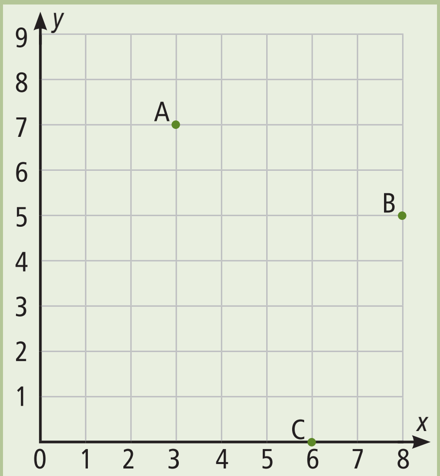
Solving practical problems

Many problems in practical contexts can be solved using coordinate geometry methods.

Example

Three girls Anna, Beth and Catelyn are practising netball throws. Their positions on the court are shown in the diagram below and have coordinates A(3,7), B(8,5) and C(6,0)



1. Calculate the distance between Anna and Beth if the grid lines are 2 m apart.
2. Beth throws the ball directly to Catelyn. Give the equation of the path for the ball.
3. Catelyn aims her throw at a point midway between Anna and Beth. Give the coordinates of this point.

Solution

1. $AB = \sqrt{(8-3)^2 + (5-7)^2}$ [distance formula]

   $= \sqrt{25 + 4}$

   $= \sqrt{29}$

   Distance $= \sqrt{29} \times 2$ [grid lines 2 m apart]

   $= 10.77$ m (2 dp)

2. Gradient BC $= \dfrac{0-5}{6-8}$ $\left[m = \dfrac{y_2 - y_1}{x_2 - x_1}\right]$

   $= \dfrac{-5}{-2}$ or 2.5

   Equation of line with $m = 2.5$, point (6,0) is

   $y - 0 = 2.5(x - 6)$ $[y - y_1 = m(x - x_1)]$

   $y = 2.5x - 15$

3. Midpoint $= \left(\dfrac{3+8}{2}, \dfrac{7+5}{2}\right)$ $\left[\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)\right]$

   $= (5.5,6)$

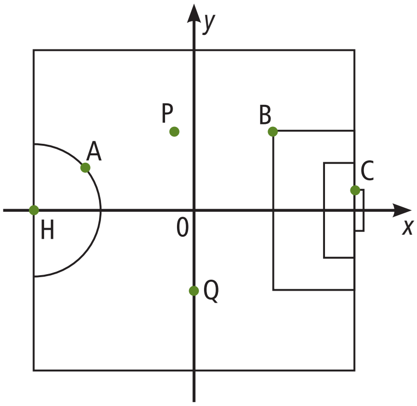
Exercise I: Solving practical problems with coordinate geometry

Ans. p. 256

Note that the markings on the pitch are not quite correct according to those recommended by the Football Association.

A football pitch has length 120 m. Half of the football pitch is shown. Coordinate axes are placed centrally over the pitch so that marked points on the pitch have the following coordinates (in metres):

A(–20, 8), B(15,15), C(30,4), P(–5,15), Q(0,–15)



Apply coordinate geometry formulae to solve the following problems.

1. A pass is made from player P to player Q which is intercepted at its midpoint M. Find the coordinates of M.

The general term of a geometric sequence

A rule can be found for the general term of a **geometric sequence** by working out how each term is calculated.

Example

In the geometric sequence 2, 6, 18, 54, 162, ... the first term is $a = 2$, and the common ratio is $r = 3$.

The first six terms are formed as follows:

Term	$t_1 = 2$	$t_2 = 6$	$t_3 = 18$
Rule	$t_1 = 2 \times 3^0$	$t_2 = 2 \times 3^1$	$t_3 = 2 \times 3^2$
Term	$t_4 = 54$	$t_5 = 162$	$t_6 = 486$
Rule	$t_4 = 2 \times 3^3$	$t_5 = 2 \times 3^4$	$t_6 = 2 \times 3^5$

In the rule the power of 3 is always 1 less than the term number, so the rule for the *n*th term is

$t_n = 2 \times 3^{n-1}$

Using this rule the 8th term, say, can be worked out to be:

$t_8 = 2 \times 3^7$ which is 4 374

In general, a geometric sequence is of the form
$a, ar, ar^2, ar^3, ar^4, \ldots$
where a is the first term and r is the common ratio.

