MATHEMATICS AND STATISTICS 2.7

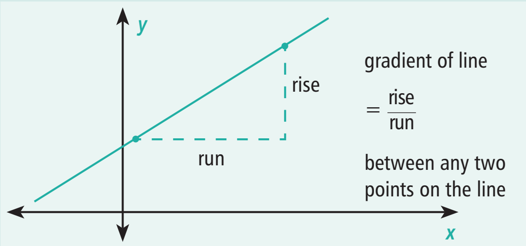
Externally assessed
5 credits

Apply calculus methods in solving problems

AS 91262

The gradient of a curve

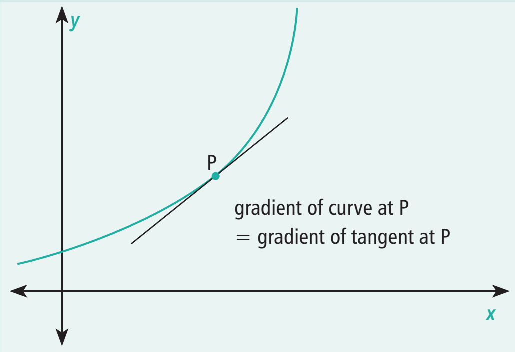
The steepness of a line is the same at all points on the line, so the **gradient of a straight line** is constant.



The gradient is a measure of the **rate of change** of *y* compared to *x*. For a straight line this rate is constant.

The **gradient of a curve** varies from point to point on the curve.

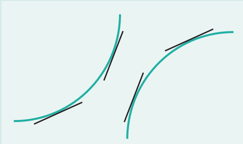
The gradient of a curve $y = f(x)$, at any point P, is defined to be the gradient of the (straight line) **tangent** to the curve at that point.



The gradient of a curve $y = f(x)$ at any point is a measure of the **instantaneous rate of change** of *y* compared with *x* at that point.

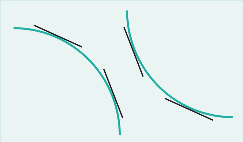
For a curve to have a gradient at any point, a unique tangent to the curve must exist at that point.

- A curve has a positive gradient at points on the curve where the tangent has a positive gradient. For these points the function is said to be **increasing**.



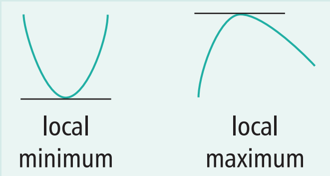
positive gradients

- A curve has a negative gradient at points on the curve where the tangent has a negative gradient. For these points the function is said to be **decreasing**.



negative gradients

- Some curves have **stationary points** where the gradient of the tangent to the curve is zero, e.g. **turning points** where the graph 'turns around'.



turning points

At a minimum turning point a curve changes smoothly from decreasing to increasing; and at a maximum turning point a graph changes smoothly from increasing to decreasing. At the turning point itself the curve is neither decreasing nor increasing.

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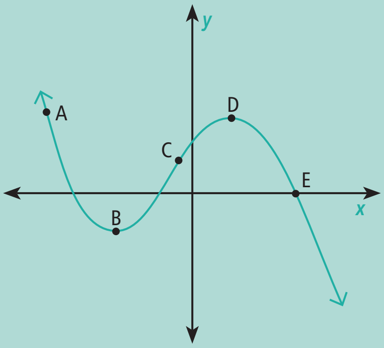
Example

The graph of a function is drawn below.



At A the gradient of the curve is negative and the function is decreasing.

At B the gradient of the curve is zero. The gradient of the curve is changing from negative to positive as it passes through B, so B is a local minimum turning point.

At C the gradient of the curve is positive and the function is increasing.

At D the gradient of the curve is zero and the gradient of the curve is changing from positive to negative as it passes through D, so D is a local maximum turning point.

At E the gradient of the curve is negative and the function is decreasing.

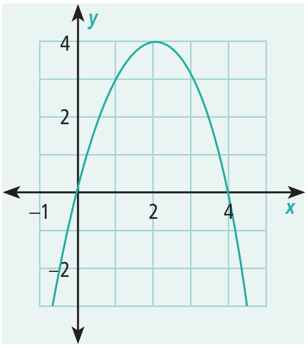
Note: If the graph had accurate scales marked on the axes, then the values of the gradient at the points A, C and E could be estimated by drawing a (straight line) tangent to the curve at each point and estimating the gradient of each tangent.

Exercise A: The gradient of a curve

Ans. p. 55

1. State whether the gradient of the curve is positive, zero or negative at each of the given points. (Only the *x*-coordinate of each point is given.)

a.



