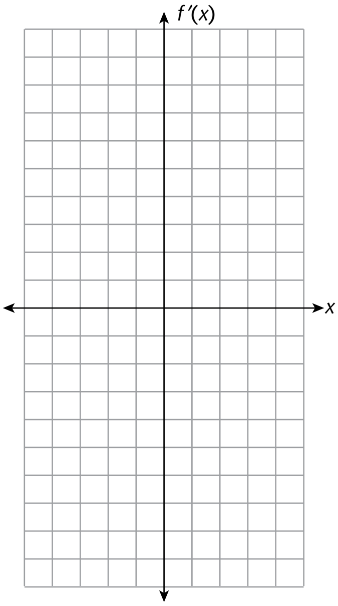
Ans. p. 104

7. The diagram below shows the graph of the function $y = f(x)$



Sketch the graph of the gradient function $y = f'(x)$ on the axes below.
Both sets of axes have the same scale.



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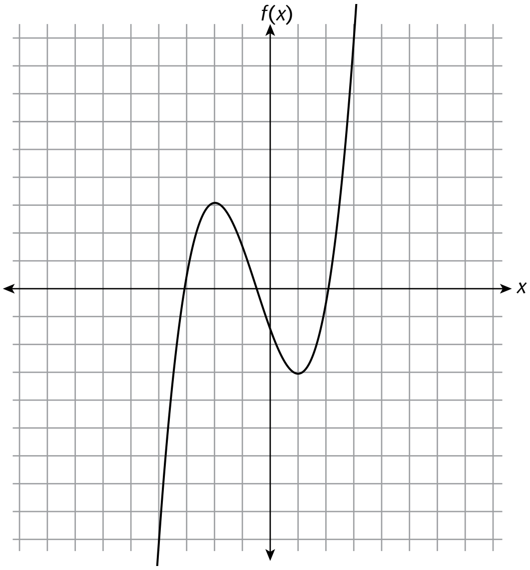
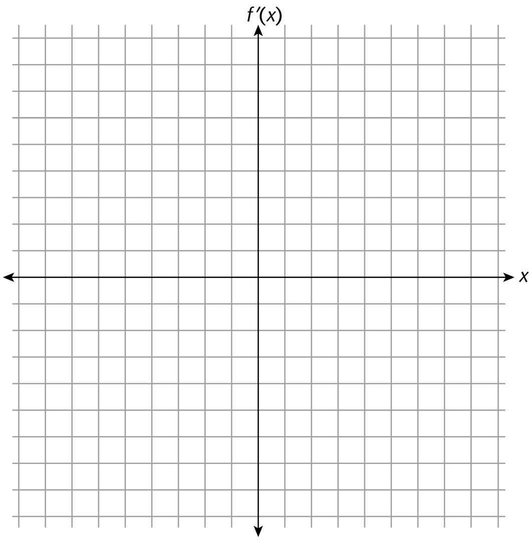
8. The diagram below shows the graph of the function $y = f(x)$.



On the axes below sketch the gradient function $y = f'(x)$.



9. For a function $y = -ax^2 + bx + c$,

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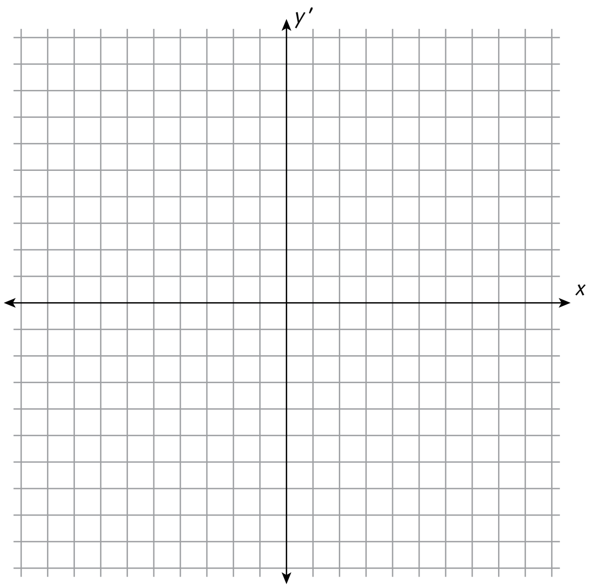
a, b, and c are positive numbers and $b = 2a$.

On the grid below, sketch the gradient function.

Show the value of all intercepts. The y'-intercept should be given in terms of b.



10. $f(x) = 8 + 3x + x^2 - \frac{x^3}{3}$

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Ans. p. 105

For what values of x is f a decreasing function?

Justify your answer.

You must show the use of calculus.