The **general term** for a geometric sequence is:

$$t_n = ar^{(n-1)}$$

- n = the position of the term within the sequence
- a = the first term of the sequence
- r = the common ratio
- t_n = the *n*th term

Example

Q. Calculate the 10th term of the geometric sequence: 1, 3, 9, 27, …

A. $a = 1$ $r = 3$ $n = 10$

[the first term is 1, the common ratio is 3 and the 10th term is required]

Substituting these values into the formula $t_n = ar^{(n-1)}$ gives:

$t_{10} = 1 \times 3^{(10-1)}$

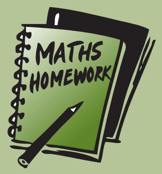
$= 3^9$

$t_{10} = 19\,683$

Many practical problems involve geometric sequences.

Example

Q. A student records the number of minutes she spends each day on a homework project. For the first three days the number of minutes spent are 200, 160 and 128. If this pattern follows a geometric sequence, how many minutes will she spend on the project on the 7th day?



A. $n = 7$

$a = 200$

$r = \dfrac{160}{200} = 0.8$

Substituting into $t_n = ar^{(n-1)}$ gives:

$t_7 = 200 \times 0.8^{(7-1)}$

$= 200 \times 0.8^6$

$t_7 = 52.43$

She will spend about 53 minutes on her project on the 7th day.

Exercise F: The general term of a geometric sequence

Ans. p. 256

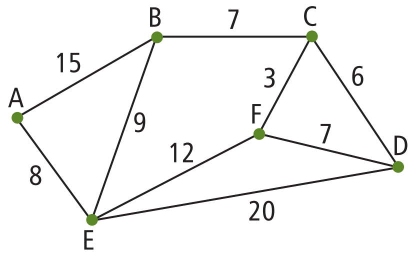
1. Calculate the required term for the following geometric sequences:

   a. 1, 4, 16, 64, …, t_9

   b. 1, 2, 4, 8, 16, …, t_{20}

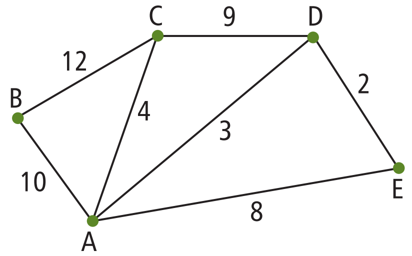
   c. 1, 5, 25, 125, …, t_7

5. Jack owns two jam factories, one at Town A and the other at Town D. Jack has to transport some jars of marmalade from A to D. A map of the district is drawn here showing the two towns A and D with some other towns (B, C, E, F) included.



The distances in kilometres between the towns are marked on the map. What is the shortest distance to transport the marmalade from A to D? Describe this shortest route and give its length.

6. Olivia is a traveller for a car importer. In a certain week she visits all the company's outlets A, B, C, D and E, which are marked on the map drawn here.



The distances in kilometres between the outlets are marked on the map.

Find the shortest distance she will travel to visit them all if she starts at home and finishes at home. Her home town is at A.

Traversable networks

A network is **traversable** if it can be drawn by travelling over every edge of the network just once, without taking the pencil from the paper (the pencil may pass through a node any number of times). The network is traversable whether you return to your starting node or end at some other node.

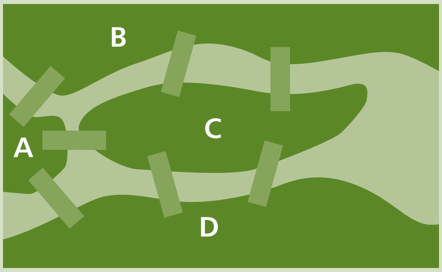
The Swiss mathematician Leonhard Euler discovered the following rule:

> A network is traversable if it has either two odd nodes or it has no odd nodes. It may have any number of even nodes.

A famous problem from history was solved using Euler's result.

Konigsberg bridge problem

In the year 1736, Euler solved a problem which interested the residents of the town of Konigsberg, in Prussia. The town was close to the river Pregel, which divided and flowed around an island. There were seven bridges over the river as shown in the figure (land is drawn shaded dark green).



Problem: Starting at any point, can you walk over every bridge only once, without getting your feet wet?

Solution

Euler reduced the bridge problem to a geometry problem. He constructed a network in which edges represented the routes AC, BC, CD and the points A,B,C,D were the nodes.

