APPENDIX-A

A.1 SI Units

Base Quantity	SI Base Unit Name	SI Base Unit Symbol
Length	metre	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	kelvin	K
Amount of substance	mole	mol
Luminous intensity	candela	cd

A.2 Some Derived SI Units

Quantity	Name	Symbol	Expression in Terms of Base Units	Expression in Terms of SI Units
Plane angle	radian	ω	m/m	
Frequency	hertz	Hz	s^{-1}	
Force	newton	N	$kg \cdot m/s^2$	J/m
Pressure	pascal	Pa	$kg/m \cdot s^2$	N/m^2
Energy	joule	J	$kg \cdot m^2/s^2$	$N \cdot m$
Power	watt	W	$kg \cdot m^2/s^3$	J/s
Electric charge	coulomb	C	$A \cdot s$	
Electric potential	volt	V	$kg \cdot m^2/A \cdot s^3$	W/A
Capacitance	farad	F	$A^2 \cdot s^4/kg \cdot m^2$	C/V
Electric resistance	ohm	Ω	$kg \cdot m^2/A^2 \cdot s^3$	V/A
Magnetic flux	weber	ϕ	$kg \cdot m^2/A \cdot s^2$	$T \cdot m$
Magnetic field	tesla	B	$kg/A \cdot s^2$	Nb/m^2 or tesla
Inductance	henry	L	$kg \cdot m^2/A^2 \cdot s^2$	$T \cdot m^2/A$

APPENDIX-B

B.1 Conversion Factors

Length

	m	cm	km	in.	ft	mi
1 meter	1	10^2	10^{-3}	39.37	3.281	6.214×10^{-4}
1 centimeter	10^{-2}	1	10^{-5}	0.3937	3.281×10^{-2}	6.214×10^{-6}
1 kilometer	10^3	10^5	1	3.937×10^4	3281×10^3	0.6214
1 inch	2.540×10^{-2}	2.540	2.540×10^{-5}	1	8.333×10^{-2}	1.578×10^{-5}
1 foot	0.3048	30.48	3.048×10^{-4}	12	1	1.894×10^{-4}
1 mile	1609	1.609×10^5	1.609	6.336×10^4	5280	1

Mass

	kg	g	slug	u
1 kilogram	1	10^3	6.854×10^{-2}	6.024×10^{26}
1 gram	10^{-3}	1	6.852×10^{-5}	6.024×10^{23}
1 slug	14.59	1.459×10^4	1	8.789×10^{27}
1 atomic mass unit	1.660×10^{-27}	1.660×10^{-24}	1.137×10^{-28}	1

Note : 1 metric ton = 1000 kg.

Time

	s	min	h	day	yr
1 second	1	1.667×10^{-2}	2.778×10^{-4}	1.157×10^{-5}	3.169×10^{-8}
1 minute	60	1	1.667×10^{-2}	6.944×10^{-4}	1.901×10^{-6}
1 hour	3600	60	1	4.167×10^{-2}	1.141×10^{-4}
1 day	8.640×10^{4}	1440	24	1	2.738×10^{-5}
1 year	3.156×10^{7}	5.259×10^{5}	8.766×10^{3}	365.2	1

Speed

	m/s	cm/s	ft/s	mi/h
1 meter per second	1	10^{2}	3.281	2.237
1 centimeter per second	10^{-2}	1	3.281×10^{-2}	2.237×10^{-2}
1 foot per second	0.3048	30.48	1	0.6818
1 mile per hour	0.4470	44.70	1.467	1

Note : 1 mi/min = 60 mi/h = 88 ft/s.

Force

	N	lb
1 newton	1	0.2248
1 pound	4.448	1

Energy, Energy Transfer

	J	ft · lb	eV
1 joule	1	0.7376	6.242×10^{18}
1 foot-pound	1.356	1	8.464×10^{18}
1 electron volt	1.602×10^{-19}	1.182×10^{-19}	1
1 calorie	4.186	3.087	2.613×10^{19}
1 British thermal unit	1.055×10^{3}	7.779×10^{2}	6.585×10^{21}
1 kilowatt-hour	3.600×10^{6}	2.655×10^{6}	2.247×10^{25}

	cal	Btu	kWh
1 joule	0.2389	9.481×10^{-4}	2.778×10^{-7}
1 foot-pound	0.3239	1.285×10^{-3}	3.766×10^{-7}
1 electron volt	3.827×10^{-20}	1.519×10^{-22}	4.450×10^{-26}
1 calorie	1	3.968×10^{-3}	1.163×10^{-6}
1 British thermal unit	2.520×10^{2}	1	2.930×10^{-4}
1 kilowatt-hour	8.601×10^{5}	3.413×10^{2}	1

Pressure

	Pa	atm
1 pascal	1	9.869×10^{-6}
1 atmosphere	1.013×10^{5}	1
1 centimeter mercury[a]	1.333×10^{3}	1.316×10^{-2}
1 pound per square inch	6.895×10^{3}	6.805×10^{-2}
1 pound per square foot	47.88	4.725×10^{-4}

	cm Hg	lb/in.^{2}	lb/ft^2
1 pascal	7.501×10^{-4}	1.450×10^{-4}	2.089×10^{-2}
1 atmosphere	76	14.70	2.116×10^{3}
1 centimeter mercury[a]	1	0.1943	27.85
1 pound per square inch	5.171	1	144
1 pound per square foot	3.591×10^{-2}	6.944×10^{-3}	1

At 0°C and at a location where the free-fall acceleration has its "standard" value, 9.80665 m/s^2.

B.2 Conversions of useful physical quantities

Length

1 in. = 2.54 cm (exact)
1 m = 39.37 in. = 3.281 ft
1 ft = 0.3048 m
12 in. = 1 ft
3 ft = 1 yd
1 yd = 0.9144 m
1 km = 0.621 mi
1 mi = 1.609 km
1 mi = 5280 ft
1 μm = 10^{-6} m = 10^3 nm
1 light-year = 9.461×10^{15} m

Area

1 m^2 = 10^4 cm^2 = 10.76 ft^2
1 ft^2 = 0.0929 m^2 = 144 $in.^2$
1 $in.^2$ = 6.452 cm^2

Volume

1 m^3 = 10^6 cm^3 = 6.102×10^4 $in.^3$
1 ft^3 = 1 728 $in.^3$ = 2.83×10^{-2} m^3
1 L = 1000 cm^3 = 1.0576 qt = 0.0353 ft^3
1 ft^3 = 7.481 gal = 28.32 L = 2.832×10^{-2} m^3
1 gal = 3.786 L = 231 $in.^3$

Mass

1000 kg = 1 t (metric ton)
1 slug = 14.59 kg
1 u = 1.66×10^{-27} kg = 931.5 MeV/c^2

Force

1 N = 0.2248 lb
1 lb = 4.448 N

Velocity

1 mi/h = 1.47 ft/s = 0.447 m/s = 1.61 km/h
1 m/s = 100 cm/s = 3.281 ft/s
1 mi/min = 60 mi/h = 88 ft/s

Acceleration

1 m/s^2 = 3.28 ft/s^2 = 100 cm/s^2
1 ft/s^2 = 0.3048 m/s^2 = 30.48 cm/s^2

Pressure

1 bar = 10^5 N/m^2 = 14.50 lb/$in.^2$
1 atm = 760 mm Hg = 76.0 cm Hg
1 atm = 14.7 lb/$in.^2$ = 1.013×10^5 N/m^2
1 Pa = 1 N/m^2 = 1.45×10^{-4} lb/$in.^2$

