HIGH SCHOOL MATHEMATICS CONTESTS

Math League Press, P.O. Box 17, Tenafly, New Jersey 07670-0017

All official participants must take this contest at the same time.

Contest Number 1 *Any calculator without a QWERTY keyboard is allowed.* Answers must be exact *or* have 4 (or more) significant digits, correctly rounded. **October 25, 2005**

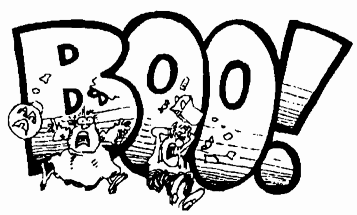
Name ______________________ **Teacher** ___________ **Grade Level** _____ **Score** ____

Time Limit: 30 minutes NEXT CONTEST: NOV. 29, 2005 *Answer Column*

1-1. If $a < b$, then $3^2+4^2+5^2+12^2 = a^2+b^2$ is satisfied by only one pair of positive integers (a,b). What is the value of $a+b$?

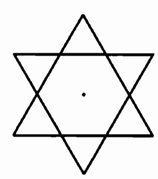
1-1.

1-2. This coming Halloween, Tom plans to scare twice as many people as Sam, and Sam plans to scare three times as many people as Roz. In all, they plan to scare at most 2005 people. If no one is scared more than once, at most how many people does Sam plan to scare?



1-2.

1-3. When two congruent equilateral triangles share a common center, their union can be a star, as shown. If their overlap is a regular hexagon with an area of 60, what is the area of one of the original equilateral triangles?

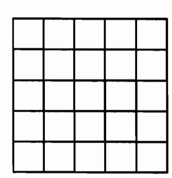


1-3.

1-4. For how many different positive integers n does $\sqrt{n}$ differ from $\sqrt{100}$ by less than 1?

1-4.

1-5. Counting every possible square of each size from 1×1 to 5×5, what is the total number of distinct squares which can be traced out along the lines of the accompanying grid?



1-5.

1-6. The four numbers $a < b < c < d$ can be paired in six different ways. If each pair has a different sum, and if the four smallest sums are 1, 2, 3, and 4, what are all possible values of d?

1-6.

Contest # 1 — *Answers & Solutions* — 10/25/05

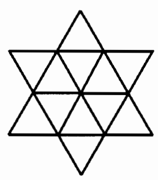
Problem 1-1

$3^2+4^2 \;+\; 5^2+12^2 = 5^2+13^2$, so $a+b = \boxed{18}$.

Problem 1-2

If Roz plans to scare n people, then Sam plans to scare $3n$ people and Tom plans to scare double that, $6n$. Altogether, $n+3n+6n \le 2005$; so $n \le 200.5$. Since n is an integer, n is at most 200. Since Sam plans to scare $3n$ people, that's at most $\boxed{600}$ people.

Problem 1-3



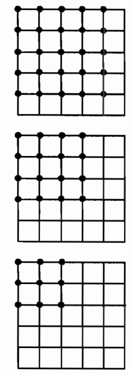
Method I: Draw the 3 diagonals of the hexagon, as shown, to partition the figure into 12 congruent small equilateral triangles. Since the overlap consists of 6 of these triangles, with a total area of 60, each small equilateral triangle has an area of 10. Since each original (large) equilateral triangle consists of 9 small ones, the area of one large equilateral triangle is $\boxed{90}$.

Method II: Join the common center to any two consecutive vertices of the hexagon. The equilateral triangle inside the hexagon is congruent to each equilateral triangle outside the hexagon. Since six of these "inside" triangles make up the hexagon, a large equilateral triangle consists of these six triangles plus three additional triangles. Therefore, the area of a large equilateral triangle is $60 + (1/2)(60) = 90$.

Problem 1-4

If the integer n is greater than $9^2 = 81$ but less than $11^2 = 121$, then $\sqrt{n}$ differs from 10 by less than 1. The set $\{82, 83, \ldots, 119, 120\}$ contains $\boxed{39}$ different integers.

Problem 1-5



Each 1×1 square is determined by its upper left vertex, for which there are 5×5 choices. Each 2×2 square is determined by its upper left vertex, for which there are 4×4 choices. Each 3×3 square is determined by its upper left vertex, for which there are 3×3 choices. In general, each $d \times d$ square is determined by its upper left vertex, for which there are $(6-d)(6-d) = (6-d)^2$ choices. Thus, the number of 1×1 squares is 5^2, the number of 2×2 squares is $4^2, \ldots,$ and the number of 5×5 squares is 1^2. The total of the number of squares of all sizes is $5^2+4^2+3^2+2^2+1^2 = \boxed{55}$.

