STATISTICS 3.14

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AS 91586

Apply probability distributions in solving problems

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Random variables

A **random variable** X is a variable whose values are determined by the outcome of a probability experiment. X can be:

- a **discrete random variable** – where values are found by *counting* (e.g., X = the number of telephone calls at an office in a day, or X = the number of sunspots in a month)
- a **continuous random variable** – where values are found by *measuring* (e.g., X = the weight of a student's schoolbag; or X = the length of time a house fire burns, etc.).

Discrete probability distributions

A discrete random variable can take on various values, and each value has an associated probability. The probability that the variable X takes on a value x is written $P(X = x)$ or $p(x)$.

For example, if X = the number uppermost when a fair 6-sided die is rolled, then X can take on the values {1, 2, 3, 4, 5, 6}. The probability that the number uppermost is 5 is written $P(X = 5)$ or $p(5)$, which is $\frac{1}{6}$

The **probability distribution** of a discrete random variable lists all the values the variable can take, along with the probabilities of these values.

For example, the probability distribution of X as described above is:

X	1	2	3	4	5	6
$P(X = x)$ or $p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

For any discrete probability distribution, the sum of the probabilities is 1.

$$\Sigma p(x) = 1$$

Expected value and variance

A random variable X has an **expected value** $E(X)$. This gives the mean value of X for a long-run experiment, i.e. the theoretical mean (μ) of the variable.

$$E(X) = \Sigma x \times p(x)$$

The spread of the values X can take can be measured using the **variance** $\text{Var}(X)$, which has the formula:

$$\text{Var}(X) = E(X^2) - \mu^2$$

The **standard deviation** SD(X) is another measure of spread for X. It is found by taking the square root of the variance.

$$SD(X) = \sqrt{\text{Var}(X)}$$

Example

The discrete random variable X has probability distribution as shown.

x	1	2	3	4	5
$p(x)$	0.1	c	0.2	0.1	0.3

Find:

1. the value of c
2. $E(X)$, the expected value of X
3. $\text{Var}(X)$, the variance of X
4. $SD(X)$, the standard deviation of X.

Solution

1. $0.1 + c + 0.2 + 0.1 + 0.3 = 1$

   $[\Sigma p(x) = 1]$

   $c + 0.7 = 1$

   $c = 0.3$

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2. $E(X) = 1 \times 0.1 + 2 \times 0.3 + 3 \times 0.2 + 4 \times 0.1 + 5 \times 0.3$ $[\Sigma x \times p(x)]$

$= 3.2$

3. By definition, $Var(X) = E(X^2) - \mu^2$

$E(X^2) = 1^2 \times 0.1 + 2^2 \times 0.3 + 3^2 \times 0.2 + 4^2 \times 0.1 + 5^2 \times 0.3$ $[\Sigma x^2 \times p(x)]$

$= 12.2$

$\mu = 3.2$ $[\mu = E(X)]$

So $Var(X) = 12.2 - 3.2^2$

$= 1.96$

4. $SD(X) = \sqrt{1.96}$ $[SD(X) = \sqrt{Var(X)}]$

$= 1.4$

Note: Avoid premature rounding of the variance to ensure your calculations for standard deviation are as accurate as possible.

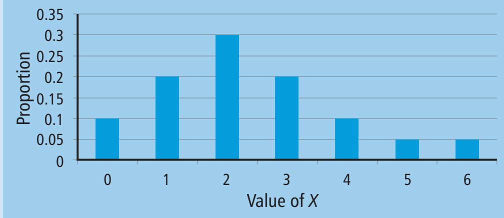
Note: Graphics calculators or spreadsheets may also be used to calculate expected values and standard deviations.

Experimental probability distributions can be used to approximate the probability distributions of discrete random variables. An experimental probability distribution is formed by conducting a series of trials and calculating the **proportion** of times each value of the variable occurs (the **relative frequency** of each value) .

The mean and standard deviation of a discrete random variable can be estimated from its experimental probability distribution.

Example

The discrete random variable X has an experimental probability distribution as shown in the bar chart.



1. State the lowest and highest value that X can take.
2. Comment on the shape of the distribution of X.
3. Determine the mode. Is X uni-modal or bi-modal?
4. Estimate the value of the mean μ of the random variable X.
5. Estimate the value of the standard deviation σ of the random variable X.

Solution

1. Lowest value is $x = 0$; highest value is $x = 6$
2. The probability distribution is skewed to the right
3. Value with highest probability is 2, so mode = 2.

