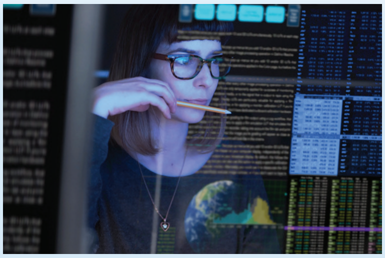
Linear programming

Linear programming is a method of **optimisation**, in which the maximum or minimum value of a quantity is found, given a set of **constraints** (restrictions expressed as **inequations**).

This decision-making process is used in many areas of life, e.g. maximising profit or minimising costs.



Drawing straight-line graphs

The equation of a linear graph will usually be given in one of two forms: gradient-intercept form or general form.

Gradient-intercept form $y = mx + c$

In equations of this form, m is the **gradient** ($\frac{\text{rise}}{\text{run}}$) and c is the **y-intercept** (the point where the line cuts the y-axis).

To draw the line, plot the y-intercept $(0,c)$ then use the gradient m to 'step out' to another point on the line.

Example

Draw the graph of $y = 2x + 4$

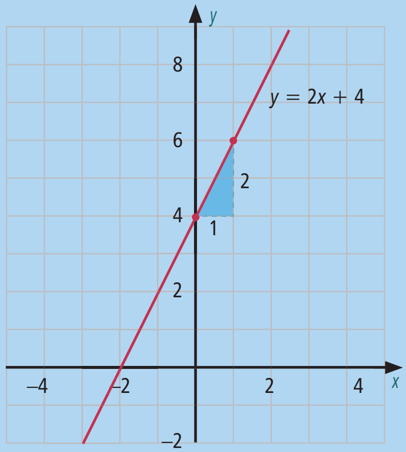
Solution

y-intercept $= 4$ so plot the point (0,4)

gradient $= 2$ so the $\frac{\text{rise}}{\text{run}} = \frac{2}{1}$

From the y-intercept move 1 step to the right then 2 steps up (to reach the point (1,6))

Join (0,4) and (1,6) with a straight line as shown.

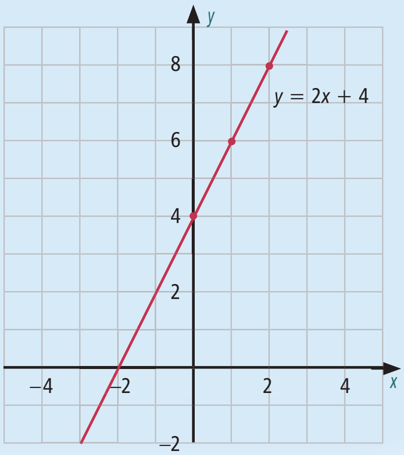


Alternatively, it may be more efficient to draw a line by calculating, plotting and joining points on the line.

For example, the table below shows three points on the line $y = 2x + 4$, which are calculated by substituting selected values for x into the equation.

x	$y = 2x + 4$	Point (x,y)
0	$2 \times 0 + 4 = 4$	(0,4)
1	$2 \times 1 + 4 = 6$	(1,6)
2	$2 \times 2 + 4 = 8$	(2,8)

These points are then plotted and joined to draw the line $y = 2x + 4$



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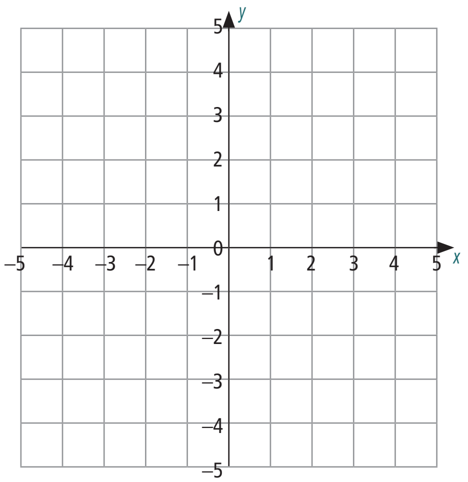
Answers p. 58

