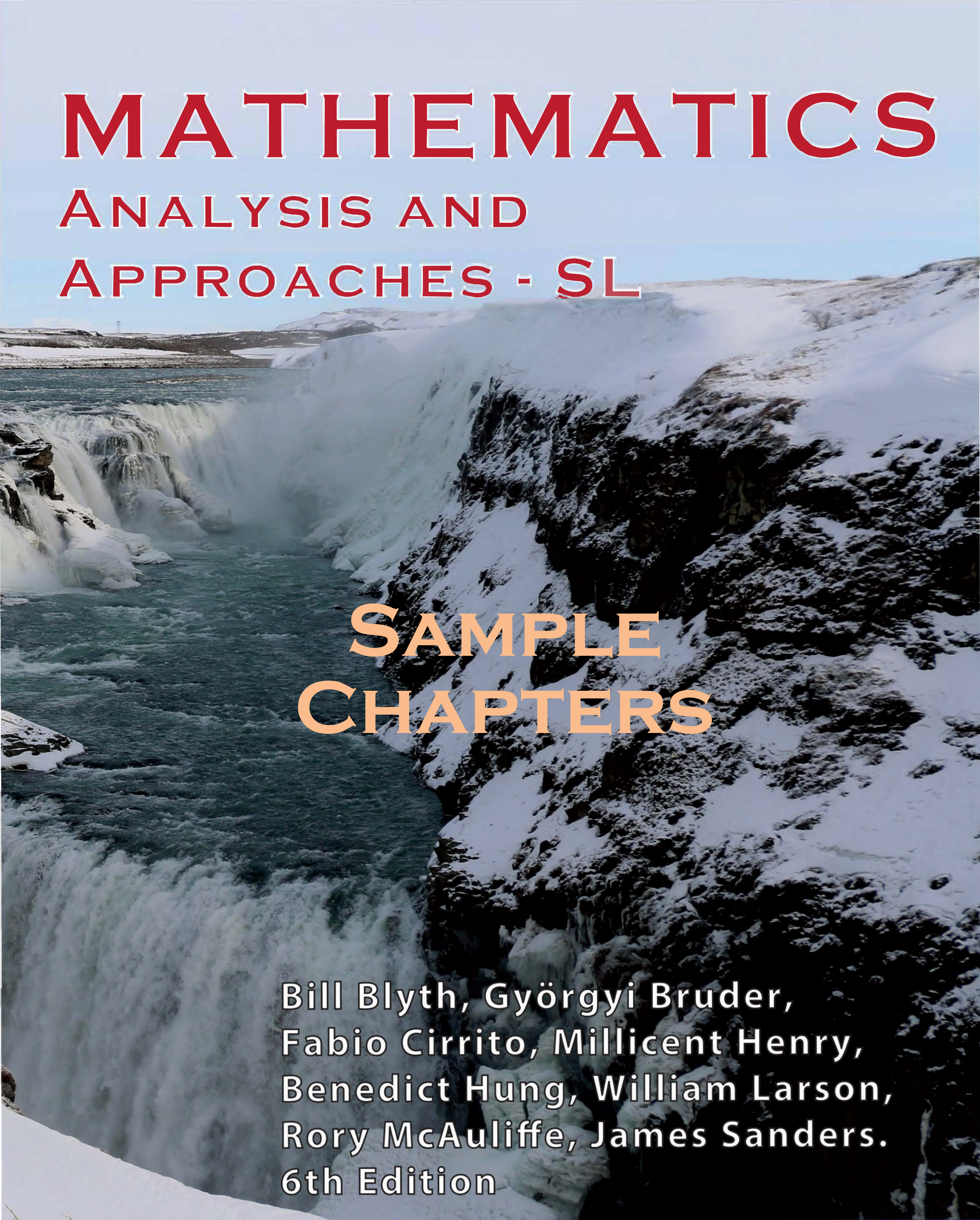

MATHEMATICS

Analysis and Approaches - SL

Sample Chapters

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Axioms

Mathematics is based on axioms. These are 'facts' that are assumed to be true. An axiom is a statement that is accepted without proof. Early sets of axioms contained statements that appeared to be obviously true. Euclid postulated a number of these 'obvious' axioms.

An example of an axiom is:

> 'Things equal to the same thing are equal to each other'; That is, if $y = a$ and $x = a$ then $y = x$.

Euclid was mainly interested in geometry and we still call plane geometry 'Euclidean'. In Euclidean space, the shortest distance between two points is a straight line. We will see later that it is possible to develop a useful, consistent mathematics that does not accept this axiom.

Most axiom systems have been based on the notion of a 'set', meaning a collection of objects. An example of a set axiom is the 'axiom of specification'. In crude terms, this says that if we have a set of objects and are looking at placing some condition or specification on this set, then the set thus specified must exist. An example of this axiom is:

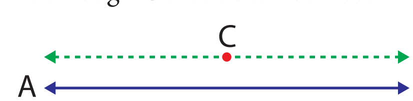
Assume that the set of citizens of China is defined. If we impose the condition that the members of this set must be female, then this new set (of Chinese females) is defined.

As a more mathematical example, if we assume that the set of whole numbers exists, then the set of even numbers (multiples of 2) must also exist.

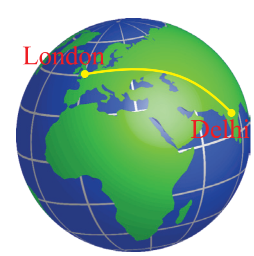
Mathematics has, in some sense, been a search for the smallest possible set of consistent axioms. It is an unusual pursuit in this respect. Pure mathematics is concerned with absolute truth only in the sense of creating a self-consistent structure of thinking.

As an example of some axioms that may not seem to be sensible, consider a geometry in which the shortest path between two points is the arc of a circle and all parallel lines meet. These 'axioms' do not seem to make sense in 'normal' geometry. The first mathematicians to investigate non-Euclidean geometry were the Russian, Nicolai Lobachevsky (1792–1856) and the Hungarian, Janos Bolyai (1802–60).

Independently, they developed self-consistent geometries that did not include the so called parallel postulate which states that for every line AB and point C outside AB there is only one line through C that does not meet AB.