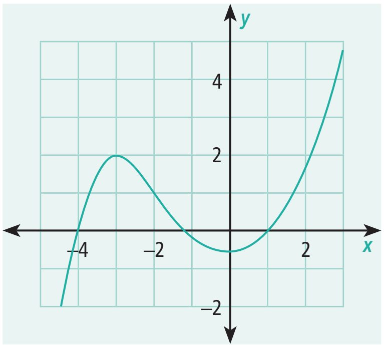
i. $x = 0$ ______

ii. $x = 1$ ______

iii. $x = 2$ ______

iv. $x = 4$ ______

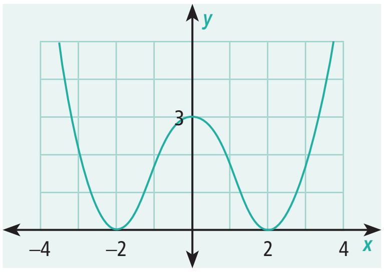
b.



i. $x = -3$ ______ **ii.** $x = -1$ ______

iii. $x = 0$ ______ **iv.** $x = 1$ ______

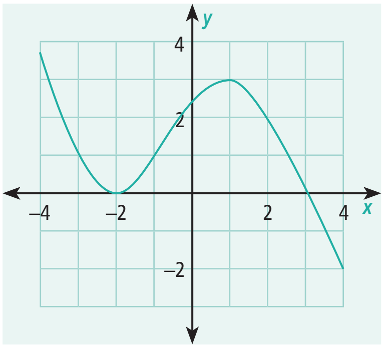
c.



i. $x = -1$ ______ **ii.** $x = 0$ ______

iii. $x = 1$ ______ **iv.** $x = 3$ ______

2. In the graph shown, give the values of x for which the graph is:



a. increasing ______________________

b. decreasing ______________________

____

c. stationary ______________________

3. For the graph in question **2.**, give the coordinates of:

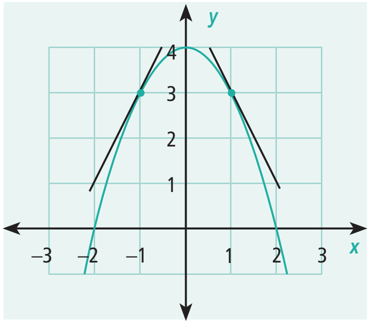
a. the local maximum turning point

____

b. the local minimum turning point

____

4. On the grid below a graph of the function $y = -x^2 + 4$ is drawn. A tangent to the curve is drawn at the point (1,3) and at the point (–1,3).



a. Estimate the gradients of the tangents to the curve at these two points.

____

____

____

____

b. For what values of x is the curve

i. increasing? ______________________

ii. decreasing? ______________________

The gradient function

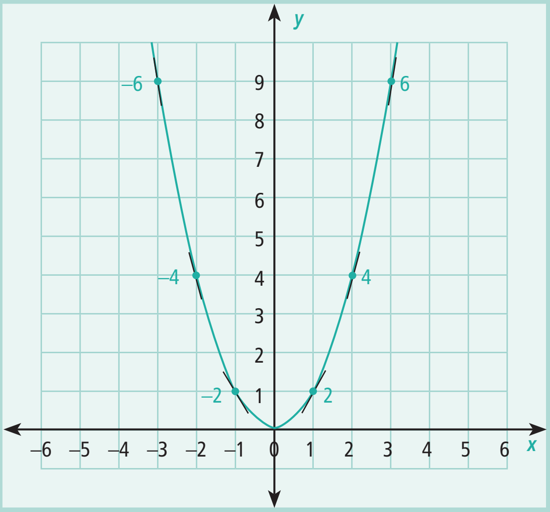
The **gradient function** gives the gradient of the curve at any point on the curve.

One way of working out the rule for a gradient function is to set up a table of values for the gradient at various points on the curve, then find a pattern.

Example

The curve $y = x^2$ is drawn.

At various points, a tangent to the curve is drawn and the gradient of the tangent to the curve is calculated (gradient values shown in colour on graph).



The x-coordinate of each point and the value of the gradient at that point are shown in the table.

x	–3	–2	–1	0	1	2	3
gradient	–6	–4	–2	0	2	4	6

The patterns in the table confirm that the gradient at any point on the curve $y = x^2$ is double the x-coordinate of the point.

The rule for the gradient function is gradient is $2x$.

Using this rule, the gradient of the curve $y = x^2$ at the point (4,16) will be $2 \times 4 = 8$ (doubling the x-coordinate).

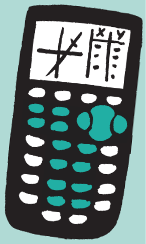
If the gradient of the curve is –5, then $2x = -5$ and $x = -2.5$

When $x = -2.5$, $y = (-2.5)^2 = 6.25$

So at the point (–2.5,6.25) the gradient of the curve is –5

The gradient of the curve is zero when $x = 0$ so (0,0) is a stationary point.

