Probability using permutations and combinations

Multiplication principle

If A can be done in n ways, and B can be done in m ways, then A followed by B can be done in $n \times m$ ways.

Example

1. A die and a coin are thrown together. How many results are possible?

   There are 6 outcomes for the die {1, 2, 3, 4, 5, 6} and 2 outcomes for the coin {H, T}

   So there are $6 \times 2 = 12$ possible outcomes for the die and the coin together.

   {1H, 2H, 3H, 4H, 5H, 6H, 1T, 2T, 3T, 4T, 5T, 6T}

2. Three 6-sided dice are thrown together.

   Each dice has six possible outcomes, so the number of possible outcomes for the three dice is:

   $6 \times 6 \times 6 = 216$

Example

Q. How many number plates are possible if each number plate is made up of 3 letters followed by a number less than 10 000 (excluding zero).

A. The number of plates is:
$26 \times 26 \times 26 \times 9\,999 = 175\,742\,424$
[for each letter there are 26 to choose from and there are 9 999 numbers less than 10 000 to choose from]

Exercise A: The multiplication principle

1. How many outfits (of shoes, skirt and top) does a female Year 13 student have for school if she has the choice of 2 pairs of shoes, 4 skirts and 5 tops in her wardrobe?

2. A male student wants to wear a 1960s outfit (trousers, shirt and shoes) to the school ball. He goes to the local costume hire shop and they have the following items he thinks would be suitable:
   - trousers (lime green, fluorescent pink, vivid orange, purple)
   - shirts (lurex, paisley, electric blue, black with rhinestones, metallic gold)
   - shoes (platform heels, winklepickers, pointed toes).

   How many possible choices of outfit does the student have?

3. There are eight doors in a hall. In how many different ways can they be left either open or closed?

4. In the UK, number plates have 2 letters followed by 2 digits then 3 letters, e.g. SW43QPA. The first digit cannot be 0. How many possible number plates are there?

5. Number plates in Hong Kong are 2 letters followed by up to 4 digits, where the first digit cannot be a zero. In China the number plates are two letters followed by five digits or letters (first digit cannot be zero). How many more possible number plates are there in China than in Hong Kong?

Factorials

If n is a whole number, then n **factorial** (written $n!$) is the product of all whole numbers from 1 to n.

For example,

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

$4! = 4 \times 3 \times 2 \times 1 = 24$ [which is $\frac{5!}{5}$]

$3! = 3 \times 2 \times 1 = 6$ [which is $\frac{4!}{4}$]

$2! = 2 \times 1 = 2$ [which is $\frac{3!}{3}$]

$1! = 1$ [which is $\frac{2!}{2}$]

Continuing this pattern, $0! = \frac{1!}{1} = 1$

Note: There is a factorial key on scientific calculators.

Using factorials to count arrangements

The three objects A, B, C can be arranged in 6 ways: ABC, ACB, BAC, BCA, CAB, CBA

The number of arrangements of 3 objects can be worked out using the following reasoning.

The first place can be filled in 3 ways, the second place in 2 ways, and the final place in 1 way.

So there are $3! = 3 \times 2 \times 1 = 6$ possible arrangements [by the multiplication principle]

This result can be generalised:

> n objects can be arranged in a line in $n!$ ways where
> $n! = n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1$

Example

Q. 8 people are waiting to go into a room, one at a time. In how many different orders can they go into the room?

A. $8! = 40\,320$ ways

When arranging n objects in a circle, one object is effectively held fixed and the remaining $(n - 1)$ objects arranged around it.

> n objects can be arranged in a circle in $(n - 1)!$ ways where
> $(n - 1)! = (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1$

Example

Q. In how many ways can 8 guests be seated around a circular dinner table?

A. $(8 - 1)! = 7! = 5\,040$ ways

There may be various *restrictions* involved in a situation, such as identical objects in the group being arranged (which reduces the number of arrangements).

Example

Q. How many ways can the letters in the word HARSH be arranged?

A. If the H's are labelled H_1 and H_2 (and treated as being different) then there would be $5! = 120$ arrangements.

Each pair of arrangements such as ASH_1RH_2 and ASH_2RH_1 (in which the positions of H_1 and H_2 are reversed) would then be counted as two different arrangements.

However, since the H's are in fact the same ($H_1 = H_2$) this pair of arrangements is really only the single arrangement ASHRH.