Since both lines extend to infinity in both directions, this seems to be 'obvious'. Non-Euclidean geometries do not include this postulate and assume either that there are no lines through C that do not meet AB or that there is more than one such line. It was the great achievement of Lobachevsky and Bolyai that they proved that these assumptions lead to geometries that are self consistent and thus acceptable as 'true' to pure mathematicians. In case you are thinking that this sort of activity is completely useless, one of the two non-Euclidean geometries discussed above has actually proved to be useful; the geometry of shapes drawn on a sphere. This is useful because it is the geometry used by the navigators of aeroplanes and ships.



Proof

Proof has a very special meaning in mathematics. We use the word generally to mean 'proof beyond reasonable doubt' in situations such as law courts when we accept some doubt in a verdict. For mathematicians, proof is an argument that has no doubt at all. When a new proof is published, it is scrutinized and criticized by other mathematicians and is accepted when it is established that every step in the argument is legitimate. Only when this has happened does a proof become accepted.

Technically, every step in a proof rests on the axioms of the mathematics that is being used. As we have seen, there is more than one set of axioms that could be chosen. The statements that we prove from the axioms are known as theorems. Once we have a theorem, it becomes a statement that we accept as true and which can be used in the proof of other theorems. In this way we build up a structure that constitutes a 'mathematics'. The axioms are the foundations and the theorems are the superstructure. In the previous section we made use of the idea of consistency.

This means that it must not be possible to use our axiom set to prove two theorems that are contradictory.

There are a variety of methods of proof. This section will look at some of these in detail. We will mention others.

Rules of Inference

All proofs depend on rules of inference. Fundamental to these rules is the idea of 'implication'.

As an example, we can say that $2x = 4$ (which is known as a proposition) implies that $x = 2$ (provided that x is a normal real number and that we are talking about normal arithmetic). In mathematical shorthand we would write this statement as $2x = 4 \Rightarrow x = 2$.

This implication works both ways because $x = 2$ implies that $2x = 4$ also. This is written as $x = 2 \Rightarrow 2x = 4$. or the fact that the implication is both ways can be written as $x = 2 \Leftrightarrow 2x = 4$. The symbol $\Leftrightarrow$ is read as 'If and only if ' or simply as 'Iff ', i.e. If with two fs.

There are four main rules of inference:

1. The **rule of detachment**: from a is true and a $\Rightarrow$ b is true we can infer that b is true where a and b are propositions.

Example

If the following propositions are true:

It is raining.

If it is raining, I will take an umbrella.

We can infer that I will take an umbrella.

2. The **rule of syllogism**: from a $\Rightarrow$ b is true and b $\Rightarrow$ c is true, we can conclude that a $\Rightarrow$ c is true. a, b and c are propositions.

Example:

If we accept as true that:

if x is an odd number then x is not divisible by 4 (a $\Rightarrow$ b) and,

if x is not divisible by 4 then x is not divisible by 16 (b $\Rightarrow$ c)

We can infer that the proposition:

if x is an odd number then x is not divisible by 16 (a $\Rightarrow$ c) is true.

3. The **rule of equivalence**: at any stage in an argument we can replace any statement by an equivalent statement.

Example:

If x is a whole number, the statement x is even could be replaced by the statement x is divisible by 2.

4. The **rule of substitution**: If we have a true statement about all the elements of a set, then that statement is true about any individual member of the set.

Example:

If we accept that all lions have sharp teeth then Benji, who is a lion, must have sharp teeth.

Now that we have our rules of inference, we can look at some of the most commonly used methods of proof

Proof by Exhaustion

This method can be, as its name implies, exhausting! It depends on testing every possible case of a theorem.

Example

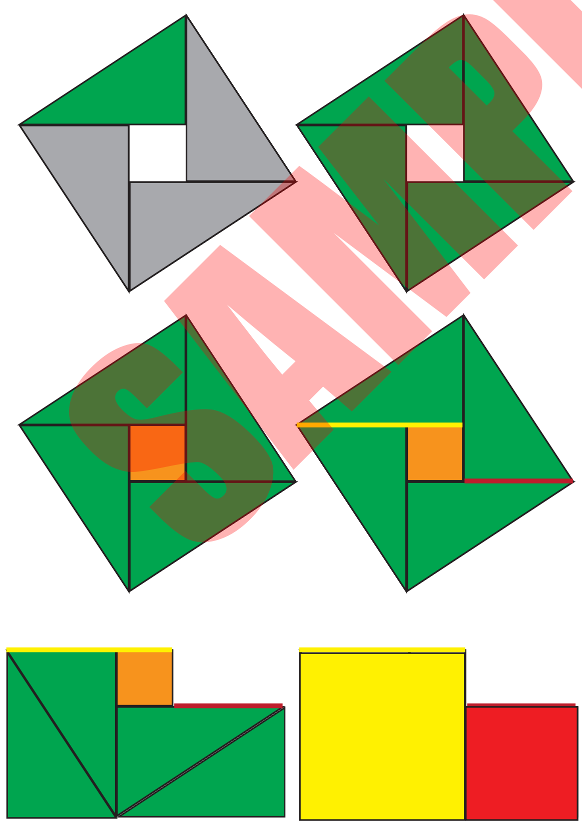
Consider the theorem: Every year must contain at least one 'Friday the thirteenth'.

There are a limited number of possibilities as the first day of every year must be a Monday or a Tuesday or a Wednesday ... or a Sunday (seven possibilities). Taking the fact that the year may or may not be a leap year (with 366 days) means that there are going to be fourteen possibilities.

Once we have established all the possibilities, we would look at the calendar associated with each and establish whether or not it has a 'Friday the thirteenth'. If, for example, we are looking at a non-leap year in which January 1st is a Saturday, there will be a 'Friday the thirteenth' in May. Take a look at all the possibilities (an electronic organizer helps!). Is the theorem true?

Direct Proof

The diagrams below represent a proof of the theorem of Pythagoras described in *The Ascent of Man* (*Bronowski*, pp. 158–61). The theorem states that the area of a square drawn on the hypotenuse of a right-angled triangle is equal to the sum of the areas of the squares drawn on the two shorter sides. The method is direct in the sense that it makes no assumptions at the start. Can you follow the steps of this proof and draw the appropriate conclusion?



