Mathematics and Statistics 2.1

Internally assessed
2 credits

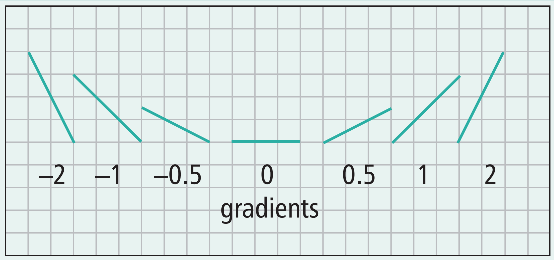
Apply coordinate geometry methods in solving problems

Gradients of lines

The **gradient** of a line is a numerical description of its steepness.

- The closer the gradient is to zero, the flatter the line is.
- The further away the gradient is from zero, the steeper the line is.

Some examples of gradients are shown below.



Notice that a line with a negative gradient goes 'downhill' from left to right and a line with a positive gradient goes 'uphill' from left to right.

Cartesian coordinates

Cartesian coordinates give the position of a point in a plane relative to two perpendicular number lines, called **axes**.

- One number line is **horizontal** and called the ***x*-axis**.
- The other number line is **vertical** and called the ***y*-axis**.

The point (0,0) where the two axes intersect is called the **origin**.

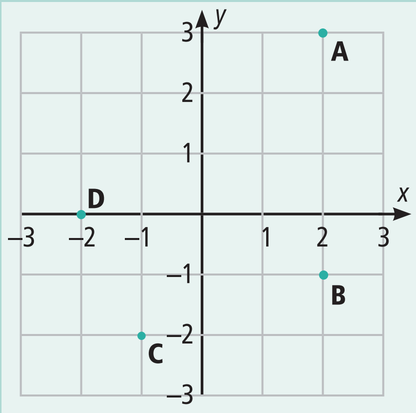
Example

On the Cartesian plane below:

The coordinates of A are (2,3)
[A is 2 right and 3 up from the origin]

The coordinates of B are (2,–1)
[B is 2 right and 1 down from the origin]

The other two points are C(–1,–2) and D(–2,0).



The gradient of a line through two points is given by the formula:

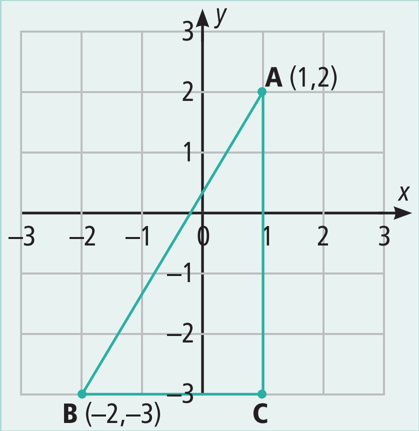
$$m = \frac{\text{rise}}{\text{run}}$$

One way of working this out is to draw a diagram.

Example

Q. A straight line is drawn through two points A(1,2) and B(–2,–3). Find the gradient of the line.

A. Points A, B are plotted on a grid. A right-angled triangle ABC is drawn on the diagram.



The length AC is found by subtracting the *y*-coordinates:

AC = 2 – –3
[(*y*-coordinate of A) – (*y*-coordinate of B)]

AC = 5 units [this is the rise]

Similarly,

BC = 1 – –2
[(*x*-coordinate of A) – (*x*-coordinate of B)]

BC = 3 units [this is the run]

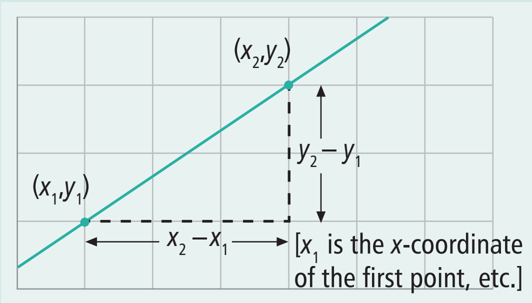
Gradient of AB is

$m = \frac{5}{3}$ [using the gradient formula $m = \frac{\text{rise}}{\text{run}}$].

Note: Subtract coordinates in the same order for both lengths (A-coordinate – B-coordinate).

Calculating the gradient of a line

If the coordinates of two points on a line are known, then the gradient can be calculated:



The **formula** that is used to find the gradient is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where m is used to represent the gradient.

Note: Either point can be the 'first' and labelled (x_1,y_1). The other point will then be labelled (x_2,y_2).