1. Graph the following lines on the grid below. Label each line with its letter.

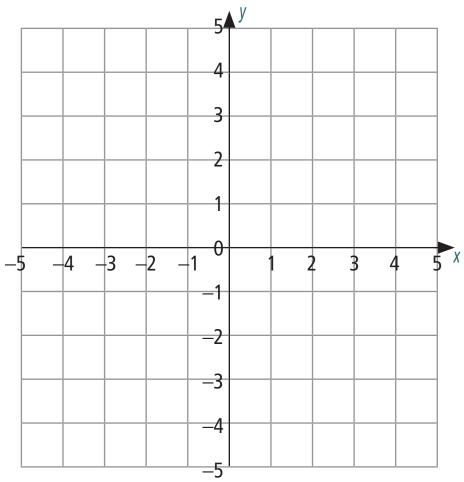
a. **i.** $y = 3x - 2$

ii. $y = 4 - 2x$



b. **i.** $y = \frac{1}{2}x + 4$

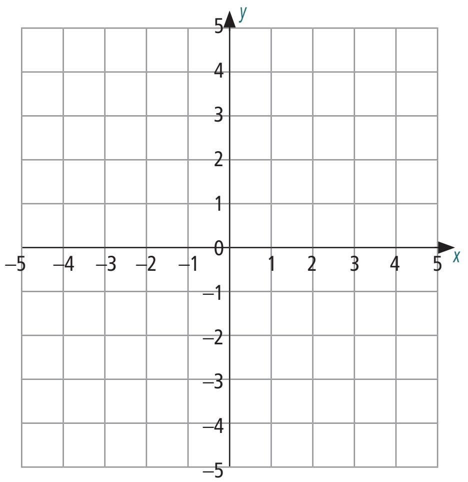
ii. $y = x$



2. Graph the following lines on the grid below.

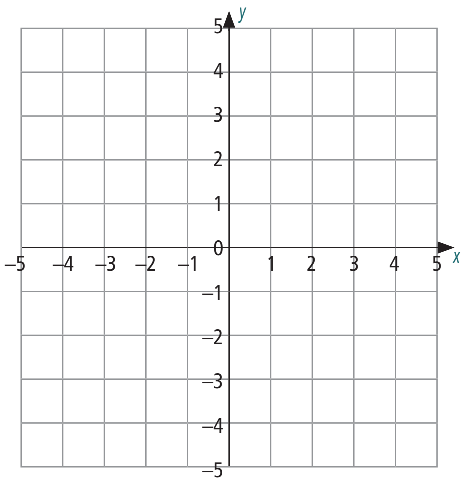
a. **i.** $y = -\frac{1}{4}x$

ii. $y = 3$



b. **i.** $x = -2$

ii. $y = 1 - \frac{2x}{3}$



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Example

Find the region that satisfies the inequations:

$3x + 2y \leq 12$ ...(1)

$y \geq 2$ ...(2)

$y \leq 2x + 4$ ...(3)

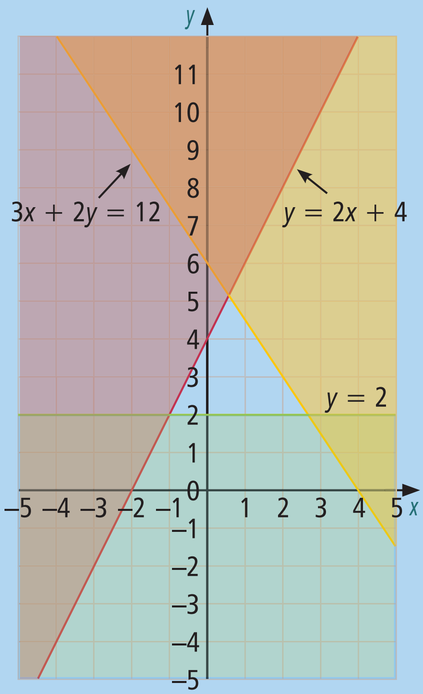
Equation (1)

Draw $3x + 2y = 12$ (plot and join (0,6) and (4,0))

Test (0,0) in the inequation: this gives $0 \leq 12$ which is true.

So the required region includes the point (0,0): this is the region lying below the line.

Shade out the region above the line (which does not include (0,0)).



Equation (2)

Draw $y = 2$ (horizontal line passing through 2 on the y-axis).

Test (0,0) in the inequation: this gives $0 \geq 2$ which is false (so shade (0,0) out).

So the required region does not include (0,0): this is the region lying above the line.

Shade out the region below the line.

Equation (3)

Draw $y = 2x + 4$ (y-intercept is (0,4), gradient is 2)

Test (0,0) in the inequation: this gives $0 \leq 4$ which is true.

So the required region includes the point (0,0): this is the region lying below the line.

Shade out the region above the line (which does not include (0,0)).

The feasible region is the central unshaded triangle along with its boundary lines.

