Statistics 3.13

Externally assessed
4 credits

Apply probability concepts in solving problems

AS 91585

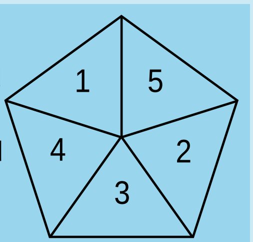
Probability concepts

A **probability experiment** is a process whose results depend on chance. The following terms are commonly used:

- **trial** – one 'run through' of an experiment
- **outcome** – the result of a trial
- **sample space** (**S**) – the set of all possible outcomes
- **event** (**E**) – a subset of the sample space

Example

In an experiment a 5-sided spinner (with face marked 1–5 as shown) is spun and the number recorded.



Spinning the spinner once is a trial.

Getting a 2 is an outcome.

Sample space of all possible outcomes is S = {1, 2, 3, 4, 5}

Events can be labelled with letters and defined in words or as a set, for example:

A = number is less than 4	A = {1, 2, 3}
B = number is divisible by 2	B = {2, 4}

Probability

The probability, p, of an event is a measure of how likely the event is to occur. A probability always lies between 0 and 1 inclusive ($0 \le p \le 1$).

- An **impossible event** has probability zero ($p = 0$)
- A **certain event** has probability one ($p = 1$)

If the probability that an event occurs is p then the probability that the event does not occur is $1 - p$.

True, theoretical and experimental probability

The **true probability** of an event is unique to a situation and is unknown. For example, when a coin is flipped, the actual chance of getting a head from that coin is unknown.

Theoretical probability is a model estimate of the true probability. When a 'fair' coin is flipped, the model treats the chance of getting a Head as equal to the chance of getting a Tail, so P(H) = P(T) = 0.5

Experimental probability is calculated using the data from an experiment or a **simulation**. For example, you may wish to know the probability of a drawing pin landing point up when it is tossed, so an experiment is conducted in which the drawing pin is tossed a large number of times and the results recorded.

If the drawing pin lands point up 67 times out of 150 throws then the **long-run relative frequency** is calculated as $\frac{67}{150}$ which is an estimate of the true probability that the drawing pin lands point up when thrown.

Example

When tossing a fair two-sided die, define the two events:

A = getting a 5 and B = not getting a 5

The die is fair, so a model for this situation is that each number on the die is equally likely. This gives:

$P(A) = \frac{1}{6}$ and $P(B) = \frac{5}{6}$

P(A) + P(B) = 1 so P(B) = 1 – P(A)

Note: B is the event 'not A' so A and B are called **complementary events**.

The **complement of A** is written **A'** and means 'not A' (A did not occur)

$P(A') = 1 - P(A)$

Random variables

A **random variable** is a variable whose values result from a probability experiment (these values depend on chance and may not be known until the experiment has been completed).

- A random variable can be described in words and is denoted by a capital letter, such as X.
- The actual values taken on by the random variable are denoted by lowercase letters, such as x.
- The probability that the random variable X takes on a value x is written $P(X = x)$ or $p(x)$
- The sum of the probabilities of all values a random variable can take is 1 so $\Sigma P(X = x) = 1$ (or $\Sigma p(x) = 1$)

Example

In an experiment, two 6-sided dice, each labelled 1–6, are thrown together.

There are 36 possible outcomes, as shown in the table where 1,2 means 1 on the first die and 2 on the second die, etc.

	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

A random variable X is defined for this experiment:

X = the sum of the numbers facing upwards

For example if the outcome is 3,2 then $X = 3 + 2 = 5$

The values that X can take (all totals possible) are:

$x = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

Probabilities can be worked out using the table:

$P(X = 2) = \frac{1}{36}$ [only 1,1 gives a sum of 2]

$P(X = 7) = \frac{6}{36}$ [outcomes 1,6; 2,5; 3,4; 4,3; 5,2; 6,1 have sum of 7]

$= \frac{1}{6}$ [simplifying]

Probability distribution tables

A **probability distribution table** displays all the values that a random variable can take, along with their associated probabilities. The table has two important properties:

- the values of the probabilities are never negative, nor greater than 1
- the sum of all the values of the probabilities is 1.

