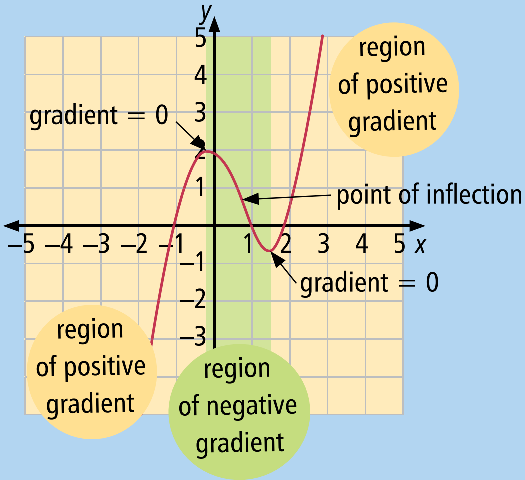
Example

The graph alongside is of the polynomial $y = x^3 - 2x^2 - x + 2$

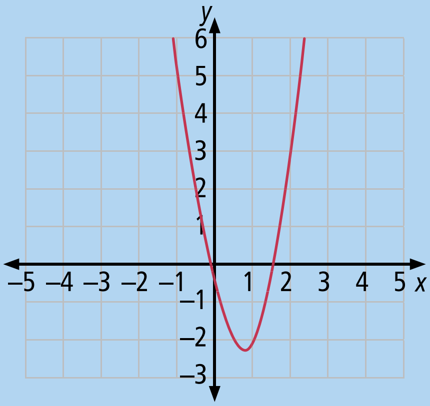
The decreasing section of the graph is shown on a green shaded background.



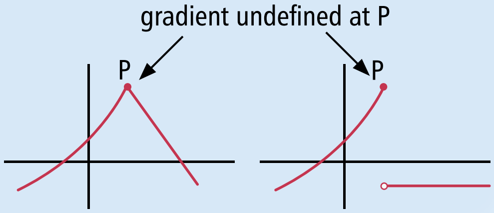
Using this graph the following information can be found:

Region	Effect on gradient
x is less than about –0.2	Positive gradient
x is equal to about –0.2	Gradient equals zero
x is between about –0.2 and 1.5	Negative gradient
x is equal to about 1.5	Gradient equals zero
x is greater than about 1.5	Positive gradient
x is equal to about 0.7	Gradient has a local minimum value

It is now possible to sketch the graph of the gradient function for the given polynomial.



Note: The gradient of a function is undefined if its graph has a break or a sharp change of steepness (making the drawing of a unique tangent impossible). The graph of the gradient function will be undefined at these points (there will be a break in the graph of the gradient).



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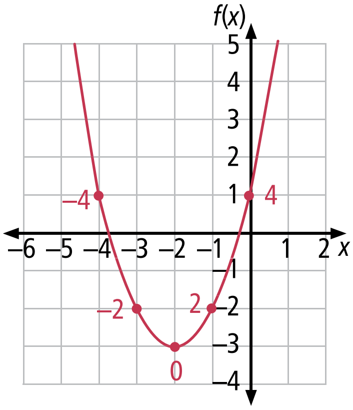
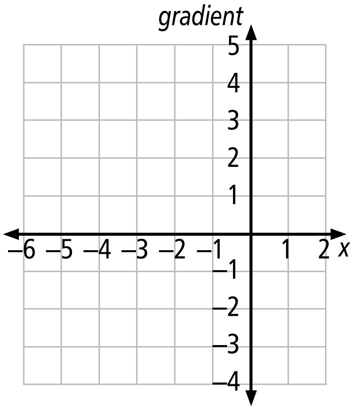
1. The graph of a function $f(x)$ is drawn. Draw the graph of its gradient function on the axes alongside. **Hint:** The red numbers show the gradient of the curve at selected points.

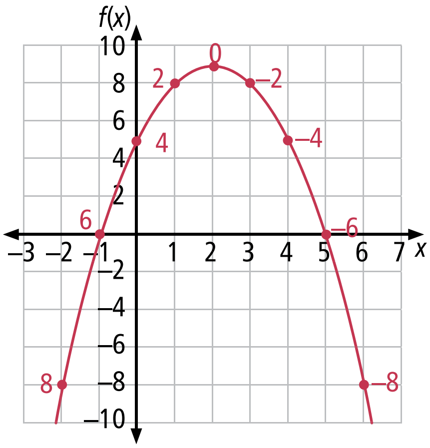
Answers p. 77

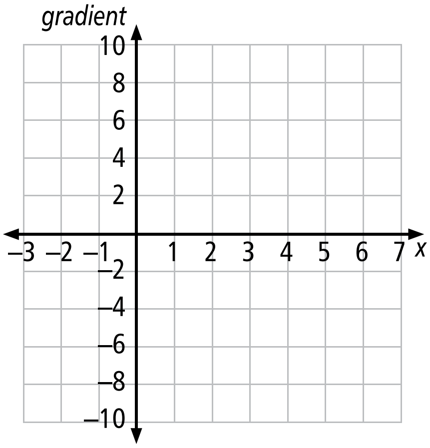
a.