This formula may also be written in words as:

$$m = \frac{\text{change in } y}{\text{change in } x}$$

Example

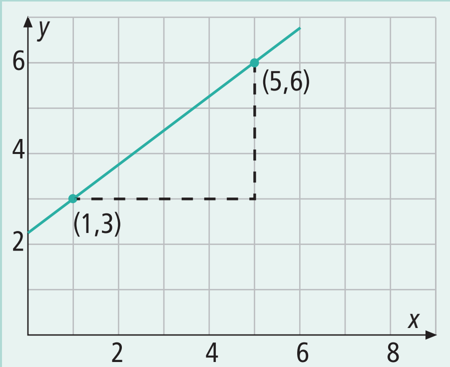
Q. Find the gradient of the line passing through the points (1,3) and (5,6)

A. Here $x_1 = 1$, $y_1 = 3$, $x_2 = 5$ and $y_2 = 6$

$m = \frac{6-3}{5-1}$ [substituting in $\frac{y_2 - y_1}{x_2 - x_1}$]

$= \frac{3}{4}$ [simplifying]

The gradient of the line is $\frac{3}{4}$.



The diagram shows the situation.

Note: If the points were labelled in reverse, so that $x_1 = 5$, $y_1 = 6$, $x_2 = 1$ and $y_2 = 3$ then the gradient is

$m = \frac{3-6}{1-5} = \frac{-3}{-4}$ or $\frac{3}{4}$, as above.

Take care with negative signs.

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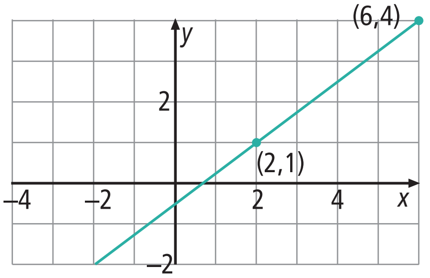
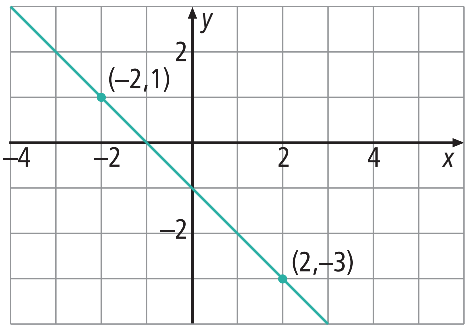
Exercise A: Gradients of lines

1. Find the gradients of the lines shown.

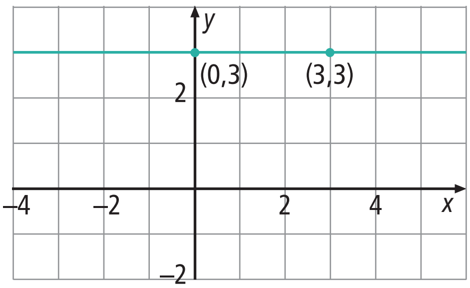
a.



b.



c.



2. Find the gradient of the line AB where

a. A = (14,20), B = (20,25)

b. A = (–3,–7), B = (2,5)

c. A = (4,–10), B = (2,–12)

d. A = (3,–5), B = (7,–5)

e. A = (0, −2), B = (–5,1)

f. A = (1,4), B = (−3,0)

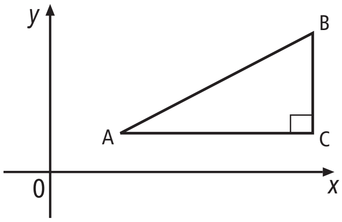
g. A(2a,–a), B(−4a,a)

11. A boat is moored at a point S, 4.5 km east and 2.3 km north of a point P. From S, the boat sails in a straight line to T, a point 1.5 km west and 2 km south of P. It then sails directly to P. What is the total length of the two legs of the trip, ST and TP?

12. A circle has centre G(2,–1). A point H(a,3) on the circumference of the circle is joined to the centre of the circle. If the radius of the circle is 8, find a.

13. For three points P, Q, R, the length of PQ is $5k$, the length of QR is $8k$ and the length of PR is $13k$. What must be true about P, Q and R?

14. In the following diagram, the coordinates of A are (x,y) and the coordinates of B are (a,b).



a. Write down the following:

i. the coordinates of point C.

ii. the length of AC.

iii. the length of BC.

b. Find a formula for the length of AB in terms of x, y, a and b.