Time

1 yr = 365 days = 3.16×10^7 s
1 day = 24 h = 1.44×10^3 min = 8.64×10^4 s

Energy

1 J = 0.738 ft · lb
1 cal = 4.186 J
1 Btu = 252 cal = 1.054×10^3 J
1 eV = 1.602×10^{-19} J
1 kWh = 3.60×10^6 J

Power

1 hp = 550 ft · lb/s = 0.746 kW
1 W = 1 J/s = 0.738 ft · lb/s
1 Btu/h = 0.293 W

Some Approx imations Useful for Estimation Problems

1 m ≈ 1 yd	1 m/s ≈ 2 mi/h
1 kg ≈ 2 lb	1 yr ≈ $\pi \times 10^7$ s
1 N ≈ $\frac{1}{4}$ lb	60 mi/h ≈ 100 ft/s
1 L ≈ $\frac{1}{4}$ gal	1 km ≈ $\frac{1}{2}$ mi

APPENDIX-C

C.1 Important Constants

Symbol	Meaning	Best Value	Approximate Value
c	Speed of light in vacuum	2.99792458×10^8 m/s	3.00×10^8 m/s
G	Gravitational constant	$6.67408(31) \times 10^{-11}$ N . m^2 /kg^2	6.67×10^{-11} N . m^2 /kg^2
N_A	Avogadro's number	$6.02214129(27) \times 10^{23}$	6.02×10^{23}
k	Boltzmann's constant	$1.3806488(13) \times 10^{-23}$ J/K	1.38×10^{-23} J/K
R	Gas constant	8.3144621(75) J/ mol . K	8.31 J/ mol · K = 1.99 cal / mol . K = 0.0821 atm . L/mol . K
σ	Stefan-Boltzmann constant	$5.670373(21) \times 10^{-8}$ W/m^2 . K	5.67×10^{-8} W/m^2 · K
k	Coulomb force constant	$8.987551788\ldots \times 10^9$ N . m^2/ C^2	8.99×10^9 N . m^2/ C^2
q_e	Charge on electron	$-1.602176565(35) \times 10^{-19}$ C	-1.60×10^{-19} C
ε_0	Permittivity of free space	$8.854187817\ldots \times 10^{-12}$ C^2/N . m^2	8.85×10^{-12} C^2/N . m^2
μ_0	Permeability of free space	$4\pi \times 10^{-7}$ T . m/A	1.26×10^{-6} T . m/A
h	Planck's constant	$6.62606957(29) \times 10^{-34}$ J . s	6.63×10^{-34} J . s

C.2 Submicroscopic Masses

Symbol	Meaning	Best Value	Approximate Value
m_e	Electron mass	$9.10938291(40) \times 10^{-31}$ kg	9.11×10^{-31} kg
m_p	Proton mass	$1.672621777(74) \times 10^{-27}$ kg	1.6726×10^{-27} kg
m_n	Neutron mass	$1.674927351(74) \times 10^{-27}$ kg	1.6749×10^{-27} kg
u	Atomic mass unit	$1.660538921(73) \times 10^{-27}$ kg	1.6605×10^{-27} kg

C.3 Solar System Data

Sun	mass	1.99×10^{30} kg
	average radius	6.96×10^{8} m
	Earth-sun distance (average)	1.496×10^{11} m
Earth	mass	5.9736×10^{24} kg
	average radius	6.376×10^{6} m
	orbital period	3.16×10^{7} s
Moon	mass	7.35×10^{22} kg
	average radius	1.74×10^{6} m
	orbital period (average)	2.36×10^{6} s
	Earth-moon distance (average)	3.84×10^{8} m

C.4 Metric Prefixes for Powers of Ten and Their Symbols

Prefix	Symbol	Value	Prefix	Symbol	Value
tera	T	10^{12}	deci	d	10^{-1}
giga	G	10^{9}	centi	c	10^{-2}
mega	M	10^{6}	milli	m	10^{-3}
kilo	k	10^{3}	micro	μ	10^{-6}
hecto	h	10^{2}	nano	n	10^{-9}
deka	da	10^{1}	pico	p	10^{-12}
–	–	10^{0} (= 1)	femto	f	10^{-15}

C.5 Selected British Units

Length	1 inch (in.) = 2.54 cm (exactly)
	1 foot (ft) = 0.3048 m
	1 mile (mi) = 1.609 km
Force	1 pound (ld) = 4.448 N
Energy	1 British thermal unit (Btu) = 1.055×10^{3} J
Power	1 horsepower (hp) = 746 W
Pressure	1 lb / in^2 = 6.895×10^{3} Pa

C.6 Other Units

Length	1 light year (ly) = 9.46×10^{15} m
	1 astronomical unit (au) = 1.50×10^{11} m
	1 nautical mile = 1.852 km
	1 angstrom (Å) = 10^{-10} m
Area	1 acre (ac) = 4.05×10^{3} m^2
	1 square foot (ft^2) = 9.29×10^{-2} m^2
	1 barn (b) = 10^{-28} m^2

Volume	1 liter (L) = 10^{-3} m^3
	1 U.S. gallon (gal) = 3.785×10^{-3} m^3
Mass	1 solar mass = 1.99×10^{30} kg
	1 metric ton = 10^{3} kg
	1 atomic mass unit (u) = 1.6605×10^{-27} kg
Time	1 year (y) = 3.16×10^{7} s
	1 day (d) = 86,400 s
Speed	1 mile per hour (mph) = 1.609 km/h
	1 nautical mile per hour (naut) = 1.852 km/h
Angle	1 degree (°) = 1.745×10^{-2} rad
	1 minute of arc (′) = 1 / 60 degree
	1 second of arc (″) = 1 / 60 minute of arc
	1 grad = 1.571×10^{-2} rad
Energy	1 kiloton TNT (kT) = 4.2×10^{12} J
	1 kilowatt hour (kW . h) = 3.60×10^{6} J
	1 food calorie (kcal) = 4186 J
	1 calorie (cal) = 4.186 J

	1 electron volt (eV) = 1.60×10^{-19} J
Pressure	1 atmosphere (atm) = 1.013×10^{5} Pa
	1 millimeter of mercury (mm Hg) = 133.3 Pa
	1 torricelli (torr) = 1 mm Hg = 133.3 Pa
Nuclear decay rate	1 curie (Ci) = 3.70×10^{10} Bq

C.7 Useful Formulae

Circumference of a circle with radius r or diameter d	$C = 2\pi r = \pi d$
Area of a circle with radius r or diameter d	$A = \pi r^2 = \pi d^2/4$
Area of a sphere with radius r	$A = 4\pi r^2$
Volume of a sphere with radius r	$V = (4/3)(\pi r^3)$

C.8 The Greek Alphabet

Alpha	Α	α	Eta	Η	η	Nu	Ν	ν	Tau	Τ	τ
Beta	Β	β	Theta	Θ	θ	Xi	Ξ	ξ	Upsilon	ϒ	υ
Gamma	Γ	γ	Iota	Ι	ι	Omicron	Ο	ο	Phi	Φ	ϕ
Delta	Δ	δ	Kappa	Κ	κ	Pi	Π	π	Chi	Χ	χ
Epsilon	Ε	ε	Lambda	Λ	λ	Rho	Ρ	ρ	Psi	Ψ	ψ
Zeta	Ζ	ζ	Mu	Μ	μ	Sigma	Σ	σ	Omega	Ω	ω