Notation

Written Mathematics tends to use a lot of 'notation' or 'shorthand'.

Here is a short list of symbols frequently used in proofs:

Equality

The symbol = means that two quantities are equal. It was invented by a Welshman, Robert Recorde (1512–1558).

Thus $2x + 3 = 7$ is a statement of equality.

One important feature of equalities is that they are not necessarily always true. Our statement is true if $x = 2$ but false if $x = 1$.

Identity

The statement $2(x + 3) \equiv 2x + 6$ is an identity. It is true for all values of x. The triple barred equals sign emphasises that it is a stronger statement of equality than =.

'Not equal to' is written $\neq$. So $5 \neq 7$.

Implication

There is a commonly used notation that is used when we want to say that one true statement implies the truth of another.

It is raining therefore I will wear a raincoat is shortened to:

'It is raining' $\Rightarrow$ 'I will wear a raincoat'. The symbol $\Rightarrow$ is read 'implies'.

Note that an implication does not necessarily work in reverse. If I wear a raincoat, it is not necessarily raining.

If an implication is two way, we use a double headed arrow.

If $x = 2$ then $2x = 4$ works both ways. If $2x = 4$ then $x = 2$. This is shortened to $x = 2 \Leftrightarrow 2x = 4$

Example A.8.1

Classify the relationship between these pairs of expressions as $\neq$, = or $\equiv$.

a	$\sqrt{x^2+1}, x+1$	b	$(x+1)^2, x^2+2x+1$
c	$\frac{1}{3}, 0.\dot{3}$	d	$\sqrt{x^2+2x+1}, x+1$

a $\sqrt{x^2+1}, x+1$

It is common to think that these two expressions are either equal or identical. If they are, they must be equal whatever value of x we choose. Suppose we choose 2.

$\sqrt{2^2+1} = \sqrt{5}$ and $2+1=3$

$\sqrt{x^2+1} \neq x+1$

b $(x+1)^2, x^2+2x+1$

This is true for all x. $(x+1)^2 \equiv x^2+2x+1$

c $\frac{1}{3}, 0.\dot{3}$

These two quantities are identically equal $\frac{1}{3} \equiv 0.\dot{3}$.

d $\sqrt{x^2+2x+1}, x+1$

This is a more complex example. We need to be careful about the fact that 'square root' returns two answers. If we allow that, then $\neq$ is the correct answer.

Suppose we only take the positive value of the square root. This will be true for $x \geq 0$. What about negative values? Taking particular values does not provide a general proof. It can, however get things started.

$\sqrt{(-2)^2+2\times(-2)+1} = \sqrt{4-4+1} = 1$ and

$-2+1=-1$.

Example A.8.2

Place either $\Rightarrow$ or $\Leftrightarrow$ between these pairs of statements as appropriate.

a $x \in \mathbb{N}$ x is even, $x \in \mathbb{N}$ x is divisible by 2.

b $a > 7$, $a > 5$.

c $p > 2$ is prime, p is odd.

d $x^2 = 49$, $x = 7$.

a x is specified as being a Natural Number, 1, 2, 3, 4, ... so we do not need to worry about whether or not -2 is even. This is an implication that runs both ways:

$x \in \mathbb{N}$ x is even $\Leftrightarrow$ $x \in \mathbb{N}$ x is divisible by 2.

b Numbers bigger than 7 are also bigger than 5 but not the other way round.

$a > 7 \Rightarrow a > 5$

c 2 is the only even prime and is included. This is a one-way implication.

$p > 2$ is prime $\Rightarrow p$ is odd

d This only works one way (because of the negative option for square root).

$x = 7 \Rightarrow x^2 = 49$

Exercise A.8.1

1. Classify these statements as true/false:

   a x is odd $\Rightarrow$ x is divisible by 3.

   b x is odd $\Rightarrow$ x^2 is odd.

   c x is even $\Rightarrow$ x^2 is even.

   d x is prime $\Leftrightarrow$ x^2 has exactly two factors.

   e $\frac{1}{x+1} - \frac{2}{x} \equiv \frac{-(x+2)}{x(x+1)}, x > 0$

   f x is divisible by 2 & 3 $\Rightarrow$ x is divisible by 6.

   g $(x+1)^3 - (x-1)^3 \equiv 2$.

2. Which of these statements of equality are also identities?

   a $2x + 3 = 7$

   b $x(x-2) = x^2 - 2x$

   c $x = \sqrt{x^2}$

   d $\frac{1}{x-1} - \frac{2}{x+1} \equiv \frac{-(x-3)}{x^2-1}, x > 0$

   e $(a^x)^y = a^{xy}$

   f $\sqrt[a]{b} = b^a$

   g $\frac{1}{a} - \frac{1}{b} + \frac{1}{c} = \frac{a-b+c}{abc}$

Geometric Proofs

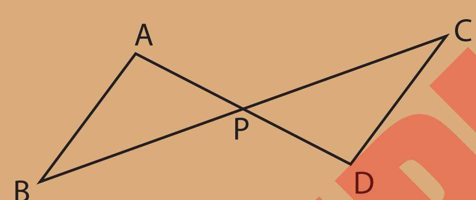
One of the best ways of practising the construction of direct proofs is to look at geometric examples. These depend on the axioms of Euclid and a number of geometric theorems.

Whilst they are not strictly on the syllabus, these proofs also played an important part in the long history of Mathematics. The following problems use Mathematics that you should have encountered during your Middle School Years.

A summary of some important facts can be found here:

Example A.8.3

Given that AP = DP and BP = CP, prove that the triangles ABP and CDP are congruent.



Proofs should proceed from accepted truths through a chain of implication to the desired statement.

In this case, 'accepted truth' includes the data in the question as well as previously proved theorems such as Pythagoras.

The data in this case AP = DP and BP = CP.

$\angle APB = \angle BPC$ (Vertically Opposite Angles)

$\Rightarrow$ ΔABP and ΔCDP are congruent (Side-Angle-Side criterion).

Notice that both of the implication have justifications. This is a key feature of all mathematical proofs - no statements are accepted as true unless they can be justified.