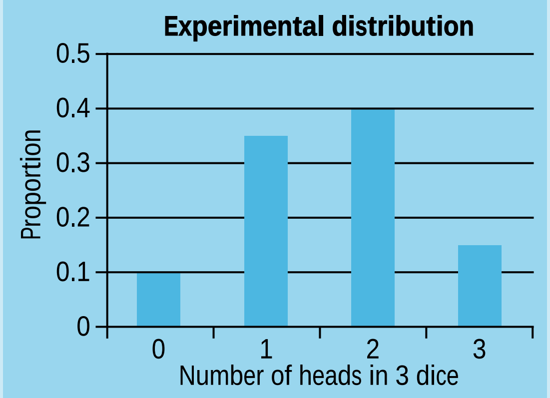
A **bar graph** of the probability distribution may also be drawn.

Example

Let X = the number of heads obtained when a fair coin is tossed 3 times.

When each coin is tossed, the result is a head (H) or a tail (T).

A probability experiment (or simulation) could be run to investigate the likelihood of the numbers of heads that could result from three throws of a coin. For example, after 100 throws of three coins the following bar graph is drawn to display outcomes and their probabilities.



The **experimental distribution** had 100 trials, and because of natural variability, a different experiment of 100 trials would result in a different experimental distribution. However, in an experiment with this many trials, there will be less variability than there would be with a smaller number of trials, so the experimental distribution should be a reasonable approximation to the actual probability distribution.

Alternatively, a theoretical model could be applied to the situation.

When 3 coins are tossed, there are 8 possible outcomes, each with a theoretical probability of $\frac{1}{8}$.

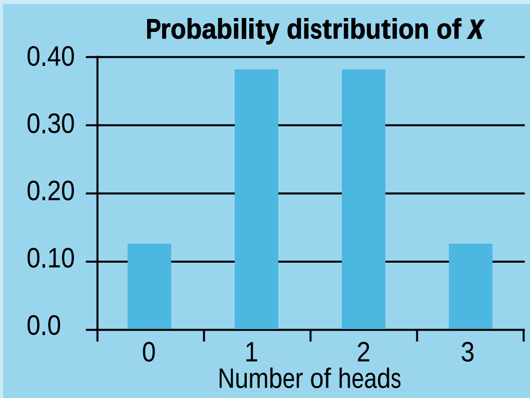
Outcome	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
Number of heads	3	2	2	1	2	1	1	0
$P(X = x)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

So X can take on the values $x = 0, 1, 2$ or 3.

The probability distribution table for X is shown below.

x	1	2	3	0
$P(X = x)$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

The graph of the probability function is shown in the next column.



Note: $P(X = 0) = \frac{1}{8}$ means that the probability of 0 heads when a fair coin is tossed 3 times is $\frac{1}{8}$

$P(X = 1) = \frac{3}{8}$ means that the probability of 1 head is $\frac{3}{8}$, and so on.

Notice that all the probabilities add up to 1:

$$\Sigma P(X = x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$$

Comparing the experimental distribution with the theoretical distribution there are many similarities in shape, with similar low probabilities for 0 and 3 heads, and similar higher probabilities for 1 and 2 heads in both graphs. For example, the experimental probability of no heads is 0.1 compared with 0.125 in the theoretical model, and the experimental probability of two heads is 0.4 compared with 0.375 in the theoretical model.

Exercise A: Probability and random variables

Answers p. 57

1. A coin (head or tail) and a six-sided die (with sides labelled 1–6) are tossed together.
   a. List the sample space.
   b. What is the probability of getting a head and an odd number?
   c. What is the probability of getting a head or an odd number?
   d. If a tail is tossed, what is the probability of getting an even number?
   e. If a number less than 5 is tossed, what is the probability of getting a tail?

2. A spinner with 5 equal sectors labelled 1–5 is spun twice.
   a. List the sample space.
   b. The outcomes for each pair of spins are added together. What is the probability that the total is:
      i. less than 5?
      ii. an even number?
      iii. at least 7?
   c. What is the probability that one number is double the other number?

3. Two six-sided dice, with sides labelled 1–6, are tossed together.
   a. List the sample space.
   b. What is the probability of getting a double (both numbers the same)?
   c. What is the probability of getting a 3 and a 4?
   d. What is the probability of getting a 3 or a 4?

9. It was found that of a group of 85 students in Ashburton:

- 29 were at least 10 years old
- 66 lived in a two-parent family
- 12 did not live with both parents and were under 10 years old

a. Calculate the probability that a randomly selected person from this group lives in a two-parent family and is under 10 years old.