Problem 1-6

Method I: The six possible sums are $a+b$, $a+c$, $b+c$, $a+d$, $b+d$, and $c+d$. Since $a < b < c < d$, the smallest sums are $a+b = 1$ and $a+c = 2$. From these, $c = b+1$. Of the other four sums, the largest are $b+d$ and $c+d$. Of the remaining sums, $b+c$ and $a+d$, one equals 3 and the other equals 4. Since $c = b+1$, if $b+c = 2b+1 = 3$, then $b = 1$. Then, since $a+b = 1$, $a = 0$; and since $a+d = 4$, $d = 4$. If, instead, $b+c = 2b+1 = 4$, then $b = 3/2$. Then, since $a+b = 1$, $a = -1/2$; and since $a+d = 3$, $d = 7/2$. Finally, the two possible values of d are $\boxed{7/2,\ 4}$.

Method II: The six possible sums are $a+b$, $a+c$, $b+c$, $a+d$, $b+d$, and $c+d$. Since $a < b < c < d$, the smallest two sums are $a+b = 1$ and $a+c = 2$. From these, $b = 1-a$ and $c = 2-a$. Of the other four sums, the largest are $b+d$ and $c+d$. Of the remaining sums, $b+c$ and $a+d$, one equals 3, the other equals 4. If $b+c = 3$, then $(1-a)+(2-a) = 3$. Solving, $a = 0$. Since $a+d = 4$, $d = 4$. If $b+c = 4$ and $a+d = 3$, then $(1-a)+(2-a) = 4$, so $a = -1/2$ and $d = 7/2$.

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Contest Number 2 *Any calculator without a QWERTY keyboard is allowed.* Answers must be exact *or* have 4 (or more) significant digits, correctly rounded. **November 13, 2012**

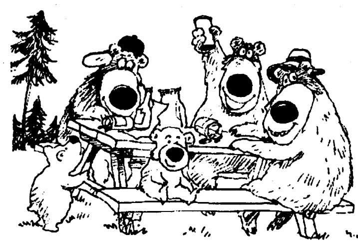
Name ______________________ **Teacher** ___________ **Grade Level** _____ **Score** ____

Time Limit: 30 minutes NEXT CONTEST: DEC. 11, 2012 *Answer Column*

2-1. What is the largest prime divisor of every 3-digit number with 3 identical non-zero digits?

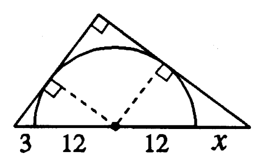
2-1.

2-2. If 3 adult bears ate an average of 16 hot dogs each, and 2 bear cubs ate an average of 6 hot dogs each, then (for these 5 bears) what was the average number of hot dogs eaten per bear?



2-2.

2-3. A semicircle is tangent to both legs of a right triangle and has its center on the hypotenuse. The hypotenuse is partitioned into 4 segments, with lengths 3, 12, 12, and x, as shown. What is the value of x?



2-3.

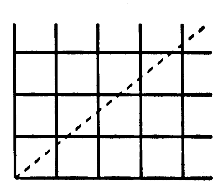
2-4. What are all 3 ordered triples of integers (a,b,c), with $0 < a \le b \le c$, for which $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$?

2-4.

2-5. A distribution consists of the integers from 1 through 100, inclusive, such that the frequency of each integer n is 2^{n-1}. What is the median of this distribution?

2-5.

2-6. If S is a 2012×2012 square split into unit squares, a diagonal of S will pass through the interior of 2012 unit squares. If R is a 2012×2015 rectangle split into unit squares, a diagonal of R will pass through the interior of how many unit squares?



2-6.

Contest # 2 — *Answers & Solutions* — 11/13/12

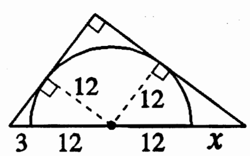
Problem 2-1

Every such 3-digit number *ddd* can be written as $d \times 111 = d \times 3 \times 37$. Since d is a digit, $1 \le d \le 9$, so the largest prime divisor is $\boxed{37}$.