Investigate the derivative function on your calculator.



For example, to find the gradient of $y = x^2$ at the point (4,16), using a CASIO CFX-9750:

- select RUN then OPTN then CALC then d/dx
- enter the function followed by a comma and the x-value of interest – the display shows d/dx(x^2,4)
- press EXE to get the required gradient, which is 8.

Note: The final bracket in d/dx(x^2,4) is optional

Exercise B: Finding the gradient function

Ans. p. 55

1. By considering how the gradients of the following graphs relate to the gradient of the graph of $y = x^2$ in the example above:
   - complete the table for the gradient functions of the following curves
   - give the equation of the gradient function in terms of x.

a. $y = x^2 + 1$

x	–3	–2	–1	0	1	2	3
gradient							

gradient = ______________________

b. $y = -x^2$

x	–3	–2	–1	0	1	2	3
gradient							

gradient = ______________________

c. $y = (x - 1)^2 = x^2 - 2x + 1$

x	–2	–1	0	1	2	3	4
gradient							

gradient = ______________________

d. $y = (x + 2)^2 = x^2 + 4x + 4$

x	–4	–3	–2	–1	0	1
gradient						

gradient = ______________________

e. $y = -(x + 1)^2 = -x^2 - 2x - 1$

x	–3	–2	–1	0	1	2
gradient						

gradient = ______________________

2. Tangents are drawn to the curve $y = \frac{1}{2}x^2 + x + 3$ at several points on the curve.

   The gradients of the tangents at the x-coordinates of these points are shown in the table below.

x	–4	–2	0	2	4
gradient	–3	–1	1	3	5

a. Look carefully at the values in the table. What is the rule for the gradient of the curve?

____

b. What would be the gradient of the curve at the point (6,27)?

____

____

c. At which point on the curve would the gradient be –7?

d. What is special about the tangent to the curve at the point where $x = -1$?

e. Will any other point on the curve have this gradient?

3. Tangents are drawn to the curve $y = x^2 + 3x + 4$ at several points on the curve.

The gradients of the tangents at these points are shown in the table below.

x	–2	–1	0	1	2
gradient	–1	1	3	5	7

a. What is the rule for the gradient of the curve?

b. What would be the gradient of the curve at the point (–3,4)?

c. At which point on the curve would the gradient be 11?

d. For what x-value does the curve have a stationary point?

e. Give the y-coordinate of the point where the gradient of the tangent to the curve is zero.

4. a. Use a graphics calculator to complete the table below for the function $y = \frac{x^3}{3}$.

x	–2	–1	0	1	2	3
gradient						

b. What seems to be the rule for the gradient function? Test your theory using the graphics calculator derivative function.

5. Extra for experts

a. The letter e stands for the irrational number 2.718 281 828…. Some values of e^x (to 4 d.p.) are shown in the table below. Use a graphics calculator to work out the values of the gradient of function $y = e^x$ at these x-values.

x	–2	–1	0	1	2	3
e^x	0.1353	0.3679	1	2.7183	7.3891	20.0855
gradient						

b. What seems to be the rule for the gradient function? Test your theory using the graphics calculator derivative function.

c. Predict the gradient of the curve $y = e^x$ when $x = 4$. Check your answer using a graphics calculator.

Sketching gradient functions

By considering important features of the gradient of a function, the graph of its gradient function can be sketched.

The important features of the graph of a gradient function that need to be shown are the following.

Points of intersection with the x-axis

Local maximum or **minimum** points of the graph have a gradient of zero, so in the sketch of the gradient function these are the points of intersection with the x-axis (i.e. the ***x*-intercepts**).

Regions of positive and negative values

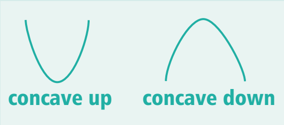
Increasing sections of the graph (which have a positive gradient) appear in the sketch of the gradient function as positive y-values.

- The steeper the slope in the original graph, the larger the size of the y-value in the gradient graph.
- The shallower the slope in the original graph, the closer the y-value in the gradient graph is to zero.

Sections of the graph of the original function which have a negative gradient transfer to the sketch of the gradient function as negative y-values.

Any local maximum and/or minimum points

Sections of a curve are either **concave up** or **concave down**.



Some functions have a **point of inflection**. This is a point where a curve changes from being concave down to concave up, or vice versa.

