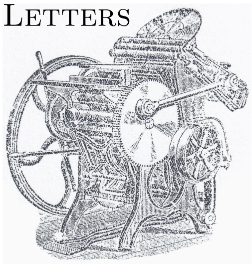
Math
Letters



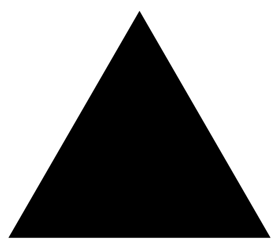
Fractals

Letter 7

Those who have learned to walk on the threshold of the unknown worlds, by means of what are commonly termed par excellence the exact sciences, may then, with the fair white wings of imagination, hope to soar further into the unexplored amidst which we live.
-Ada Lovelace

Fractals are one of the most extraordinary creations of modern mathematics. While properties of the infinitely complex shapes have been studied for at least a century, only with the advent of computers have we been able to visualize them. The mathematics of fractals matured in the 1970s with the work of the brilliant mathematician Benoit Mandelbrot. His influential publications brought these strange objects to the forefront of mainstream mathematics. To understand where fractals come from it is best to start with an example.

Consider an equilateral triangle.

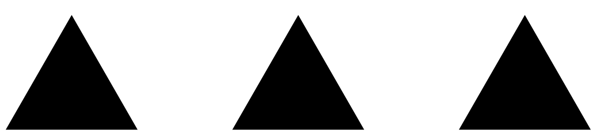


Equilateral Triangle

Three simple geometric operations will be applied to it:

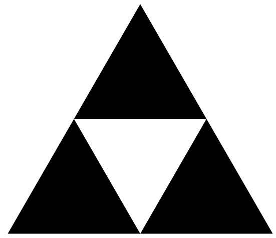
1. Shrink the sides to half their original length.
2. Make 3 copies.
3. Stack the 3 copies to make one larger triangle the same size as the original.

The first two steps are straightforward and lead to 3 identical triangles—each one with half the side length of the original.



Three Half Size Copies

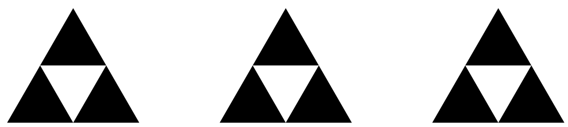
The final step is to stack them into the shape of the original triangle:



Three Half Size Copies Stacked

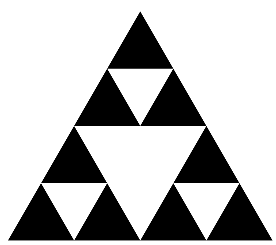
This composite triangle is exactly the same size and shape as the one we started with.

Now using *this* triangle, the process can be repeated. The first two steps yield three identical copies:



Three Half Size Copies

The last step (stacking) creates the new triangle shown below:



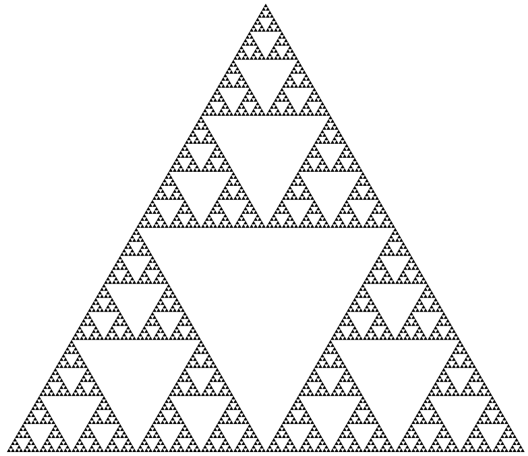
Three Half Size Copies Stacked

Once again, the new triangle is the same size and shape as the original.

This process can be continued indefinitely; shrink, copy, stack, and the steps can be applied to the resulting triangle once again.

What kind of object emerges if these steps are repeated many times? The first two iterations show that the shading of the composite triangle certainly changes. After the first application an upside-down white triangle appears in the center, and after the second, three smaller upside-down triangles surround the larger one in the center. Will the black shading eventually disappear? Or could the triangle end up completely shaded in black once again?

What actually happens is rather remarkable. The next image approximates the resulting triangle after repeating these operations many times, and is enlarged to show more detail. This famous fractal is known as the *Sierpiński triangle* after the Polish mathematician Wacław Sierpiński who described it in 1915.



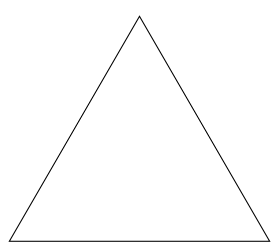
Sierpiński Triangle

Notice the beautiful symmetry and *self-similarity* of the smaller sub-triangles—each one resembles the overall image. Focusing on any one of the 3 main sub-triangles again reveals even smaller triangles resembling the whole. Continued magnification will show that every composite triangle is a mini version of the overall image.

Applying the shrink, copy, stack steps yet again will produce virtually the same image. In a sense, the triangle has *converged* to an object that is impervious to the three steps. Recalling our study of chaos and feedback loops in a previous letter, this is the geometric equivalent of a feedback loop converging to a single point. This makes sense if the steps are viewed as a kind of geometric feedback loop. The output of one application of the rules is used as input for the next, and just as some feedback loops eventually converge to a single number, this geometric feedback loop eventually converges to a single shape.

Roughly speaking, a figure qualifies as a fractal if it displays aspects of infinite self-similarity. In other words, zooming into certain areas of a fractal will expose shapes that bear a close resemblance to the overall image. This isn't our first encounter with the concept of self-similarity. Similar curiosities surfaced in the feedback loop plots presented in a previous letter. Systems that produce fractals usually share a common trait; the repeated application of rules. The rules can take the form of a mathematical function, or a set of geometric operations.

Let's consider another example. The geometric operations will only affect the perimeter, but the starting shape will be an equilateral triangle as before:



Equilateral Triangle