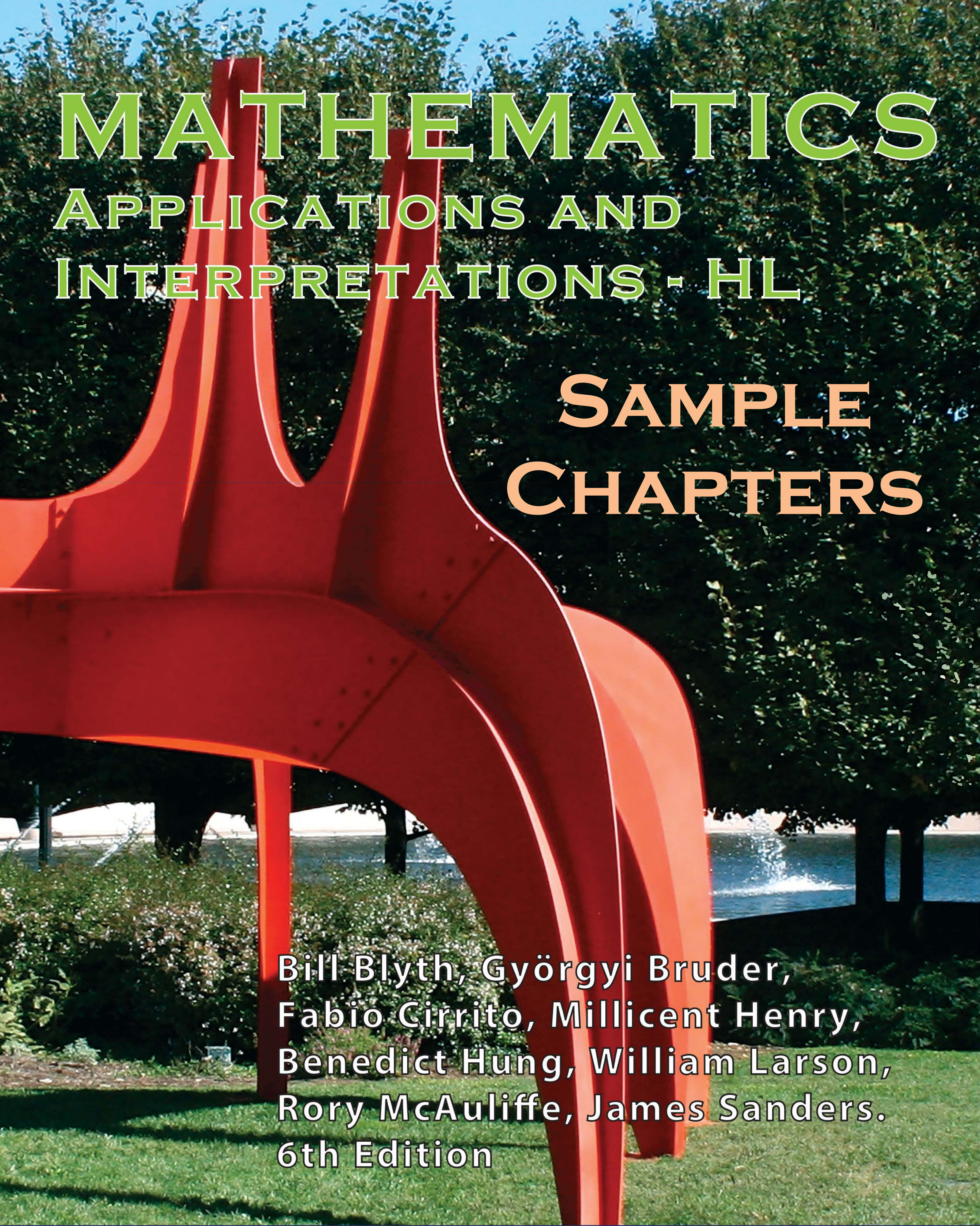
MATHEMATICS

Applications and Interpretations - HL

Sample Chapters



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D.12 Poisson Distribution

AHL 4.17

The Poisson distribution was first brought to light by the eminent French mathematician Simeon Denis Poisson (1781–1840) in his 1837 work *Recherches sur la probabilite de Judgement*, where he included a limit theorem for the binomial distribution. At the time, this was viewed as little more than a welcome approximation for the difficult computations required when using the binomial distribution. However, this was the embryo from which grew what is now one of the most important of all probability models.