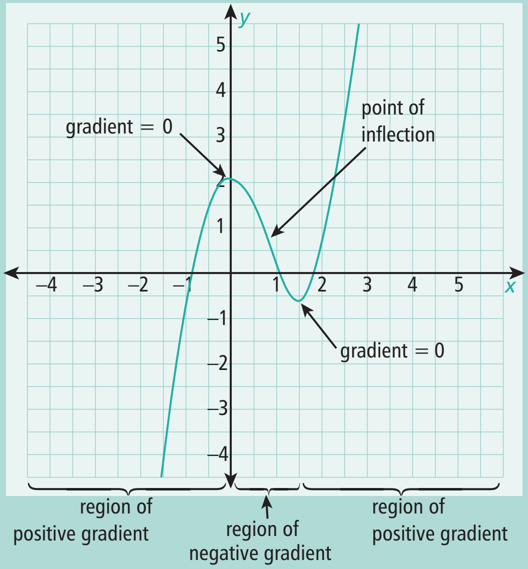
At points of inflection on the original graph the gradient has a local minimum or maximum value, so there will be a turning point on the gradient graph.

In some cases the point of inflection on the original graph has a gradient of zero. (The second point of inflection diagram shown above illustrates this case.) This point will be an x-intercept which is also a turning point on the gradient graph.

When sketching gradient graphs, the locations of these turning points on the gradient graph need only be approximate.

Example

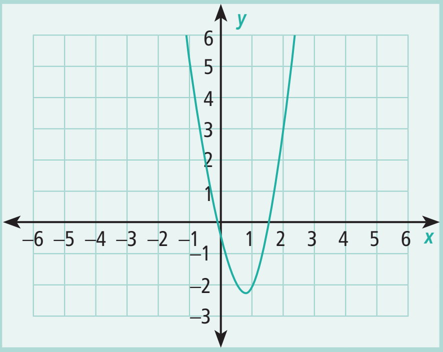
The graph below is of the polynomial
$y = x^3 - 2x^2 - x + 2$



Using this graph the following information can be found:

Region	Effect on gradient
x is less than about –0.2	Positive gradient
x is equal to about –0.2	Gradient equals zero
x is between about –0.2 and 1.5	Negative gradient
x is equal to about 1.5	Gradient equals zero
x is greater than about 1.5	Positive gradient
x is equal to about 0.7	Gradient has a local minimum value

It is now possible to sketch the graph of the gradient function for the given polynomial.



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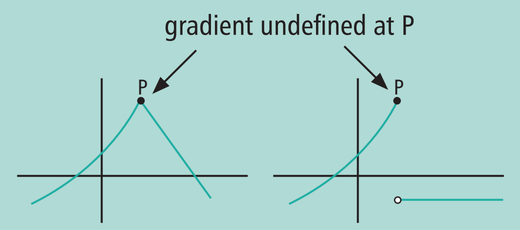
Note: The gradient of a function is undefined if its graph has a break or a sharp change of steepness (making the drawing of a unique tangent impossible). The graph of the gradient function will be undefined at these points (there will be a break in the graph of the gradient).