Problem 2-2

The average of 5 numbers is their sum divided by 5. Since the sum of these 5 numbers is $(16+16+16) + (6+6) = 60$, their average is $60/5 = \boxed{12}$.

Problem 2-3



The two small right $\triangle$s are similar to each other (and the large right $\triangle$). A radius of the circle is 12. Thus the longer leg of the right $\triangle$ at the lower left is 12. Since its hypotenuse is 15, its dimensions are 9, 12, 15. The shorter leg of the right $\triangle$ at the lower right is 12, so its dimensions are 12, 16, 20. Since $12+x = 20$, $x = \boxed{8}$.

Problem 2-4

Clearly, $(a,b,c) = (3,3,3)$ is a solution. In any other solution, at least one fraction must exceed $\frac{1}{3}$, which means one fraction must equal $\frac{1}{2}$. Since $0 < a \le b \le c$, it follows that, in any other solution, $a = 2$. Now, solve $\frac{1}{b} + \frac{1}{c} = \frac{1}{2}$ in positive integers. This is a simpler version of the original equation. This time, an obvious solution is $(b,c) = (4,4)$. In any other solution, one fraction must exceed $\frac{1}{4}$. That means that one fraction must equal $\frac{1}{3}$. Thus, $\frac{1}{c} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. Finally, the positive integer solutions are the ordered triples $\boxed{(3,3,3),\ (2,4,4),\ (2,3,6)}$.

Problem 2-5

The integers range from 1 through 100. How many 1's are there? There's $2^{1-1} = 1$ of them. How many 2's? There are $2^1 = 2$ of those. Similarly, there are 2^2 3's, 2^3 4's, . . . , 2^{98} 99's. The total number of all these integers is $2^0+2^1+2^2+2^3+ \ldots +2^{98} = 2^{99}-1$. The number of 100's is 2^{99}, so we can pair one 100 with every other integer—and we'll still have one 100 left over. So, if the numbers are ordered from least to greatest, the middle number will be the extra $\boxed{100}$.

Problem 2-6

Place the rectangle on the coordinate axes with vertices (0,0), (2015,0), (2015,2012), and (0,2012). The diagonal is $y = \frac{2012}{2015}x$, with $0 \le x \le 2015$. The key observation is that the diagonal enters a new square each time the diagonal crosses a vertical line of the form $x = a$, with $a = 1, 2, 3, \ldots, 2014$, or a horizontal line of the form $y = b$, with $b = 1, 2, 3, \ldots, 2011$. (Since the greatest common divisor of 2012 and 2015 is 1, the diagonal never passes through any point with integral coordinates—where the grid lines cross—that is interior to the rectangle.) Start with the unit square that has one vertex at (0,0), then go through another 2014 squares horizontally and 2011 squares vertically. The total number of such squares is $1+2014+2011 = \boxed{4026}$.

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Contest Number 5 *Any calculator without a QWERTY keyboard is allowed.* Answers must be exact *or* have 4 (or more) significant digits, correctly rounded. **February 12, 2019**

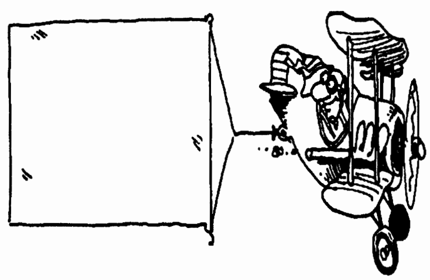
Name ________________ **Teacher** __________ **Grade Level** _____ **Score** ____

Time Limit: 30 minutes NEXT CONTEST: MAR. 19, 2019 *Answer Column*

5-1. If N is a 3-digit integer, and given that the result of reversing the digits of N is the number M, what is the maximum value of $N-M$? [Note: Integers written with 1 or more leading zeros can also be written with every leading zero removed.] — 5-1.

5-2. If a and b are integers whose product is 5, what is the least possible value of a^b? — 5-2.

5-3. Al's biplane can fly two different types of square advertising signs. Both signs have integer side-lengths. Their areas are $n+1$ and $2n+1$. What is the least positive integer n for which each sign's area is the square of an integer? — 5-3.



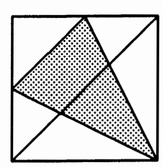
5-4. In a group of 100, each a Borg or a Corg, every Borg has 2 Corg friends and no Corg has more than 1 Borg friend. If exactly 13 Corgs have no Borg friends, then how many Corgs are in the group? — 5-4.