APPENDIX-D

Symbols, Dimensions, and Units of Physical Quantities

Quantity	Common Symbol	Unit	Dimensions	Unit in Terms of Base SI Units
Acceleration	$\vec{a}$	m/s^2	L/T^2	m/s^2
Amount of substance	n	MOLE		mol
Angle	θ, ϕ	radian (rad)	1	
Angular acceleration	$\vec{\alpha}$	rad/s^2	T^{-2}	s^{-2}
Angular frequency	ω	rad/s^2	T^{-1}	s^{-1}
Angular momentum	$\vec{L}$	kg.m^2/s	ML^2/T	kg . m^2/s
Angular velocity	$\vec{\omega}$	rad/s	T^{-1}	s^{-1}
Area	A	m^2	L^2	m^2
Atomic number	Z			
Capacitance	C	farad (F)	Q^{2}T^2/ML^2	A^2.s^4/kg.m^2
Charge	q, Q, e	coulomb (C)	Q	A.s
Line Charge density	λ	C/m	Q/L	A·s/m
Surface Charge density	σ	C/m^2	Q/L^2	A·s/m^2
Volume Charge density	ρ	C/m^3	Q/L^3	A·s/m^3
Conductivity	σ	1/Ω· m	Q^{2}T/ML^3	A^2·s^3/kg·m^3
Current	I	AMPERE	Q/T	A
Current density	J	A/m^2	Q/TL^2	A/m^2
Density	ρ	kg/m^3	M/L^3	kg/m^3
Dielectric constant	κ			

Electric dipole moment	$\vec{P}$	C · m	QL	A·s·m
Electric field	$\vec{E}$	V/m	ML/QT^2	$kg{\cdot}m/A{\cdot}s^3$
Electric flux	Φ_E	V · m	ML^3/QT^2	$kg{\cdot}m^3/A{\cdot}s^3$
Electromotive force	ε	volt (V)	ML^2/QT^2	$kg{\cdot}m^2/A{\cdot}s^3$
Energy	E, U, K	joule (J)	ML^2/T^2	$kg{\cdot}m^2/s^2$
Entropy	S	J/K	ML^2/T^2K	$kg{\cdot}m^2/s^2{\cdot}K$
Force	$\vec{F}$	newton (N)	ML/T^2	$kg{\cdot}m/s^2$
Frequency	f / v	hertz (Hz)	T^{-1}	s^{-1}
Heat	Q	joule (J)	ML^2/T^2	$kg{\cdot}m^2/s^2$
Inductance	L	henry (H)	ML^2/Q^2	$kg{\cdot}m^2/A^2{\cdot}s^2$
Length	ℓ, L	Meter	L	m
Displacement	Δx, $\Delta\vec{r}$			
Distance	d, h			
Position	$x, y, z, \vec{r}$			
Magnetic dipole moment	$\vec{\mu}$	N · m/T	QL^2/T	$A{\cdot}m^2$
Magnetic field	$\vec{B}$	tesla (T) (=Wb/m^2)	M/QT	$kg/A{\cdot}s^2$
Magnetic flux	Φ_B	weber (Wb)	ML^2/QT	$kg{\cdot}m^2/A{\cdot}s^2$
Mass	m, M	Kilogram	M	kg
Molar specific heat	C	J/mol · K		$kg{\cdot}m^2/s^2{\cdot}mol{\cdot}K$
Moment of inertia	I	$kg \cdot m^2$	ML^2	$kg{\cdot}m^2$
Momentum	$\vec{P}$	kg · m/s	ML/T	kg·m/s
Time Period	T	S	T	s
Permeability of free space	μ_0	N/A^2 (=H/m)	ML/Q^2	$kg{\cdot}m/A^2{\cdot}s^2$
Permittivity of free space	ε_0	$C^2/N \cdot m^2$ (=F/m)	Q^2T^2/ML^3	$A^2{\cdot}s^4/kg{\cdot}m^3$
Potential	V	volt (V) (=J/C)	ML^2/QT^2	$kg{\cdot}m^2/A{\cdot}s^3$
Power	P	watt (W)(=J/s)	ML^2/T^3	$kg{\cdot}m^2/s^3$
Pressure	P	pascal (Pa)(=N/m^2)	M/LT^2	$kg/m{\cdot}s^2$
Resistance	R	ohm (Ω)(=V/A)	ML^2/Q^2T	$kg{\cdot}m^2/A^2{\cdot}s^3$
Specific heat	c	J/kg · K	L^2/T^2K	$m^2/s^2{\cdot}K$
Speed	v	m/s	L/T	m/s
Temperature	T	Kelvin	K	K
Time	t	Second	T	s
Torque	$\vec{\tau}$	N · m	ML^2/T^2	$kg{\cdot}m^2/s^2$
Velocity	$\vec{v}$	m/s	L/T	m/s
Volume	V	m^3	L^3	m^3
Wavelength	λ	m	L	m
Work	W	joule (J) (=N· m)	ML^2/T^2	$kg{\cdot}m^2/s^2$

APPENDIX-E

Indian Space Research Organisation (ISRO) - - - - [1975 to 2020]

India has been successfully launching satellites of many types since 1975.

Satellites have been launched from various vehicles, including those launched by American, Russian and European rockets, as well as those launched indigenously by India.

The organization responsible for India's satellite program is the **Indian Space Research Organisation (ISRO).**

Satellites	Launch Date	Launch Vehicle
Aryabhata	19-Apr-75	u-11 Interkosmos
Bhaskara-I	07-Jun-79	C-1 Interkosmos
Rohini Technology Payload	10-Aug-79	SLV-3
Rohini RS-1	18-Jul-80	SLV-3
Rohini RS-D1	31-May-81	SLV-3
Ariane Passenger Payload Experiment	19-Jun-81	Ariane-1 (V-3)
Bhaskara-II	20-Nov-81	C-1 Intercosmos
INSAT-1A	10-Apr-82	Delta 3910 PAM-D
Rohini RS-D2	17-Apr-83	SLV-3
INSAT-1B	30-Aug-83	Shuttle [PAM-D]
Stretched Rohini Satellite Series (SROSS-1)	24-Mar-87	ASLV
IRS-1A	17-Mar-88	Vostok
Stretched Rohini Satellite Series (SROSS-2)	13-Jul-88	ASLV
INSAT-1C	21-Jul-88	Ariane-3
INSAT-1D	12-Jun-90	Delta 4925
IRS-1B	29-Aug-91	Vostok
INSAT-2DT	26-Feb-92	Ariane-44L H10
Stretched Rohini Satellite Series (SROSS-C)	20-May-92	ASLV
INSAT-2A	10-Jul-92	Ariane-44L H10
INSAT-2B	23-Jul-93	Ariane-44L H10+
IRS-1E	20-Sep-93	PSLV-D1
Stretched Rohini Satellite Series (SROSS-C2)	04-May-94	ASLV
IRS-P2	15-Oct-94	PSLV-D2
INSAT-2C	07-Dec-95	Ariane-44L H10-3
IRS-1C	29-Dec-95	Molniya
IRS-P3	21-Mar-96	PSLV-D3
INSAT-2D	04-Jun-97	Ariane-44L H10-3
IRS-1D	29-Sep-97	PSLV-C1
INSAT-2E	03-Apr-99	Ariane-42P H10-3
Oceansat-(IRS-P4)	26-May-99	PSLV-C2
INSAT-3B	22-Mar-2000	Ariane-5G
GSAT-1	18-Apr-01	GSLV-D1
Technology Experiment Satellite (TES)	22-Oct-01	PSLV-C3
INSAT-3C	24-Jan-02	Ariane-42L H10-3
Kalpana-1(METSAT)	12-Sep-02	PSLV-C4
INSAT-3A	10-Apr-03	Ariane-5G
GSAT-2	08-May-03	GSLV-D2
INSAT-3E	28-Sep-03	Ariane-5G
RESOURCESAT-1(IRS-P6)	17-Oct-03	PSLV-C5
EDUSAT	20-Oct-04	GSLV-F01
HAMSAT	05-May-05	PSLV-C6
CARTOSAT-1	05-May-05	PSLV-C6
INSAT-4A	22-Dec-05	Ariane-5GS
INSAT-4C	10-Jul-06	GSLV-F02
CARTOSAT-2	10-Jan-07	PSLV-C7
Space Capsule Recovery Experiment(SRE-1)	10-Jan-07	PSLV-C7
INSAT-4B	12-Mar-07	Ariane-5ECA
INSAT-4CR	02-Sep-07	GSLV-F04
CARTOSAT-2A	28-Apr-08	PSLV-C9

