Math
Letters

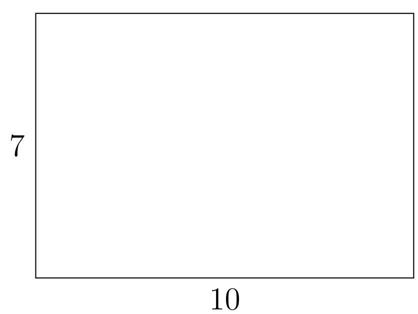
Random Sampling & Buffon's Needle

Letter 4

The moving power of mathematical invention is not reasoning but imagination.
-Augustus de Morgan

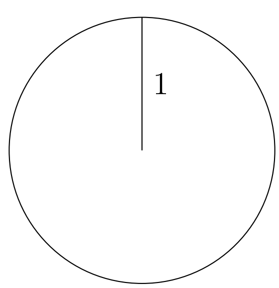
Randomness is normally viewed as an undesirable aspect of our daily lives. It's hard to plan ahead when unpredictable events like traffic jams, illness, or flight delays disrupt our plans. Scientists have a similar disdain for randomness. To make accurate inferences, experiments need to be repeatable—with as few variables left to chance as possible. However, there are some situations where randomness can be harnessed to produce both useful and unexpected results.

To begin, we'll focus on a simple practical problem—computing the area of a shape. Let's review a situation where the formula is easy to work out. If a rectangle measures 10 by 7, then multiplying length times width gives a total area of $10 \cdot 7 = 70$. This formula is straightforward and intuitive.



Rectangle with Area $10 \cdot 7 = 70$

If we move away from straight lines, the task of finding the area is a bit more complicated. Consider a circle with radius 1.

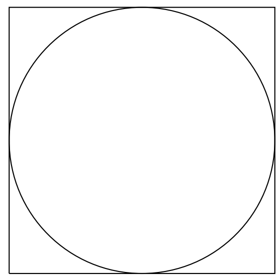


Circle with Radius 1

Unlike rectangles, computing the area of a curved shape isn't so obvious, and usually requires the use of a memorized formula. In this case, the formula for the area of a circle is $\pi \cdot r^2$ where r is the radius. So the area of the circle above is simply $\pi \cdot 1^2 = \pi = 3.14159$ (rounded to five decimal places).

Now suppose we didn't know this formula. How could the area of the circle be determined?

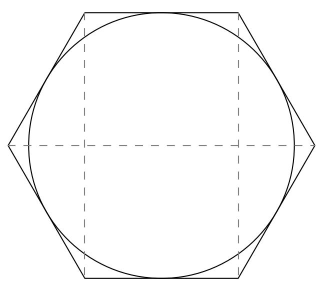
One approach is to approximate it with something more manageable, like a square:



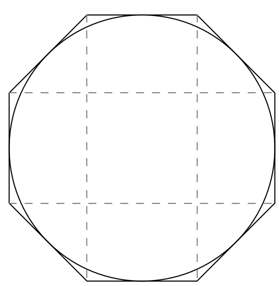
Circle Enclosed by Square

The square has side lengths equal to the diameter of the circle. Recall that the diameter is just twice the radius of 1, so the diameter is 2 and therefore the square has sides of length 2. We already know how to compute the area of a square—it is $2 \cdot 2 = 4$. This isn't a very good approximation, but since the circle resides within the square, it must have an area less than 4.

To obtain a better approximation we can use polygons with more sides, like a hexagon (six sides) or an octagon (eight sides):



Circle Enclosed in Hexagon



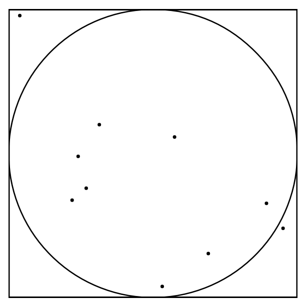
Circle Enclosed in Octagon

The polygons can be subdivided into shapes made of straight lines, like the rectangles and right triangles delineated by the dashed lines above. We know how to compute the area of these straightlined shapes, and by computing the area of each one we can work out a better approximation for the area of the circle. As polygons with more sides are used the approximation becomes more precise. Unfortunately, this is tedious work, and the work increases with the number of sides.

There exists an entirely different approach, which despite its unintuitive nature, uses randomness to estimate a fixed quantity.

Let's use the same idea of a circle enclosed in a square. Now imagine randomly tossing a small coin onto the surface of the square. When the coin lands, pick it up and mark where it fell with a small dot. This will be done repeatedly, and in the end we'll count up how many times the coin landed within the circle.

Suppose the coin is tossed onto the square 10 times. For each toss, the landing place is marked with a small dot as shown below:



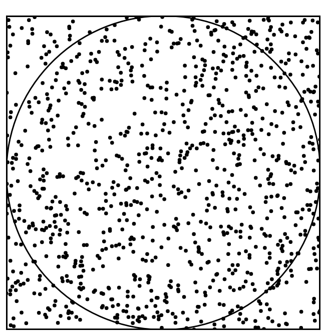
10 Random Points

In this particular example, the coin landed within the circle 8 times. Now take this number, divide by 10 (the total number of tosses), and multiply by 4. Dividing by the number of tosses and multiplying by 4 will be explained later.

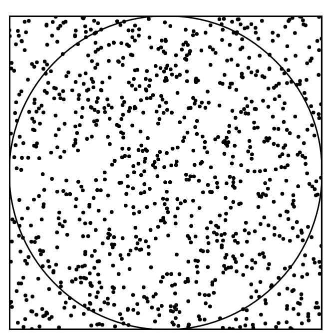
$$\left(\frac{8}{10}\right) \cdot 4 = 3.2$$

The resulting number 3.2 isn't too far from the true value of π. What is rather surprising is that this number will move closer to π the more times the coin is tossed.

If the number of tosses is increased to 1000, the result is a decent approximation to π. It's important to remember that this is a *random* process, and tossing the coin another 1000 times would yield a slightly different number. The images below show two trials of 1000 random tosses each:



1000 Random Points
Trial 1



1000 Random Points
Trial 2