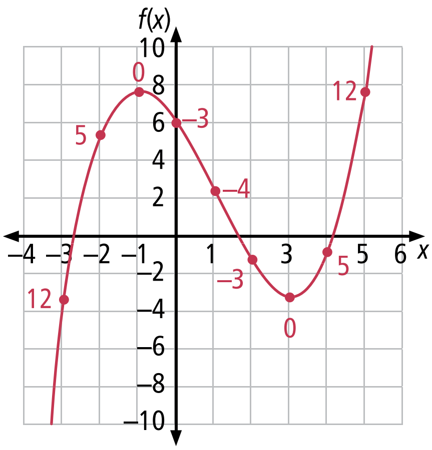


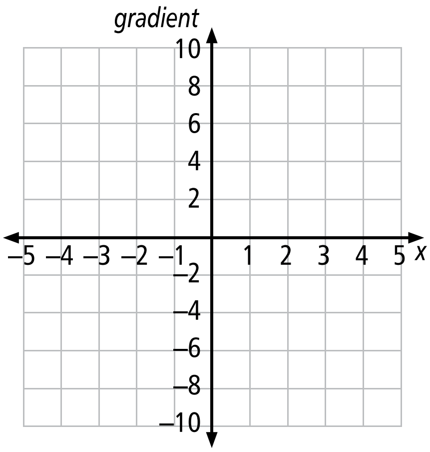
b.





c.





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Nature of the turning points

By considering the sign of the gradient in the neighbourhood of the turning points, it can be determined whether a turning point is a local maximum or a local minimum.

Example

For the curve $y = x^3 + 2x^2 - 4x + 5$, the gradient function is $\frac{dy}{dx} = 3x^2 + 4x - 4$ and the turning points are at $x = \frac{2}{3}$ and $x = -2$ (see previous example).

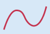
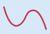
The **sign table** shows the sign of the gradient for points either side of these values.

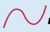
x	-3	-2	-1	0	$\frac{2}{3}$	1
$\frac{dy}{dx}$	+	0	−	−	0	+

for example, when $x = -3$, $\frac{dy}{dx} = 3(-3)^2 + 4(-3) - 4 = 11$, a positive value (+)

- In the neighbourhood of $x = -2$, the gradient changes from positive (increasing) to negative (decreasing) therefore the turning point at $x = -2$ is a local maximum.
- In the neighbourhood of $x = \frac{2}{3}$, the gradient changes from negative (decreasing) to positive (increasing) therefore the turning point at $x = \frac{2}{3}$ is a local minimum.

Thus (–2,13) is a local maximum and $\left(\frac{2}{3}, 3\frac{14}{27}\right)$ is a local minimum.

Note: For known curves, such as the cubic, the shape of the curve  or  can be used to determine the nature of the turning points.

In the above example, the shape of the curve is , so the local maximum occurs to the left of the local minimum. It follows that the local maximum is at $x = -2$ and the local minimum is at $x = \frac{2}{3}$ (since –2 is left of $\frac{2}{3}$).

Identifying the nature of turning points using the second derivative

It can be seen from the example above that at the local maximum turning point ($x = -2$) the gradient is decreasing (in the neighbourhood of –2, the gradient is changing from a positive value to a negative value). This means that at a local maximum turning point, the gradient of the gradient (the **second derivative**) is negative.

For example, in the worked example above, the gradient function is $3x^2 + 4x - 4$. Differentiating the gradient function gives the rule for the gradient of the gradient (the second derivative) as $\frac{d^2y}{dx^2} = 6x + 4$.

Substituting $x = -2$ into $6x + 4$ gives –-8 (a negative result so the gradient is decreasing and the turning point is a local maximum).

Similarly, at a local minimum, the gradient is increasing so the gradient of the gradient (the second derivative) is positive.