IMS-1 (Third World Satellite – TWsat)	28-Apr-08	PSLV-C9
Chandrayaan-1	22-Oct-08	PSLV-C11
RISAT-2	20-Apr-09	PSLV-C12
ANUSAT	20-Apr-09	PSLV-C12
Oceansat-2(IRS-P4)	23-Sep-09	PSLV-C14
GSAT-4	15-Apr-10	GSLV-D3
CARTOSAT-2B	12-Jul-10	PSLV-C15
StudSat	12-Jul-10	PSLV-C15
GSAT-5P /INSAT-4D	25-Dec-10	GSLV-F06
RESOURCESAT-2	20-Apr-11	PSLV-C16
Youthsat	20-Apr-11	PSLV-C16
GSAT-8 / INSAT-4G	21-May-11	Ariane-5VA-202
GSAT-12	15-Jul-11	PSLV-C17
Megha-Tropiques	12-Oct-11	PSLV-C18
Jugnu	12-Oct-11	PSLV-C18
RISAT-1	26-Apr-12	PSLV-C19
SRMSAT	26-Apr-12	PSLV-C18
GSAT-10	29-Sep-12	Ariane-5VA-209
SARAL	25-Feb-13	PSLV-C20
IRNSS-1A	01-Jul-13	PSLV-C22
INSAT-3D	26-Jul-13	Ariane-5
GSAT-7	30-Aug-13	Ariane-5
Mars Orbiter Mission (MOM)	05-Nov-13	PSLV-C25
GSAT-14	05-Jan-14	GSLV-D5
IRNSS-1B	04-Apr-14	PSLV-C24
IRNSS-1C	16-Oct-14	PSLV-C26
GSAT-16	07-Dec-14	Ariane-5
IRNSS-1D	28-Mar-15	PSLV-C27
GSAT-6	27-Aug-15	GSLV-D6
Astrosat	28-Sep-15	PSLV-C30
GSAT-15	11-Nov-15	Ariane 5 VA-227
IRNSS -1E	20-Jan-16	PSLV-C31
IRNSS -1F	10-Mar-16	PSLV-C32
IRNSS-1G	28-Apr-16	PSLV-C33
Cartosat-2C	22-Jun-16	PSLV-C34
CartoSat-2E	8 September 2016,	INSAT-3DR
Pratham	26 September 2016,	PSLV-C35
GSAT-18	6 October 2016,	Ariane-5 ECA
ResourceSat-2A	7 December 2016,	PSLV-C36
CartoSat-2D	15 February 2017,	PSLV-C37
South Asia Satellite (GSAT-9)	5 May 2017,	GSLV Mk.II[3
GSAT-19	05-Jun-17	GSLV Mk.III-D1
NIUSat[	23-June 2017,	PSLV-C38
GSAT-17	29 June 2017,	Ariane-5 ECA
IRNSS-1H	02-Sep-17	PSLV–C39
CartoSat-2F	10-January 2018,	PSLV-C40
GSAT-6A	29-Mar-18	GSLV-F08
IRNSS-11	12-Apr-18	GSLV-F08, PSLV-C41
GSAT-29	01-Nov-18	GSLV Mk III D2
HySIS	29-Nov-18	PSLV-C43
GSAT-7A	19-Dec-18	GSLV Mk.II-F11
Microsat-R	23-Jan-19	PSLV-C44
EMISAT	01-Apr-19	PSLV-C45
PS4 Stage attached with ExseedSat-2, AMSAT, ARIS and AIS payloads	01-Apr-19	PSLV-C45
Risat-2B	21-May-19	PSLV-C46
Chandrayaan-2	22-Jul-19	Chandrayaan-2

APPENDIX-A

A Table of Greek letters

Upper case	Lower case	In English
Α	α	alpha
Β	β	beta
Γ	γ	gamma
Δ	δ	delta
Ε	ε	epsilon
Ζ	ζ	zeta
Η	η	eta
Θ	θ	theta
Ι	ι	iota
Κ	κ	kappa
Λ	λ	lambda
Μ	μ	mu
Ν	υ	nu
Ξ	ξ	csi
Ο	ο	omicron
Π	π	pi
Ρ	ρ	rho
Σ	σ	sigma
Τ	τ	tau
ϒ	υ	upsilon
Φ	ϕ	phi
Ψ	ψ	psi
Χ	χ	chi
Ω	ω	omega

APPENDIX-B

Useful Physical Constant in Chemistry

Constant	Symbol	Value
Acceleration due to gravity	g	$9.8\ \text{m s}^{-2}$
Atomic mass unit	amu, m_u or u	1.66×10^{-27} kg
Avogadro's Number	N, N_A	$6.022 \times 10^{23}\text{mol}^{-1}$
Bohr radius	r_o	0.529×10^{-10}m
Boltzmann constant	k	$1.38 \times 10^{-23}\ \text{J K}^{-1}$
Electron charge to mass ratio	$\frac{-e}{m_e}$	$-1.7588 \times 10^{11}\ \text{C kg}^{-1}$
Electron classical radius	r_e	2.818×10^{-15} m
Electron mass energy(J)	m_ec^2	8.187×10^{-14} J
Electron mass energy (MeV)	m_ec^2	0.511 MeV
Electron rest mass	m_e	9.109×10^{-31} kg
Faraday constant	F	$9.649 \times 10^{4}\ \text{Cmol}^{-1}$
Fine-structure constant	α	7.297×10^{-3}
Gas constant	R	$8.314\ \text{J mol}^{-1}\ \text{K}^{-1}$
Gravitational constant	G	$6.67 \times 10^{-11}\ \text{Nm}^2\ \text{Kg}^{-2}$
Neutron mass energy	m_nc^2	1.505×10^{-10} J or 939.565 MeV
Neutron rest mass	m_n	1.675×10^{-27} kg
Neutron-electron mass ratio	$\frac{m_n}{m_e}$	1838.68
Neutron-proton mass ratio	$\frac{m_n}{m_p}$	1.0014
Permeability of a vacuum	μ_0	$4\pi \times 10^{-7}\text{NA}^{-2}$
Permittivity of a vacuum	ε_0	$8.854 \times 10^{-12}\ \text{F m}^{-1}$
Planck constant	h	6.626×10^{-34} J s
Proton mass energy	m_pc^2	1.503×10^{-10} J or 938.272 MeV
Proton rest mass	m_p	1.6726×10^{-27} kg
Proton-electron mass ratio	$\frac{m_p}{m_e}$	1836.15
Rydberg constant	R_e	$1.0974 \times 10^{7}\ \text{m}^{-1}$
Speed of light in vacuum	c	2.9979×10^{8} m/s

APPENDIX-C

Most Popular Chemists and their Contributions

AMEDEO AVOGADRO 1776 – 1856
The first scientist to realize that elements could exist in the form of molecules rather than as individual atoms; originator of Avogadro's law.