This is the region of points that satisfy all three inequalities.

For example, the point (1,3) lies in the feasible region. Substituting $x = 1$ $y = 3$ into each inequation gives:

- $3 \times 1 + 2 \times 3 \leq 12$ which simplifies to $9 \leq 12$ (true) — substituting into $3x + 2y \leq 12$
- $3 \geq 2$ (true) — substituting into $y \geq 2$
- $3 \leq 2 \times 1 + 4$ which simplifies to $3 \leq 6$ (true) — substituting into $y \leq 2x + 4$

So, all inequations are satisfied for (1,3).

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Forming equations and inequations

When writing equations, first define the two **decision variables** which will be used to express the quantity to be optimised. Represent each decision variable with a different letter (often x and y).

Translate the information given in words into mathematical statements. Certain types of statement need to be considered carefully so that the resulting equation is not the 'wrong way around'. It may be useful to list a few related values to see if they satisfy your equation.

Example

In a problem involving the numbers of students and teachers in a school, the following decision variables are defined:

S = the number of students in the school

T = the number of teachers in the school.

If the information states: 'There are six times as many students as teachers at this school', it may be tempting to write $6S = T$

However this would be incorrect (in this statement the number of teachers is 6 times the number of students!).

Teachers ***T***	Students ***S***
1	6
2	12
3	18

Listing a few related values to check the actual relationship gives the table above.

From this it can easily be seen that $S = 6T$

In linear programming problems, you may be required to form inequations. Again, testing a list of acceptable values can be useful.

Example

1. 'Jack can't buy more than 4 boxes of shapes.'

If Jack can't buy more than 4 boxes, he cannot buy 5, 6, 7, ... boxes

If x is the number of boxes Jack can buy then $x = 4, 3, 2, 1, 0$

So the inequation is $x \leq 4$

2. 'The total weight of the apples and bananas is less than 20 kg.'

If a = the weight of the apples and b = the weight of the bananas, then the total weight of the apples and bananas is $a + b$. So the inequation is $a + b < 20$

When forming inequalities it is often useful to look for any totals given.

Example

A manufacturer makes chairs and tables.

Each chair takes 4 hours to manufacture, while each table takes 6 hours.

A chair costs $25 to make, while a table costs $60 to make.

A total of 50 hours and $1 000 is available for the manufacture of the items.

Define the variables as x = number of chairs, y = number of tables, and look for 'totals'.

The time required to make x chairs is $4x$ hours; the time required to make y tables is $6y$ hours.

Linking total hours with hours required per item gives: $4x + 6y \leq 50$

Similarly, linking total cost with cost per item gives: $25x + 60y \leq 1\,000$

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5. At the gym Fiona spends time on two of the exercise machines – the stationary bike and the rowing machine. Fiona spends no more than one hour at the gym. She spends at least 15 minutes, but no more than 25 minutes, on the stationary bike. For every 10 minutes she spends on the rowing machine, she spends 15 minutes on the stationary bike.

If x = minutes spent on the stationary bike and y = minutes spent on the rowing machine, write down which of the statements above produced each of the following equations or inequations.

a. $x + y \leq 60$

b. $x \geq 15$

c. $x \leq 25$

d. $3y = 2x$

6. Theresa's statistics teacher gave her the following problem and asked her to translate the sentences into equations.

A car yard sells three models of car: hatchbacks, saloons and four-wheel-drives. In order to best utilise the yard, the following requirements must be met:

- the total number of saloons and four-wheel-drives must not exceed the number of hatchbacks
- the number of saloon cars must not exceed 10% of the total number of cars
- there must be at least as many saloons as four-wheel-drives
- the number of four-wheel-drives must be less than 65% of the number of hatchbacks.

Theresa decided to let:

x = the number of hatchbacks purchased

y = number of saloons purchased

z = number of four-wheel-drives purchased.

She writes the following equations. Determine if the equations are correct and if not, rewrite correctly.

a. $x \leq y + z$

b. $9y \leq x + z$

c. $y \leq z$

d. $z < 0.65x$

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Optimisation

Linear programming is an **optimisation** technique. Optimisation involves finding maximum or minimum values for an expression (the **objective function**) while satisfying a set of inequations called the **constraints**.

The constraints define which values of the variable are permitted. The constraints are graphed to form the **feasible region** – the area where all constraints hold true simultaneously. This area is highlighted by **shading out** the areas that do not satisfy the constraints.