We were not asked to do this but some further inferences can be drawn:

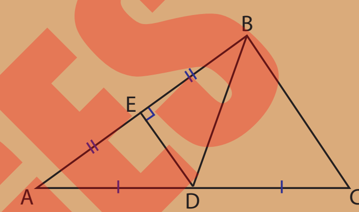
Since the triangles are congruent, it follows that $\angle ABP = \angle DCP$

Note that we must be careful to pair these angles correctly. It is this pair that are equal because they are opposite AP and DP and AP = DP..

$\angle ABP = \angle DCP \Rightarrow$ AB is parallel to CD (alternate angles).

Example A.8.4

Prove that ΔDBC is isosceles



In ΔAED and ΔBDE:

AE = EB (given)

ED is common to both triangles

$\angle AED = \angle BED = 90^\circ$ (given) $\Rightarrow$ ΔAED and ΔBDE are congruent (side-angle-side criterion)

AD = BD

AD = DC (given) $\Rightarrow$

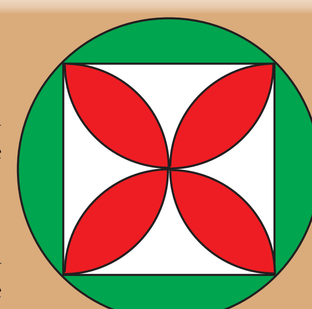
BD = DC (rule of syllogism) $\Rightarrow$

ΔBDC is isosceles

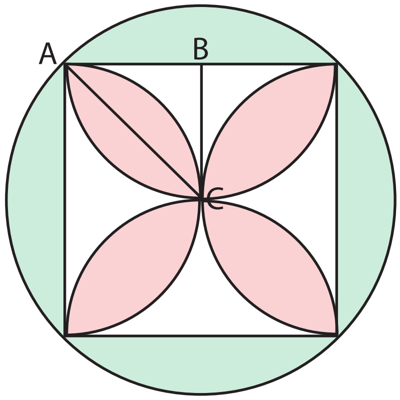
Example A.8.5

A square is inscribed in a circle. The red areas are bounded by semicircles.

Prove that the red shaded area is equal in magnitude to the green shaded area.



Be careful of falling for superficial proofs such as "There are four red bits and four green bits so the areas are the same".



Let the square by 2 by 2.

The area of the square is $2 \times 2 = 4$ square units (u^2).

$AB = BC = 1$ by symmetry.

$AC^2 = AB^2 + BC^2$ (Pythagoras)

$AC^2 = 1^2 + 1^2$

$\Rightarrow AC = \sqrt{2}$

Area of the large circle $= \pi r^2$

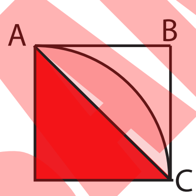
$= \pi \times \left(\sqrt{2}\right)^2$

$= 2\pi$ (u^2)

Green shaded area = area of circle – area of square

$2\pi - 4$ (u^2)

To find the area of the red 'petal shapes'.



The area with the two types of red shading is a quarter circle of radius 1.

Red shaded area $= \frac{1}{4}\pi r^2$

$= \frac{1}{4}\pi \times 1^2$

$= \frac{1}{4}\pi$

The surrounding square is 1 by 1 and has area 1 (u^2)

The white area bounded above $= 1 - \frac{1}{4}\pi$

Returning to the first diagram the red shaded area is the big square (2 by 2) minus 8 of these white areas.

The red area $= 4 - 8 \times (1 - \frac{1}{4}\pi)$

$= -4 + 2\pi$

$= 2\pi - 4.$

It follows that the red and green areas are the same.

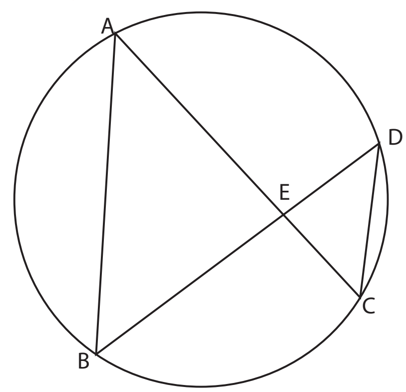
If you feel that the step we made right at the start (assuming a size for the square) was not legitimate, you are probably right.

This can all be put right by letting the size of the square be x.

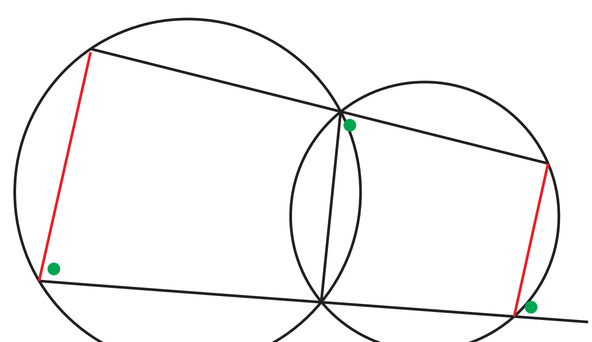
The whole proof can then proceed as before. The technique of tackling a difficult proof by simplifying it first, developing a proof and then using the same strategy to do the general case can be very useful.

Exercise A.8.2

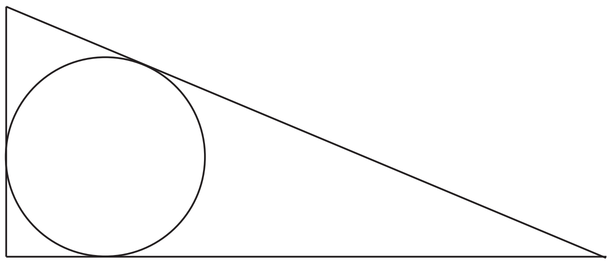
1. Prove that if $\angle BAC = \angle ACD$, triangles ABE and CDE are similar.



2. Prove that if the angles marked with the green dots are equal, then the red lines are parallel.

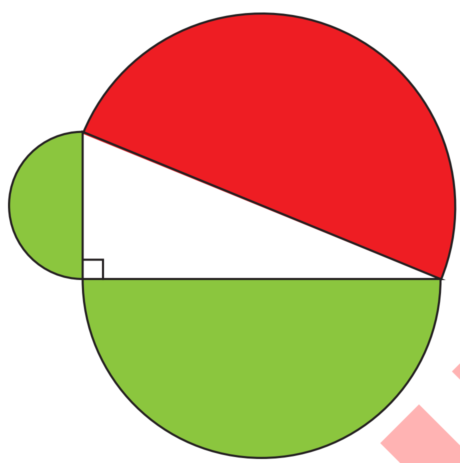


3. A circle is inscribed in a 5, 12, 13 right triangle



Prove that the radius of the circle = 2.