It was also found that of this group:

- 22 live in a rented house
- Everyone is either at least 10 years old, or lives in a 2-parent family, or lives in a rented house
- 58 live in a 2-parent family in a house that is not rented
- 1 is at least 10 years old, lives in a rented house and with 2 parents.

b. Calculate the probability that a randomly selected person from this group doesn't live in a rented house and is at least 10 years old.

10. In a sample of 25 shoppers all bought at least one of the products A, B and C.

- 13 bought product A
- 8 bought product B
- 9 bought product C

No shopper bought all three products, but 3 shoppers bought both product A and product B and 2 shoppers bought both product A and product C. No shoppers bought both product B and product C.

What is the probability that a randomly selected shopper from this sample bought exactly one product out of A, B and C?

Probability of unions of events

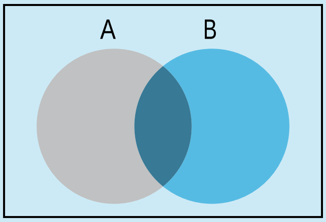
The **union** of two events (A OR B) is written $A \cup B$ (note that OR means A or B or *both*).

Note: $A \cup B$ can also be described as the event that at least one of events A or B occurs.

The probability of the union of A and B can be found using a formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Note: When the probability of A is added to the probability of B, the probability of $A \cap B$ is added in twice, so $P(A \cap B)$ needs to be subtracted once from $P(A) + P(B)$ to get $P(A \cup B)$.



The double shaded intersection area shows that the sum $P(A) + P(B)$ adds this area twice.

Example

The probability a man visiting a supermarket buys bread is 0.5, the probability he buys both milk and bread is 0.13 and the probability he buys milk or bread is 0.66. What is the probability he buys milk?

Solution

Let B = 'man buys bread' and
M = 'man buys milk'

From the given information,
$P(B) = 0.5$,
$P(M \cap B) = 0.13$
and $P(M \cup B) = 0.66$

Using the formula

$0.66 = P(M) + 0.5 - 0.13$

$[P(M \cup B) = P(M) + P(B) - P(M \cap B)]$

$P(M) = 0.66 + 0.13 - 0.5$ [rearranging]

$= 0.29$

The probability he buys milk is 0.29

Exercise C: Probabilities of unions of events

Answers p. 58

1. The probability a boy visiting a dairy buys a drink is 0.63, the probability he buys both a drink and a chocolate bar is 0.37 and the probability he buys a drink or a chocolate bar is 0.66. What is the probability he buys a chocolate bar?

2. In the town of Newton, the probability a household buys a newspaper is 0.57, the probability a household buys a magazine is 0.69 and the probability a household buys a newspaper and a magazine is 0.51. What is the probability a household in Newton buys a newspaper or a magazine?

4. A survey was taken of 992 students at a city co-ed school. Each student was asked whether they usually walk to school from home. The table gives the results of the survey.

	Walk to school	Does not walk to school	Totals
Boys	320	192	512
Girls	300	180	480
Totals	620	372	992

Are the events ‘a student walks to school’ and ‘a student is female’ independent? Justify your answer fully.

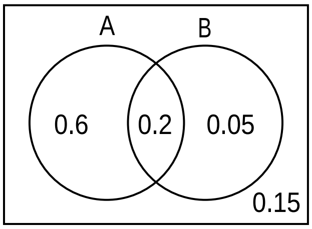
5. A and B are events with
$P(A) = 0.3$ and
$P(B) = 0.15$
$P(A \cap B) = 0.045$
Are A and B independent events?

6. E and F are independent events. $P(E) = 0.4$ and $P(F) = 0.2$. Find:

a. $P(E \cap F)$

b. $P(E \cup F)$

7. Events A and B have probabilities as shown in the Venn diagram.



Are events A and B independent?

8. Events C and D are independent. The following probabilities are known:
$P(C \cap D') = 0.5$ and $P(C \cap D) = 0.3$
Find $P(C' \cap D)$
You may wish to draw a Venn Diagram in the space below.

Conditional probability

Conditional probabilities are worked out using a reduced sample space. There will be a restriction to take into consideration, such as the knowledge that another event has already taken place, before finding the probability.

Often the words 'if' or 'given that' are used.

Example

If the throw of a fair 6-sided die is even, what is the probability the number thrown is divisible by 3?