So 5! needs to be divided by 2 (which is 2!, the number of ways the two H's can be arranged).

So the number of different arrangements is $\frac{5!}{2} = 60$

Probabilities with factorials

If p is the probability that an arrangement with a given restriction occurs, then

$$p = \frac{\text{Number of arrangements with restriction}}{\text{Number of arrangements without restriction}}$$

Example

Q. If 4 girls and 1 boy are arranged randomly in a line, find the probability that
1. the boy is at one end of the line.
2. the boy is in the middle

A. 1. P(boy is at one end)

$$= \frac{\text{Number of arrangements with boy at one end}}{\text{Number of arrangements without restriction}}$$

AS 91585

There are 2 ways of placing the boy at one end of the line (at the front or at the back).

For each placement of the boy, there are 4! ways of arranging the girls.

So the number of arrangements with boy at one end $= 2 \times 4! = 48$ [multiplication principle]

Number of arrangements without restriction is $5! = 120$

$$\text{P(boy is at one end)} = \frac{48}{120} = \frac{2}{5}$$

2. $$\text{P(boy in middle)} = \frac{4 \times 3 \times 1 \times 2 \times 1}{120} = 0.2$$

Exercise B: Factorials and probability

1. Simplify

   a. $\frac{8!}{7!}$

   b. $\frac{36!}{34!}$

   c. $\frac{2\,640!}{2\,639!}$

   d. $\frac{n!}{(n+1)!}$

   e. $\frac{(n-5)!}{(n-6)!}$

   f. $\frac{5(n+3)!}{(n+2)!}$

   g. $\frac{n!}{(n+1)!}$

   h. $\frac{(n-2)!}{n!}$

2. 11 people are lining up for the school canteen. In how many ways could they line up?

3. A new upholstery fabric is being created that is made of 10 differently coloured stripes. In how many different ways can the stripes be arranged?

4. A primary school teacher has taken a group of 10 students outside to play a game in which they need to be in a circle. In how many different ways could the circle of students be arranged?

5. There are 4 people in a family and one bathroom. In how many different orders could they use the bathroom in the morning if:

   a. there is no restriction?

   b. one particular family member always gets up and uses the bathroom first?

6. A new school logo is being developed and the students want to have 7 different symbols arranged in a circle. In how many ways could the symbols be arranged?

7. a. How many different 'words' can be made from the word SURFIE?

   b. Find the probability that a 'word' begins and ends with a vowel.

8. a. How many different 'words' can be made from the word SKATEBOARD?

b. Find the probability that a 'word' begins and ends with an A.

9. a. How many different 'words' can be made from the word PROBABILITY?

b. Find the probability that a 'word' has the two B's together and the two I's together.

10. The letters of the word TEAMS are arranged to make 'words'. Find the probability that the 'word':

a. has alternating consonants and vowels

b. has the two vowels together.

Permutations

A **permutation** is an arrangement in which order matters.

For example, if two letters are selected from the word THE and put in order, then there are 6 permutations possible: TH, HT, TE, ET, HE, EH

Note: The number of arrangements can be worked out using the multiplication principle: there are 3 ways of choosing the first letter and 2 ways of choosing the second letter, so altogether there are $3 \times 2 = 6$ ways of arranging two letters from three.

The number of permutations (ordered arrangements) of 2 objects from 3 available objects is written ${}^3P_2$

So ${}^3P_2 = 6$

The number of permutations of r objects selected from a group of n different objects is written ${}^nP_r$

Example

Q How many ways can 4 letters of the word **iPhone** be arranged so that no letter is used more than once in each arrangement?

A. There are 6 letters from which to choose 4, so the number of permutations is ${}^6P_4$

There are 6 ways of choosing the first letter, 5 ways of choosing the second letter (the first letter cannot be used again), 4 ways of choosing the third letter (the first two letters cannot be used again) and 3 ways of choosing the fourth letter (the first three letters cannot be used again), so

${}^6P_4 = 6 \times 5 \times 4 \times 3 = 360$

Note: ${}^6P_4 = 6 \times 5 \times 4 \times 3$

$$= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1}$$

$$= \frac{6!}{2!}$$

which is $\frac{6!}{(6-4)!}$

The formula for the number of permutations of r objects from n objects is:

$${}^nP_r = \frac{n!}{(n-r)!}$$

Scientific calculators have a $\boxed{{}^nP_r}$ ${}^nP_r$ key which can be used for calculating permutations.