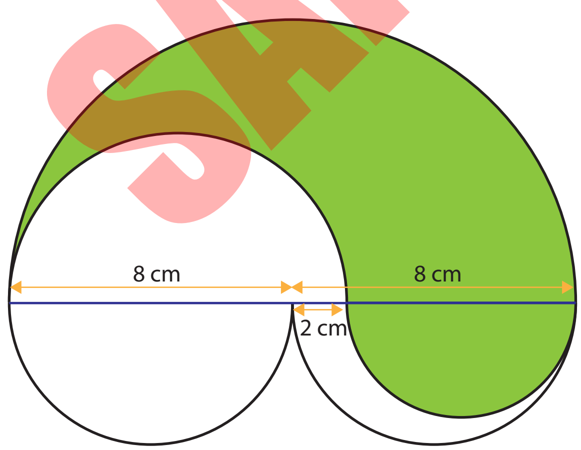
4. Three semicircles are drawn on the sides of a right angled triangle. Prove that the sum of the areas of the two smaller circles (green) is equal to the area of the large semicircle (red).



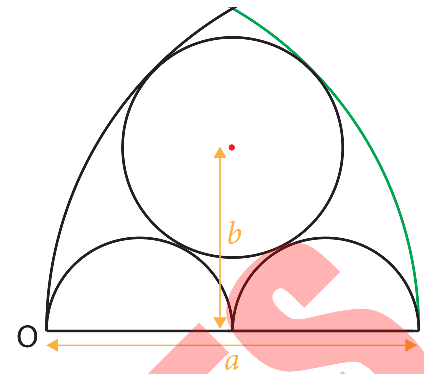
5. The figure consists of five semicircles all centred on the blue line. Prove that the white and green areas have:

i Equal areas

ii Equal perimeters.



6. Gothic Tracery. The diagram shows a pattern common in gothic window designs.



The main arch consists of the arcs of two circles of radius a. The green arc is centred at O.

Prove that $b = \frac{a\sqrt{6}}{5}$.

7. A right triangle has area A and perimeter $2P$. Prove that the hypotenuse is given by:

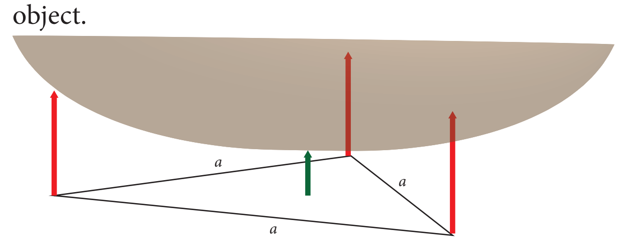
$$P - \frac{A}{P}$$

8. Two solids are made from twelve congruent equilateral triangles.

The first is a tetrahedron of volume T and the second is an octahedron of volume O.

Prove that $O = 4T$.

9. A spherometer is a device for measuring the radius of curvature of objects such as lenses. It consists of a triangular device with three prongs arranged in an equilateral triangle. A fourth prong can be screwed up and down so that all four make contact with the object.



The green prong makes contact with the lens (which has radius r) when it is h below the plane of the red prongs. Prove that:

$$r = \frac{a^2}{6h} + \frac{h}{2}$$

Numerical Proofs

Proofs of theorems involving numbers almost always deal with infinite sets. This means that 'proof by exhaustion' is seldom an option.

It is also the case that there are some superficially simple statements that have been very resistant to proof. Some examples are:

Twin Primes

Pairs of prime numbers such as 11 & 13 that are separated by one even number are said to be 'twin primes'. The truth of the statement 'There are an infinite number of twin primes' is unresolved.

Goldbach's Conjecture

Every even number greater than 2 can be written as the sum of two prime numbers. For example: 6=3+3, 8=3+5 etc.

This is unproved.

Fermat's Last Theorem

$a^n + b^n = c^n, a, b, c, n \in \mathbb{N}$ has no solutions for $n > 2$.

This has been proved recently. It was proposed in 1637 so it took over 300 years of trying!

The approaches we have been using (working from the data through a string of inferences to the required statement) work for these problems too.

Example A.8.6

Prove that for any four consecutive integers a, b, c, d:

$ab + ac + ad + bc + bd + cd + 1$ is divisible by 12.

Let:

$$a = n-1, b = n, c = n+1, d = n+2$$
$$ab + ac + ad + bc + bd + cd + 1$$
$$= (n-1)n + (n-1)(n+1) + (n-1)(n+2) + n(n+1)$$
$$\ldots + n(n+2) + (n+1)(n+2) + 1$$
$$= n^2 - n + n^2 - 1 + n^2 + n - 2 + n^2 + n + n^2 + 2n + n^2 + 3n + 2 + 1$$
$$= 6n^2 + 6n$$
$$= 6n(n+1)$$

$n(n+1)$ is the product of an odd and even number and so is even.

$6 \times$ an even number is divisible by 12, so:

$ab + ac + ad + bc + bd + cd + 1$ is divisible by 12.

Example A.8.7

The test for divisibility by 9 is:

Add the digits to get a digit sum. If the digit sum is divisible by 9, then the original number is divisible by 9.

We have suggested that a good way of approaching a general proof of this sort is to look at a particular example, frame a proof for that and use the strategy to complete the full truth.

Let us look at the problem of the divisibility by 9 of 567 (which has a digit sum of $5 + 6 + 2 = 18$).

According to the test, since 18 is divisible by 9, then 567 is divisible by 9.

In constructing the proof, write 567 as:

$567 = 5 \times 10^2 + 6 \times 10^1 + 7 \times 10^0$

Consider what happens if we divide every term in this equation by 9. What we are interested in is what the remainders will be.