   Distribution has a single peak so X is uni-modal.
4. $\mu = E(X)$

Using the experimental proportions as estimates of the probabilities in the formula for the expected value:

$E(X) \approx 0 \times 0.1 + 1 \times 0.2 + 2 \times 0.3 + 3 \times 0.2 + 4 \times 0.1 + 5 \times 0.05 + 6 \times 0.05$

$= 2.35$

5. Using experimental proportions as estimates of the probabilities in the formula for the standard deviation:

$E(X^2) \approx 0^2 \times 0.1 + 1^2 \times 0.2 + 2^2 \times 0.3 + 3^2 \times 0.2 + 4^2 \times 0.1 + 5^2 \times 0.05 + 6^2 \times 0.05$

$= 7.85$

$Var(X) = E(X^2) - [E(X)]^2$

$\approx 7.85 - 2.35^2$

$= 2.3275$

$SD(X) = \sqrt{Var(X)}$

$\approx \sqrt{2.3275}$

$\sigma = 1.53$

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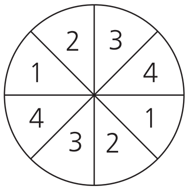
Ans. p. 75

Exercise A: Discrete probability distributions

1. Find the expected value, variance and standard deviation for the following probability distribution.

x	1	2	3	4
$P(X = x)$	0.2	0.1	0.4	0.3

2. A wheel, with face as shown in the diagram below, is used for a quickfire raffle.



Complete the following probability distribution for X = the number showing after a spin, and use it to find the expected value, variance and standard deviation of X.

X				
$P(X = x)$				

3. The probability distribution for a discrete random variable X is shown. Use it to find a and then calculate the mean, variance and standard deviation of X.

x	1	2	3	4
$P(X = x)$	0.1	0.25	0.35	a

4. The probability distribution for a discrete random variable Y is shown. Use it to find b and then calculate the mean, variance and standard deviation of Y.

y	1	2	3	4
$P(Y = y)$	$\frac{1}{8}$	$\frac{1}{4}$	b	$\frac{1}{2}$

5. The probability distribution for a discrete random variable W is shown. Use it to find c and then calculate the mean, variance and standard deviation of W.

w	1	2	3	4
$P(W = w)$	3c	2c	2c	c

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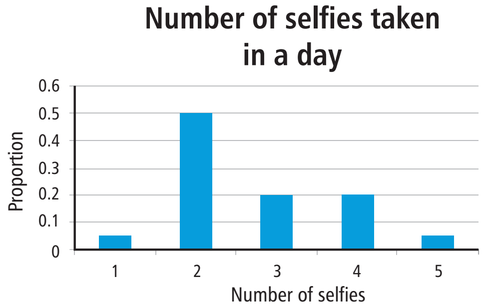
6. In each graph below, an experimental probability distribution is drawn for a discrete random variable *X*. For each graph:

- **i.** state the lowest and highest values the random variable *X* can take
- **ii.** comment on the shape of the distribution of *X*
- **iii.** determine a mode and state if the distribution is uni- or bi-modal
- **iv.** estimate the mean.

a.



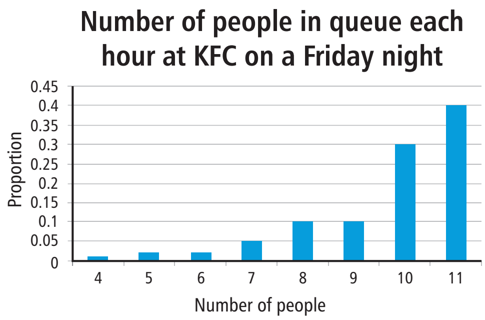
i. ____________________

ii. ____________________

iii. ____________________

iv. ____________________

b.



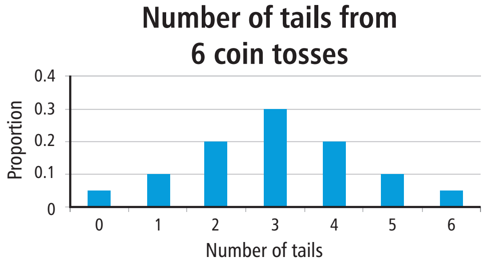
i. ____________________

ii. ____________________

iii. ____________________

iv. ____________________

c.