Poisson's distribution attracted little attention until the publication in 1898 of *Das Gesetz der keinem Zahlen*, in which the author, Ladislaus Bortkiewicz (1868 – 1931), showed how the distribution could be used to explain statistical regularities in the occurrence of rare events. His most striking example was the number of deaths from horse kicks in 14 units of the Prussian Army in different years between 1875 & 1894. There was comprehensive data on this and Poisson's predictions matched them closely.

Number of Deaths	Number of Years	Poisson Prediction
1	91	92
2	32	34
3	11	8
4	2	1
5+	0	0

This table means that in the 20 years of the study and over the 14 cavalry units, there were 91 unit-months in which one soldier died from horse kicks. The Poisson Predictions are striking in their match to the data.

Our heading picture is of one of the horses of the Spanish Riding School in Vienna - which would, of course, never dream of kicking anyone.

However, a more general (and useful) use of the Poisson distribution (as opposed to only seeing it as an approximation to the binomial under certain conditions) is to define the distribution as the distribution of the number of 'events' in a 'random process'.

The key in identifying a Poisson distribution, then, is to be able to identify the 'random process' and the 'event'. As we shall see, the event can be distributed over time, or distance, or length, or area, or volume, or …

Examples of 'random processes' and their corresponding 'events' are:

Random process	Event
Telephone calls in a fixed time interval.	Number of wrong calls in an hour. (Time dependent)
Accidents in a factory.	Number of accidents in a day. (Time dependent)
Flaws in a glass panel.	Number of flaws per square centimetre (Area dependent)
Flaws in a string.	Number of flaws per 5 metres. (Length dependent)
Bacteria in milk.	Number of bacteria per 2 litres. (Volume dependent)

The above examples serve to highlight the properties associated with the Poisson distribution. These can be best summarized as:

Step 1. An event is as likely to occur in one given interval as it is in another (equally likely).

Step 2. The occurrence of an event at a 'point' – be it a time interval, an area, etc. – is independent of when (or where) other events have occurred.

Step 3. Events occur uniformly, ie. the expected number of events in a given time interval, or area, or, … is proportional to the size of the time interval, or area, or, …

Note how similar these conditions are to those of the binomial distribution. However, one main difference between the two distributions is that there is, at least theoretically, no upper limit to the number of times an event may occur!

With this in mind, we now provide a statement for the Poisson distribution, incorporating the distribution function.

If $X(t)$ is the number of events in a time interval of length t, corresponding to a random process, with rate λ per unit time, then, we say that $X(t) \sim \text{Pn}(t)$– read as 'the random variable X has a Poisson distribution with parameter λt'.

Setting $\mu = \lambda t$, we define the Poisson probability distribution as:

$$P(X=x)=\frac{e^{-\mu}.\mu^{x}}{x!}, x=0,1,2,...$$

Note that the rate λ can be specified as the number of events per unit time, or per unit area, or per unit of volume, or unit of length, etc.

The best way to see how this works is through the following examples.

Example D.12.1

Cars have been observed to pass a given point on a back road at a rate of 0.5 cars per hour. Find the probability that no cars pass this point in a two-hour period.

The description of the situation fits the conditions under which a Poisson distribution can be assumed. From the information given we have that $\lambda = 0.5$.

Next we define the random variable X as the number of cars that pass the given point in a two-hour period.