For example, substituting $x = \frac{2}{3}$ into $6x + 4$ gives 8 (positive so the turning point is a local minimum).

In summary:

At a maximum turning point the second derivative is negative $\left(\frac{d^2y}{dx^2} < 0\right)$

At a minimum turning point, the second derivative is positive $\left(\frac{d^2y}{dx^2} > 0\right)$

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Answers p. 79

1. Find the x-coordinates of the turning points of the following curves.

a. $y = x^2 - 3x + 1$

b. $y = x^2 + 8x - 4$

c. $y = -\frac{1}{2}x^2 - 6x + 2$

d. $y = -x^2 + 4x$

e. $y = x^3 + 6x^2 + 2$

f. $y = x^3 - 10x^2$

g. $y = x^2 - \frac{x^3}{3} + 15x - 4$

h. $y = -3x^3 + \frac{3x^2}{2} + 12x - 7$

2. By considering the shape of the curve, identify the nature of the turning points in question 1.

a. $y = x^2 - 3x + 1$

b. $y = x^2 + 8x - 4$

c. $y = -\frac{1}{2}x^2 - 6x + 2$

d. $y = -x^2 + 4x$

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5. Heather claims that the resale value of a car, t years after it is sold, can be modelled by the function $R = 120t^2 - 2\,320t + 45\,000$, where R is the resale value in dollars.

a. At what rate is the value of the car dropping after 5 years?

b. How many years after it was sold was the car's value dropping by \$400 per year?

6. The amount of coal being mined is recorded over a 60-day period. The following equation models the amount of coal mined (in tonnes) t days after recording began $A(t) = 1.325 + 0.084t - 0.001t^2$

Describe the rate of change of the amount of coal mined with respect to t, the number of days since recording began.

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Kinematics

Kinematics is the study of motion, which is a particular type of rate of change.

The **displacement**, s, of an object is its position relative to a point of reference (this can be positive or negative, like a point on a number line).

- Displacement is positive if right of the origin.
- Displacement is negative if left of the origin.

For vertical motion, heights above 'ground level' are positive, and heights below are negative.

The **velocity** (speed), v, of an object (the rate at which the displacement is changing with respect to time) is found by differentiation.

$$v = \frac{ds}{dt}$$

Positive velocity means movement right (increasing s) or up; negative velocity means movement left (decreasing s) or down; zero velocity means s is neither increasing nor decreasing (stationary value).

The **acceleration**, a, of the object is the rate at which the velocity is changing as time passes.

$$a = \frac{dv}{dt}$$

Positive acceleration means velocity is increasing; negative acceleration (**deceleration**) means velocity is decreasing; zero acceleration means velocity is constant.

Example

The displacement of an object from a reference point at time t is given by the formula $s = 3t^2 - 13t + 12$, where t is in seconds and s is in centimetres. Find each of the following:

1. The initial velocity of the object.
2. The acceleration of the object after 2 seconds.
3. The displacement of the object when the velocity is 5 cm per second.

Solution

1. $v = 6t - 13$ — differentiating ($v = \frac{ds}{dt}$)

   When $t = 0$, $v = 6 \times 0 - 13 = -13$ — substituting $t = 0$ in $v = 6t - 13$

   Initially the velocity is -13 cm s^{-1} (travelling at 13 cm/s to the left).

2. $a = \frac{dv}{dt} = 6$

   The velocity is increasing by 6 cm/s each second. As this is a constant, it is true for all values of t. So the acceleration after 2 seconds is 6 cm s^{-2}.

3. $6t - 13 = 5$ — setting $v = 5$

   $6t = 18$

   $t = 3$

   When $t = 3$,

   $s = 3(3)^2 - 13(3) + 12$

   $= 0$

   When the velocity is 5 cm/s, the object is at the reference point.

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Calculus problem solving

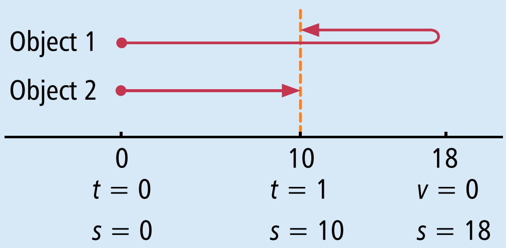
In order to solve calculus problems, you will need to select and use differentiation and/or antidifferentiation techniques that are appropriate to the problem, possibly both within the one problem.

Displacement versus distance travelled

There is a difference between the **displacement** of a moving object (i.e. its position after a certain period of time) and the actual **distance** travelled by the object (if the sign of the velocity changes over the time period then the object will have travelled forward and backward to reach its final position).