The **vertices** of the feasible region are the 'corners' where the boundary lines intersect (the coordinates of the vertices are found by inspection, or by solving the equations of the boundary lines simultaneously).

The coordinates of the vertices are substituted into the objective function to determine which point gives the maximum (or minimum) value of this function.

If a context involves only whole-number values of the variables, a **vertex** with fractional values is inappropriate. In this case, select and test points with whole-number coordinates which lie in the feasible region close to this vertex.

Do not assume you know which vertex will give the maximum/minimum – you must test at least two to three points (depending on the total number of vertices).

To find the maximum or minimum value for a function:

- identify the objective function (this will be what you are trying to maximise and is NOT plotted on the graph)
- identify the constraints, writing these as inequations
- graph the constraints, using *x*- and *y*-axes and shade out to find the feasible region
- find the coordinates of all the vertices of the feasible region (if possible, read the coordinates of the vertices from the graph; alternatively you may need to solve simultaneously the equations of the boundary lines)
- substitute the coordinates into the objective function to determine which point gives the maximum or minimum value.

Example

A luxury ceramics factory has two popular items: an urn and a planter. An urn can be made in 2 hours and painted in 1 hour while a planter can be made in 3 hours and painted in 3 hours.

There are up to 12 hours a day available for making urns and planters. There are up to 9 hours a day available for painting urns and planters.

The profit on each completed urn is $60 and the profit on each completed planter is $100.

What combination of urns and planters should be made and painted in order to achieve the maximum possible daily profit?

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13. A farmer wishes to plant two types of crop on 50 hectares of his farm: wheat and barley.

To plant a hectare of wheat requires 4 hours of labour, while each hectare of barley requires 2 hours of labour.

Wheat seed costs $2 per hectare and barley seed costs $5 per hectare.

The farmer can afford to pay for up to 160 hours of labour and a maximum of $220 for seed.

a. If x is the number of hectares of wheat and y is the number of hectares of barley, write inequations to represent the above constraints.

b. The farmer must pay his labourers $15 per hour. Write an expression for the total cost of labour and seeds.

c. The farmer can sell wheat for $100 per hectare cultivated, and barley for $120 per hectare cultivated.

i. Use this information to write an expression for total income.

ii. Hence find an expression for the farmer's profit.

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The objective function

An **objective function** of the form $P = ax + by$ can be graphed as a line. Drawing this line for various values of P results in a set of parallel lines, some of which lie across the feasible region.

In simple cases, the optimal value of P will be reached at a corner of the feasible region.

Example

Consider the following inequations and objective function, P. The value of the objective function is to be maximised.

$x \geq 2$ $y \geq 1$ $x + y \leq 10$ $P = 2x + 3y$

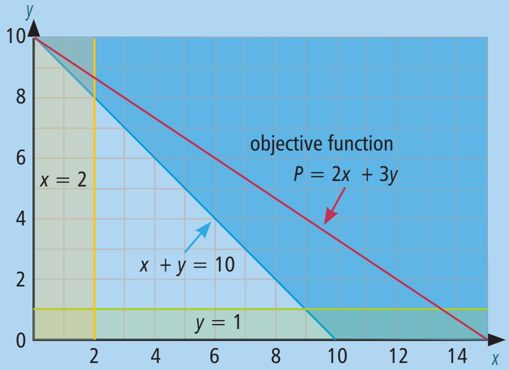
The constraints are drawn on a graph as shown.

The unshaded region is the feasible region.

The objective function $P = 2x + 3y$ is also shown for one value of P, namely $P = 30$.

As the value of P reduces, the objective function 'moves' down (parallel to this line) towards the feasible region. The line will first cross the feasible region at the vertex (2,8), so this point gives the single optimal solution:

$P = 2 \times 2 + 3 \times 8 = 28$



Note: As the line 'moves' further down into the feasible region, the value of P reduces, so substituting the coordinates of other points in the feasible region will result in values of P which are not optimal.

Linear progamming with multiple solutions

Generally when solving a linear programming problem, a single solution is found. However, **multiple solutions** result when the objective function has the same gradient as the boundary line of a constraint.