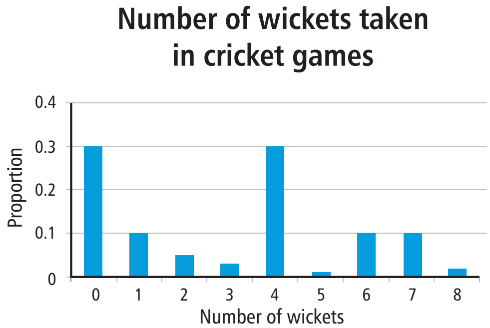
i. ____________________

ii. ____________________

iii. ____________________

iv. ____________________

d.



i. ____________________

ii. ____________________

iii. ____________________

iv. ____________________

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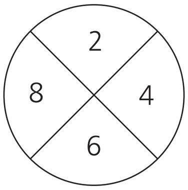
7. A spinner has its face marked as shown. Find the expected value, variance and standard deviation of the number resulting from a spin.



8. Car trips between home and school vary from 7 minutes to 11 minutes for a very busy Statistics teacher (who knows that every extra minute spent at school counts). Over a period of a year she has timed her travel and has drawn up the following probability distribution for T = the time (min) taken to travel between home and school.

t	7	8	9	10	11
$P(T = t)$	0.09	0.42	0.31	0.15	0.03

For journeys between home and school for this statistics teacher, find:

a. the expected time of travel

b. the variance of the time of travel

c. the standard deviation of the time of travel.

Expected value applications

Expected gain

In probability, the word **gain** means how much a person has *won or lost* after playing a game. For example, if it costs \$2 to play a game and you lose, then your gain is –\$2 (negative two dollars).

Example

Mark pays \$1 to enter a game where the prize is \$5. If Mark has a 1 in 10 chance of winning, find his expected gain.

Solution

If X = Mark's gain in dollars after one game, then either Mark wins and his gain is $5 - 1 =$ \$4, or he loses and his gain is –\$1

The probability distribution of X is as shown.

X	4	–1
$P(X = x)$	0.1	0.9

Mark's expected gain is:

$E(X) = 4 \times 0.1 - 1 \times 0.9$

$= -\$0.50$

[include units so answer is in context]

This means that if a person played this game a large number of times he/she would lose an average of 50 cents per game.

Expected winnings

Winnings means how much the person won, disregarding the entry fee.

In Mark's game, he either wins \$5 or \$0, so the probability distribution of W = Mark's winnings in dollars after one game is:

W	5	0
$P(W = w)$	0.1	0.9

Mark's expected winnings are:

$E(W) = 5 \times 0.1 + 0 \times 0.9$

$= \$0.50$

This means that if a person played this game a large number of times he would win an average of 50 cents per game. (Since he has paid \$1 per game to enter, the expected gain per game is –\$0.50)

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Games that are fair

If a game is **fair**, then the expected gain is zero. One way of making a game fair is by altering the cost of playing the game, or by changing the value of the prize(s) for winning.

Example

In Mark's game he has a 1 in 10 chance of winning \$5, and it costs \$1 to play. Adjust the prize value so that the game is fair.

Solution

Let the value of the prize for winning a fair game be k dollars.

Let X = Mark's gain (in dollars) after playing the game.

This could be –\$1 if he loses, and $(k - 1)$ dollars if he wins.

The probability distribution of X is shown below.

X	$k - 1$	-1
$P(X = x)$	0.1	0.9

Mark's expected gain is:

$E(X) = (k - 1) \times 0.1 - 1 \times 0.9$ $[E(X) = \Sigma x \times p(x)]$

$= 0.1k - 0.1 - 0.9$ [expanding]

$= 0.1k - 1$ [simplifying]

The value of the prize in a fair game can be found by setting Mark's expected gain to zero and solving for k.

$0.1k - 1 = 0$

$0.1k = 1$ [rearranging]

$k = 10$ [dividing by 0.1]

If the game is to be fair the prize should be \$10.

Another method of making a game fair is by altering the chance of winning.

Example

Make Mark's game fair by altering the probability of winning. Remember the person pays \$1 to enter the game and the prize is \$5.

Solution

Let p be the new probability of Mark winning a game, and X be Mark's gain (in dollars) from playing a game.

Since probabilities add to 1, the probability of losing a game is $1 - p$

The probability distribution of X is as shown.