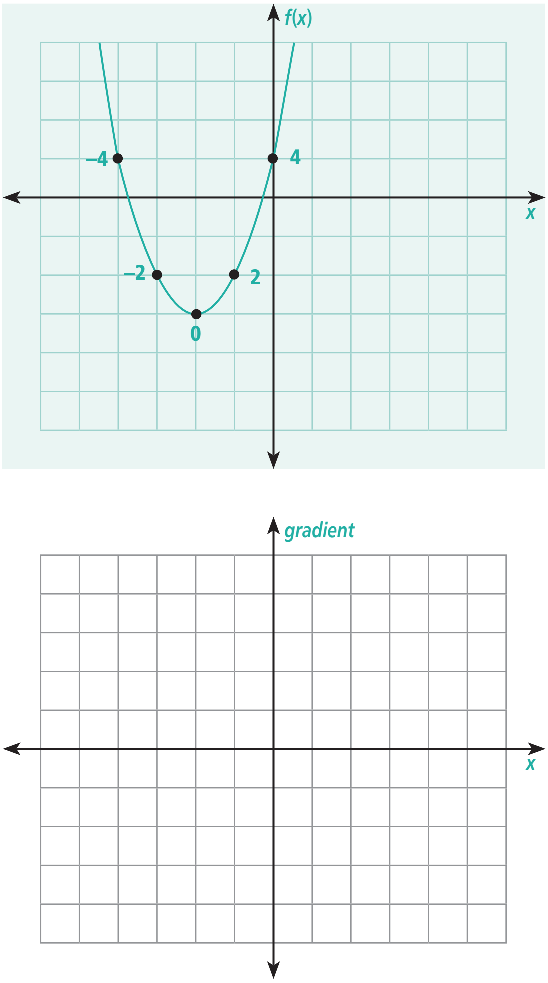


Exercise C: Sketching gradient functions

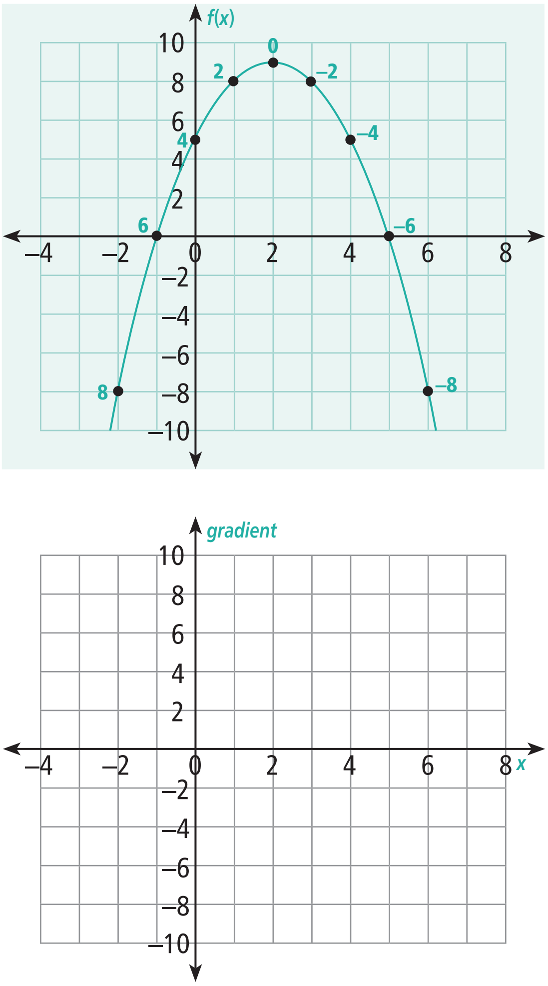
Ans. p. 55

1. The graph of a function $f(x)$ is drawn. Draw the graph of its gradient function on the axes below.
Hint: The small numbers show the gradient of the curve at selected points.

a.



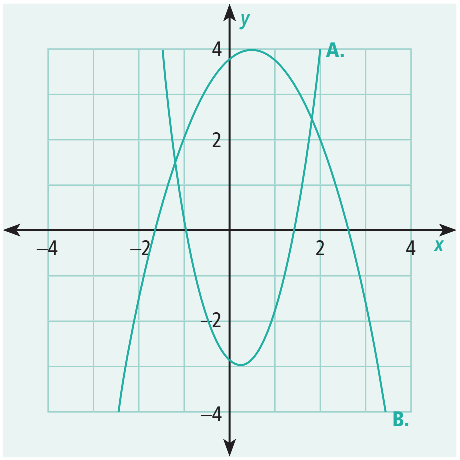
b.

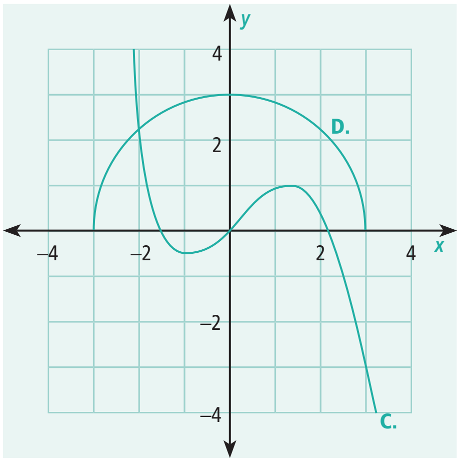


2. Match each graph A, B, C, D with the correct graph of its gradient function a, b, c, d.

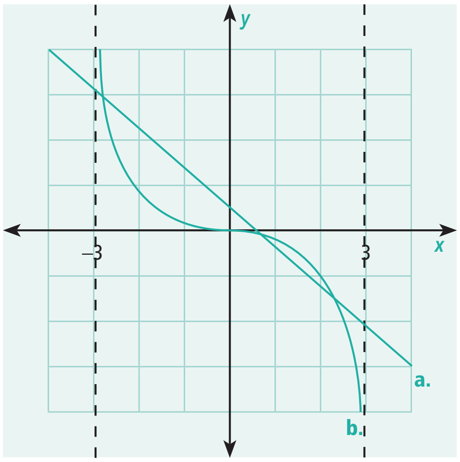
Functions

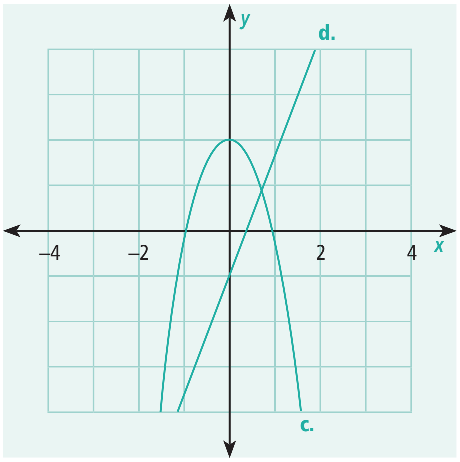
A. ________ **B.** ________ **C.** ________ **D.** ________