Example

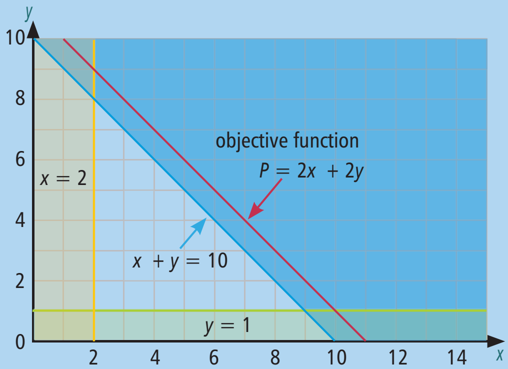
Consider the previous problem, with a different objective function.

$x \geq 2$ $y \geq 1$ $x + y \leq 10$ $P = 2x + 2y$

The graph is as shown.

In this case, as the objective function 'moves' down towards the feasible region, the line meets the region along one boundary line of the unshaded area. This means that there are multiple solutions: all points along this boundary line will give the optimal solution.

This situation arises because the objective function $P = 2x + 2y$ is parallel to the boundary line of the constraint $x + y \leq 10$.



The coefficients of x and y are in the same proportion, resulting in multiple solutions.

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5. A craftsman in Nepal creates pashminas (large scarves) made either of 100% cashmere or of a mixture of cashmere and silk. The raw materials cost the craftsman 50 cents per 100% cashmere pashmina, but only 30 cents per mixed-material pashmina. The craftsman has only $10 per week to spend on materials. Each 100% cashmere pashmina takes 2 hours to make, while a mixed-material pashmina takes 1.5 hours to make. The craftsman spends no more than 45 hours per week making pashminas.

a. The craftsman sells his pashminas in the local market. He can sell each 100% cashmere pashmina for $15 and each mixed-material pashmina for $10. What mixture of 100% cashmere and mixed-material pashminas should he produce to maximise his income?

b. As the number of tourists in Nepal has declined recently, the craftsman has had to reduce his prices. The price for the 100% cashmere pashminas is reduced to $10 and the price for the mixed-material pashminas is reduced to $7.50. What is the effect of this price decrease on the craftsman's income and how many of each pashmina should he now make?

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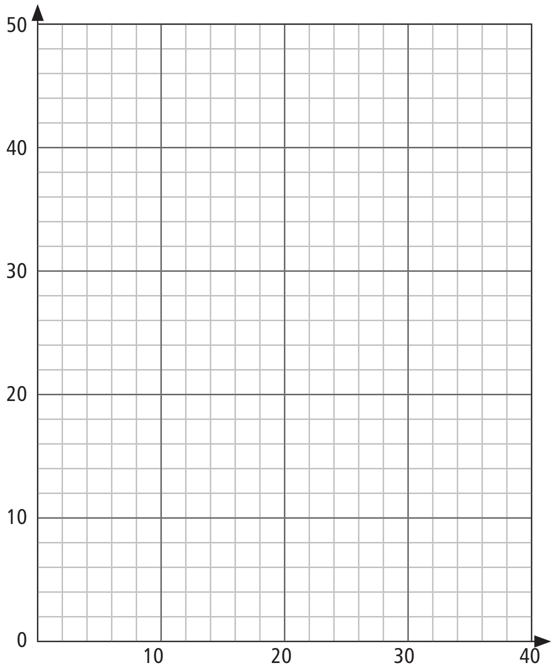
12. *Beds4U* design and manufacture beds. Its best-selling items are the king set (mattress and base) and the single set. Each week *Beds4U* produces x king sets and y single sets and sells all it produces, with a maximum number of 24 king sets able to be sold. The company sells at least $\frac{1}{3}$ more king sets than single sets.

Beds4U has 90 m² available to store the beds before sale. Each king set requires 3 m² of space and each single set requires 2 m² of space.

The profit on each king set is $500 and the profit on each single set is $300. This profit is reduced by manufacturing costs of $3 000.

a. Determine the number of each type of bed set *Beds4U* should manufacture in order to maximise profits.



b. *Beds4U* is going to increase its storage capacity so that it is no longer limited by this aspect. What is the minimum area it will need so that storage is no longer a restriction on its profit-making ability?