Since 9, 99, 999, 9 999 etc. are all divisible by 9, it follows that 10^0, 10^1, 10^2, 10^3 etc, leave a remainder of 1 on division by 9.

Likewise, 5 leaves a remainder of 5 on division by 9, 6 leaves a remainder of 6 on division by 9 etc.

So the remainders of the right hand side are:

$5 \times 1 + 6 \times 1 + 7 \times 1 = 5 + 6 + 7 = 18.$

If the remainder is divisible by 9, then there is no remainder at all.

Generalising:

If $x = a_0 \times 10^0 + a_1 \times 10^1 + a_2 \times 10^2 + a_3 \times 10^3 + \ldots$

The remainders on division by 9 are:

$a_0 \times 1 + a_1 \times 1 + a_2 \times 1 + a_3 \times 1 + \ldots$ (ie. the digit sum).

So, x is divisible by 9 if and only if the digit sum is divisible by 9.

Note that this is a theorem that can be successively applied.

383, 942 has a digit sum of 29 which has a digit sum of 11 which has a digit sum of 2 which is not divisible by 9. Hence,

383, 942 is not divisible by 9.

Example A.8.8

Prove that between any two rational numbers there is at least one rational number.

What further can you conclude?

Again, we look at a specific example.

A simplified version is:

Find a rational number that is between $\frac{2}{3}$ and $\frac{3}{4}$.

The mean of two numbers is always between them.

The mean of $\frac{2}{3}$ and $\frac{3}{4}$ is $\dfrac{\frac{2}{3}+\frac{3}{4}}{2}=\dfrac{\frac{2\times4}{3\times4}+\frac{3\times3}{4\times3}}{2}$

$$=\frac{2\times4+3\times3}{2\times3\times4}$$

$$=\frac{17}{24}$$

If we want to check that this fraction is between $\frac{2}{3}$ and $\frac{3}{4}$, we need to use LCMs.

$$\frac{2}{3}:\frac{17}{24}:\frac{3}{4}$$

$$\frac{2\times8}{3\times8}:\frac{17}{24}:\frac{3\times6}{4\times6}$$

$$\frac{16}{24}:\frac{17}{24}:\frac{18}{24}$$

It is evident that: $\frac{16}{24}<\frac{17}{24}<\frac{18}{24}$

Thus, we have found a rational number between $\frac{2}{3}$ and $\frac{3}{4}$.

Now, follow this pattern to prove the general case:

Let the two rational numbers be: $\frac{a}{b},\frac{c}{d},a,b,c,d\in\mathbb{Z}$.

with - $\frac{a}{b}<\frac{c}{d}$.

The mean of these two numbers is: $\dfrac{\frac{a}{b}+\frac{c}{d}}{2}=\dfrac{\frac{ad}{bd}+\frac{ac}{bd}}{2}$

$$=\frac{ad+ac}{2bd}$$

Since a, b, c, d are all integers, both the numerator and denominator of this expression are integers.

Thus $\frac{ad+bc}{2bd}$ is a rational number between $\frac{a}{b},\frac{c}{d}$.

If you accept that a mean of two numbers must lie between them, the proof ends here.

Note that we can conclude that there must be another rational number between the lower number and the mean. There is also yet another rational between the mean and the larger number.

Successive use of this theorem means that we can infer that, between any two rational numbers, there are an infinite number of other rational numbers.

The check (which is not really necessary) is a bit more complex as we may be dealing with cases in which some of a, b, c, d are negative.

If a, b, c, $d > 0$: $\frac{a}{b}<\frac{ad+bc}{2bd}<\frac{c}{d}$

$$\frac{2ad}{2bd}<\frac{ad+bc}{2bd}<\frac{2bc}{2bd}$$

$$2ad<ad+bc<2bc$$

This contains two propositions: $2ad<ad+bc$

$$ad<bc$$

and $ad+bc<2bc$

$$ad<bc$$

That is, there is a single proposition here: $ad < bc$.

We have to be careful multiplying both sides of the inequality by $2bd$ in case this is negative. We have already specified that it is not.

Since we started with: $\frac{a}{b}<\frac{c}{d}$, it follows that: $\frac{ad}{bd}<\frac{bc}{bd}$.

Since bd is positive: $ad < bc$, which is what we need.

If bd is negative, inequality signs must be reversed if there is multiplication by it.

$$\frac{2ad}{2bd} < \frac{ad+bc}{2bd} < \frac{2bc}{2bd} \text{ becomes } 2ad > ad + bc > 2bc\,.$$

As before, this resolves to the single statement: $ad > bc$.

But the original premise also leads to this:

$$\frac{a}{b} < \frac{c}{d}$$
$$\frac{ad}{bd} < \frac{bc}{bd}$$
$$ad > bc$$

and the proof is complete.

Exercise A.8.3

1. Prove that every number that ends in 0 or 5 is divisible by 5.

2. Prove that every square number can be written as the sum of two triangle numbers.

3. Does this pattern continue?

   $1^3 = 1^2$

   $1^3 + 2^3 = (1 + 2)^2$

   $1^3 + 2^3 + 3^3 = (1 + 2 + 3)^2$

   $1^3 + 2^3 + 3^3 + 4^3 = (1 + 2 + 3 + 4)^2$

4. Generalise and prove that this pattern continues:

   $1 + 1 + 1 \qquad = 3$

   $1 + 2 + 3 + 2 + 1 \qquad = 9$

   $1 + 3 + 6 + 7 + 6 + 3 + 1 = \quad = 27$

5. Generalise and prove that this pattern continues:

   $1 - 1 + 1 \qquad = 1$

   $1 - 2 + 3 - 2 + 1 \qquad = 1$

   $1 - 3 + 6 - 7 + 6 - 3 + 1 = \quad = 1$

6. Prove that x is divisible by 11 if and only if alternating sum of its digits $a_0 - a_1 + a_2 - a_3 + a_4 + \cdots + a_m(-1)^m$ is divisible by 11.