JACOB BERZELIUS 1779 – 1848
A founder of modern chemistry : the first person to measure accurate atomic weights for the chemical elements; discovered three elements: cerium, thorium and selenium; devised the modern symbols for elements; described how chemical bonds form by electrostatic attraction.

NIELS BOHR 1885 – 1962
Founded quantum mechanics when he remodeled the atom that electrons occupied 'allowed' orbits around the nucleus while all other orbits were forbidden; architect of the Copenhagen interpretation of quantum mechanics.

ROBERT BOYLE 1627 – 1691
Transformed chemistry from a field bogged down in alchemy and mysticism into one based on measurement. He defined elements, compounds, and mixtures; and he discovered the first gas law – Boyle's Law.

LAWRENCE BRAGG 1890 – 1971
Discovered how to locate the positions of atoms in solids using X-ray diffraction, enabling scientists to build 3D models of the atomic arrangements in solids. The discovery was arguably the most significant experimental breakthrough of twentieth century science.

HENNIG BRAND 1630 – 1710
Discovered phosphorus, becoming the first named person in history to discover a chemical element.

GEORG BRANDT 1694 – 1768
The first named person in history to discover a new metal – cobalt; was one of the first scientists to condemn alchemy, publicly demonstrating tricks used by alchemists to make people think they could make gold.

ROBERT BUNSEN 1811 – 1899
Discovered cesium and rubidium; discovered the antidote to arsenic poisoning; invented the zinc-carbon battery and flash photography; discovered how geysers operate.

ERWIN CHARGAFF 1905 – 2002
Chargaff's rules paved the way to the discovery of DNA's structure.

MARIE CURIE 1867 – 1934
Co-discovered the chemical elements radium and polonium; made numerous pioneering contributions to the study of radioactive elements; carried out the first research into the treatment of tumors with radiation.

JOHN DALTON 1766 – 1844
Dalton's Atomic Theory is the basis of chemistry; discovered Gay-Lussac's Law relating gases' temperature, volume, and pressure; discovered the law of partial gas pressures.

DEMOCRITUS (C. 460 — C. 370 BC)
Devised an atomic theory featuring tiny particles always in motion interacting through collisions; advocated a universe containing an infinity of diverse inhabited worlds governed by natural, mechanistic laws rather than gods; deduced that the light of stars explains the Milky Way's appearance; discovered that a cone's volume is one-third that of the cylinder with the same base and height.

EMPEDOCLES (C. 490 – C. 430 BC)
An ancient theory of natural selection; mass conservation; and the four elements which are now often misattributed to Aristotle.

MICHAEL FARADAY 1791 – 1867
Discovered electromagnetic induction; devised Faraday's laws of electrolysis; discovered the first experimental link between light and magnetism; carried out the first room-temperature liquefaction of a gas; discovered benzene.

ROSALIND FRANKLIN 1920 – 1958
Provided much of the experimental data used to establish the structure of DNA; discovered that DNA can exist in two forms; established that coal acts as a molecular sieve.

WILLARD GIBBS 1839 – 1903
Gibbs invented vector analysis and founded the sciences of modern statistical mechanics and chemical thermodynamics.

GEORGE DE HEVESY 1885 – 1966
Discovered element 72, hafnium. Pioneered isotopes as tracers to study chemical and biological processes; discovered how plants and animals utilize particular chemical elements after they are taken in as nutrients.

FRED HOYLE 1915 – 2001
Established that most of the naturally occurring elements in the periodic table were made inside stars and distributed through space by supernova explosions.

IRENE JOLIOT-CURIE 1897 – 1956
Co-discovered how to convert stable chemical elements into 'designer' radioactive elements; these have saved millions of lives and are used in tens of millions of medical procedures every year.

MARTIN KLAPROTH 1743 – 1817
Discovered the chemical elements uranium, zirconium, and cerium – naming the first two of these elements; verified the discoveries of titanium, tellurium and strontium, again naming the first two.

STEPHANIE KWOLEK 1923 – 2014
Invented kevlar, the incredibly strong plastic used in applications ranging from body armor to tennis racquet strings.

ANTOINE LAVOISIER 1743 – 1794
A founder of modern chemistry; discovered oxygen's role in combustion and respiration; discovered that water is a compound of hydrogen and oxygen; proved that diamond and charcoal are different forms of the same element, which he named carbon.

ERNEST LAWRENCE 1901 – 1958
Invented the cyclotron, used by scientific teams in his laboratories to discover large numbers of new chemical elements and isotopes. Founded big science.

JANE MARCET 1769 – 1858.
Author of *Conversations on Chemistry*, a unique textbook for its time written for people with little formal education, such as girls and the poor. The book inspired Michael Faraday to overcome his poor origins to become a great scientist.

DMITRI MENDELEEV 1834 – 1907
Discovered the periodic table in a dream. Utilized the organizing principles of the periodic table to correctly predict the existence and properties of six new chemical elements.

HENRY MOSELEY 1887 – 1915
Proved that every element's identity is uniquely determined by its number of protons, establishing this is the true organizing principle of the periodic table; correctly predicted the existence of four new chemical elements; invented the atomic battery.

GIULIO NATTA 1903 – 1979
Discovered how to produce polymer chains with orderly spatial arrangements – i.e. stereoregular polymers.

ALFRED NOBEL 1833 – 1896
Invented dynamite, the blasting cap, gelignite, and ballistite; grew enormously wealthy manufacturing explosives; used his wealth to bequeath annual prizes in science, literature, and peace.

HANS CHRISTIAN OERSTED 1777 – 1851
Discovered electromagnetism when he found that electric current caused a nearby magnetic needle to move; discovered piperine and achieved the first isolation of the element aluminum.

LOUIS PASTEUR 1822 – 1895
The father of modern microbiology; transformed chemistry and biology with his discovery of mirror-image molecules; discovered anaerobic bacteria; established the germ theory of disease; invented food preservation by pasteurization.

LINUS PAULING 1901 – 1994
Maverick giant of chemistry; formulated valence bond theory and electronegativity; founded the fields of quantum chemistry, molecular biology, and molecular genetics. Discovered the alpha-helix structure of proteins; proved that sickle-cell anemia is a molecular disease.

MARGUERITE PEREY 1909 – 1975
Discovered francium, the last of the naturally occurring chemical elements to be discovered – all elements since have been produced artificially.

WILLIAM PERKIN 1838 – 1907
At age 18 started the synthetic dye revolution when his discovery of mauveine brought the once formidably expensive color purple to everyone. Perkins' revolution took the world by storm, transforming textiles, foods and medicine.

C. V. RAMAN 1888 – 1970
Discovered that light can donate a small amount of energy to a molecule, changing the light's color and causing the molecule to vibrate. The color change acts as a 'fingerprint' for the molecule that can be used to identify molecules and detect diseases such as cancer.

WILLIAM RAMSAY 1852 – 1916
Predicted the existence of the noble gases and discovered or was first to isolate every member of the group; created the world's first neon light.