The pilot survey

Having created your **draft questionnaire**, you need to test it on a small group. This is called a **pilot survey**. This will help identify any questions that give unexpected answers or questions that seem to be commonly misunderstood. If you don't get the answers you expected, you may need to reword or add extra questions.

Example

On conducting a pilot survey, a student found that the question, 'How many times a week do you participate in sport: 0–1, 2–3, 4–5, over 5', resulted in no one answering in the category over 5, but many answering in the category 0–1.

It was therefore decided to change the response categories to '0, 1, 2, 3, 4, over 4', as it was felt it was important to separate the students who participated in no sport at all, from those who did, and less important to distinguish between those doing sport over 4 times per week and those doing sport over 5 times per week.

You should keep a record of all stages in the process of finalising your questionnaire, noting how and why various questions were changed.

The final questionnaire

Once adjustments have been made to the questionnaire, based on the pilot survey findings, your **final questionnaire** is ready to be tried out on a **sample** of people from your **target population**.

In order for your results of your survey to be meaningful, the sample will need to be randomly chosen and suitably large so that it is **representative** of the views of the population of interest.

Exercise E: The pilot survey and the final questionnaire

1. Scientists have discovered that excessive internet use may cause parts of teenagers' brains to waste away. The Ministry of Health has asked you to create a questionnaire in order to determine the health effects of excessive internet usage on teenagers.

a. Write a purpose statement for your questionnaire, including who will use the data. Give a brief overview of what sort of information you would hope to receive from your questionnaire.

b. Research the topic of internet usage to define what is meant by the phrase 'excessive internet usage' and to find out what claims are made about the effects of excessive internet usage. Summarise your main findings.

Ans. p. 262

2. Each table gives the 5 summary statistics for a random sample. For each sample calculate an informal confidence interval for the population median and explain what this interval means.

a. Weekly hours of homework for 50 Year 11 students at a school.

Min	LQ	Med	UQ	Max
2	8	12	14	18

b. Waiting times for service (in minutes) for 36 customers in a restaurant.

Min	LQ	Med	UQ	Max
1	5	11	13	19

c. Lengths of 40 phone calls (min) in a household.

Min	LQ	Med	UQ	Max
3	5	12	14	18

3. Maree wondered if butterfly was a slower swimming stroke than backstroke for Year 10 swimmers.

At a swim carnival, Maree took a sample of times taken by 30 Year 10 competitors to swim a 25 m butterfly race. She created an informal confidence interval for the median time taken by all Year 10 competitors in the carnival to swim this race:

21.3 sec < median time (butterfly) < 25.1 sec

a. What does this interval mean?

Maree also created an informal confidence interval for the median length of time taken by 30 Year 10 competitors in the carnival to swim a 25 m backstroke race. Her interval was:

17.8 sec < median time (backstroke) < 21.0 sec

b. Do the times in these two confidence intervals overlap?

c. Do you think there tends to be a difference between the median times taken to swim butterfly and backstroke for all Year 10 competitors at this carnival?

d. How confident are you about your answer to part **c.**? Explain your answer.

AS 91264

The actual difference between the two measurements can be calculated for each individual, and presented on a dot plot or single box-and-whisker plot. If most of the plot is to one side of zero (e.g. the box is to the right of zero), then there is evidence that there is a difference between the two measurements.

You should also discuss what is observed in terms of the centre, spread, unusual features and percentages (25%, 50%, 75% of participants).

Example

The reaction times of a group of students were measured while driving using a car simulator, where 'reaction time' (the time a student took to brake when an obstacle appeared) was measured, first while the student was driving without texting on a cell phone, then while the student was driving while texting. The results are shown in the table alongside.

Subject number	Reaction time without texting (seconds)	Reaction time while texting (seconds)
1	2.5	3.0
2	3.8	3.6
3	2.6	2.9
4	2.7	2.8
5	2.2	2.4
6	1.8	1.9
7	1.9	2.2
8	1.5	1.8
9	2.4	2.8
10	2.8	3.0
11	1.7	2.9
12	2.3	2.8
13	2.1	2.7
14	2.6	2.9
15	3.2	3.0
16	1.7	2.7
17	3.1	2.7
18	2.2	2.8
19	1.9	2.5
20	2.5	2.9

By calculating statistics and drawing appropriate graphs, what conclusions can you draw about how texting affects reaction times?

