Moessner's Theorem.

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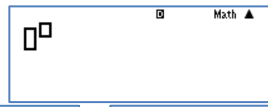
Moessner's Theorem describes a procedure for generating a n integer sequences that lead unexpectedly to the sequence of nth powers 1^n, 2^n, 3^n, … for $n > 1$.
To generate the first sequence, write down the positive integers 1, 2, 3, … , then cross out every n^{th} element. For the second sequence, compute the prefix sums of the first sequence, ignoring the crossed-out elements, then cross out every $(n-1)^{th}$ element. For the third sequence, compute the prefix sums of the second sequence, then cross out every $(n-2)^{th}$ element, and so on until you have only one number for each 'n-group' that you started with.

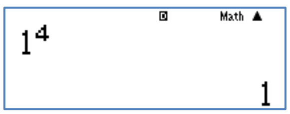
For example, for $n = 4$:

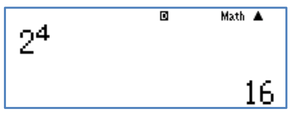
1 2 3 ~~4~~	5 6 7 ~~8~~	9 10 11 ~~12~~	13 14 15 ~~16~~	17 18 19 ~~20~~	21 22 23 ~~24~~
1 3 ~~6~~	11 17 ~~24~~	33 43 ~~54~~	67 81 ~~96~~	113 131 ~~150~~	171 193 ~~216~~
1 ~~4~~	15 ~~32~~	65 ~~108~~	175 ~~256~~	369 ~~500~~	671 ~~864~~
1	16	81	256	625	1296

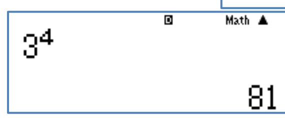
Moessner's theorem says that the final sequence is 1^n, 2^n, 3^n, …

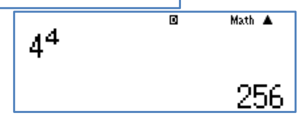
Checking: $1 = 1^4$, $16 = 2^4$, $81 = 3^4$, $256 = 4^4$, $625 = 5^4$, $1296 = 6^4$











Try for *n* = 3

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27

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_   _   _   _   _   _   _   _   _

Try for *n* = 5

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

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_ _   _ _   _ _   _ _   _ _

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Try for *n* = 6

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30

Questions:

1. What happens when $n = 1$?
2. What is the value of n so that the 3rd and 4th number in the last sequence generated are 729 and 6 561 respectively?
3. In the construction of Moessner's Theorem, the initial step size n is constant.
   (a) What would happen if you increase it at each step? Repeat the construction starting with a step size of one and increasing the step size by one each time. Thus, in the first sequence, we cross out 1, 3, 6, 10, … What sequence of numbers are generated now?
   (b) What would happen if you increased n as an arithmetic sequence? Thus, in the first sequence, we cross out 1, 3, 5, 7, … What sequence of numbers are generated now?

Hint: Factorials. $3! = 3\times2\times1 = 6$ Hint: $a\times1^{n-1}$, $(a+d)\times2^{n-1}$, $(a+2d)\times3^{n-1}$, …

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