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Example

Q. There are 220 Year 13 students eligible to be the chairperson, treasurer and secretary of the ball committee. How many possible ways can these offices be filled?

A. ${}^{220}P_3 = 10\,503\,240$ possible ways [from 220 students, select and arrange 3 students]

Note: By the formula this is $\frac{220!}{(220-3)!}$ which is too large to be worked out directly on a calculator. Simplifying gives:

$$\frac{220!}{(220-3)!} = \frac{220 \times 219 \times 218 \times 217!}{217!}$$
$$= 220 \times 219 \times 218$$
$$= 10\,503\,240$$

Probability with permutations

Probability problems may deal with **restrictions** on the permutations of objects.

Example

5-letter words are being formed from the word COMPUTER. Find the probability that a 'word' begins with C and ends with M.

Solution:

P(word begins with C and ends with M)

$$= \frac{\text{Number of 'words' beginning with C and ending with M}}{\text{Number of 4-letter 'words' without restriction}}$$

The 'word' is of the form C _ _ _ M

The number of ways of placing the C and the M is 1 [C must be first and M last]

This leaves 6 letters from which to choose and arrange the remaining 3 letters.

This can be done in ${}^{6}P_3 = 120$ ways.

So the number of 'words' beginning with C and ending with M is $1 \times 120 = 120$

The number of 5-letter words (without restriction) is ${}^{8}P_5 = 6\,720$

P(word begins with C and ends with M) is

$$\frac{120}{6\,720} = \frac{1}{56}$$

Exercise C: Permutations and probability

1. Students in a secondary school are asked to fill out a questionnaire ranking the five things they are most satisfied with about their school. The choices they have are: teachers, working environment, facilities, computer access, fellow students, uniform, discipline, buildings, sports facilities, activities and canteen. The ranking is to be shown by the numbers 1, 2, 3, 4, 5 where 1 is the thing they are most satisfied with and 2 the next most satisfied. How many different ways can a student complete the questionnaire?

2. A combination lock has 10 different digits (0, 1, 2, … , 9) and in order to open the lock a 'combination' of three different digits must be selected in the correct order. How many different 'combinations' could be selected for this lock?

3. A school has 224 Year 13 students. The principal of the school wants to select a group of three Year 13 students to work together on a committee to fundraise for World Vision. One student will be the leader of the group, another will take charge of the money and the third will be the publicist. In how many ways could the principal select this committee:
   a. without restriction?
   b. if the leader of the group must be Bethany, the Head Girl?

4. a. How many different 4-letter 'words' can be made from the word TRIUMPH?

b. What is the probability that a 4-letter 'word' made from the word TRIUMPH starts with a P?

5. Find the probability that a 5-letter 'word' made from the word PROBLEMS has first letter P and last letter S.

Combinations

A **combination** is a selection in which order *doesn't* matter.

For example, if two letters are to be selected from the four letters A, B, C, D, without order of selection mattering ('A then B' is the same selection as 'B then A'), then there are six combinations possible:

A,B   A,C   A,D   B,C   B,D   C,D

The number of combinations (unordered selections) of 2 objects from 4 available objects is written ${}^4C_2$

So ${}^4C_2 = 6$

Example

Q. A school selects 3 girls from a group of 8 girls to play tennis against another school. How many combinations of team members are possible?

A. The number of permutations would be ${}^8P_3 = 336$

But … since the order doesn't matter there will be fewer combinations

If the three girls chosen are called Aroha (A), Beatrice (B) and Carissa (C) then there would be $3! = 6$ permutations possible for this selection (ABC, ACB, BAC, BCA, CAB, CBA). All six permutations correspond to a single combination {A, B, C}.

So there are 6 times as many permutations as there would be combinations, so the number of permutations needs to be divided by 6:

Number of possible tennis teams is:

$$\frac{{}^8P_3}{3!} = \frac{336}{6}$$
$$= 56$$

The formula for the number of combinations of *r* objects from *n* objects is:

$$\begin{aligned} {}^nC_r &= \frac{{}^nP_r}{r!} \\ &= \frac{n!}{(n-r)!r!} \end{aligned}$$

${}^nC_r$ can also be written as $\binom{n}{r}$

Scientific calculators have a $\boxed{{}^nC_r}$ ${}^nC_r$ key which can be used for calculating combinations.