ERNEST RUTHERFORD 1871 – 1937
The father of nuclear chemistry and nuclear physics; discovered and named the atomic nucleus, the proton, the alpha particle, and the beta particle; discovered the concept of nuclear half-lives; achieved the first laboratory transformation of one element into another.

GLENN SEABORG 1912 TO 1999
Took part in the discovery of ten of the periodic table's chemical elements. His work on the electronic structure of elements led to the periodic table being rewritten.

HERMANN STAUDINGER 1881 – 1965
Founded macromolecular chemistry when he established that molecules made of hundreds of thousands of atoms exist; demonstrated that synthetic polymers can make fibers similar to natural fibers; discovered polyoxymethylene; discovered pyrethroid natural insecticides.

J.J. THOMSON 1856 – 1940
Discovered the electron; invented one of the most powerful tools in analytical chemistry – the mass spectrometer; obtained the first evidence for isotopes of stable elements.

HAROLD UREY 1893 – 1981
Discovered deuterium; showed how isotope ratios in rocks reveal past Earth climates; founded modern planetary science; the Miller-Urey experiment demonstrated that electrically sparking simple gases produces amino acids – the building blocks of life.

ALESSANDRO VOLTA 1745 – 1827
Pioneer of electrical science; invented the electric battery; wrote the first electromotive series; isolated methane for the first time; discovered a methane-air mixture could be exploded using an electric spark – the basis of the internal combustion engine.

SERGEI WINOGRADSKY 1856 – 1953
Founded microbial ecology; discovered chemosynthetic life forms which obtain energy from chemical reactions rather than from sunlight; discovered nitrogen-fixing bacteria in soil that make nitrates available to green plants.

APPENDIX-D

Periodic Table of the Elements

1 1IA 1A	2 IIA 2A	3 IIIB 3B	4 IVB 4B	5 VB 5B	6 VIB 6B	7 VIIB 7B	8	9 VIII 8	10	11 IB 1B	12 IIB 2B	13	14	15	16	17	18
1 H Hydrogen 1.008																	2 He Helium 4.003
3 Li Lithium 6.941	4 Be Beryllium 9.012											5 B Boron 10.811	6 C Carbon 12.011	7 N Nitrogen 14.007	8 O Oxygen 15.999	9 F Fluorine 18.998	10 Ne Neon 20.180
11 Na Sodium 22.99	12 Mg Magnesium 24.305											13 Al Aluminium 26.982	14 Si Silicon 28.086	15 P Phosphorus 30.974	16 S Sulphur 32.056	17 Cl Chlorine 35.453	18 Ar Argon 39.948
19 K Potassium 39.098	20 Ca Calcium 40.078	21 Sc Scandium 45.956	22 Ti Titanium 47.867	23 V Vanadium 50.942	24 Cr Chromium 51.996	25 Mn Manganese 54.938	26 Fe Iron 55.845	27 Co Cobalt 58.933	28 Ni Nickel 58.693	29 Cu Copper 63.5	30 Zn Zinc 65.38	31 Ga Gallium 69.723	32 Ge Germanium 72.631	33 As Arsenic 74.922	34 Se Selenium 78.971	35 Br Bromine 79.904	36 Kr Krypton 83.789
37 Rb Rubidium 85.468	38 Sr Strontium 87.62	39 Y Yttrium 88.906	40 Zr Zirconium 91.224	41 Nb Niobium 92.906	42 Mo Molybdenum 95.95	43 Tc Technetium 98.907	44 Ru Ruthenium 101.07	45 Rh Rhodium 102.906	46 Pd Palladium 106.42	47 Ag Silver 107.9	48 Cd Cadmium 112.414	49 In Indium 114.818	50 Sn Tin 118.711	51 Sb Antimony 121.760	52 Te Tellurium 127.6	53 I Iodine 126.904	54 Xe Xenon 131.3
55 Cs Cesium 132.905	56 Ba Barium 137.328	57-71	72 Hf Hafnium 178.49	73 Ta Tantalum 180.948	74 W Tungsten 183.84	75 Re Rhenium 186.207	76 Os Osmium 190.23	77 Ir Iridium 192.217	78 Pt Platinum 195.085	79 Au Gold 197.0	80 Hg Mercury 200.592	81 Ti Thallium 204.383	82 Pb Lead 207.2	83 Bi Bismuth 208.980	84 Po Polonium 208.982	85 At Astatine 209.987	86 Rn Radon 222.018
87 Fr Francium 223.020	88 Ra Radium 226.025	89-103	104 Rf Rutherfordium (26.1)	105 Db Dubnium (262)	106 Sg Seaborgium (266)	107 Bh Bohrium (264)	108 Hs Hassium (269)	109 Mt Meitnerium (278)	110 Ds Darmstadtium (281)	111 Rg Roentgenium (280)	112 Cn Copernicium (285)	113 Nh Nihonium (286)	114 FI Flerovium (289)	115 Mc Moscovium (286)	116 Lv Livermorium (293)	117 Ts Tennessine (294)	118 Og Oganesson (294)

Lanthanide Series	57 La Lanthanum 138.905	58 Ce Cerium 140.116	59 Pr Praseodymium 140.908	60 Nd Neodymium 144.243	61 Pm Promethium 144.913	62 Sm Samarium 160.36	63 Eu Europium 151.964	64 Gd Gadolinium 157.25	65 Tb Terbium 158.925	66 Dy Dysprosium 162.500	67 Ho Holmium 164.9	68 Er Erbium 167.259	69 Tm Thulium 168.934	70 Yb Ytterbium 173.055	71 Lu Lutetium 174.967
Actinide Series	89 Ac Actinium 227.028	90 Th Thorium 232.038	91 Pa Protactinium 231.036	92 U Uranium 238.029	93 Np Neptunium 237.064	94 Pu Plutonium 244.064	95 Am Americium 243.061	96 Cm Curium 247.070	97 Bk Berkelium 247.070	98 Cf Califomium 251.080	99 Es Einsteinium (254)	100 Fm Femium 257.095	101 Md Mendelevium 258.1	102 No Nobelium 259.101	103 Lr Lawercium (262)

Alkali Metal	Alkaline Earth	Transition Metal	Basic Metal	Semi-metal	Non-metal	Halogen	Noble Gas	Lantha-nide	Actinide

APPENDIX-E

Important Organic Chemical Reactions

Sandmeyer Reaction

The Sandmeyer reaction is a chemical reaction which is used to synthesize aryl halides from aryl diazonium salts. This reaction is a method for substitution of an aromatic amino group by preparing diazonium salt that is followed by its displacement and copper salts often catalyze it.

$$ArN_2^+X^- \xrightarrow{CuCl/HCl} ArCl + N_2$$
$$ArN_2^+X^- \xrightarrow{CuBr/HBr} ArBr + N_2$$
$$ArN_2^+X^- \xrightarrow{CuCN/KCN} ArCN + N_2$$

Gattermann Reaction

Bromine and Chlorine can be present in the benzene ring by preparing the benzene diazonium salt solution with similar halogen acid present with copper powder. This is the Gattermann Reaction.

$$ArN_2^+X^- \xrightarrow{Cu/HCl} ArCl + N_2 + CuX$$
$$ArN_2^+X^- \xrightarrow{Cu/HBr} ArBr + N_2 + CuX$$

Balz-Schiemann Reaction

When arene-diazonium chloride is prepared with fluoroboric acid, arene diazonium fluoroborate is precipitated and decomposes to yield aryl fluoride which on heating.