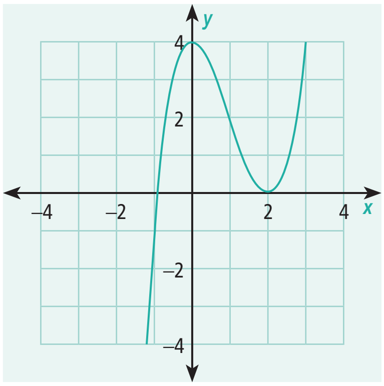
Gradient functions

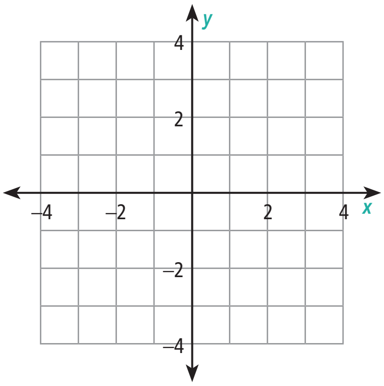




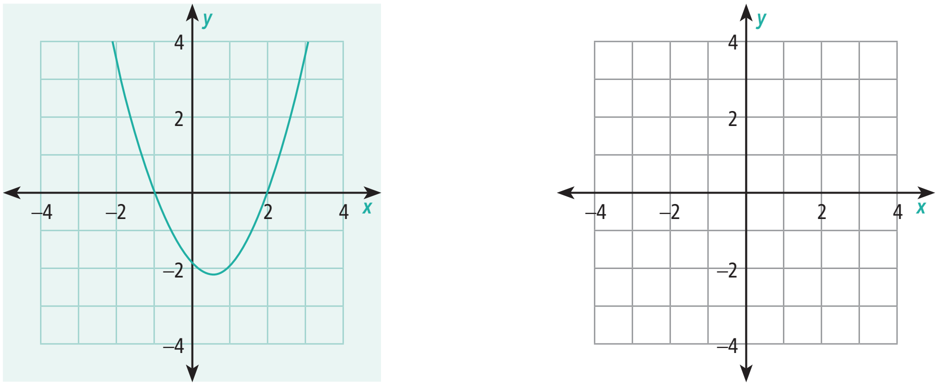
3. Sketch a graph of the gradient function for each of the polynomial functions graphed below.

a.

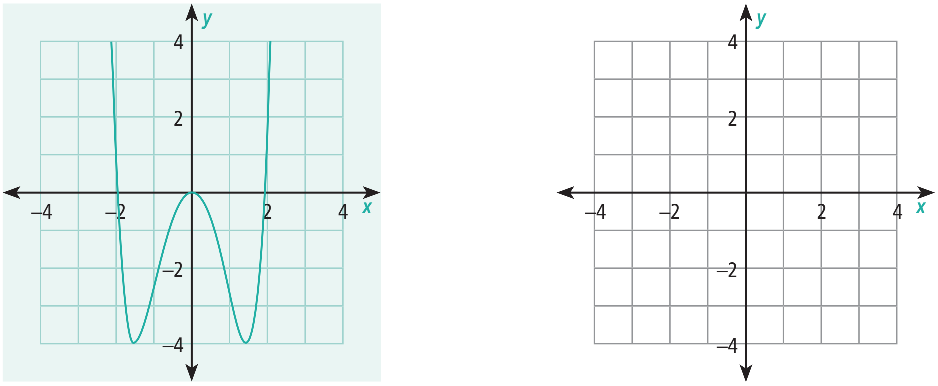




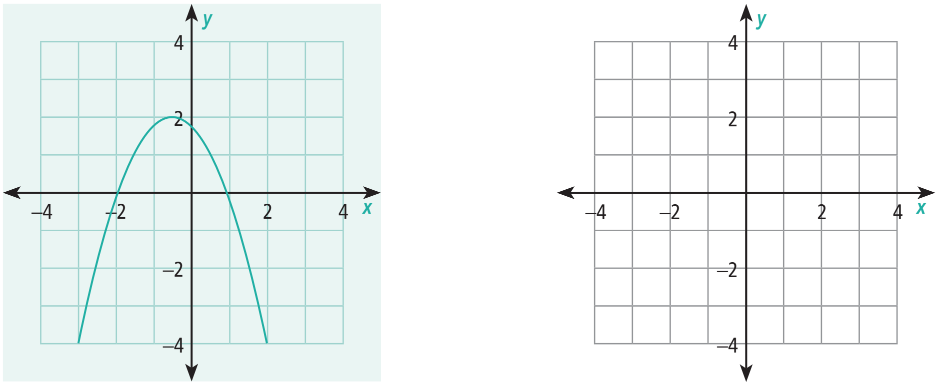
b.