Solution

When throwing a die, $S = \{1, 2, 3, 4, 5, 6\}$

The throw is even, so the reduced sample space is $B = \{2, 4, 6\}$

Let A be the event 'getting a number divisible by 3'

So the probability of getting a number divisible by 3 if the number is even, is

$P(A \text{ given } B) = \frac{1}{3}$ [only 6 is divisible by 3]

The probability of A given B is written $P(A|B)$ and has the following formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example

Using this formula in the example above:

$P(A \cap B) = P(\text{getting a number that is divisible by 3 and even}) = \frac{1}{6}$

$P(B) = P(\text{getting an even number}) = \frac{3}{6}$

So the probability of getting a number divisible by 3 if the number is even is

$$P(A|B) = \frac{\frac{1}{6}}{\frac{3}{6}} \qquad \left[P(A|B) = \frac{P(A \cap B)}{P(B)}\right]$$

$$= \frac{1}{3} \text{ as before.}$$

Rearranging the formula for conditional probability gives the general formula for the probability of 'A and B'

$$P(A \cap B) = P(A) \times P(B|A)$$

Note that if A and B are independent then $P(B|A) = P(B)$ (the occurrence of A has no effect on the likelihood of B occurring) and the formula becomes $P(A \cap B) = P(A) \times P(B)$, which is the condition for independence.

It is important to pay close attention to the wording of a problem; it is easy to confuse 'A and B' with 'A given B'.

Example

The probability that a married man watches the news on television is 0.4 and the probability that a married woman watches it is 0.5. The probability that a married man watches the news if his wife does is 0.7.

Find the probability that

1. a husband and wife both watch the news
2. a wife watches the news if her husband does
3. at least one person of a married couple watches the news
4. a man watches the news if his wife does not.

Solution

1. Let M = 'married man watches the news'

   W = 'married woman watches the news'

   From the given information:

   $P(M) = 0.4$,

   $P(W) = 0.5$,

   $P(M|W) = 0.7$

   Require $P(M \cap W)$

Practice Assessment Task

Answers p. 61

1. A study of 1 500 employees found:
   - 960 were over 40 years old
   - 386 were unhappy in their job
   - 74 were unhappy in their job and not over 40 years old

   a. What proportion of the employees were happy in their job and were over 40 years old?

   b. Two different employees from the study were randomly selected and both were found to be unhappy in their job. Calculate the probability that both these employees are also over 40 years old.

   c. Consider the events 'an employee is unhappy in their job' and 'an employee is over 40 years old'. Explain whether these events are independent.

   d. Further investigation in this study found that:
   - 230 of the employees were middle managers
   - 13 of the employees were over 40 years old, were middle managers and were unhappy in their job
   - 16 of the employees were not over 40 years old, were middle managers and were unhappy in their job
   - 94 of the employees were over 40 years old, were middle managers and were happy with their jobs.

   An employee is selected randomly from the study population.

   Calculate the probability that this employee is happy in their job, is not a middle manager and is not over 40 years old.

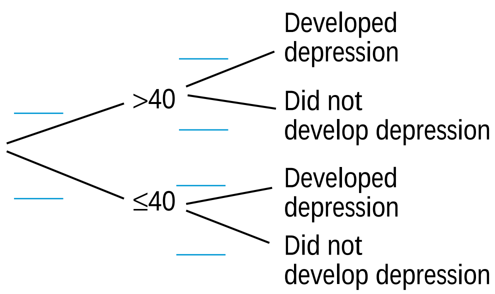
2. Clinical depression is a mental disorder characterised by a pervasive and persistent low mood that is accompanied by low self-esteem and by a loss of interest or pleasure in normally enjoyable activities.

a. The table shows the proportions of employees who are unhappy in their jobs who developed clinical depression. The data is from an observational study of 600 employees.

	Over 40 years old	Not over 40 years old
Developed depression	0.26	0.175
Did not develop depression	0.375	0.19

i. Determine the number of employees in this study who developed depression or were over 40 years old.

ii. Use the information provided to calculate the necessary probabilities to complete the probability tree shown below, rounding the probabilities to 3 decimal places.



iii. Compare the risk of developing depression for an employee who is not over 40 years old and unhappy in their job with the risk of developing depression for an employee who is over 40 years old and unhappy in their job.

b. Medical records of overweight married couples show that 23.2% of overweight husbands are depressed. In instances where an overweight husband is depressed, there is a 76.4% chance their overweight wife is also depressed. For 33.5% of overweight married couples, neither person is depressed.

