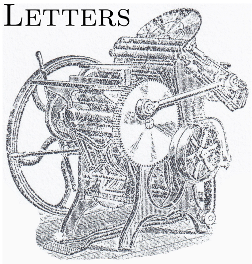
Math
Letters



Sum of the First n Integers

Letter 3

Many who have had an opportunity of knowing any more about mathematics confuse it with arithmetic, and consider it an arid science. In reality, however, it is a science which requires a great amount of imagination.
-Sofia Kovalevskaya

Quick, try to solve the following sums in your head:

$$1 + 2 + 3 = ?$$
$$1 + 2 + 3 + 4 + 5 = ?$$
$$1 + 2 + 3 + 4 + 5 + 6 + 7 = ?$$

How long did it take to compute these?

What about this one?

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 = ?$$

This last sum is certainly more time-consuming. Even with a calculator, it would take several seconds to punch in all the numbers. The answer is 210.

What if you were asked to sum up the integers 1 through 100? As the numbers grow, the time it takes to compute the sum increases. Summing the numbers 1 through 1000 would take at least 10 times as long. Remarkably, there is a famous formula that collapses this problem into a single computation. It allows these sums to be computed quickly, regardless of how many terms there are.

Our goal is to find the sum of the first n integers.

The symbol n is being used as a placeholder for any positive integer. For instance, choosing $n = 5$ gives the sum of the first 5 integers $1 + 2 + 3 + 4 + 5$, and choosing $n = 20$ gives the sum of the first 20 integers (which we wrote out above).

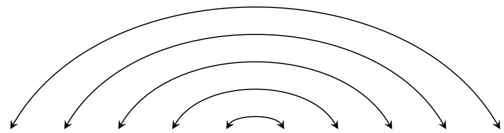
To begin, let's find the sum with $n = 5$ using a methodology that may seem strange at first sight. We'll write the problem out *twice* and add both copies together, like so:

$$(1 + 2 + 3 + 4 + 5) + (1 + 2 + 3 + 4 + 5)$$

Summing these numbers will of course yield twice the answer we seek.

Note that the terms can be freely rearranged without changing the final answer. That is, the order of the numbers is inconsequential—the sum $1+2$ is the same as $2+1$. The trick is to rearrange the sum above in a special way. Starting with the outermost two integers, we'll work our way inward forming pairs along the way.

The strategy is illustrated below:



$$(1+2+3+4+5)+(1+2+3+4+5)$$

Pairing the numbers in this way yields the following rearranged sum:

$$(1+5)+(2+4)+(3+3)+(4+2)+(5+1)$$

The final sum is precisely the same as before; it still yields twice the answer to $1+2+3+4+5$.

Is there anything significant about the terms in parentheses? Each pair of integers sums to exactly 6. How many copies of 6 are there? There is one for each pair, and there is a pair for each of the integers 1 through 5—so there are 5 copies of 6. Summing each pair above gives:

$$6+6+6+6+6$$

An alternative way to view these pairs is to write the problem out twice as we did before, but arrange the terms as two sequences of numbers one above the other. One sequence will increment from 1 to 5, and the other will decrement from 5 to 1. Now the pairs can be visualized like this:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \\ \hline 6 & 6 & 6 & 6 & 6 \end{array}$$

When viewed from this angle it makes sense that all pairs add to the same number; each time the top row increases by 1, the bottom row decreases by 1. This offset is why the sum of each pair remains constant.

As a second step toward our goal, recall that multiplication is a convenient way to sum several copies of the same number together. Summing 4 copies of the number 3 can be represented as $3+3+3+3 = 4 \cdot 3$. Another example is $5+5 = 2 \cdot 5$. The multiplier, then, is the number of times that the integer is repeated in the sum.

Applying this concept to the sum of 6s above gives:

$$6+6+6+6+6 = 5 \cdot 6 = 30$$

At this point, with $n = 5$, the answer to *twice* the problem is $5 \cdot 6 = 30$. To correct for this we need to *divide* by 2, producing a final answer of:

$$\frac{5 \cdot 6}{2} = \frac{30}{2} = 15$$

The solution can be confirmed by manually summing $1 + 2 + 3 + 4 + 5 = 15$.

This may seem like a lot of unnecessary work to arrive at an answer; however, we've discovered a very useful pattern that can potentially be used for *any* value of n. Let's try another choice of n to verify.

Using the same trick as before, if $n = 6$ the pairs now add to 7.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ \hline 7 & 7 & 7 & 7 & 7 & 7 \end{array}$$

There are 6 copies of the number 7, and summing them gives:

$$7 + 7 + 7 + 7 + 7 + 7 = 6 \cdot 7 = 42$$

This is twice the answer to the original problem, so we need to divide by 2.

$$\frac{6 \cdot 7}{2} = \frac{42}{2} = 21$$

Once again, the solution can be confirmed by manually summing $1 + 2 + 3 + 4 + 5 + 6 = 21$.

Now for the leap of faith—will this pattern work for *any* n? With this question, we arrive at a core tenet of mathematics, which is the discovery of a pattern that holds true for any input. To convince ourselves that the pattern does indeed hold true for any n, let's imagine writing out two rows of numbers just as we've done for $n = 5$ and $n = 6$. The first row will count up from 1 to n and the second row will count down from n to 1. In every case each column will add up to $n + 1$. With $n = 5$ the columns added to (substituting 5 in place of n) $5 + 1 = 6$, and with $n = 6$ the columns added to $6 + 1 = 7$. For any n, each column will *always* add to $n + 1$.

Now imagine summing all resulting copies of the number $n + 1$. There is one copy for each of the n columns. Adding them all together gives $n \cdot (n + 1)$. Remember that with $n = 5$ each column summed to $5 + 1 = 6$. There were 5 columns, and adding all the 6s together gave $5 \cdot 6$. This is precisely what the formula $n \cdot (n + 1)$ gives when n is replaced with 5. Using yet another example from above, when $n = 6$ each column summed to $6 + 1 = 7$. There were 6 columns, so summing all the 7s gave $6 \cdot 7$. Replacing n with 6 in the formula $n \cdot (n + 1)$ gives the same answer. This formula works for any integer n.

The last step, like before, is to correct for the fact that we arrive at twice the answer to the original problem when all the numbers are summed. In other words, the formula $n \cdot (n + 1)$ for the sum of all columns is double the solution we seek; therefore, we must divide $n \cdot (n + 1)$ by 2.

The famous solution for the sum of the first n integers can finally be assembled:

$$1 + 2 + 3 + ... + n = \frac{n \cdot (n + 1)}{2}$$

In mathematics, the three dots between the 3 and the n above signify the continuation of a pattern. In this case, they stand in for "continue adding together all the integers up to n".

This formula is so useful because it works for *any* integer n, as can be verified with a few examples. Earlier, we asked for the sum of the first 20 integers. Plugging $n = 20$ into the formula gives:

$$1 + 2 + 3 + ... + 20 = \frac{20 \cdot (20 + 1)}{2} = \frac{20 \cdot 21}{2} = \frac{420}{2} = 210$$