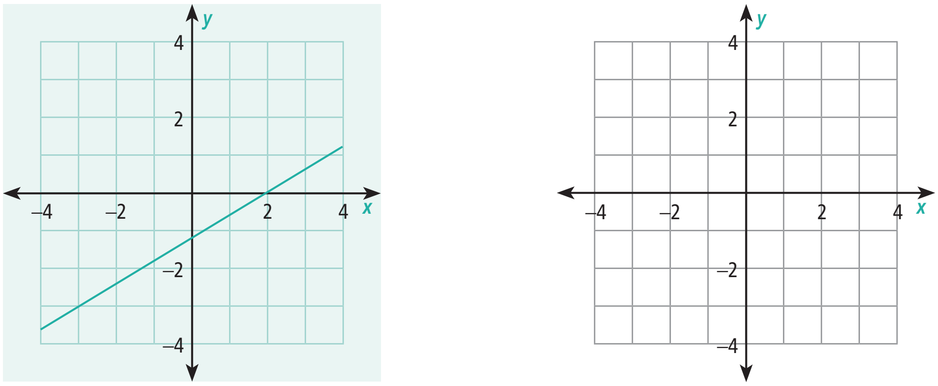
c.



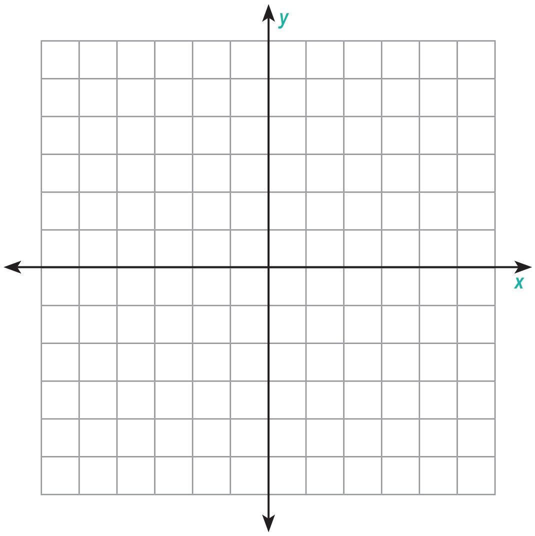
d.



e.

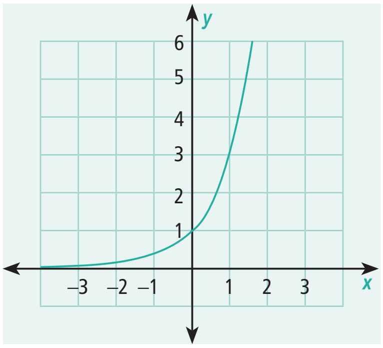


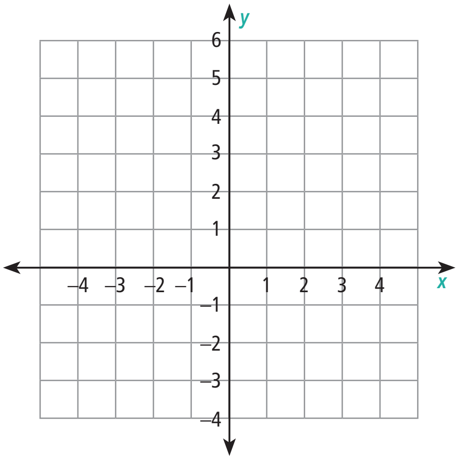
4. A polynomial graph has maximum turning points at (–4,3) and (4,3) and a minimum turning point at (0,–6). The graph is increasing for $x < -4$ and $0 < x < 4$, and is decreasing elsewhere. On the same set of axes, draw a sketch of this function and its gradient function.



5. Extra for experts

a. The graph of a growth curve is shown. Sketch the graph of its gradient function on the axes below.