This means that our parameter $\mu = \lambda \times 2 = 0.5 \times 2 = 1$.

$$P(X=x)=\frac{e^{-1}.1^{x}}{x!}=\frac{e^{-1}}{x!}, x=0,1,2,...$$

And so, $P(X=0)=\frac{e^{-1}}{0!}\approx 0.3679$

Example D.12.2

Faults occur on a piece of string at an average rate of one every three metres. Bobbins, each containing 5 metres of this string, are to be used. What is the probability that a randomly selected bobbin will contain:

a two faults. b at least two faults.

The description of the situation fits the conditions under which a Poisson distribution can be assumed.

From the information given we have that $\lambda=\frac{1}{3}$ (i.e. one in three metres).

Next we define the random variable X as the number of faults in a string 5 metres long. That is, number of faults per bobbin.

This means that our parameter $\mu=\lambda\times 5=\frac{1}{3}\times 5=\frac{5}{3}$ so that

the probability function for X is given by:

$$P(X=x)=\frac{e^{-5/3}}{x!}\times\left(\frac{5}{3}\right)^{x}, x=0,1,2,...$$

a $$P(X=2)=\frac{e^{-5/3}}{2!}\times\left(\frac{5}{3}\right)^{2}\approx 0.2623$$

b At first sight, $P(X\geq 2)$ would seem to require us to compute an infinite set of probabilities for X = 2, 3, 4,... however, we can use the fact that all probability distributions sum to 1. It is easier to calculate:

$$\begin{aligned}P(X\geq 2)&=1-P(X<2)\\&=1-P(X=0)-P(X=1)\end{aligned}$$

$$P(X \geq 2) = 1 - \frac{e^{-5/3}}{0!} \times \left(\frac{5}{3}\right)^0 - \frac{e^{-5/3}}{1!} \times \left(\frac{5}{3}\right)^1$$
$$= 1 - e^{-5/3} - \frac{5}{3} e^{-5/3}$$
$$\approx 0.4963$$

Video demonstration of this using calculators.

Based on these two examples we can set out a general approach to handling questions that require the use of the Poisson distribution:

1. Identify that scenario which fits the requirements of a Poisson distribution.
2. Determine the 'base' rate, λ.
3. Define the random variable.
4. Determine the parameter, μ, that corresponds to the random variable in Step 3.

Example D.12.3

A radioactive source emits particles at an average rate of one every 12 seconds. Find the probability that at most 5 particles are emitted in one minute.

The description of the situation fits the conditions under which a Poisson distribution can be assumed.

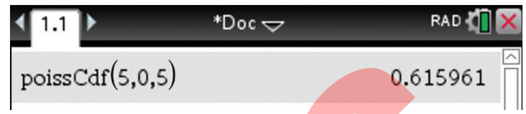
From the information given we have that $\lambda = \frac{1}{12}$ (i.e. 1 in 12 seconds).

Next we define the random variable X as the number of particles emitted in 1 minute (or 60 seconds).

This means that our parameter $\mu = \lambda \times 60 = \frac{1}{12} \times 60 = 5$ so that the probability function for X is given by:

$$P(X = x) = \frac{e^{-5} 5^x}{x!}, x = 0, 1, 2, \ldots$$

$$P(X \leq 5) = \frac{e^{-5} 5^0}{0!} + \frac{e^{-5} 5^1}{1!} + \ldots + \frac{e^{-5} 5^5}{5!}$$
$$= e^{-5}\left(1 + 5 + \frac{5^2}{2} + \frac{5^3}{6} + \frac{5^4}{24} + \frac{5^5}{120}\right)$$
$$= \frac{1097}{12} e^{-5}$$
$$\approx 0.6160$$



Mean and Variance of the Poisson Distribution

For a random variable X having a Poisson distribution with parameter μ, then:

$$E(X) = \mu \text{ and } Var(X) = \mu$$

These are remarkable and distinctive properties of the Poisson Distribution. Their value is illustrated in these examples:

Example D.12.4

A data entry operative finds that, on average, they make two mistakes every three screens. Assuming that the number of errors per screen follows a Poisson distribution, what are the chances that there will be 2 mistakes on the next screen they enter?