Solution

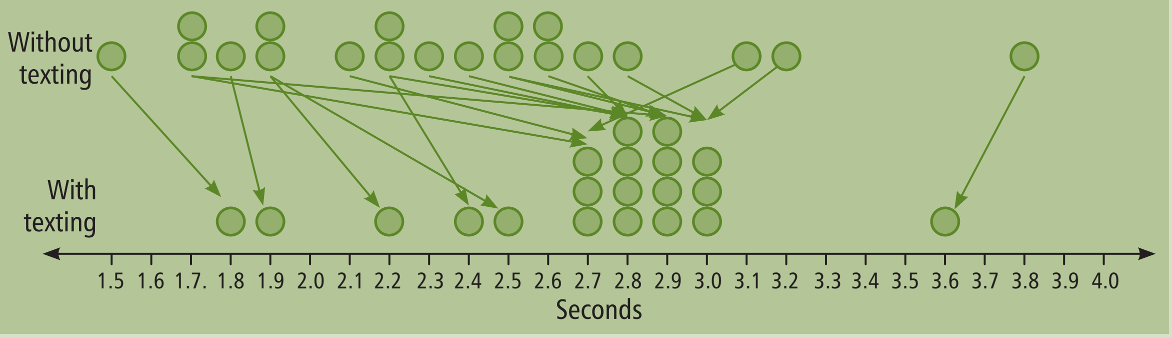
From the above data the following statistics (to 1 dp) can be calculated:

Reaction times (seconds)			
	Mean	Standard deviation	Range
Without texting	2.4	0.6	2.3
With texting	2.7	0.4	1.8

Note: The mean and standard deviation can easily be calculated using a scientific or graphics calculator. The standard deviation is a measure of how spread out the data is about the mean (the larger the standard deviation, the more widely scattered the data values are about the mean).

We can see that 'reaction times without texting' have a larger standard deviation and so are more spread out and hence less consistent than 'reaction times with texting'.

As this is paired data, the most appropriate graph to display the raw data is a paired dot plot.



AS 91265

Experimental distributions

Data from a statistical or probability experiment can be presented in a histogram so its **distribution** (shape and other features) can be seen.

Example

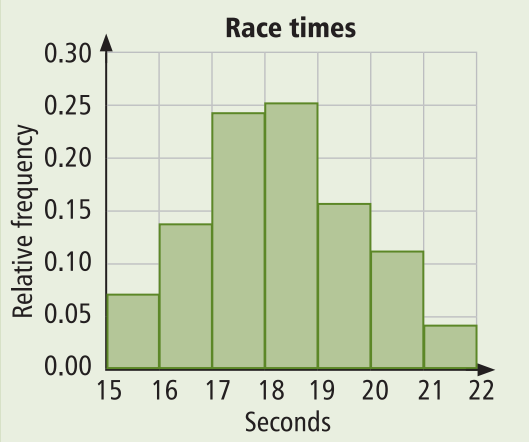
In an inter-school swimming competition, the times taken for a random sample of 200 swimmers to swim a 25 m freestyle race are shown in the table below.

Times are put into intervals of width one second, and the relative frequency of each interval is calculated.

For example, the interval 15- means times of at least 15 seconds but less than 16 seconds, e.g. a time of 15.4 seconds would fall in this interval. 14 swimmers had times in this interval, so the relative frequency of the interval is $\frac{14}{200}$ which is 0.07.

Time (sec)	Frequency	Relative frequency
15-	14	0.07
16-	27	0.14
17-	48	0.24
18-	50	0.25
19-	31	0.16
20-	22	0.11
21-	8	0.04
Totals	200	1.00

A relative-frequency histogram is drawn for the data in the sample.



The distribution of the race times can be seen to be **unimodal** (it has one peak which is the interval 18- seconds) with nearly 50% of race times between 17 and 19 seconds.

The shape of the distribution is approximately symmetrical with race times close to the peak (near 18 to 19 seconds) more likely than race times further from the peak. In the tails, only 7% of race times were below 16 seconds and only 4% of race times were above 21 seconds.

Always remember that random samples **vary** from sample to sample (each new sample usually results in a different set of values of the variable). So the distribution of the relative frequencies from a random sample is only an approximation of the population probability distribution (for *all* race times in this event at this competition).