X	4	-1
$P(X = x)$	p	$1 - p$

Mark's expected gain is:

$E(X) = 4 \times p - 1 \times (1 - p)$

$= 4p - 1 + p$

$= 5p - 1$

For a fair game, require the expected gain to be zero.

$5p - 1 = 0$ [substituting $E(X) = 0$]

$p = 0.2$ [rearranging and solving]

If the game is to be fair the probability of winning should be 0.2 (and the probability of losing should be 0.8).

Ans. p. 75

Exercise B: Expected value applications

1. Supreme College is running a fundraising quickfire raffle where 20 tickets are sold for \$5 each. Prize winners are decided using a wheel divided into 20 equal sectors numbered 1 to 20. The wheel is spun three times: the first number spun wins \$50; the second number wins \$20; and the third number wins \$10.

a. If G = gain (in dollars) for a person buying a ticket, complete the probability distribution for G.

g	45	15	5	-5
$P(G = g)$				

b. i. What is the expected gain for a person buying a ticket?

ii. How many tickets would need to be sold for the school to make $1000?

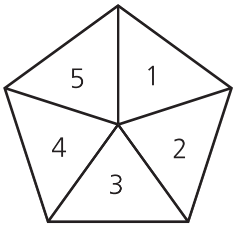
c. i. Explain why this game is not fair.

ii. How could the price of tickets be adjusted to make the raffle fair? Complete the following probability distribution to help you answer the question (k dollars is the new price of a ticket).

g	$50 - k$	$20 - k$	$10 - k$	$-k$
$P(G=g)$				

2. Tiopira and Jade are playing a game with two spinners, each with a face labelled as shown.

The spinners are spun and the resulting two numbers added. Find the expected sum of the numbers from the two spins.



First complete the probability distribution of T, the sum of the numbers from the two spins.

t	2	3	4	5	6	7	8	9	10
$P(T = t)$									

3. Students in a Year 10 class are playing a game where the teacher tosses two dice and calls out the total. Students bet $2 (with play money) on whether or not the two dice will come up with a double (two equal numbers). If a double is thrown the student will win $10.

a. Draw up a probability distribution for this situation. Define your variable clearly.

b. If a student bets on a double coming up for nine games, what are her expected winnings?

c. If a student bets on a double coming up for nine games, what is her expected gain?

d. Is the game fair? If not, how could the prize be altered to make the game fair?

4. In a game where the entry is 50c, two coins are tossed. If either 2 Heads or 2 Tails come up there is a prize of $1. Is this a fair game? Explain your answer.

5. A game costs $5 to enter. In the game a coin and a die are thrown together. If a Head and a 6 are thrown there is a prize of $10. If a Tail and a 6 are thrown the prize is $20.

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a. What is the expected gain to someone playing the game?

b. What is the expected gain to the person running the game?

c. If a person played the game twenty times what their expected gain be?

d. What should the entry fee be to make this a fair game?

Expectation algebra

If a random variable X is multiplied by a constant a, then a new random variable aX is formed, with expected value, standard deviation and variance as shown below.

$$E(aX) = a \times E(X)$$
$$Var(aX) = a^2 \times Var(X)$$
$$SD(aX) = |a| \times SD(X)$$

Note that $|a|$ means the size of a (a positive value).

If a constant, b, is added to a random variable X, then a new random variable is formed, with expected value, variance and standard deviation as shown below.

$$E(X + b) = E(X) + b$$
$$Var(X + b) = Var(X)$$
$$SD(X + b) = SD(X)$$

These rules can be extended further.

$$E(aX + b) = a \times E(X) + b$$
$$Var(aX + b) = a^2 \times Var(X)$$
$$SD(aX + b) = |a| \times SD(X)$$

Example

If $E(X) = 4$ and $Var(X) = 2$ find:

1. $E(7X)$ and $Var(7X)$

2. $E(X + 19)$ and $Var(X + 19)$

3. $E(6X + 3)$ and $Var(6X + 3)$

Solution

1. $E(7X) = 7 \times E(X)$
$= 28$ $[7 \times 4]$
$Var(7X) = 7^2 \times Var(X)$
$= 98$ $[49 \times 2]$

2. $E(X + 19) = E(X) + 19$
$= 23$ $[4 + 19]$
$Var(X + 19) = Var(X)$
$= 2$

3. $E(6X + 3) = 6 \times E(X) + 3$
$= 27$ $[6 \times 4 + 3]$
$Var(6X + 3) = 6^2 \times Var(X)$
$= 72$ $[36 \times 2]$

Ans. p. 75

Exercise C: Expectation algebra

1. The table below shows the probability distribution for X = the number obtained by rolling a standard die (labelled 1–6).