Read the the problem carefully to determine if permutations or combinations are required.

Example

Q. From a club of 12 members, how many ways are there of selecting:

1. a president, vice-president and a treasurer?
2. a committee of 3 people?

A. 1. The order matters so there are ${}^{12}P_3 = 1\,320$ ways

2. Order doesn't matter so there are ${}^{12}C_3 = 220$ ways

Probabilities with combinations

Remember to multiply for AND and add for OR.

Example

Q. 1. In how many ways can a committee of 5 be chosen from 3 women and 7 men if the committee must contain at least 2 women?

2. If a committee of 5 is chosen randomly from 3 women and 7 men, what is the probability the committee has:

a. at least 2 women?

b. no women?

AS 91585

A. 1. The committee must have either 2 women (and 3 men) OR 3 women (and 2 men)

The number of ways of forming a committee with 2 women AND 3 men is:

${}^{3}C_{2} \times {}^{7}C_{3} = 105$

[multiplication principle]

The number of ways of forming a committee with 3 women AND 2 men is:

${}^{3}C_{3} \times {}^{7}C_{2} = 21$

[multiplication principle]

So the total number of committees possible with at least 2 women is

$105 + 21 = 126$

2. a. P(committee has at least 2 women) =

$\frac{\text{Number of committees with at least 2 women}}{\text{Number of committees without restriction}}$

There are $7 + 3 = 10$ people available from which to select a committee of size 5.

The number of committees that can be formed without restriction is

${}^{10}C_{5} = 252$

P(committee has at least 2 women) is

$\frac{126}{252} = \frac{1}{2}$

b. P(no women) $= \frac{{}^{3}C_{0} \times {}^{7}C_{5}}{{}^{10}C_{5}}$

$= \frac{1}{12}$

Exercise D: Combinations and probability

1. A catering student wants to make a fruit salad with 8 fruits so he goes to the supermarket and finds there are 23 different fruits to choose from. How many different possible varieties of fruit salad could he make?

2. At the local ice-cream parlour there are 8 different flavours to choose from. If Maree wants to buy a sundae with one scoop of each of three different flavours of ice-cream, how many possible sundaes could she choose?

3. A school has 140 Year 13 students. The principal of the school wants to select a group of 6 students to work together on a committee to fundraise for World Vision. In how many ways could the Principal select this committee?

4. There are 15 girls and 13 boys in 13Pmm. The school want to send four students from the class to represent the school at the Anzac Day service.
   a. How many different ways are there to select the four representatives?
   b. i. If the class decides it wants to send 2 girls and 2 boys, how many different ways can the four representatives be chosen?
      ii. If the class decides to send at least 1 girl, how many different ways can the four representatives be chosen?
   c. Calculate the the probability that a randomly selected group of four students from 13Pmm is made up of:
      i. 2 girls and 2 boys
      ii. at least 1 girl.

5. One of the teachers at school likes to give students *Fruitbursts* sweets if they have been working very hard. Misbah always works hard so the teacher decided to give her two *Fruitbursts*. The teacher

held one of each colour in his hand (purple, red, green, yellow and orange) and asked her to choose two.

a. How many different combinations of colour could Misbah choose?

b. What is the probability her selection does not include a purple *Fruitburst*?

6. A class of 20 students includes a pair of twins.

a. A group of 4 students is selected from the class. What is the probability that the group comprises:

i. both twins?

ii. neither twin?

iii. exactly one of the twins?

b. Let the random variable X = the number of twins in a selection of 4 students from this class of 20 students. Draw up a probability distribution table for X.

7. A coach has a squad of 15 football players, of whom 9 are boys and 6 are girls. The coach decides to interview a random group of 6 of these players, to find out their opinion on training schedules.

a. In how many ways could the coach select this group to interview?

The coach makes a selection and the group is made up completely of boys.

b. Do you consider it likely that this group of six was randomly selected? Justify your answer with suitable probability calculations.

8. A group of 5 Year 13 mathematics students is studying simple random sampling. Each student chooses a sample of 4 jelly snakes from a population of 12 differently coloured jelly snakes. The students notice that all 5 samples are different and they wonder what the chances are of two students choosing exactly the same colours in their sample of 4 snakes. They also wonder how the situation would change if the group of students was a different size, e.g. a whole class of 30 students performing this sampling process? Discuss this situation. Include some appropriate probability calculations with your answer.