However, the larger the sample, the more reliable the approximation, so with a random sample of size 200 as in the example above, we should be confident that the distribution of relative frequencies drawn above should be quite similar to the population probability distribution.

Comparing experimental distributions with theoretical distributions

Some experimental distributions may appear to have features of a **normal distribution**. To conclude that a variable is normally distributed, its sample distribution should:

Distributions of probabilities

It can be useful to graph the **distributions** of the experimental probabilities (the long-run relative frequencies) that you have worked out from a simulation. This gives us an idea about the way the *actual* probabilities of outcomes are distributed (the **population distribution**).

It is important to keep in mind that the results of simulations are different each time a simulation is run. So simulation distributions will also vary each time the simulation is run.

Because of this **variability** the simulation distribution is only an **estimate** of the actual distribution of probabilities.

Example

Q. A DVD store is running a competition.

- A scratch card is given to each customer with each payment.
- Each card has a letter W, I or N.
- If a customer collects all three letters (to spell WIN) they win a prize.

The quantities printed of the letters W, I and N are in the ratio 1:2:5.

Design a simulation to work out the average number of cards a customer would need to collect in order to win a prize.

A. The ratio has $1 + 2 + 5 = 8$ parts so use a calculator to generate random numbers from 1 to 8 (8 Ran# + 1). Assign the numbers proportionally, as in the context, for example:

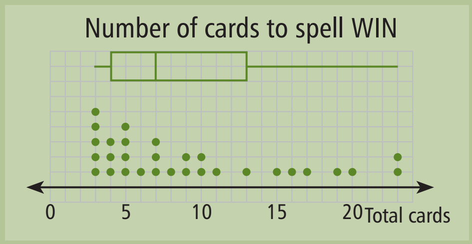
- 1 – letter is W
- 2, 3 – letter is I
- 4, 5, 6, 7, 8 – letter is N

Each trial will involve generating numbers until all three letters are represented.

Results are recorded in a table such as the one following (more columns for the rolls will be needed in some instances).

Trial	Roll										Number of cards
	1	2	3	4	5	6	7	8	9	10	
1	2 I	1 W	6 N								3
2	7 N	5 N	1 W	1 W	7 N	6 N	3 I				7
3	4 N	6 N	8 N	8 N	5 N	1 W	6 N	6 N	3 I		9
4	4 N	7 N	2 I	8 N	5 N	7 N	4 N	1 W			8
5	6 N	2 I	8 N	5 N	1 W						5

In one simulation of the above situation, consisting of 30 trials, the results were as shown in the dot plot and box-and-whisker plot below.



From the plots it can be seen that the distribution of the number of cards resulting from the simulation is **positively skewed**. Just over half the time the word WIN was spelled after 3 to 7 cards were purchased. Of note are several extreme values present (three trials required 20 or more cards before WIN was spelt).

The mean number of cards required to spell WIN is calculated to be 9.2 (adding up all thirty 'total numbers of cards' values and dividing by 30).

However, the mean has been 'pushed up' by a few very large 'total numbers of cards'. As a result, the mean number of cards does not seem typical of the average 'total number of cards' needed (nearly two-thirds of the sample had 'total numbers of cards' below the mean).

4. Ben makes flower boxes in two sizes, small and large, for a local garden shop. He needs to supply a weekly minimum of 10 small and 5 large flower boxes.

   The small boxes use 5.5 m of wood and the large boxes use 15 m of wood. The wood comes pre-cut to the correct sizes and Ben has at least 165 m of wood available per week.

   It takes Ben 15 minutes to assemble the small boxes and 20 minutes to assemble the large boxes. Each week Ben has up to 10 hours available for making his flower boxes.

   a. Use this information to find the feasible region for the numbers of boxes Ben can supply.

   b. What is the largest number of small flower boxes Ben could make in a week, and still satisfy all of the above constraints?

   The garden shop decides to change the weekly minimum number of flower boxes made to 12 boxes of each size.

   c. How does this affect the feasible region? You may wish to use a new grid to explain your answer.

   d. What is the maximum number of small flower boxes Ben could now make in a week?

See ▶ ESA Online for a practice assessment.