In the diagram below, both object 1 and object 2 are at position 0 units on the scale initially ($s = 0$). After 1 second of motion (as shown by the arrows), both particles are at position 10 on the scale ($s = 10$).

In the first second of motion, object 1 moved with a positive velocity to position 18 on the scale (where its velocity changed from positive to negative) then returned to position 10 on the scale, covering a total distance of $18 + 8 = 26$ units.



In the first second of motion, the second object travelled directly to position 10 so its distance covered in that time was 10 units.

Example

Q. A particle travels at a velocity of $\pi - 0.4t$ centimetres per second, where t is the number of seconds since timing began. How far does the particle travel in the first 10 seconds of movement?

A. Displacement is found by antidifferentiating velocity:

$s = \frac{-0.4t^2}{2} + \pi t + c$ antidifferentiating

$s = -0.2t^2 + \pi t + c$

Distance travelled is measured from the initial position of the particle, so when $t = 0$, $s = 0$.

So $c = 0$ and $s = -0.2t^2 + \pi t$

The velocity of the particle is zero when

$0 = -0.4t + \pi$ substituting $v = 0$ in $v = -0.4t + \pi$

$t = \frac{\pi}{0.4}$ or 7.854 rearranging

- From $t = 0$ to $t = 7.854$ seconds, the velocity of the particle is positive.
  After $t = 7.854$ seconds, the displacement of the particle from its initial position is
  $s = -0.2 \times 7.854^2 + \pi \times 7.854$
  $s = 12.337$ cm
- From $t = 7.854$ to $t = 10$ seconds, the velocity of the particle is negative.
  After 10 seconds the displacement of the particle from its initial position is
  $s = -0.2 \times 10^2 + \pi \times 10$
  $s = 11.416$ cm
  So between $t = 7.854$ and $t = 10$ seconds, the particle travelled $12.337 - 11.416 = 0.921$ cm.

Total distance covered by the particle in the first 10 seconds is

$12.337 + 0.921 = 13.26$ cm (2 d.p.)

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Answers p. 82

1. The graph of $f(x) = -kx^2 + x - 5$ has a turning point at $x = -2$. Find the gradient of the function at the point where $x = 2$.

2. The gradient of a curve is given by $f'(x) = ax + b$. The curve has a turning point at (1,3) and cuts the y-axis at 4. Use calculus to find the equation of the curve.

3. A vat holds 100 L of water when full. A full vat of water starts leaking water at the rate of $0.15t$ litres per hour. After how many hours since it first started leaking will the vat be half full?

4. The depth (d metres) of a diver below the surface of the water t seconds after he enters the water is given by $d(t) = 1.4t^2 - kt$ The diver reaches his maximum depth 1.2 seconds after entering the water. What was this maximum depth?

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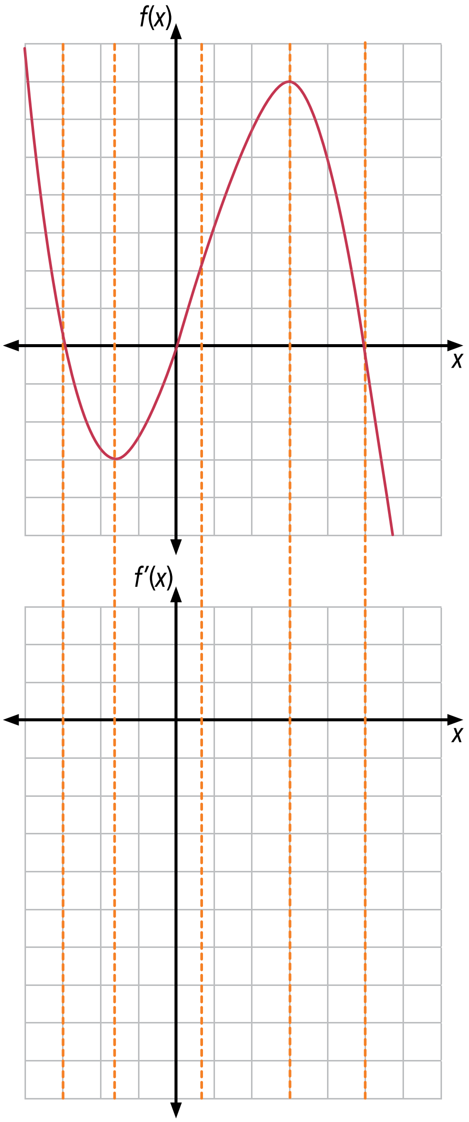
Practice assessment task

