MATHEMATICS

ANALYSIS AND APPROACHES - HL

SAMPLE CHAPTERS

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***Note that this sample chapter has an additional 16 pages of fully worked solutions and an extra page of exercises available through QR links or from the IBID website.

E.8 Further Limits

AHL 5.13

AHL 5.13 L'Hôpital's Rule

French mathematician Guillaume François Antoine, Marquis de l'Hôpital (1661 – 1704) is chiefly remembered for a limits rule that bears his name. The name is also frequently spelled l'Hôspital.

L'Hôpital's Rule is particularly useful in evaluating limits that involve expressions that resolve to $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

L'Hôpital's Rule is usually stated as:

If $\lim\limits_{x \to c} \frac{f(x)}{g(x)}$ takes the indeterminate form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

The full proof of this result is quite complex. We will show that the result holds true for the indeterminate form when $f(c) = g(c) = 0$.

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f(x)-0}{g(x)-0}$$

$$= \lim_{x \to c} \frac{f(x)-f(c)}{g(x)-g(c)}$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c}}{\lim\limits_{x \to c} \frac{g(x)-g(c)}{x-c}}$$

$$= \frac{\lim\limits_{x \to c} f'(x)}{\lim\limits_{x \to c} g'(x)}$$

$$= \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

So, as long as $g'(c) \neq 0$, the result is complete.

If the quotient of the derivatives is still of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ we have to apply L'Hopital's rule again and calculate the quotient of the second, third,…. derivatives at $x = c$ until the quotient yields a properly defined value.

The first of our examples deals with a very important limit that is crucial in the first principles differential of the trigonometric functions.

Example E.8.1

Use L'Hôpital's Rule to evaluate: $\lim\limits_{x \to 0} \frac{\sin(x)}{x}$

As $\frac{\sin(0)}{0}$ is of the form $\frac{0}{0}$ we can apply L'Hopital's Rule.

Letting: $f(x)=\sin(x), g(x)=x$, we use calculus:

$$f'(x)=\cos(x), g'(x)=1$$

$$\lim_{x\to 0}\frac{\sin(x)}{x}=\lim_{x\to 0}\frac{\cos(x)}{1}$$

$$=\frac{\cos(0)}{1}$$

$$=1$$

Example E.8.2

Evaluate: $\lim_{x\to 1}\frac{\ln(x)}{x-1}$

Let: $f(x)=\ln(x), g(x)=x-1$

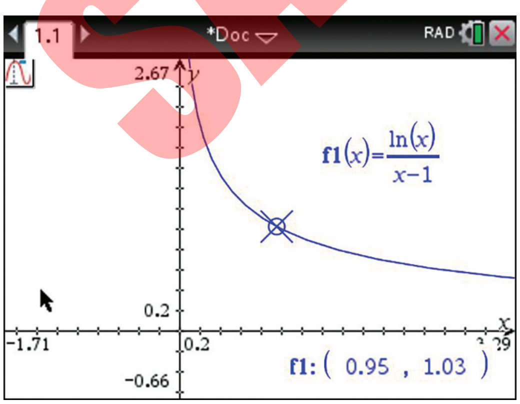
$$f'(x)=\frac{1}{x}, g'(x)=1$$

$f(1)=\ln(1)=0, g(1)=1-1=0$ and so L'Hopital's rule is applicable.

$$\lim_{x\to 1}\frac{\ln(x)}{x-1}=\lim_{x\to 1}\frac{\frac{1}{x}}{1}$$

$$=1$$

This cannot be fully checked using a calculator as any attempt to evaluate the expression at $x = 1$ will give an error message. However, plotting the graph and using trace will support our answer:



Example E.8.3

Evaluate: $\lim_{x\to 0}\frac{\cot(x)}{\ln(x)}$

Let: $f(x)=\cot(x), g(x)=\ln(x)$

We have a $\frac{\infty}{-\infty}$ limit and can use L'Hopital's Rule.

$$\lim_{x\to 0}\frac{\cot(x)}{\ln(x)}=\lim_{x\to 0}\frac{\left(-\frac{1}{\sin^2(x)}\right)}{\left(\frac{1}{x}\right)}$$

$$=-\lim_{x\to 0}\frac{x}{\sin^2(x)}$$

This is $\frac{0}{0}$ and we apply the rule a second time.

$$-\lim_{x\to 0}\frac{x}{\sin^2(x)}=-\lim_{x\to 0}\frac{1}{2\sin(x)\cos(x)}$$

$$=-\lim_{x\to 0}\frac{1}{\sin(2x)}$$

$$=-\infty$$

Example E.8.4

Evaluate: $\lim_{x\to \infty}\frac{\ln(x)}{x^n}$

We have a $\frac{\infty}{\infty}$ limit and can use L'Hopital's Rule.

$$\lim_{x\to \infty}\frac{\ln(x)}{x^n}=\lim_{x\to \infty}\frac{\frac{1}{x}}{nx^{n-1}}$$

$$=\lim_{x\to \infty}\frac{0}{nx^{n-1}}$$

$$=0$$

Example E.8.5

Evaluate: $\lim_{x\to 0} x\ln(x)$

This product is of the form $0 \times -\infty$ and so the expression must be rewritten as:

$$\lim_{x\to 0} x\ln(x) = \lim_{x\to 0} \frac{\ln(x)}{\frac{1}{x}}$$

Next, use the rule:

$$\lim_{x\to 0} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$
$$= \lim_{x\to 0} (-x)$$
$$= 0$$

Example E.8.6

Evaluate: $\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right)$

This product is of the form $0 \times \infty$ and so the expression must be rewritten as:

$$\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}}$$

This is of the form $\frac{0}{0}$ so we can apply L'Hopital's Rule.

$$\lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{\pi}{x^2}\cos\left(\frac{\pi}{x}\right)}{-\frac{1}{x^2}}$$
$$= \lim_{x\to\infty} \pi\cos\left(\frac{\pi}{x}\right)$$
$$= \pi$$

Example E.8.7

Evaluate: $\lim_{x\to\frac{\pi}{4}} \frac{1-\tan(x)}{\cos(2x)}$

This is of the form $\frac{0}{0}$ so we can apply L'Hopital's Rule.