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

It can be shown that
$E(X) = 3.5$ and $Var(X) = 2.916$ (4 sf)

Find the mean, variance and standard deviation of $3X$, which would be the new mean, variance and standard deviation if each number on the die was multiplied by 3 (i.e. the numbers on the dice became: 3, 6, 9, 12, 15, 18)

a. $E(3X)$

b. Var(3X)

c. SD(3X)

2. Using the same probability distribution given in question 1 for throwing a die, find

a. E(X + 5)

b. Var(X + 5)

c. SD(X + 5)

d. E(3X + 7)

e. Var(3X + 7)

f. SD(3X + 7)

g. SD(–X + 1)

3. A taxi company decides to have a flat rate for local fares. It charges:

$10 for journeys up to 10 km;

$15 for journeys over 10 km and up to 15 km;

$20 for journeys over 15 km and up to 20 km.

From long experience the taxi firm knows the likelihood of each type of journey. If F = the amount spent per taxi hire, then F has the probability distribution given below.

f	$10	$15	$20
P(F = f)	0.35	0.47	0.18

a. Find the mean fare (the mean amount spent per taxi hire) E(F).

b. Calculate the variance of the amount spent per taxi hire.

c. Calculate the standard deviation of the amount spent per taxi hire.

The company decides to investigate the effect of increasing its prices by 25%. The random variable for the new fare would be $N = 1.25F$

d. Use your answer from part **a.** to find the new mean fare, E(1.25F).

e. Calculate the new variance of fares, Var(1.25F).

f. Calculate the new standard deviation of fares, SD(1.25F).

Another possibility is to increase each original fare by $2.50. In this case the new random variable for the fare paid is $X = F + 2.5$

g. Use your answer from part **a.** to find the new mean fare, E(F + 2.5)

h. Calculate the new variance of fares, Var(F + 2.5)

i. Calculate the new standard deviation of fares, SD(F + 2.5)

A final option the taxi firm looks at is increasing the fares by 20% and adding 50 cents. The random variable for the new fare would be $Y = 1.2F + 0.5$

j. Use your answer from part **a.** to find the new mean fare, E(1.2F + 0.5)

k. Calculate the new variance of fares, Var(1.2F + 0.5)

l. Calculate the new standard deviation of fares, SD(1.2F + 0.5)

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Combinations of random variables

If X and Y are random variables then the sum, $X + Y$, and the difference, $X - Y$, have the following expected values and variances (note that variances are always added):

$$E(X + Y) = E(X) + E(Y)$$

$$Var(X + Y) = Var(X) + Var(Y)$$

(if X and Y are independent)

$$E(X - Y) = E(X) - E(Y)$$

$$Var(X - Y) = Var(X) + Var(Y)$$

(if X and Y are independent)

It is very important to note that the standard deviation of the sum of two independent variables must be found using the variance of the sum of the two variables.

$$SD(X + Y) = \sqrt{Var(X + Y)}$$

$SD(X + Y)$ *cannot* be found directly using $SD(X)$ and $SD(Y)$.

These results can be extended to sums of more than two variables.

Example

Mikara is filling up his trailer with wheelbarrow loads of mulch. Each wheelbarrow load has a mean volume of 0.1 m^3 and standard deviation of 0.02 m^3. The trailer can take 30 wheelbarrow loads of mulch.

Find the mean and the standard deviation of the volume of mulch in a full trailer.

Solution

If W_1 = the volume (m^3) of mulch in the first wheelbarrow load, W_2 = the volume of mulch in the second wheelbarrow, and so on, then the volume of mulch in a full trailer load (T) is:

$$T = W_1 + W_2 + W_3 + \ldots + W_{30}$$

$$\begin{aligned} E(T) &= E(W_1 + W_2 + W_3 + \ldots + W_{30}) \\ &= E(W_1) + E(W_2) + E(W_3) + \ldots + E(W_{30}) \\ &= 0.1 + 0.1 + 0.1 + \ldots + 0.1 \\ &= 0.1 \times 30 = 3 \text{ m}^3 \quad \text{[30 terms]} \end{aligned}$$

