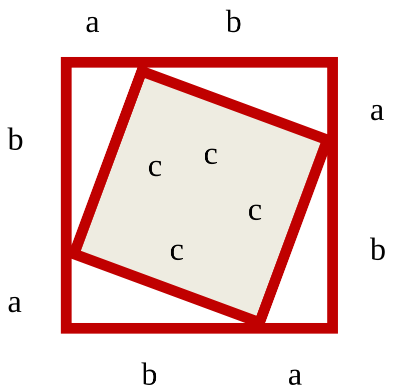
Quadratics and Pythagorean Triples

Pythagoreas's Theorem can be derived from two squares and 4 triangles using area calculations of these shapes.



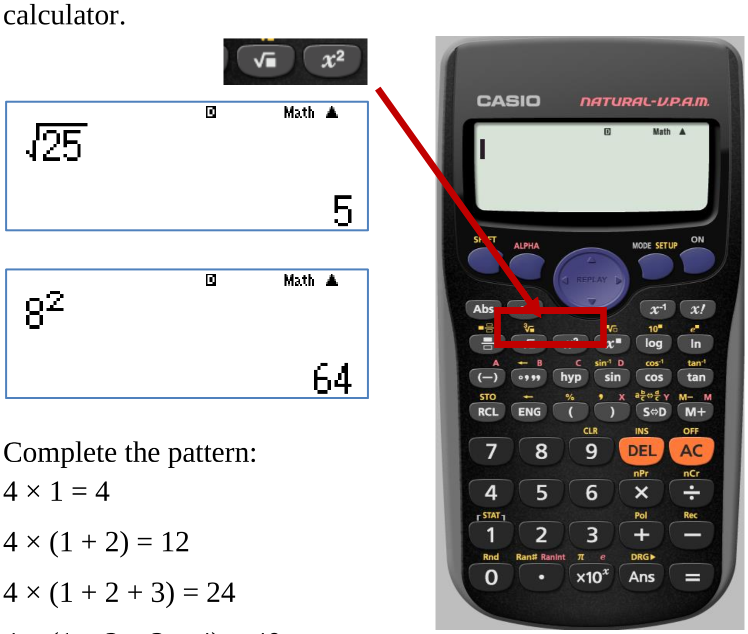
The area of the large square = $(a + b)^2$ and the area of the small square = c^2, and each of the triangles area are ½ab respectively.

Equating the areas gives: $(a + b)^2 = c^2 + 4 \times ½ab$

Expanding: $a^2 + 2ab + b^2 = c^2 + 2ab$

Simplifying: $a^2 + b^2 = c^2$

Squares and square roots buttons on the scientific calculator.



Complete the pattern:

4 × 1 = 4

4 × (1 + 2) = 12

4 × (1 + 2 + 3) = 24

4 × (1 + 2 + 3 + 4) = 40

4 × (1 + 2 + 3 + 4 + 5) = _______

4 × (1 + 2 + 3 + 4 + 5 + 6) = _______

4 × (1 + 2 + 3 + 4 + 5 + 6 + 7) = ____________________

4 × (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8) = _________________

4 × (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = _______________

4 × (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10) = ___________

What do you notice?

Complete the table below.

Pythagorean Triple (a, b, c)			
a	**b**	**c**	**$a^2 + b^2 = c^2$**
3	4	5	$3^2 + 4^2$ = 25 [= 5^2]
5	12	13	
7	24	25	
9	40	41	
11	60	61	
13	84	85	
15	112	113	
17	144	145	
19	180	181	
21	220	221	
23			
25			

What do you notice?

Now, consider the values of **a** + **b**, which values are in the first two columns in the table above.

Complete the table below.

n	a	b	a + b	$2(n + 1)^2 - 1$
1	3	4	7	$2 \times 2^2 - 1 = 7$
2	5	12	17	$2 \times 3^2 - 1 = 17$
3	7	24	31	$2 \times 4^2 - 1 = 31$
4	9	40		
5	11	60		
6	13	84		
7	15	112		
8	17	144		
9	19	180		
10	21	220		
11	23			
12	25			

What do you notice?

These Pythagorean Triples are known as **Primitive Pythagorean Triples** as the difference between one of the short sides and the hypotenuse is **always 1**.

Facts:

1. In any Pythagorean Triple one of the sides of the triangle is divisible by 3.
2. There are an infinite number of Pythagorean Triples where the hypotenuse is one unit more that one of the short sides.
3. Every integer except 1, 2 can be the shortest side of a Pythagorean Triple.
4. If a circle is inscribed inside a Primitive Pythagorean Triple, then the radius is always a natural number.
5. In a Primitive Pythagorean Triple the sum of the hypotenuse and the odd short side is always twice a square number.
6. The famous French Mathematician, Pierre de Fermat (1601 – 1665) proved that it was impossible for the area of a Pythagorean Triple to equal a square number.

Can you find, show examples or prove of any of the above facts?

What else can you find out about Pythagorean Triples?

Can you find a relationship between the short sides and hypotenuse for a Pythagorean Triple where the hypotenuse is two units more that one of the short sides? Or three or 4? ...

If you make a = $m^2 - n^2$, b = $2mn$ and c = $m^2 + n^2$, check that $a^2 + b^2 = c^2$ is true, i.e. $(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$

If m and n are both ≥ 1, and one is an even number and the other is an odd number respectively then complete the table below.

		a	b	c				
m	n	$m^2 - n^2$	$2mn$	$m^2 + n^2$	→	a	b	c
2	1	$2^2 - 1^2$	2×2×1	$2^2 + 1^2$	→	3	4	5
3	2				→			
4	3				→			
5	4				→			
6	5				→			
7	6				→			
8	7				→			
9	8				→			
10	9				→			

What do you notice?