Let: $f(x) = 1-\tan(x), f'(x) = -\sec^2(x)$
$g(x) = \cos(2x), g'(x) = -2\sin(2x)$

$$\lim_{x\to\frac{\pi}{4}} \frac{1-\tan(x)}{\cos(2x)} = \lim_{x\to\frac{\pi}{4}} \frac{-\sec^2(x)}{-2\sin(2x)}$$
$$= \lim_{x\to\frac{\pi}{4}} \frac{-\sec^2\left(\frac{\pi}{4}\right)}{-2\sin\left(\frac{\pi}{2}\right)}$$
$$= \frac{\frac{1}{\cos^2\left(\frac{\pi}{4}\right)}}{2\sin\left(\frac{\pi}{2}\right)}$$
$$= \frac{\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2}}{2}$$
$$= \frac{2}{2}$$
$$= 1$$

Exercise E.8.1

1. Determine the following limits.

a $\lim_{x\to 0} \left(\frac{x+\sin 2x}{x-\sin 2x}\right)$

b $\lim_{x\to\pi} \left(\frac{x-\pi}{\sin 2x}\right)$

c $\lim_{x\to\frac{\pi}{2}} \left(\frac{\sin 2x}{\cos x}\right)$

2. Determine the following limits.

a $\lim\limits_{x\to\infty}\left(\dfrac{x}{e^{2x}}\right)$

b $\lim\limits_{x\to\infty}\left(\dfrac{\ln x}{x}\right)$

c $\lim\limits_{x\to\infty}\left(\dfrac{2x}{x+\ln x}\right)$

3. Determine the following limits.

a $\lim\limits_{x\to 0}\left(\dfrac{2x}{x+\sin x}\right)$

b $\lim\limits_{x\to 0}\left(\dfrac{\cos x-1}{x^2}\right)$

c $\lim\limits_{x\to 0}\left(\dfrac{x-\sin x}{x^3}\right)$

4. Determine the following limits.

a $\lim\limits_{x\to\frac{\pi}{2}}\left(\dfrac{\sin x-1}{\cos x}\right)$

b $\lim\limits_{x\to 0^+} x\ln\left(1+\dfrac{1}{x}\right)$

c $\lim\limits_{x\to 1}\left(\dfrac{\ln x-(x-1)}{x-1}\right)$

5. Determine the following limits, if they exist.

a $\lim\limits_{x\to\frac{\pi}{2}}(\tan x+\sec x)$

b $\lim\limits_{x\to 1}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)$

c $\lim\limits_{x\to 1}\left(\dfrac{\ln x}{x^2-x}\right)$

6. What is wrong in the calculation:

$$\lim_{x\to 0}\left(\frac{\cos x}{x^2}\right)=\lim_{x\to 0}\left(\frac{-\sin x}{2x}\right)$$
$$=\lim_{x\to 0}\left(\frac{-\cos x}{2}\right)$$
$$=-\frac{1}{2}$$

7. Determine the following limits, if they exist.

a $\lim\limits_{x\to\infty}\left(\dfrac{1}{x}e^x\right)$

b $\lim\limits_{x\to\infty}\left(\dfrac{x^2}{e^x}\right)$

8. Evaluate the following limits, if they exist.

a $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^2e^x}$

b $\lim\limits_{x\to 0}\dfrac{1-\cos x}{\sin^2 x}$

c $\lim\limits_{x\to 1}\left(\dfrac{x^4-7x^3+8x^2-2}{x^3+5x-6}\right)$

Extra questions

Answers

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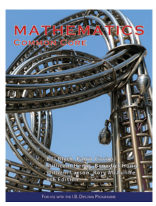
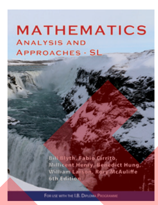
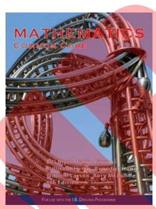
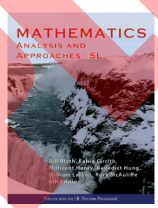
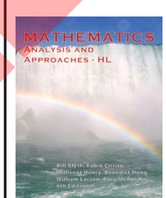
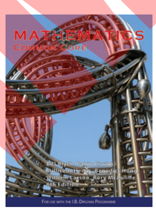
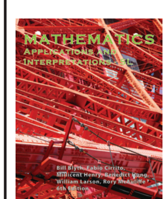
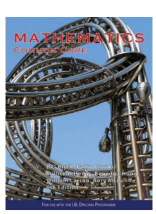
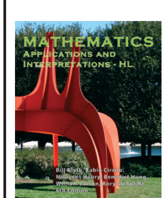
Mathematics: Analysis and Approaches (SL)						Pure(SL)
Mathematics: Analysis and Approaches (HL)						Pure(HL)
Mathematics: Applications and Interpretations (SL)						Applied(SL)
Mathematics: Applications and Interpretations (HL)						Applied(HL)

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