In this case we are given that the average is $^2/_3$ errors per screen. Then, if we let the random variable N denote the number of errors per screen we have that $E(N) = {}^2/_3$ and $\mu = {}^2/_3$

$$P(N = 2) = \frac{e^{-2/3}}{2!}\left(\frac{2}{3}\right)^2 \approx 0.1141$$

Example D.12.5

The frequency distribution shows the number of cars that drive over a bridge in a country area over a period of 100 days. Verify that this follows approximately a Poisson distribution.

Number of cars passing over bridge	0	1	2	3	4
Number of days observed	58	29	10	2	1

Let the random variable X denote the number of cars that pass over the bridge per day. We first determine the average number of cars that pass over the bridge over the 100 days.

Cars (X)	0	1	2	3	4
Days (f)	58	29	10	2	1
Cars × Days	0	29	20	6	4

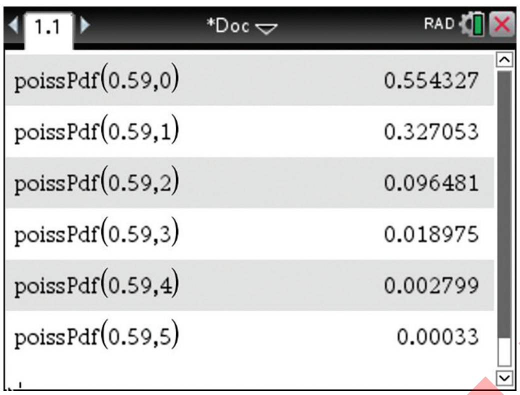
Total number of cars = 0 + 29 + 20 + 6 + 4 = 59.

μ = average number of cars per day = 59 ÷ 100 = 0.59

If this is a Poisson Distribution, it will be:

$$P(X=x)=\frac{e^{-0.59}.(0.59)^x}{x!}, x=0,1,2,...$$

Using a calculator to find the first few probabilities:



The actual number of cars predicted by the model is found by multiplying by 100.

Cars (X)	0	1	2	3	4
Days (f)	58	29	10	2	1
P(X)	0.554	0.327	0.096	0.019	0.003
Prediction	55	33	10	2	0

The actual data (row 2) and the expected number from the Poisson model (row 4) are in good agreement.

We can, therefore, be reasonably sure that the Poisson distribution is an appropriate model for the number of cars that pass over this bridge.

Example D.12.6

The number of flaws in metal sheets 100 cm by 150 cm is known to follow a Poisson distribution. On inspecting a large number of these metal sheets it is found that 20% of these sheets contain at least one flaw.

a Find the average number of flaws per sheet.

b Find the probability of observing one flaw in a metal sheet selected at random.

a Let the random variable X denote the number of flaws per 100 cm by 150 cm metal sheet.

We have that $X \sim \text{Pn}(\mu)$ where μ is to be determined.

Knowing $\text{P}(X \geq 1) = 0.2$ we have, $1 - \text{P}(X = 0) = 0.2$

$$P(X=0)=0.8$$
$$e^{-\mu}=0.8$$
$$\mu \approx 0.2231$$

i.e. average number of flaws per sheet is 0.2232.

b $P(X=1)=0.8(0.2231)^1 \approx 0.1785$

Sum of two Poisson Distributions

If we have two independent Poisson Distributions:

$X \sim \text{P}(\lambda)$ and $Y \sim \text{P}(\mu)$, what about the distribution of the sum of the two variables?

We might hope that $X + Y$, the sum of the two variables, will also follow a Poisson distribution. Further more, we would hope also that its mean will be the sum of the independent means. In mathematical terms $X + Y \sim \text{P}(\lambda + \mu)$

This is an applied course so we will not prove this result (yet), but will use an example to illustrate it.

Example D.12.7

A manufacturer of electrical components finds that, in the long run, 1.5% of the components have defective semiconductors and 1% have defective casings. The inspection procedure requires that batches of 50 components are tested. What is the probability distribution of the defectives in each batch?