$$Ar\overset{+}{N}_2Cl^- + \underset{\text{Fluoro boric acid}}{HBF_4} \longrightarrow Ar-\overset{+}{N}_2BF_4^- \xrightarrow{Heat} Ar-F + BF_3 + N_2$$

Finkelstein Reaction

In the Finkelstein Reaction Alkyl iodides are prepared easily by the reaction of alkyl chlorides with NaI in dry acetone.

$$R-X + NaI \rightarrow R-I + NaX \quad (X = Cl, Br)$$

Swarts Reaction

When alkyl chloride is heated in the presence of a metallic fluoride like AgF, Hg_2F_2, SbF_3 or CoF_2, we get alkyl fluoride The reaction is specifically used to prepare alkyl fluorides.

$$H_3C-X + AgF \rightarrow H_3C-F + AgX \quad (X = Cl, Br)$$

Wurtz Reaction

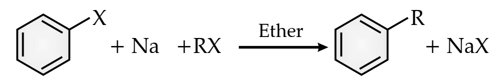
When Alkyl halides get reacted with sodium with dry ether, we get hydrocarbons that include the double number of carbon atoms present in the halide. This is known as the Wurtz Reaction.

$$\underset{\text{Bromomethane}}{CH_3Br} + 2Na + BrCH_3 \xrightarrow{\text{dry ether}} \underset{\text{Ethane}}{CH_3-CH_3} + 2NaBr$$

$$\underset{\text{Bromoethane}}{CH_3Br} + 2Na + BrC_2H_5 \xrightarrow{\text{dry ether}} \underset{\text{n-Butane}}{C_2H_5-C_2H_5}$$

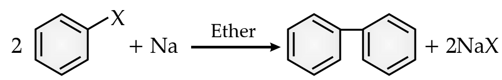
Wurtz-Fittig Reaction

When a mixture of alkyl halide and aryl halide gets treated with sodium in dry ether, we get an alkyl arene.



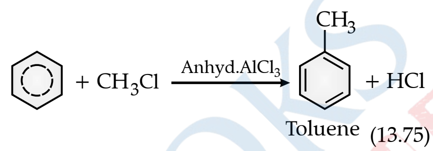
Fittig Reaction

Aryl halides prepared with sodium in dry ether to give analogous compounds where two aryl groups joined.



Friedel-Crafts Alkylation Reaction

Benzene is prepared with an alkyl halide in the presence of anhydrous aluminum chloride to give alkylbenzene.



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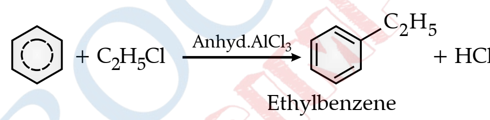
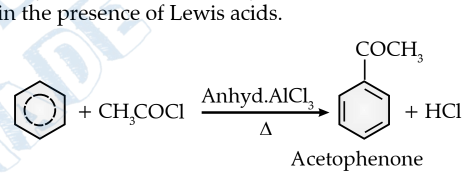
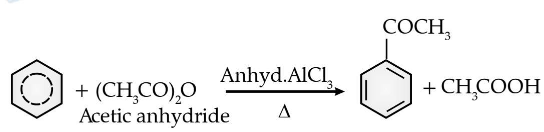


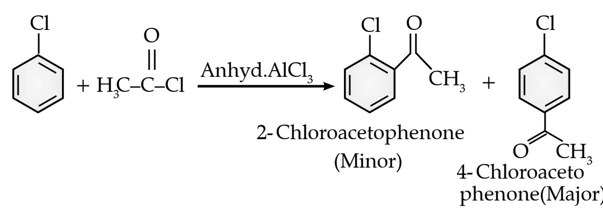
Friedel-Crafts Acylation Reaction

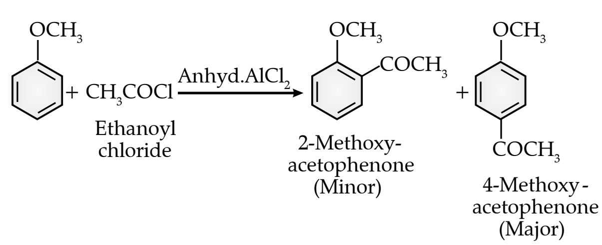
We get acyl benzene when an acyl halide is reacted with benzene in the presence of Lewis acids.



(13.77)

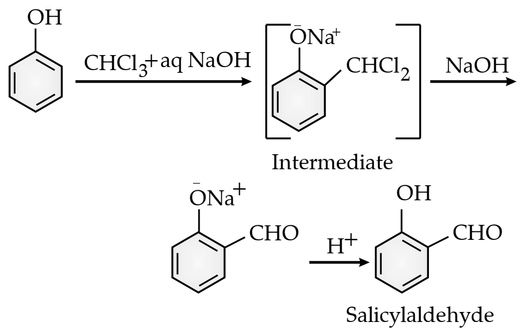






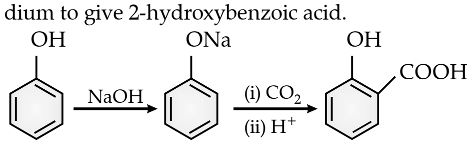
Reimer-Tiemann Reaction

When preparing phenol with chloroform in the presence of sodium hydroxide, –CHO group is present at the ortho position of the benzene ring which results into salicylaldehyde.



Kolbe's Reaction

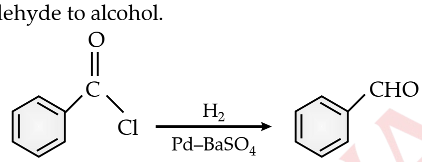
Phenol reacts with sodium hydroxide to give sodium phenoxide which then reacts with carbon dioxide in acidic medium to give 2-hydroxybenzoic acid.



2-Hydroxybenzoic acid
(Salicylic acid)

Rosenmund Reduction

When Acyl chloride is hydrogenated to an aldehyde over a catalyst, known as Rosenmund catalyst which is either palladium or barium sulfate. The catalyst is poisoned with either sulphur or quinoline in order to prevent further reduction of aldehyde to alcohol.



Benzoyl chloride Benzaldehyde

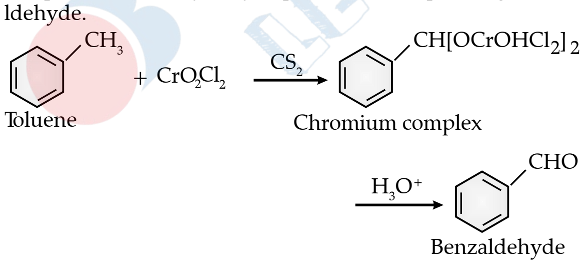
Stephen Reaction

Nitriles with stannous chloride in the presence of hydrochloric acid reduced to the corresponding imine and give the corresponding aldehyde after hydrolysis.

$$RCN + SnCl_2 + HCl \longrightarrow RCH = NH \xrightarrow{H_3O^+} RCHO$$

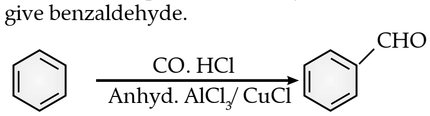
Etard Reaction

Chromyl chloride oxidizes methyl group to get chromium complex which on hydrolysis provides corresponding benzaldehyde.



Gatterman – Koch Reaction

Benzene is prepared with carbon monoxide and hydrogen chloride in the presence of anhydrous aluminium chloride to give benzaldehyde.