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Coordinates of vertices	Harvesting days = $3x + 2y$
(0,60)	120
(10,40)	110
(37.5,12.5)	137.5
(100,0)	300

Minimum amount of labour required is 110 days which occurs when there are 10 hectares of apricots and 40 hectares of peaches.

b. The intersection occurs at (37.5,12.5). In order for the objective function to touch this point first the function must be steeper than the line $x + 5y = 100$ (or $y = -\frac{1}{5}x + 20$) and less steep than $x + y = 50$ (or $y = -x + 50$)

Objective function is: $H = ax + 5y$ or $y = -\frac{a}{5}x + \frac{H}{5}$

Require that gradient of objective function satisfies:

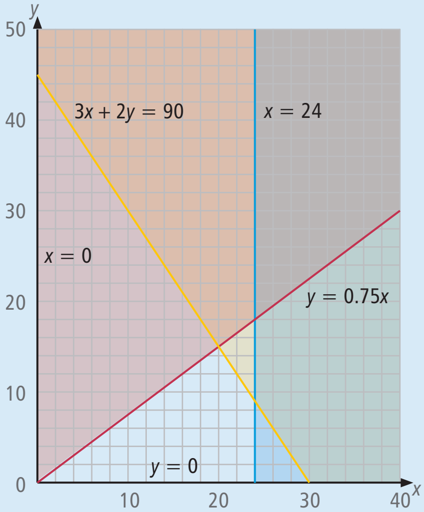
$-\frac{a}{5} < -\frac{1}{5}$ i.e. $a > 1$ and $-\frac{a}{5} > -1$ i.e. $a < 5$

Hence $1 < a < 5$

12. a. If x = number of king sets and y = number of single sets then the constraints are:

$3x + 2y \le 90$; $y \le 0.75x$; $x \le 24$; $x \ge 0$; $y \ge 0$

Objective function is: Profit = $500x + 300y - 3\,000$



Coordinates of vertices	Profit = $500x + 300y - 3\,000$
(0,0)	–\$3 000
(20,15)	\$11 500
(24,9)	\$11 700
(24,0)	\$9 000

Maximum profit of \$11 700 occurs when 24 king sets and 9 single sets are sold.

b. In order for the storage constraint to be removed, the storage line would need to pass through (or beyond) the point of intersection of the lines $x = 24$ and $y = 0.75x$ which is (24,18).

The old storage line was $3x + 2y = 90$ or $y = -\frac{3}{2}x + \frac{90}{2}$

New storage line would be $y = -\frac{3}{2}x + \frac{A}{2}$ where A is the area available.

rearranging $3x + 2y = A$

If point (24,18) is on this line, then $18 = -\frac{3}{2} \times 24 + \frac{A}{2}$

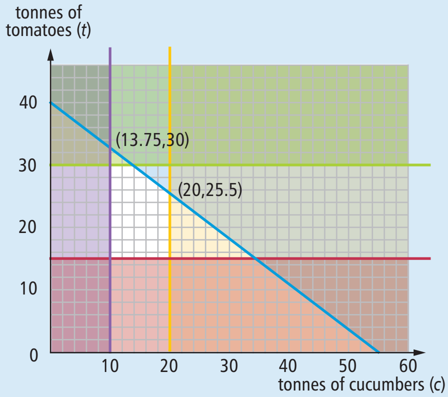
$18 = -36 + \frac{A}{2}$

$\frac{A}{2} = 54$

$A = 108$ m^2 is the area required to remove the storage constraint.

13 a. Let c represent the tonnage of cucumbers produced and t the tonnage of tomatoes.

The constraints are: $10 \le c \le 20$, $15 \le t \le 30$, $8c + 11t \le 440$



The combinations of crop tonnage which possibly give maximum income will arise at one of the selected vertices of the feasible region.

income from (13.75,30) clearly greater than from (10,30), etc.

The table below gives the incomes for these combinations.

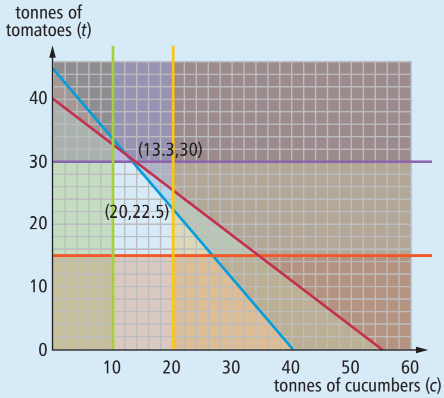
Vertex (c,t)	Income $1\,250c + 3\,000t$
(13.75,30)	\$107 187.50
$(20,25\frac{5}{11})$	\$101 363.64

exact value of $25\frac{5}{11}$ used

Thus the combination which will lead to the greatest income is 13.75 tonnes of cucumbers and 30 tonnes of tomatoes.

b. Using c and t as defined previously there is a new constraint which is $9c + 8t \le 360$

The new feasible region is shown below.



The table following gives the incomes for viable combinations at the vertices of the feasible region.

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