What can we say about the number of defectives of either or both types in our batch of 50?

If we look first at the issues with the semiconductors there is a mean number of defectives per batch of $0.015 \times 50 = 0.75$

We will call these 'Type A faults'.

Tabulating the probabilities for the number of faults per batch, we get:

Defectives	Probability A
0	0.4724
1	0.3543
2	0.1329
3	0.0332
4	0.0062
5	0.0009
6	0.0001

Next, we look at the defects of the casings which have a mean of $0.01 \times 50 = 0.5$ per batch.

We will call these 'Type B faults'.

Tabulating this distribution:

Defectives	Probability B
0	0.6065
1	0.3033
2	0.0758
3	0.0126
4	0.0016
5	0.0002
6	0.0000

Looking at the tables, what is the probability that we will have a batch with no faults of either sort? This means, a batch with no faults at all.

The two types of fault are independent and so we have that:

$$P(\text{A+B} = 0) = P(\text{A} = 0) \times P(\text{B} = 0)$$

$$= 0.4724 \times 0.6065$$

$$\approx 0.2865$$

What about the probability that we have just one fault in a batch? This is EITHER one fault of type A AND no faults of type B OR no faults of type A AND one fault of type B.

The 'either', 'and' & 'or' words in these statements have precise meanings in probability. 'And' means we can multiply probabilities, but only if the events are independent. 'Or' means we add probabilities, but only if the events are exclusive (cannot occur together). In this case a batch cannot have no type A faults and one type B at the same time as one type A fault and no type B faults. So these two events are exclusive and we can add their probabilities (without subtracting their intersection - as this is zero).

$$P(\text{A+B} = 1) = P(\text{A} = 0) \times P(\text{B} = 1) + P(\text{A} = 1) \times P(\text{B} = 0)$$

$$= 0.4724 \times 0.3033 + 0.3543 \times 0.6065$$

$$\approx 0.3582$$

What is the probability of a total of 2 faults in a batch? This is more complicated as there are three ways this can happen:

Type A	Type B	Probability
0	2	$0.4724 \times 0.0758 = 0.0358$
2	0	$0.1329 \times 0.6065 = 0.0806$
1	1	$0.3543 \times 0.3033 = 0.1075$
		Total = 0.2239

The calculation is getting progressively more complicated, but do not despair as we can use the fact that the distribution we are after is the Poisson Distribution with the sum of the two parameters.

Recall that type A faults average 0.75 per batch and type B average 0.5 per batch.

The sum of the two fault types is $0.75 + 0.5 = 1.25$

Thus, we should find that the total number of faults per batch is Poisson distributed:

$$P(N=0) = \frac{e^{-1.25}}{0!}(1.25)^0 \approx 0.2865$$

$$P(N=1) = \frac{e^{-1.25}}{1!}(1.25)^1 \approx 0.3581$$

$$P(N=2) = \frac{e^{-1.25}}{2!}(1.25)^2 \approx 0.2238$$

etc.

Give or take rounding errors, these are the results we obtained after much more work using the two separate distributions.

Example D.12.8

In a study to establish whether it will be economical to instal safety gates at a remote road/rail crossing, a road traffic survey was conducted. It was found that an average of 5.7 cars per day and 3.6 trucks per day used the crossing.

Use the Poisson Distribution to tabulate the probabilities for:

a The number of cars per day using the crossing.

b The number of trucks per day using the crossing.

c The number of vehicles per day using the crossing.

a We have a Poisson Distribution with a mean of 5.7.

$$P(N=0)=\frac{e^{-5.7}\times 5.7^{0}}{0!}\approx 3.34\times 10^{-3}$$

$$P(N=1)=\frac{e^{-5.7}\times 5.7^{1}}{1!}\approx 0.0191$$

$$P(N=1)=\frac{e^{-5.7}\times 5.7^{2}}{2!}\approx 0.0544$$

etc.