11. What is the equation of the straight line that intersects the *x*-axis at (b,0), and has a gradient of $-\frac{1}{2}$?

12. A plan of a piece of land is in the form of a triangle ABC where the points are A(–13,14), B(1,1) and C(12,20). It is decided to divide the land by running a fence between the midpoint of AC and the midpoint of BC.

a. What is the gradient of the fence line?

b. What is the equation of the fence line?

c. Compare the gradient of the fence line with the gradient of the boundary line AB.

13. A map of a bike track is drawn relative to a reference point, e.g. (24,35) is a point 24 km east and 35 km north of the reference point.

Part of the track is a straight line between two points labelled (54,38) and (72,47).

At the point (72,47) another straight track of length 20 km runs off at an angle of 90° to the first track.

A checkpoint is located at the point (80,31). Will the checkpoint lie on this second track? Justify your answer fully.

d. What is the equation of the intended pass from P to Q?

e. B passes to a player in a direction that is parallel to the line OP. What is the equation of the pass?

f. Prove that the lines AB and PQ are **not** perpendicular.

g. A is on the centre circle. Find the equation of the tangent to the semicircle at A. (**Hint:** the radius HA is perpendicular to the tangent at A)

h. Player Q has the ball. Which would be the shorter kick: from Q to A or from Q to P? Justify your answer with working.

2. William's mother is a primary school teacher and has been given the task of designing a children's playground. She has discussed the project with William who is enthusiastic about helping her.

They have drawn up a preliminary list of six activities:

- Basketball net for shooting practice (BB)
- Swinging car tyres (CT)
- Swing boats (B)
- Trampoline with safety net (T)
- Suspension bridge, with chains (SB)
- Fort with a fireman's pole for ascending and descending (FP)

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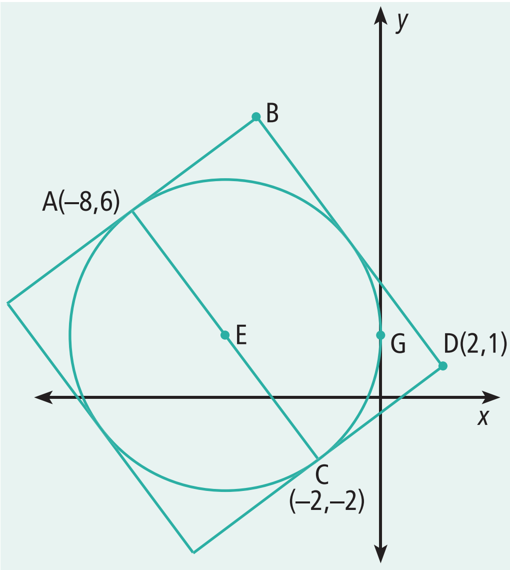
Practice assessment tasks

1. T-shirt logo

A logo is being designed as shown on the axes below. The design involves a circle, centre E, surrounded by a square. Two of the vertices of the square are marked B and D(2,1). The line from A(–8,6) to C(–2,–2) passes through the centre of the circle. The *y*-axis is a tangent to the circle at G.



The company stitching the design needs to know:

a. the gradient of CD

b. the coordinates of the centre of the circle, E

c. the length of the radius of the circle EC (or EA)

d. the coordinates of G

e. the angle between CD and the *x*-axis

f. the equation of the line BD

g. the distance from G to BD

Show the necessary working required to find this information.

Answers

Exercise A: Gradients of lines (page 4)

1. **a.** $\frac{3}{4}$ **b.** -1 **c.** 0

2. **a.** $\frac{5}{6}$ **b.** $2\frac{2}{5}$ **c.** 1 **d.** 0
e. $-\frac{3}{5}$ **f.** 1 **g.** $-\frac{1}{3}$ **h.** 3
i. -3

3. **a.** $\frac{1}{3}$ **b.** -1 **c.** 2 **d.** 1

4. **a.** 1 **b.** -2 **c.** $-\frac{2}{3}$ **d.** $-\frac{2}{3}$

5. **a.** A(0,3), B(2,–2) **b.** $-\frac{5}{2}$ **c.** $\frac{5}{2}$
d. **i.** AC is steeper (shorter ladder reaching same height)
ii. $\frac{5}{3}$ (which is less than $\frac{5}{2}$)

6. **a.** $\frac{1}{2}$ **b.** -2 **c.** BC

7. 7

8. –4.5

9. $-\frac{1}{3}$

10. True for all non-zero values of d.

11. **a.** collinear **b.** not collinear **c.** collinear

12. $k = 10$

Exercise B: Distance between two points (page 9)

1. **a.** 4.24 (2 d.p.) **b.** 5.10 (2 d.p.)
c. 5 **d.** 9.22 (2 d.p.)