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Ans. p. 105

11. Use calculus to prove that the graph of the function

$y = x^3(3 - x)$

has a local maximum when $x = \frac{9}{4}$.

Justify that the turning point is a local maximum.

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Ans. p. 106

12. A function is defined by $y = 3x^3 - 4a^2x + 5$ where a is a positive number.

Find the range of values of x in terms of a for which the function is decreasing.

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Ans. p. 106

13. The graph of a function $f(x) = 2x^3 + bx^2 - 2$ has a turning point when $x = -1$.

Find the value of b.

Answers and explanations

Achievement Standard 91261 (Mathematics and Statistics 2.6)

Apply algebraic methods in solving problems

2.6 Expanding and factorising

p. 3

1. a.	$(3x + 1)(2x + 3) = 6x^2 + 9x + 2x + 3$	expanding
	$= 6x^2 + 11x + 3$ (A)	gathering like terms
b.	$(2x + 3)^2 = (2x + 3)(2x + 3)$	writing $(2x + 3)^2$ as a product
	$= 4x^2 + 6x + 6x + 9$	expanding
	$= 4x^2 + 12x + 9$ (A)	gathering like terms
c.	$(x + 3)(x^2 + 3x + 1) = x^3 + 3x^2 + x + 3x^2 + 9x + 3$	expanding
	$= x^3 + 6x^2 + 10x + 3$ (A)	gathering like terms
d.	$(x + 1)(x + 3)(x + 5) = (x + 1)(x^2 + 8x + 15)$	expanding and simplifying $(x + 3)(x + 5)$
	$= x^3 + 8x^2 + 15x + x^2 + 8x + 15$	expanding
	$= x^3 + 9x^2 + 23x + 15$ (A)	simplifying
e.	$(3x + 1)(x - 1)(x + 2) = (3x + 1)(x^2 + x - 2)$	expanding right two brackets
	$= 3x^3 + 3x^2 - 6x + x^2 + x - 2$	
	$= 3x^3 + 4x^2 - 5x - 2$ (A)	
f.	$(x + 1)^3 = (x + 1)(x + 1)^2$	
	$= (x + 1)(x^2 + 2x + 1)$	expanding $(x + 1)^2$
	$= x^3 + 2x^2 + x + x^2 + 2x + 1$	further expansion
	$= x^3 + 3x^2 + 3x + 1$ (A)	simplifying
2. a.	$(5x - 4)(x + 1)(x + 2) = (5x - 4)(x^2 + 3x + 2)$	expanding last two brackets
	$= 5x^3 + 15x^2 + 10x - 4x^2 - 12x - 8$	expanding
	$= 5x^3 + 11x^2 - 2x - 8$ (A)	gathering like terms
b.	$2x(x^2 - 3x + 5) - x(x - 3)^2 = 2x(x^2 - 3x + 5) - x(x - 3)(x - 3)$	writing $(x - 3)^2$ as a product
	$= 2x(x^2 - 3x + 5) - x(x^2 - 6x + 9)$	expanding $(x - 3)(x - 3)$
	$= 2x^3 - 6x^2 + 10x - x^3 + 6x^2 - 9x$	expanding
	$= x^3 + x$ (A)	gathering like terms
c.	$(5x - 2)(3 - x)(x + 4) = (5x - 2)(-x + 12 - x^2)$	expanding and simplifying $(3 - x)(x + 4)$
	$= -5x^2 + 60x - 5x^3 + 2x - 24 + 2x^2$	expanding
	$= -5x^3 - 3x^2 + 62x - 24$ (A)	simplifying
d.	$(2x - 1)(x + 3)^2 = (2x - 1)(x + 3)(x + 3)$	
	$= (2x - 1)(x^2 + 6x + 9)$	expanding the second two brackets
	$= 2x^3 + 12x^2 + 18x - x^2 - 6x - 9$	multiplying each term in the second bracket by $2x$ then by -1
	$= 2x^3 + 11x^2 + 12x - 9$ (A)	simplifying

3. a. Multiplying the coefficient of x^2 by the constant term gives $6 \times -10 = -60$

Require factors of -60 that add to -11 (the coefficient of x)

These are -15 and 4, so express the term in x as a sum of terms using these numbers

$6x^2 - 11x - 10 = 6x^2 - 15x + 4x - 10$	writing $-11x$ as $-15x + 4x$
$= 3x(2x - 5) + 2(2x - 5)$	
$= (3x + 2)(2x - 5)$ (A)	

Note: Alternatively, consider factor pairs of $6x^2$ (the correct pair is $3x$ and $2x$) and factor pairs of -10 (the correct pair is 2 and -5 respectively) until the correct term in x results ($-11x$).