If we tabulate the first few results, we get:

Cars	Probability
0	0.0033
1	0.0191
2	0.0544
3	0.1033
4	0.1472
5	0.1678
6	0.1594
7	0.1298
8	0.0925
9	0.0586
10	0.0334

b We have a Poisson Distribution with a mean of 3.6.

Cars	Probability
0	0.0273
1	0.0984
2	0.1771
3	0.2125
4	0.1912
5	0.1377
6	0.0826
7	0.0425
8	0.0191
9	0.0076
10	0.0028

c We have a Poisson Distribution with a mean of 5.7 + 3.6 = 9.3.

Cars	Probability
0	0.0001
1	0.0009
2	0.0040
3	0.0123
4	0.0285
5	0.0530
6	0.0822
7	0.1091
8	0.1269
9	0.1311
10	0.1219

This road/rail crossing is one of the remotest in the World. It is at Forrest in the middle of the Nullarbor Plain. You might just be able to see that there is a gravel road crossing. What drivers need to be careful to notice is the Indian Pacific on its 4 352 km, 75 hour trip from Perth to Sydney.

It would be difficult to justify expending taxpayer funds on a full set of gates at this location!

Exercise D.12.1

1. a If $X \sim \text{Pn}(2)$ write down the probability distribution function for the random variable X.

   b Find:

   i $P(X = 0)$ ii $P(X = 2)$

   iii $P(X > 1)$ iv $P(X = 2 \mid X > 1)$

2. The flaws in a string occur at a rate of 2 every 5 metres. Find the probability that a string contains 3 flaws in:

   a 2 metres of string.

   b 10 metres of string.

3. Cars that stop at a particular petrol station during weekdays arrive at a rate of 10 cars every hour. Assuming a Poisson distribution, find the probability that:

   a there will be one car at the petrol station during any 15-minute interval.

   b there will be some cars at the petrol station during any 15-minute interval.

4. A switchboard receives an average of 100 calls per hour. Find the probability that:

   a the switchboard receives 2 calls during a one-minute time interval.

   b the switchboard receives at least 2 calls during a two-minute time interval.

5. On average a data entry operative has to correct one item in every 800 items. Each screen contains 200 items.

   a Find the probability that the operative makes more than one correction per screen.

   b If more than one correction per screen is required, the screen needs to be retyped. What is the probability that more than two attempts are needed before a screen is deemed satisfactory?

6. Cars have been observed to pass a given point on a country road at a rate of 5 cars per hour.

   a Find the probability that no cars pass this point in a 20-minute period.

   b Find the probability that at least 2 cars pass this point in a 30-minute period.

7. Bolts are produced in large quantities and it is expected that there is a 2.5% rejection rate due to thread defects and a 1.5% rejection rate due to finish defects. A batch of 40 bolts is randomly selected for inspection. Using the Poisson distribution, find the probability that:

   a the batch contains at least one defective.

   b the batch contains no defectives.

   Ten such batches are randomly selected. If it is found that at least 2 batches have at least 4 defective bolts, the total output is considered for the scrap heap to be recycled.

   c Find the probability that the total output is sent to the scrap heap.

8. Road accidents in a certain area occur at an average of 1 every 4 days. Find the probability that during a one week period there will be:

   a two accidents.

   b at least two accidents.

9. Telephone calls arrive at a switchboard at a rate of 4 every minute. Find the probability that in a two-minute interval there will be fewer than 6 incoming calls.

10. Faults in glass sheets occur at a rate of 2.1 per square metre. If a square metre glass sheet contains at least 3 faults it is returned to the manufacturer.

    a Find the probability that a square metre sheet is returned to the manufacturer.

b Six such glass sheets are inspected. What is the probability that at least half of them are returned to the manufacturer?

11. The number of faults in a glass sheet is known to have a Poisson distribution. It is found that 2% of sheets are rejected because they have chipped edges and 3% are rejected because they have scratched faces.

a Find the probability that a sheet contains at least two flaws.

b If the random variable X denotes the number of flaws per sheet, find $P(X > \mu + 2\sigma)$.