A married couple who are both overweight is selected at random from medical records. If the wife is found to be not depressed, determine the probability that the husband is depressed.

Answers

Exercise A: Probability and random variables (page 4)

1. a. S = {(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)}

b. $\frac{3}{12}$ or $\frac{1}{4}$ **c.** $\frac{9}{12}$ or $\frac{3}{4}$ **d.** $\frac{3}{6}$ or $\frac{1}{2}$ **e.** $\frac{4}{8}$ or $\frac{1}{2}$

2. a. S = {(1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), (2,3), (2,4), (2,5), (3,1), (3,2), (3,3), (3,4), (3,5), (4,1), (4,2), (4,3), (4,4), (4,5), (5,1), (5,2), (5,3), (5,4), (5,5)}

b. i. $\frac{6}{25}$ **ii.** $\frac{13}{25}$ **iii.** $\frac{10}{25}$ or $\frac{2}{5}$

c. $\frac{4}{25}$

3. a. S = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

b. $\frac{6}{36}$ or $\frac{1}{6}$ **c.** $\frac{2}{36}$ or $\frac{1}{18}$ **d.** $\frac{20}{36}$ or $\frac{5}{9}$

e.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

i. $\frac{5}{36}$ **ii.** $\frac{15}{36}$ or $\frac{5}{12}$ **iii.** $\frac{30}{36}$ or $\frac{5}{6}$

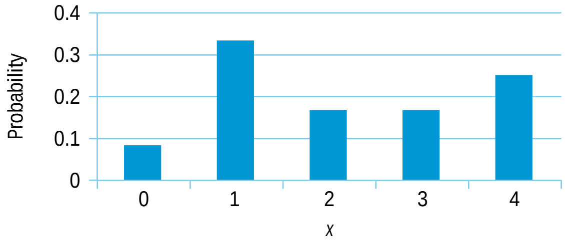
iv. $\frac{26}{36}$ or $\frac{13}{18}$ **v.** $\frac{15}{36}$ or $\frac{5}{12}$

f. $\frac{5}{11}$ **g.** $\frac{5}{18}$

4. a. 0.24 **b.** 0.91 **c.** 0.47 **d.** 0.72

5. a. $k = \frac{1}{4}$

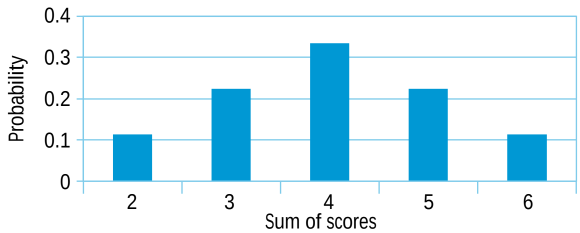
b. **Probability distribution of *X***



6. a.

x	2	3	4	5	6
$P(X = x)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{1}{9}$

b.



c. This is a theoretical probability, as the true probabilities for this spinner are unknown and the probabilities were worked out by areas taken up by each number on the spinner. (Experimental probabilities would differ each time, depending on the number of spins.)

7.

b	0	1	2
$P(B = b)$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{3}{8}$

8.

W	0	1	2	3	4
$P(W = w)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$

9.

x	1	2	3
$P(X = x)$	$\frac{3}{10}$	$\frac{3}{5}$	$\frac{1}{10}$

10. a. i. Equally likely since each number has a quarter chance of occurring, and to get a sum of zero would require four zeros, which has a probability of $\left(\frac{1}{4}\right)^4 = \frac{1}{256}$ or 0.004 (3 dp). Similarly to get a sum of 12 would require four 3s which has the same likelihood.

ii. No, since the proportion of times a sum of zero occurs is 0.02 and the proportion of times a sum of 12 occurs is zero. This is due to the variability of samples.

b. i. Proportion of times = 0.02 + 0.02 = 0.04

ii. Sara has a reasonable likelihood of being close to the true probability, but due to experimental variability, there will be a difference between her estimate and the true probability (especially since the proportion of times that the sum was 12 was zero in the experiment but is non-zero in reality). (Using a model, there are 15 equally likely ways a score of 10 or more can be achieved, so the theoretical probability is $15 \times \frac{1}{256} = 0.06$ (2 dp))

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