$SD(W) = 0.02$, so $Var(W) = 0.02^2 = 0.0004$

$$\begin{aligned} Var(T) &= Var(W_1 + W_2 + W_3 + \ldots + W_{30}) \\ &= Var(W_1) + Var(W_2) + \ldots + Var(W_{30}) \\ &= 0.0004 + 0.0004 + \ldots + 0.0004 \\ &= 0.0004 \times 30 = 0.012 \quad \text{[30 terms]} \end{aligned}$$

$$SD(T) = \sqrt{0.012} = 0.1095 \text{ (4 sf)}$$

Note: It is assumed that the volume of each wheelbarrow load, W, is independently and identically distributed with the same mean and variance. It is VERY important to note that T (the total of 30 wheelbarrow loads) is *not* $30W$. So, the variance of a total load $\neq Var(30W)$ (which is $900 \times Var(W)$).

$Var(aX) = a^2 \times Var(X)$ should only be used when a random variable is being multiplied by a constant. In summary:

- when a random variable represents the sum of a items, with each item X having its own mean and variance, then the variance of the total will be $a \times Var(X)$
- when a random variable represents a constant multiple of a random variable, aX, the variance will be $a^2 \times Var(X)$

A **linear combination** of two random variables X and Y is any random variable of the form $aX + bY$. Expected values and variances obey the following rules, where a and b are constants:

$$E(aX + bY) = aE(X) + bE(Y)$$

$$Var(aX + bY) = a^2Var(X) + b^2Var(Y)$$

if X, Y are independent

Example

In a shop Alpha pens sell for \$3 and Beta pens sell for \$2. Sales of the pens are independent. The mean number of Alpha pens sold per week is 35 with standard deviation 8.

The mean number of Beta pens sold per week is 68 with standard deviation 12.

Find the mean and standard deviation of the weekly income from sales of pens.

Solution

Let X = the number of Alpha pens sold per week.

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Answers

Achievement Standard 91586 (Statistics 3.14): Apply probability distributions in solving problems

Exercise A: Discrete probability distributions (page 3)

1. $E(X) = 2.8$, $Var(X) = 1.16$, $SD(X) = 1.077$
2.

x	1	2	3	4
$P(X = x)$	0.25	0.25	0.25	0.25

$E(X) = 2.5$, $Var(X) = 1.25$, $SD(X) = 1.118$

3. $a = 0.3$, $\mu = E(X) = 2.85$, $Var(X) = 0.9275$, $SD(X) = 0.9631$
4. $b = \frac{1}{8}$, $\mu = E(Y) = 3$, $Var(Y) = 1.25$, $SD(Y) = 1.118$
5. $8c = 1$ so $c = \frac{1}{8}$

$\mu = E(W) = 2.125$, $Var(W) = 1.109$, $SD(W) = 1.053$

6. a. i. Lowest value = 1, highest value = 5
   ii. Skewed to the right iii. Mode = 2; uni-modal
   iv. 2.7
   b. i. Lowest value = 4, highest value = 11
   ii. Skewed to the left iii. Mode = 11; uni-modal
   iv. 9.71
   c. i. Lowest value = 0, highest value = 6
   ii. Symmetrical iii. Mode = 3; uni-modal
   iv. 3
   d. i. Lowest value = 0, highest value = 8
   ii. Minimal skew iii. Modes = 0, 4; bi-modal
   iv. 3
7. Probability distribution is

x	2	4	6	8
$P(X = x)$	0.25	0.25	0.25	0.25

$E(X) = 5$, $Var(X) = 5$, $SD(X) = 2.236$

8. Probability distribution is

t	7	8	9	10	11
$P(T = t)$	0.09	0.42	0.31	0.15	0.03

a. 8.61 minutes
b. 0.8979
c. 0.9476 minutes (4 sf)

Exercise B: Expected value applications (page 6)

1. a.

g	45	15	5	–5
$P(G = g)$	0.05	0.05	0.05	0.85

b. i. E(gain) = –\$1.00 (the person could expect, on average, to lose \$1.00 per game if a large number of games were played)
   ii. 1 000 games
c. i. Game is not fair as expected gain is not zero.
   ii.

g	$50 - k$	$20 - k$	$10 - k$	$-k$
$P(G = g)$	0.05	0.05	0.05	0.85

$E(G) = 0.05(50 - k) + 0.05(20 - k) + 0.05(10 - k) - 0.85k$
$= 4 - k$

Game fair if $E(G) = 0$ giving $k = 4$

Make the price \$4.00 per game.