b.	$x^2 + 5x + 6 = (x + 3)(x + 2)$ (A)	factors of 6 that add to 5 are 2, 3
c.	$A^2 - 13A + 36 = (A - 9)(A - 4)$ (A)	factors of 36 that add to -13 are -9, -4
d.	$5x^2 - 9x - 2 = (5x + 1)(x - 2)$ (A)	check by expanding
e.	$4x^2 + 5x - 6 = (4x - 3)(x + 2)$ (A)	check by expanding

f. $3x^2 + 9x + 6 = 3(x^2 + 3x + 2)$
$= 3(x + 1)(x + 2)$ (A)
3 is a common factor of every term factorising $x^2 + 3x + 2$

g. $25x^2 - 36z^2 = (5x)^2 - (6z)^2$
$= (5x - 6z)(5x + 6z)$ (A)
writing $25x^2$ and $36z^2$ as squares
difference of two squares

h. $25x^2 + 40xy + 16y^2 = (5x)^2 + 2(5x)(4y) + (4y)^2$
$= (5x + 4y)^2$ (A)
expression in $A^2 + 2AB + B^2$ form
since $A^2 + 2AB + B^2 = (A + B)^2$

2.6 Rational expressions

p. 5

1. a. $\frac{x^2 - 4x + 4}{x - 2} = \frac{(x-2)(x-2)}{x-2}$ — factorising the numerator
$= x - 2$ (A) — cancelling the common factor $(x - 2)$

b. $\frac{(x+2)(x+3)}{x^2-9} = \frac{(x+2)(x+3)}{(x-3)(x+3)}$ — factorising $x^2 - 9$
$= \frac{x+2}{x-3}$ (A) — cancelling $(x + 3)$ top and bottom

c. $\frac{2x^2 - 50}{9x^2 - 39x - 30} = \frac{2(x^2 - 25)}{3(3x^2 - 13x - 10)}$ — common factors
$= \frac{2(x+5)(x-5)}{3(3x+2)(x-5)}$ — factorising
$= \frac{2(x+5)}{3(3x+2)}$ (M) — cancelling common factor $(x - 5)$

d. $\frac{2x^2 - 8}{x^2 - 2x - 8} = \frac{2(x^2 - 4)}{x^2 - 2x - 8}$ — factorising
$= \frac{2(x-2)(x+2)}{(x-4)(x+2)}$ — further factorising
$= \frac{2(x-2)}{(x-4)}$ (M) — cancelling $\frac{(x+2)}{(x+2)}$

2. a. $\frac{xy}{z} \times \frac{z}{3x} = \frac{y}{3}$ (A) — cancelling x, z

b. $\frac{A^2}{7} \div \frac{A}{14} = \frac{A^2}{7} \times \frac{14}{A}$ — division is the same as multiplication by reciprocal
$= 2A$ (A) — $14 \div 7 = 2$, cancelling A

c. $\left(\frac{x}{3} \times \frac{x}{4}\right) \div \frac{x}{5} = \frac{x.x}{12} \times \frac{5}{x}$ — division is the same as multiplication by reciprocal
$= \frac{5x}{12}$ (A) — cancelling x

d. $\frac{x}{3}\left(\frac{x}{2} - 1\right) = \frac{x^2}{6} - \frac{x}{3}$ — expanding
$= \frac{x^2 - 2x}{6}$ (A) — subtracting fractions

e. $\frac{A}{7} + \frac{A}{8} = \frac{8A}{56} + \frac{7A}{56}$ — converting both fractions to the common denominator of 56
$= \frac{15A}{56}$ (A) — adding numerators

f. $\frac{A}{5} - \frac{B}{3} = \frac{3A}{15} - \frac{5B}{15}$ — converting both fractions so they have a common denominator 15
$= \frac{3A - 5B}{15}$ (A) — subtracting the numerators

3. a. $\frac{2A}{A-3} + \frac{7}{A+4} = \frac{2A(A+4)}{(A-3)(A+4)} + \frac{7(A-3)}{(A-3)(A+4)}$ — converting both fractions so they have a common denominator of $(A - 3)(A + 4)$
$= \frac{2A^2 + 8A}{(A-3)(A+4)} + \frac{7A - 21}{(A-3)(A+4)}$ — expanding the numerators
$= \frac{2A^2 + 15A - 21}{(A-3)(A+4)}$ (A) — adding the numerators

b. $\frac{2A}{A-3} \div \frac{7A}{A+B} = \frac{2A}{A-3} \times \frac{A+B}{7A}$ — division is the same as multiplication by the reciprocal
$= \frac{2A(A+B)}{7A(A-3)}$ — multiplying the numerators and denominators
$= \frac{2A + 2B}{7A - 21}$ (A) — cancelling out the common factor A then expanding