Answers

Achievement Standard 91256
Mathematics and Statistics 2.1

Exercise A: Gradients of lines (page 4)

1. a. $\frac{3}{4}$ b. -1 c. 0
2. a. $\frac{5}{6}$ b. $2\frac{2}{5}$ c. 1 d. 0
   e. $-\frac{3}{5}$ f. 1
3. a. $\frac{1}{3}$ b. -1 c. 2 d. 1
4. a. 1 b. -2 c. $-\frac{2}{3}$ d. $-\frac{2}{3}$
5. a. A(0,3), B(2,–2) b. $-\frac{5}{2}$ c. $\frac{5}{2}$
   d. i. AC is steeper (shorter ladder reaching same height)
      ii. $\frac{5}{3}$ (which is less than $\frac{5}{2}$)
6. a. $\frac{1}{2}$ b. -2 c. BC

Exercise B: Distance between two points (page 8)

1. a. 4.24 (2 dp) b. 5.10 (2 dp)
   c. 5 d. 9.22 (2 dp)
2. a. 3.61 (2 dp) b. 10.44 (2 dp)
   c. 19.80 (2 dp) d. 10
   e. 7.07 (2 dp)
3. 548.8 mm (1 dp)
4. 11.66 km (2 dp)
5. a. 7 m
   b. i. 15 m ii. 20 m
   c. 12.82 m (2 dp)
6. 3.464 m (3 dp)

Exercise C: Midpoint between two points (page 11)

1. a. (3,6) b. (0,2) c. (3,6) d. (–8,3)
   e. (4.5,5) f. (1,–2)
2. (1,0)
3. a. (–1,0) b. 5
4. Midpoint AC is $(\frac{1}{2}, -\frac{1}{2})$; midpoint BD is also $(\frac{1}{2}, -\frac{1}{2})$ so diagonals bisect each other.
5. $(5\frac{1}{4}, 3\frac{3}{4})$

Exercise D: Gradient-intercept form of the equation of a straight line (page 14)

1. a. $y = -2x + 4$ b. $y = 3x + 1$
   c. $y = x - 2$ d. $y = \frac{1}{2}x$
   e. $y = -2x + 4$
2. a. $y = -2x + 1$ b. $y = x + 2$
   c. $y = -\frac{1}{2}x$ d. $y = -\frac{4}{3}x + 4$
   e. $y = \frac{3}{2}x - 3$
3. a. Line passing through (0,1) and (1,3)
   b. Line passing through (0,–2) and (1,–4)
   c. Line passing through (0,3) and (2,4)
   d. Line passing through (0,0) and (4,–3)
4. a. i. $y = -x + 4$
      ii. $y = 2x + 3$
      iii. $y = \frac{1}{4}x + 2$
      iv. $y = -2x - 4$
   b. i. Line passing through (0,4) and (1,3)
      ii. Line passing through (0,3) and (1,5)
      iii. Line passing through (0,2) and (4,3)
      iv. Line passing through (0,–4) and (–1,–2)

Exercise E: Equation of line when gradient and a point are known (page 17)

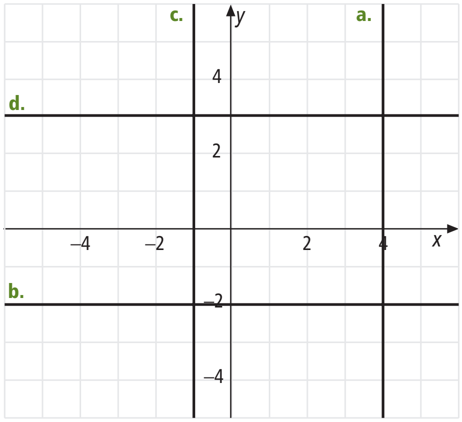
1. $y = 2x - 1$ 2. $y = -x + 7$ 3. $y = -4x - 1$
4. $y = \frac{1}{2}x + 4$ 5. $y = \frac{3}{2}x - 4$ 6. $y = -\frac{3}{4}x - 6$
7. $y = 0.5x + 1$

Exercise F: Equation of a line when two points are known (page 18)

1. $y = -\frac{1}{3}x + \frac{2}{3}$ 2. $y = \frac{7}{4}x + \frac{1}{4}$
3. $y = -x + 10$ 4. $y = -x + 2$
5. $y = x - 3$ 6. $y = \frac{2}{3}x - 2$ 7. $y = 2$
8. a. -2 b. $y = -2x + 110$
   c. $y = x + 20$ d. 113.14 m (2 dp)

Exercise G: Equations of special lines (page 22)

1.



INDEX