Answers p. 82

1. a. The diagram below shows the graph of a function $y = f(x)$. On the axes below, sketch the graph of the gradient function $y = f'(x)$.



b. The point P(–2,5) lies on the curve $g(x) = 2x^3 + 5x^2 - x - 1$.

i. Find the equation of the tangent to the curve at P.

ii. The curve has the same gradient at the point Q as it does at the point P. Find the coordinates of Q.

c. A function has rule $f(x) = x^3 - 4x^2 + 5x - 1$. Find the gradient of $f(x)$ at the point where $x = 2$.

 ISBN 978-1-988586-61-8

Answers

The gradient of a curve (page 3)

1. a. i. Positive ii. Positive
iii. Zero iv. Negative
b. i. Zero ii. Negative
iii. Zero iv. Positive
c. i. Positive ii. Zero
iii. Negative iv. Positive
2. a. $-2 < x < 1$ b. $x < -2$ and $x > 1$
c. $x = -2$ and $x = 1$
3. a. (1,3) b. (–2,0)
4. a. At (1,3) gradient = –2; at (–1,3) gradient = 2
b. i. $x < 0$ ii. $x > 0$

The gradient function (page 5)

1. a. $y = x^2 + 1$

x	–3	–2	–1	0	1	2	3
gradient	–6	–4	–2	0	2	4	6

gradient = $2x$

b. $y = -x^2$

x	–3	–2	–1	0	1	2	3
gradient	6	4	2	0	–2	–4	–6

gradient = $-2x$

c. $y = (x - 1)^2 = x^2 - 2x + 1$

x	–2	–1	0	1	2	3	4
gradient	–6	–4	–2	0	2	4	6

gradient = $2x - 2$

d. $y = (x + 2)^2 = x^2 + 4x + 4$

x	–4	–3	–2	–1	0	1
gradient	–4	–2	0	2	4	6

gradient = $2x + 4$

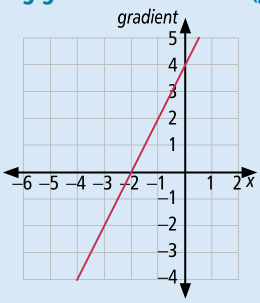
2. a. gradient = $x + 1$ b. 7
c. (–8,27) d. $\left(-\frac{1}{2}, 2\frac{5}{8}\right)$
e. i. gradient = 0 (horizontal line) ii. No
3. a. gradient = $2x + 3$ b. –3
c. (4,32) d. (–5,14)
e. $x = -1.5$ f. $y = 1.75$
4. a.

x	–3	–2	–1	0	1	2	3
gradient	9	4	1	0	1	4	9

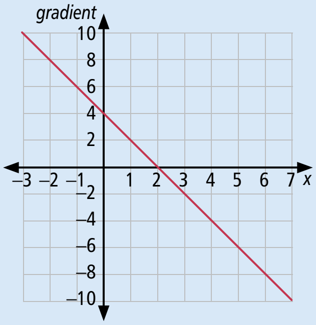
b. gradient = x^2

Sketching gradient functions (page 9)

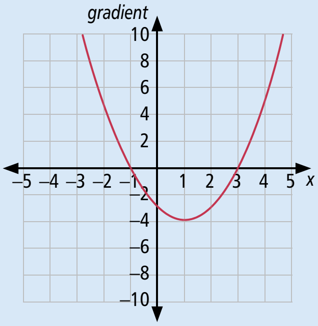
1. a.



b.

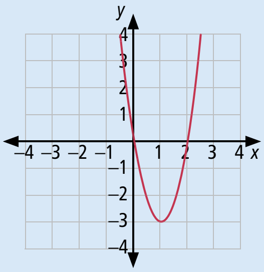


c.

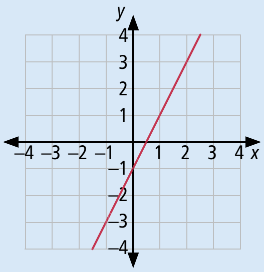


2. Ad, Ba, Cc, Db
3. *Note: Graphs are approximate.*

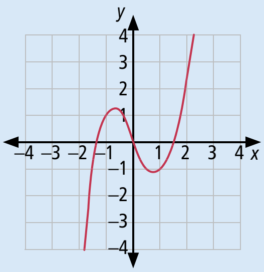
a.



b.



c.



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Index

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