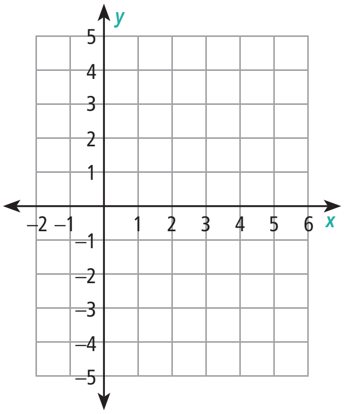




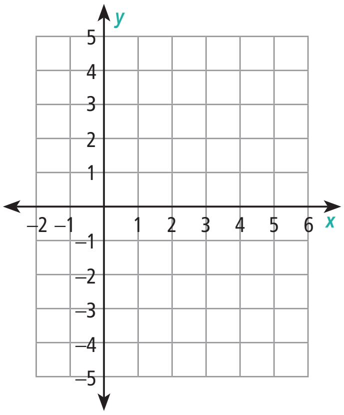
b. Compare the shapes of the two graphs.

6. A piecewise function follows the rule $y = 2x - 1$ for $x < 2$ and the rule $y = 5 - x$ for $x \geq 2$.

a. Sketch a graph of the function on the axes below.



b. Sketch a graph of the gradient function on the second set of axes below.



c. For which value of x is the gradient undefined? Explain why this occurs.

ANSWERS ACHIEVEMENT STANDARD 91262

Exercise A: The gradient of a curve (page 2)

1. a. i. Positive ii. Positive
   iii. Zero iv. Negative
   b. i. Zero ii. Negative
   iii. Zero iv. Positive
   c. i. Positive ii. Zero
   iii. Negative iv. Positive
2. a. $-2 < x < 1$ b. $x < -2$ and $x > 1$
   c. $x = -2$ and $x = 1$
3. a. (1,3) b. (–2,0)
4. a. At (1,3) gradient = –2;
   at (–1,3) gradient = 2
   b. i. $x < 0$ ii. $x > 0$

Exercise B: Finding the gradient function (page 4)

1. a. $y = x^2 + 1$

x	–3	–2	–1	0	1	2	3
gradient	–6	–4	–2	0	2	4	6

gradient = $2x$

b. $y = -x^2$

x	–3	–2	–1	0	1	2	3
gradient	6	4	2	0	–2	–4	–6

gradient = $-2x$

c. $y = (x - 1)^2 = x^2 - 2x + 1$

x	–2	–1	0	1	2	3	4
gradient	–6	–4	–2	0	2	4	6

gradient = $2x - 2$

d. $y = (x + 2)^2 = x^2 + 4x + 4$

x	–4	–3	–2	–1	0	1
gradient	–4	–2	0	2	4	6

gradient = $2x + 4$

e. $y = -(x + 1)^2 = -x^2 - 2x - 1$

x	–3	–2	–1	0	1	2
gradient	4	2	0	–2	–4	–6

gradient = $-2x - 2$

2. a. gradient = $x + 1$
   b. 7 c. (–8,27)
   d. gradient = 0 (horizontal line) e. No
3. a. gradient = $2x + 3$
   b. –3 c. (4,32)
   d. $x = -1.5$ e. $y = 1.75$
4. a. $y = \frac{x^3}{3}$

x	–2	–1	0	1	2	3
gradient	4	1	0	1	4	9

b. gradient = x^2

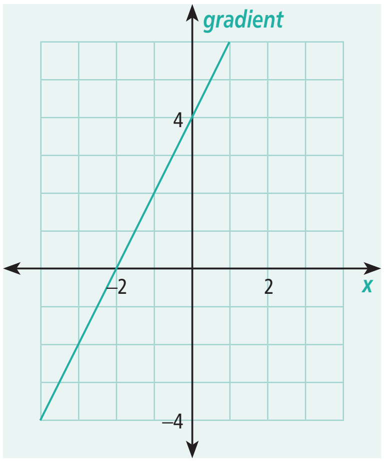
5. a.

x	–2	–1	0	1	2	3
e^x	0.1353	0.3679	1	2.7183	7.3891	20.0855
gradient	0.1353	0.3679	1	2.7183	7.3891	20.0855

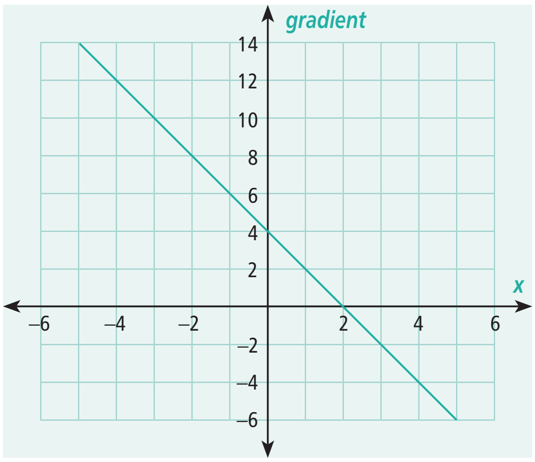
b. gradient = e^x
c. gradient = e^4 or 54.6 (1 d.p.)

Exercise C: Sketching gradient functions (page 7)

1. a.



b.



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