7. Prove that the sum of the digits of the digits of the sum of the digits of the sum of the digits of 4444^{4444} is 7.

8. Prove that there exist rational numbers A & B such that A^B is irrational.

9. Following on from the previous question, prove that there exist irrational numbers A & B such that A^B is rational.

10. If one million factorial is written in full, how many zeros does it end in?

11. If a, b and c are integers such that $a|b$ and $b|c$,then $a|c$ means a divides into c.

12. Prove that if m is an integer then 3 divides $m^3 - m$.

13. Two integers are said to be relatively prime if their greatest common divisor is 1.

    i Show that if m is a positive integer, then $3m + 2$ and $5m + 3$ are relatively prime.

    ii Show that if a and b are relatively prime integers,then the greatest common divisor of $a + 2b$ and $2a + b = 1$ or 3.

14. The diagram shows the pattern known a Leibniz's Harmonic Triangle. It is cousin to Pascal's Triangle. Can you see how it 'works'?

$$\begin{array}{ccccccccc}
 & & & & \frac{1}{1} & & & & \\
 & & & \frac{1}{2} & & \frac{1}{2} & & & \\
 & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & \\
 & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & \\
\frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5}
\end{array}$$

Use the triangle to investigate the truth, or otherwise, of these postulates.

i $\dfrac{1}{1} = \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{12} + \dfrac{1}{20} + \dfrac{1}{30} + \ldots$

ii $\dfrac{1}{2} = \dfrac{1}{3} + \dfrac{1}{12} + \dfrac{1}{30} + \dfrac{1}{60} + \dfrac{1}{105} + \ldots$

iii $\dfrac{1}{3} = \dfrac{1}{4} + \dfrac{1}{20} + \dfrac{1}{60} + \dfrac{1}{140} + \dfrac{1}{280} + \ldots$

Algebraic Proofs

We have already been using algebra to construct proofs as they relate to postulates involving numbers and geometric shapes.

Here are two further examples:

Example A.8.9

Prove that for $n \in \mathbb{N}, (3n+1)^2 - (3n-1)^2$ is a multiple of 4.

$$\begin{aligned}(3n+1)^2 - (3n-1)^2 &= 9n^2 + 6n + 1 - \left(9n^2 - 6n + 1\right) \\ &= 12n\end{aligned}$$

n is a whole number, 12 is a multiple of 4 so $12n$ is also a multiple of 4.

Example A.8.10

Prove that the sum of the squares of any two consecutive even numbers is divisible by 4.

For a natural number, $2n$ is even. The next even number is $2n + 2$.

The sum of the squares is: $(2n)^2 + (2n+2)^2 = 4n^2 + 4n^2 + 4n + 4$

$$= 4\left(2n^2 + n + 1\right)$$

Since 4 is divisible by 4 and the bracket is a whole number, the expression is divisible by 4 as required.

Exercise A.8.4

1. Show that the sum of any three consecutive even numbers is a multiple of 6.

2. Show that $n \in \mathbb{N}, (2n+3)^2 - (2n-3)^2$ is divisible by 8.

3. Compare the two series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \ldots$$

Hence prove that: $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

4. Prove that every recurring decimal can be written as a mixed number and is, hence, rational. Hence conclude that the decimal representation of π is not a recurring decimal.

5. Prove that the conjecture:

For all natural numbers n, $n^n > n!$ is false.

6. Investigate the truth, or otherwise, of Stirling's Formula:

$$n! \simeq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

7. Prove that $\frac{(k+2)!}{(k-1)!} = k^3 + 3k^2 + 2k, k > 1, k \in \mathbb{N}$.

8. Prove that a set with n members has 2^n subsets (including the set itself and the empty set).

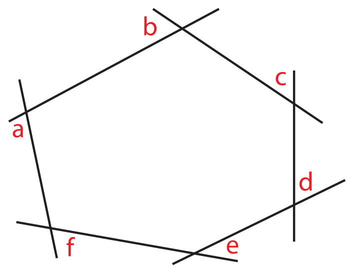
9. Use the laws of indices to prove that for non-zero a, a^0 is 1.

10. Consider $\lim_{n \to \infty} \left(\sqrt{n(n+1)} - n\right)$.

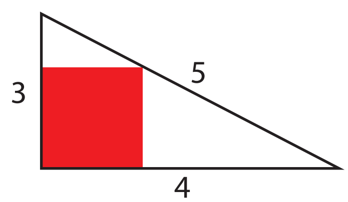
Hence prove that π can be the limit of a sequence of numbers of the form $\sqrt{n} - \sqrt{m}$.

Exercise A.8.5

1. Prove that a+b+c+d+e+f=360˚.



2. A square is inscribed in a 3, 4, 5 triangle. Prove that its side length is $1^5/_7$ units.

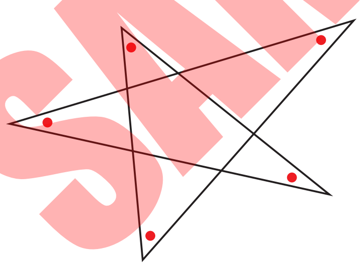


3. A '7 smooth number' is one that factorises into prime numbers less than or equal to 7.

   Prove that 86 436 000 is 7 smooth.

4. Prove that $2^n + 1$ where n is an odd number is divisible by 3.

5. Prove that the sum of the angles marked with the red dots is 180˚.



6. Prove that there are at least two positive integers a, b >5 such that $a^3 + b^3 = a^4$.

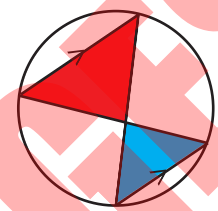
7. Prove that if n is an odd positive integer $n^4 - 18n^2 + 17$ is divisible by 64.

8. The binary operation ⊙ is defined on the set $\{a,b,c,d\}$ by this table:

⊙	a	b	c	d
a	a	c	d	b
b	c	b	a	d
c	d	a	c	a
d	b	d	a	d

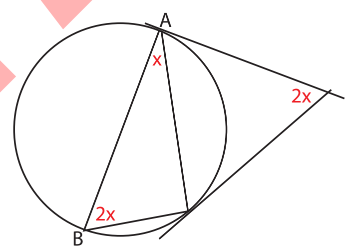
   Prove that the operation is commutative ie. $X \odot Y = Y \odot X$ for all members of the set.