2.

t	2	3	4	5	6	7	8	9	10
$P(T = t)$	0.04	0.08	0.12	0.16	0.2	0.16	0.12	0.08	0.04

Expected sum of numbers is 6

3. a. X = \$ winnings from a single game;

P(win) = P(double) = $\frac{6}{36} = \frac{1}{6}$

X	0	10
$P(X = x)$	$\frac{5}{6}$	$\frac{1}{6}$

b. $E(X) = 0 \times \frac{5}{6} + 10 \times \frac{1}{6} = 1\frac{2}{3}$ dollars

Over 9 games expected winnings $= 9 \times 1\frac{2}{3} = \$15$

c. Expected gain = 15 – 18 = –\$3 (loss of \$3)
d. Game not fair so change the entry price to \$$1\frac{2}{3}$ (hard to do!) or change the prize money to \$12

4. This is a fair game.

P(HT or TH) = P(TT or HH) = 0.5

E(gain) $= -0.50 \times 0.5 + 0.50 \times 0.5 = 0$

5. a. A loss of \$2.50 per game over a large number of games
   b. A profit of \$2.50 per game over a large number of games
   c. A loss of \$50
   d. \$2.50

Exercise C: Expectation algebra (page 8)

1. a. 10.5 b. 26.244 c. 5.123 (4 sf)
2. a. 8.5 b. 2.916 c. 1.708 (4 sf)
   d. 17.5 e. 26.244 f. 5.123 (4 sf)
   g. 1.708 (4 sf)
3. a. \$14.15 b. \$12.53 (2 dp) c. \$3.54
   d. \$17.69 (2 dp) e. \$19.58 (2 dp) f. \$4.42
   g. \$16.65 h. \$12.53 i. \$3.54
   j. \$17.48 k. \$18.04 (2 dp) l. \$4.25 (2 dp)

Exercise D: Combinations of random variables (page 11)

1. $E(T) = 80$ L, $Var(T) = 0.32$ L, $SD(T) = 0.57$ L
2. $E(T) = 60$ kg, $SD(T) = 1.342$ kg (4 sf)
3. $E(T) = 120$ m, $Var(T) = 0.003$ m, $SD(T) = 0.05477$ m (4 sf)

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4. E(T) = 100 kg, Var(T) = 0.1125 kg, SD(T) = 0.3354 kg (4 sf)
5. E(D) = 35 g, SD(D) = 0.3007 g (4 sf)
6. E(T) = 3 700 g, Var(T) = 113 g, SD(T) = 10.63 g (4 sf)
7. E(T) = 41 kg, SD(T) = 0.2417 kg (4 sf)
8. a. i. 219 ii. 161 iii. –58
   b. i. 564 ii. 524 iii. 296
9. Mean = \$238, standard deviation = \$14.66

Exercise E: Binomial distribution (page 15)

1. 0.215
2. a. 0.1272
   b. 0.0532
3. a. mean = 3, standard deviation = 1.625
   b. 0.2387
4. a. 0.0584 b. 0.0197 c. 0.9219
5. a. 0.0213 b. 0.6590 c. 0.9999
6. a. 0.1780 b. 0.8220 c. 0.4285
7. a. 0.3585 b. 0.2642 c. 90
8. There is no upper limit to the number of cellphones you can have, so there is not a limited number of trials.
9. a. $(1 - p)^{25} + 25(1 - p)^{24}p$ or $(1 - p)^{24}(1 + 24p)$
   b. i. 0.3576 ii. 0.2774
      iii. Probability that a note is defective doesn't change. If one note is defective it has no effect on whether other notes are defective (i.e. condition of each note is independent of previous notes).
10. a. 0.1211 b. 3 c. 1.4491
11. a. $P(X \geq 4) = 0.8060$
    Binomial distribution selected: has trials with two outcomes – either 'has Psa-V' or not, set number of trials = 15, assuming outcome at one orchard is independent of what happens at another orchard. This is unlikely to be correct since, if one orchard has the disease, it is more likely that neighbouring properties would also have it.
    b. 0.5797
12. 0.1536 × 0.288 = 0.044
13. 1 – P(9 ≤ number of males ≤ 11) = 0.5034
14. a. $n = 18, p = \frac{1}{3}$
    b. $P(X < 5) = 0.2311$
    c. $P(X \geq 4) = 0.8983$

Exercise F: Poisson distribution (page 19)