2. **a.** 3.61 (2 d.p.) **b.** 10.44 (2 d.p.)
c. 19.80 (2 d.p.) **d.** 10
e. 7.07 (2 d.p.) **f.** $\sqrt{10}$ or 3.1623
g. $\sqrt{52}$ or 7.2111 **h.** $5a$

3. 548.8 mm (1 d.p.) **4.** 11.66 km (2 d.p.)

5. **a.** 7 m
b. **i.** 15 m **ii.** 20 m
c. 12.82 m (2 d.p.)

6. 3.464 m (3 d.p.) **7.** 0.9

8. **a.** Student proof: (show AB = AC = $\sqrt{18}$ and BC = 6; and side lengths obey Pythagoras' rule)
b. (1,7)

9. B by 0.25

10. 41.23 m (2 d.p.)

11. 9.88 km (1 d.p.)

12. 8.93 or –4.93 (2 d.p.)

13. They must lie on the same straight line (P, Q, R collinear).

14. **a.** **i.** (a,y) **ii.** $(a-x)$ **iii.** $(b-y)$
b. $\text{AB} = \sqrt{(a-x)^2+(b-y)^2}$

Exercise C: Midpoint between two points (page 15)

1. **a.** (3,6) **b.** (0,2) **c.** (3,6) **d.** (–8,3)
e. (4.5,5) **f.** (1,–2) **g.** $(2a,3b)$
h. $(2a + 2b, 2a)$ **i.** $(a,0)$

2. (1,0)

3. **a.** (–1,0) **b.** 5

4. Midpoint AC is $(\frac{1}{2}, -\frac{1}{2})$; midpoint BD is also $(\frac{1}{2}, -\frac{1}{2})$ so diagonals bisect each other.

5. $(5\frac{1}{4}, 3\frac{3}{4})$

6. **a.** (10.4,17) **b.** (13.3,18.6)

7. (7,–4)

8. (5,0)

9. (2,–7)

10. **a.** $\left(\frac{3x_1+2x_2}{5}, \frac{3y_1+2y_2}{5}\right)$ **b.** (3.6,6)

Exercise D: Gradient-intercept form of the equation of a straight line (page 19)

1. **a.** $y = -2x + 4$ **b.** $y = 3x + 1$
c. $y = x - 2$ **d.** $y = \frac{1}{2}x$
e. $y = -2x + 4$ **f.** $y = -x - 2$

2. **a.** $y = -2x + 1$ **b.** $y = x + 2$
c. $y = -\frac{1}{2}x$ **d.** $y = -\frac{4}{3}x + 4$
e. $y = \frac{3}{2}x - 3$

3. **a.** Line passing through (0,1) and (1,3)
b. Line passing through (0,–2) and (1,–4)
c. Line passing through (0,3) and (2,4)
d. Line passing through (0,0) and (4,–3)

4. **a.** **i.** $y = -x + 4$ **ii.** $y = 2x + 3$
iii. $y = \frac{1}{4}x + 2$ **iv.** $y = -2x - 4$
b. **i.** Line passing through (0,4) and (1,3)
ii. Line passing through (0,3) and (1,5)
iii. Line passing through (0,2) and (4,3)
iv. Line passing through (0,–4) and (–1,–2)

5. **a.** -2 **b.** $\frac{5}{2}$ **c.** 4 **d.** $-\frac{2}{3}$ **e.** $\frac{1}{4}$ **f.** $-\frac{3}{4}$

6. **a.** 2 **b.** -3 **c.** -1 **d.** 0 **e.** $-\frac{1}{4}$ **f.** $\frac{2}{3}$

7. **a.** $-\frac{1}{5}$ **b.** $\frac{1}{2}$ **c.** $-\frac{2}{3}$ **d.** 4 **e.** $-\frac{3}{2}$

8. **a.** $y = -2x + 3$ **b.** $y = -\frac{4}{3}x - 2$

9. **a.** **i.** 3 **ii.** 2 **iii.** –6
b. **i.** –2.5 **ii.** –0.8 **iii.** –2
c. **i.** 2.5 **ii.** 1.4 **iii.** –3.5
d. **i.** 0.5 **ii.** 12 **iii.** –6