9. Prove that the two coloured triangles are similar.



10. Prove that if n is not divisible by 7, then either $n^3 + 1$ or $n^3 - 1$ is divisible by 7.

11. Prove that AB is a diameter of the circle.



12. Two circles have radii 1 and $3+2\sqrt{2}$ and touch externally. Prove that their common tangents are perpendicular.

13. If $p_1, p_2, p_3, \ldots p_n$ are the first n prime numbers, prove that: $p_1 \times p_2 \times p_3 \times \ldots \times p_n + 1$ is also prime and hence prove that there are an infinite number of prime numbers.

14. $a>0$, $b>0$, $c>0$ and $a+b+c=2$ prove $abc \leq 1$.

15. Prove that an integer that ends with 7 cannot be a perfect square.

16. If a and b are integers and b is odd, prove that $x^2 + 2ax + 2b = 0$ has no rational roots.

17. Prove that for $n \in \mathbb{Z}^+, n! < \left(\frac{n+1}{2}\right)^n$.

Pascal and Fibonnaci

Pascal's Triangle is the pattern shown here. It is of considerable importance in Probability Theory, The Binomial Theorem etc.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

The numbers down the side are all 1. The numbers in the body of the triangle are such that each number (eg. yellow) is the sum of the two numbers above and to either side.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

If we compare the combinatorial numbers (which we encountered when studying the Binomial Theorem) with the numbers in the Pascal Triangle, there is a superficial similarity.

$$^0C_0=1$$

$$^1C_0=1 \quad ^1C_1=1$$

$$^2C_0=1 \quad ^2C_1=2 \quad ^2C_2=1$$

$$^3C_0=1 \quad ^3C_1=3 \quad ^3C_2=3 \quad ^3C_3=1$$

$$^4C_0=1 \quad ^4C_1=4 \quad ^4C_2=6 \quad ^4C_3=4 \quad ^4C_4=1$$

There are a number of things that we need to prove if we want to establish that the resemblance is more than superficial.

The first is that the beginning and end numbers in each row are 1.

This means that, if we are to prove the whole thing, we start by proving:

$$^nC_0={}^nC_n=1, n\in\mathbb{N}$$

Hint: $^nC_r=\dfrac{n!}{(n-r)!r!}$.

The hard bit is the pattern in the body of the table.

We have advised you to tackle difficult proofs like this by taking a particular case, proving that and then using the same strategy to prove the general case.

Taking the example marked in yellow and green in the diagram:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

The numbers in green are: $^5C_2=10$ and $^5C_3=10$.

The number in yellow is: $^6C_3=20$.

Just using a calculator and observing that it 'works' is not good enough. We need to work from the definition of combinatorial numbers to prove this special case:

$^5C_2+{}^5C_3=\dfrac{5!}{(5-2)!2!}+\dfrac{5!}{(5-3)!3!}$ which has to be equal to:

$$^6C_3=\frac{6!}{(6-3)!3!}$$

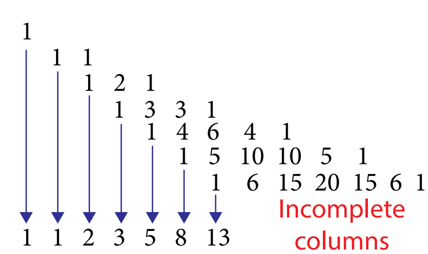
There is some fairly intricate LCM work involved here as you are working with numbers in factorial form. Remember to notice how this works as, to complete the proof, you will need to tackle the general case.

Further Patterns

The rows of Pascal's Triangle appear to sum to powers of 2. Can you prove that they all do?

The first few triangle numbers (1, 3, 6, 10, 15,...) appear in two of the diagonals. Is this a pattern that continues?

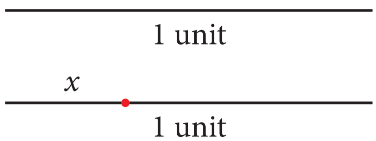
If Pascal's Triangle is written in echelon form, it appears that the Fibonacci sequence emerges.



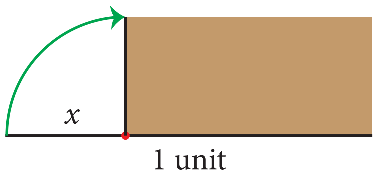
Does this pattern continue. Can you prove it?

There are many surprising patterns in the Fibonacci Sequence. For example, the limit of the ratio of successive terms is related to the Golden Mean

The Golden Mean



Place a dividing point such that the ratio of the shorter part to the longer part is the same as the ratio of the longer part to the whole.
This becomes the basis of the 'perfect rectangle'.



The Fibonacci Limit

If the nth Fibonacci Number is denoted by F_n, find:

$$\lim_{n \to \infty} \left(\frac{F_{n+1}}{F_n} \right)$$

Can you prove the relationship between this and the Golden Mean?

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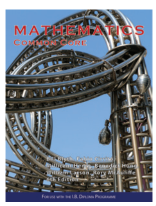
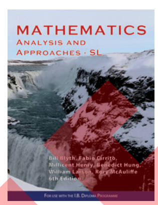
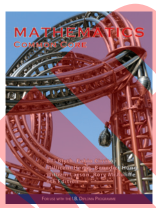
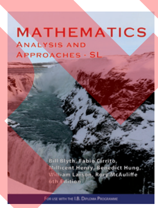
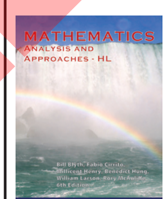
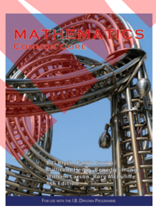
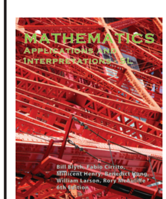
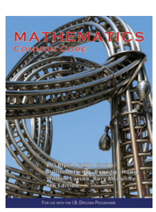
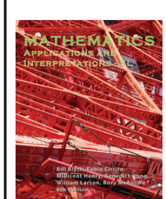
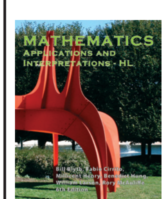
Mathematics: Analysis and Approaches (SL)						Pure(SL)
Mathematics: Analysis and Approaches (HL)						Pure(HL)
Mathematics: Applications and Interpretations (SL)						Applied(SL)
Mathematics: Applications and Interpretations (HL)						Applied(HL)

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