1. a. 0.0446 b. 0.1512 c. 0.0134
2. a. 0.8088 b. 0.0286 c. 0.2196
3. a. 0.8951 b. 0.9657 c. 0.4684
4. a. 0.6767 b. 0.8647 c. 0.1429
   d. \$70
5. a. 0.0911 b. 0.8215
6. a. 6 accidents/year b. 2.4495
7. a. 0.1680 b. 0.3528
   c. mean = 3, standard deviation = 1.7321
8. a. 0.0498 b. 0.5768 c. $(1 - 0.64723)^2 = 0.1244$
9. a. 0.7697 b. 0.1297 c. 0.3819
10. a. i. 0.0025 ii. 0.1512 iii. 0.3937
    b. mean = 6, standard deviation = 2.4495
    c. mean for week = 30 possums; $P(20 \leq X \leq 25) = 0.1865$
11. a. Poisson distribution, with mean $\lambda = 10$ (rugby patients seen by hospital per Saturday).
    Justification: Patients will be seen one at a time (so are not occurring simultaneously); patients arrive randomly, at a given rate; patients arrive independently of each other; no upper limit to the number of patients.
    b. $P(X > 11) = 0.3032$
    c. $P(X < 8$ on two consecutive Saturdays$) = (0.2202)^2 = 0.0485$
12. a. $P(X = 12) = 0.0481$
    b. $P(X = 10$ for four consecutive days$) = (0.09926)^4 = 0.0001$
    c. $P(X > 10) = 0.1841$, so using Binomial distribution with $n = 7$, $p = 0.1841$, $x = 4$, $P(X > 10$ for 4 out of 7 days$) = 0.0218$
13. a. $P(X = 2) = 0.2707$
    b. $P(X = 3$ on two consecutive sessions$) = (0.180447)^2 = 0.0326$
    c. $P(X = 4) = 0.0902$ so using the Binomial distribution with $n = 4$, $p = 0.0902$, $x = 3$, $P(X = 4$ for 3 out of 4 sessions$) = 0.0027$
14. a. $P(X = 0) = 0.0498^2$, $P(X = 0$ in two consecutive grids$) = 0.0498^2 = 0.0025$
    b. $P(X > 3) = 0.3528$, so $P(X > 3$ for 3 out of 5 of the grids$) = 0.1839$
    c. $P(X = 3 \mid X > 2) = \frac{0.22404}{0.80085} = 0.2798$
15. Standard deviation = $\sqrt{\frac{2}{3}}$ or 0.8165

Exercise G: Inverse Poisson problems (page 23)

1. 3.5 calls/hr 2. 1.14 dishes per hour
3. 1.2 workers per day 4. 0.08 purchases per hour
5. a. 1.61 patients b. 0.1355
6. 10.8 problems per hour
7. a. 1.14 spots per apricot b. 316 apricots
8. 5.27 m, i.e. 5 complete metres
9. $\lambda(1) = 2.525/10\ m^2$, $\lambda(2) = 0.7215$/area, so area = 2.86 m^2

Exercise H: Normal distribution (page 27)

1. a. 0.10564 b. 0.49379
2. a. 43.21% b. 39.29% c. 10.98%
3. a. 0.3445 b. 0.25474
   c. 0.69146 d. 0.9522
4. a. 4.3% b. 0.360 c. 6 or 7 players
5. a. 0.3571 b. 0.2895
6. a. 25.2% b. 3.3% c. 44%
   d. 9% e. 0.0011
7. a. 25.2% b. 1 305 c. 456
8. 0.17121 + 0.26339 = 0.4346, i.e. 43.5% (1 d.p.)
9. a. 0.0766 b. 161 c. 0.0174
   d. 0.0564
10. a. 0.65734 b. 0.10399
    c. 0.39184 d. 0.93123
11. a. 0.30853 b. 0.38116 c. 0.69966
12. a. 0.46679 b. 0.012547
13. a. i. 0.034518 ii. 0.96228 iii. 0.81834
    b. 0.08634 c. 0.00022579
14. 0.031959 15. 5.3% 16. 0.1661

Exercise I: Continuity corrections (page 35)

1. a. 0.094675 b. 0.51972
   c. Answer to part a. is slightly lower as the interval of values starts at a slightly lower point; answer to part b. is slightly higher as interval of required values is slightly wider.
2. a. i. 0.01861 ii. 0.99897 iii. 0.66153
   b. 0.95872
3. a. 0.058041 b. 0.93437

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