12. A shopkeeper finds that the number of orders for an electrical good averages 2 per week. At the start of the trading week, i.e. on a Monday, the shopkeeper has 5 such items in stock. Assuming that the orders follow a Poisson distribution, find the probability that during a given 5-day week:

a there are three orders.

b there are more orders than he can satisfy from his existing stock.

If and when his stock level is down to two items during the week, he orders another four items:

c what are the chances that he will order another four items?

Extra Questions

Answers

We have alluded the fact that the sum of two Poisson Distributions is also a Poisson Distribution.

Using Algebra, can you prove that if:

$$P(X=x)=\frac{e^{-\mu}.\mu^{x}}{x!}, x=0,1,2,...$$

$$P(Y=y)=\frac{e^{-\lambda}.\lambda^{y}}{y!}, y=0,1,2,...$$

then the sum of these two variables: $Z = X + Y$ is:

$$P(Z=z)=\frac{e^{-(\mu+\lambda)}.(\mu+\lambda)^{y}}{z!}, z=0,1,2,...$$

Investigations

There is a lot in the news these days about 'freak weather events'.

These are events such as tornados, hurricanes etc. There are good statistics available for the long term average occurrences of these.

Can you use these to investigate the legitimacy of sensationalist headlines such as:

Once in a Century Storm Next Week!!!

Alternatively, many of the events described in our examples and exercises refer to negatives such as faults, flaws and accidents. Bear in mind that you can apply Poisson to positive events such as prizes, lottery wins etc. You may find some good investigative topics here!

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This series of texts has been written for the IB Courses Mathematics: Analysis and Approaches and Mathematics: Applications and Interpretations that start teaching in August 2019.

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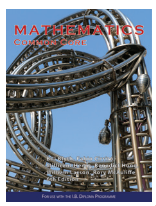
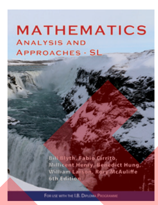
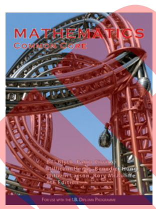
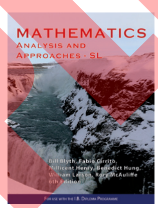
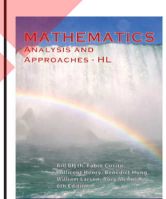
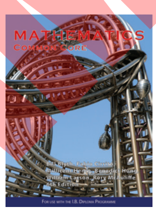
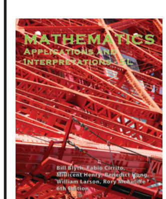
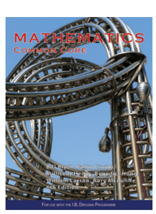
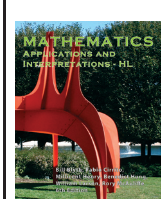
Mathematics: Analysis and Approaches (SL)						Pure(SL)
Mathematics: Analysis and Approaches (HL)						Pure(HL)
Mathematics: Applications and Interpretations (SL)						Applied(SL)
Mathematics: Applications and Interpretations (HL)						Applied(HL)

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- An HL student will receive 3 books (~700pp) for $AUS 110 (approximately $US 80 , € 70, £ 65)
- The requirements for 'Toolbox' activities are covered extensively. There is a complete chapter in the Core Text (included in this 'Sampler') that deals with Problem Solving Skills. Toolbox sections with hints and suggestions for Investigations are scattered throughout the books. This component will account for 20% of a student's assessment.
- Extra questions and answers to the exercises are complimentary extras accessible via QR codes in the book or as free downloads from the IBID website.
- Additional resources such as videos describing calculator use, experimental data etc. available via QR codes or the website.
- Organised and keyed to the 2019 Syllabi
- Full colour presentation, comprehensive explanations, examples and exercises.