5-5. If 2019! is the product of the first 2019 positive integers, for which positive integer k is $x = 2019!$ the only value of $x > 0$ that satisfies

$$\frac{1}{\log_4 x} + \frac{1}{\log_9 x} + \frac{1}{\log_{16} x} + \ldots + \frac{1}{\log_{(2019^2)} x} = k?$$

— 5-5.

5-6. Midpoints of two sides of a square are vertices of the shaded triangle shown. Drawing the diagonal of the square pictured splits this triangle into two parts, one of which is a trapezoid. If the area of the square is 192, what is the area of the trapezoid? — 5-6.



Contest # 5 — *Answers & Solutions* — 2/12/19

Problem 5-1

Since making N's hundreds' digit a 9 and its units' digit a 0 will maximize the difference, no matter what the middle digit is, $N-M = 9\underline{?}0 - 0\underline{?}9 = \boxed{891}$.

Problem 5-2

Since $ab = 5$, and a and b are integers, (a,b) must be one of (1,5), (5,1), (-1,-5), or (-5,-1). The least value of a^b is $(-1)^{-5} = \boxed{-1}$.

Problem 5-3

By trial and error, take each positive perfect square and subtract 1 to find every positive integer n for which $n+1$ is a perfect square. Find the least of those integers n for which $2n+1$ is the square of an integer. Let's try 1. That fails because $1-1 = 0$ is not positive. Try 4: $4-1 = 3$, but $2(3)+1 = 7$, which is not a perfect square. We try 9, then 16, and when we finally try 25, we get $25-1 = 24$, and $2(24)+1 = 49$. Both $n+1$ and $2n+1$ are perfect squares when $n = \boxed{24}$.

Problem 5-4

If B is the number of Borgs, and C is the number of Corgs, then the number of Corgs is $2B+13$. Therefore, $B+(2B+13) = 100$, $B = 29$, and $C = \boxed{71}$.

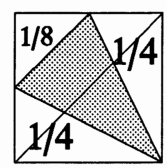
Problem 5-5

Whenever $a > 1$ and $b > 1$, $\log_a b = \dfrac{1}{\log_b a}$. We can write $\log_x 4 + \log_x 9 + \log_x 16 + \ldots + \log_x 2019^2 = k$, or $\log_x(2^2 \times 3^2 \times 4^2 \times \ldots \times 2019^2) = k$. It follows that $x^k = (2^2 \times 3^2 \times 4^2 \times \ldots \times 2019^2) = (2019!)^2$, so $k = \boxed{2}$.

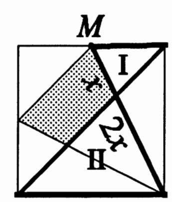
Problem 5-6

Method I: The square's side is $8\sqrt{3}$, so its diagonal is $8\sqrt{6} = d$. The trapezoid's height is $\frac{d}{4}$. Its longer base is $\frac{d}{2}$. The shaded $\triangle$'s legs trisect the diagonal (to prove this, use similar triangles I and II outlined in the middle diagram), so the shorter base is $\frac{d}{3}$. The trapezoid's area is $\frac{h}{2}(b_1+b_2) = \frac{d}{8}(\frac{d}{2}+\frac{d}{3}) = \frac{5d^2}{48} = \boxed{40}$.

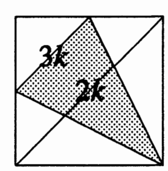
Method II: Two of the shaded triangle's vertices are midpoints of sides of the square, so the smaller unshaded region is 1/8 of the square, and the other unshaded regions are each 1/4 of the square. The shaded triangle's area is $(3/8) \times 192 = 72$.



Notice that $\triangle$I is similar to $\triangle$II. Since M is a midpoint, the parallel sides of $\triangle$s I and II are in the ratio $x{:}2x = 1{:}2$, as are all corresponding sides. Thus, the smaller shaded triangle and the larger shaded triangle are similar, with ratio of similitude $2k{:}3k = 2{:}3$.



Since the area ratio of similar triangles is the square of their ratio of similitude, the ratio of the areas of the shaded triangles is $(2/3)^2 = 4{:}9$. The smaller shaded triangle's area $= (4/9) \times 72 = 32$.



The trapezoid's area is what remains: $72-32 = 40$.