Benzene Benzaldehyde

Clemmensen Reduction

In Clemmensen reduction, Carbonyl group of aldehydes and ketones on treatment with zinc amalgam and concentrated hydrochloric acid reduced to CH_2 group.

$$>C = O \xrightarrow[HCl]{Zn\text{-}Hg} >CH_2 + H_2O$$

Wolff Kishner Reduction

Carbonyl group of aldehydes and ketones on treatment with hydrazine produces hydrazone which on heating with potassium hydroxide in a high boiling solvent (ethylene glycol) and reduce to $-CH_2-$ group.

$$>C = O \xrightarrow[-H_2O]{NH_2NH_2} >C{=}NNH_2 \xrightarrow[heat]{KOH/ethylene\ glycol} >CH_2 + N_2$$

Tollens' test

Heating an aldehyde with fresh prepared ammoniacal silver nitrate solution produces a bright silver mirror due to the formation of silver metal.

$$RCHO + 2[Ag(NH_3)]_2 + 3OH^- \longrightarrow RCO\bar{O} + 2Ag + 2H_2O + 4NH_3$$

Silver mirror

Fehling's test

Fehling's solution A (aqueous copper sulfate) and Fehling solution B (alkaline sodium potassium tartrate) are mixed in equal amounts before the test. A reddish brown precipitate is obtained when an aldehyde is heated with Fehling's reagent.

$$R\text{-}CHO + 2Cu^{2+} + 5OH^- \longrightarrow RCOO^- + Cu_2O + 3H_2O$$

Reddish-brown ppt

Aldol reaction

Aldehydes and ketones having one α-hydrogen undergo a reaction in the presence of dilute alkali as the catalyst to produce α-hydroxy aldehydes or β-hydroxy ketones.

(i) Aldol condensation

Aldol and Ketol lose water to provide α,β-unsaturated carbonyl compounds which are aldol condensation products.

$$2\,CH_3-CHO \xrightleftharpoons{dil.\ NaOH} CH_3-\underset{\displaystyle OH}{\underset{|}{CH}}-CH_2-CHO$$

Ethanal 3 –Hydroxybutanal (Aldol)

$$\xrightarrow[-H_2O]{\Delta} CH_3-CH{=}CH-CHO$$

But – 2 –enal
(Aldol condensation product)

$$2\,CH_3-CO-CH_3 \xrightleftharpoons{Ba(OH)_2} CH_3-\overset{\displaystyle CH_3}{\overset{|}{\underset{\displaystyle OH}{\underset{|}{C}}}}-CH_2CO-CH_3$$

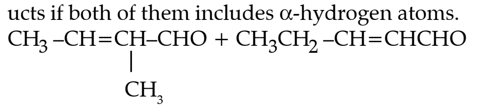
Propanone (Ketol)
4-Hydroxy-4-methylpentan-2-one

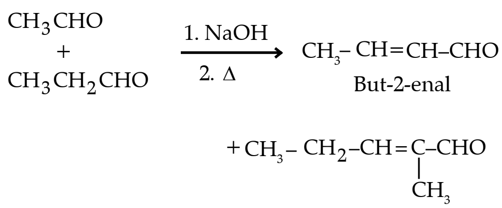
$$\xrightarrow[-H_2O]{\Delta} CH_3-\overset{\displaystyle CH_3}{\overset{|}{C}}{=}CH-CO-CH_3$$

4-Methylpent-3-en-2-one
(Aldol condensation product)

(ii) Cross aldol condensation

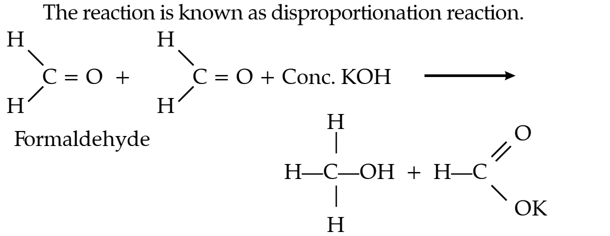
Aldol condensation is carried out between two different aldehydes and ketones. It gives a mixture of four products if both of them includes α-hydrogen atoms.

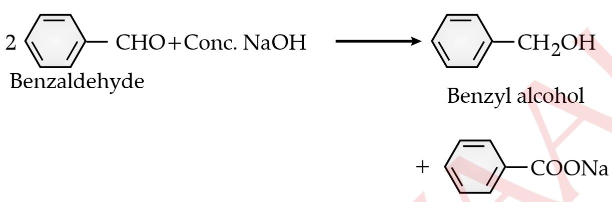




Cannizzaro Reaction

Aldehydes without α-hydrogen atom undergo self-oxidation and reduction reaction when prepared with concentrated alkali. The reaction is known as disproportionation reaction.





Kolbe electrolysis

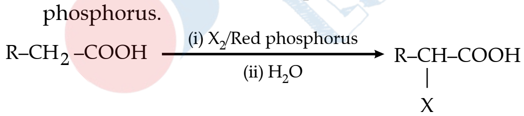
In Kolbe electrolysis, An aqueous solution of sodium or potassium salt of a carboxylic acid gives alkane containing an even number of carbon atoms on electrolysis.

$$2CH_3COONa + 2H_2O \xrightarrow{\text{Electrolysis}} CH_3-CH_3 + 2CO_2 + H_2 + 2NaOH$$

Sodium acetate

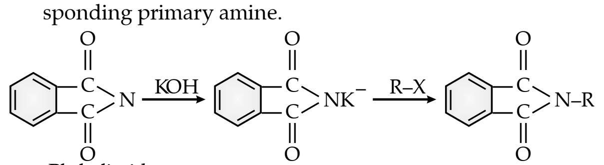
Hell-Volhard-Zelinsky (HVZ) Reaction

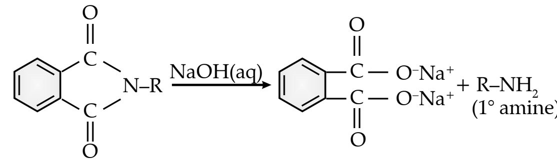
Carboxylic acids having a α-hydrogen are halogenated at the α-position give α-halo carboxylic acids on treatment with chlorine or bromine in the presence of small amount of red phosphorus.



Gabriel Phthalimide Synthesis

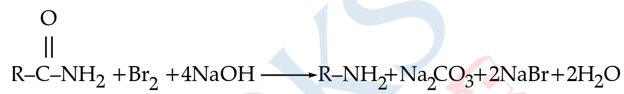
Phthalimide prepared with ethanolic potassium hydroxide produces potassium salt of phthalimide when heated with alkyl halide followed by alkaline hydrolysis forms the corresponding primary amine.





Hoffmann Bromamide Degradation Reaction

An amide upon heating with bromine in presence of sodium hydroxide produces primary amine. Migration of an alkyl or aryl group takes place from carbonyl carbon of the amide to the nitrogen atom. The amine so produced includes one carbon less than that present in the amide.



Carbylamine Reaction

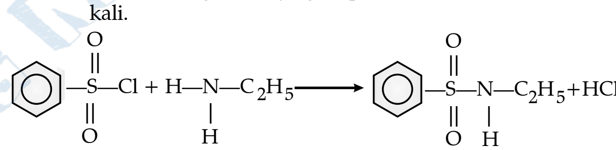
Aliphatic and aromatic primary amines when heated with chloroform and ethanolic potassium hydroxide produces isocyanides or carbyl amines which are foul smelling substances.

$$R-NH_2 + CHCl_3 + 3KOH \xrightarrow{\text{Heat}} R-NC + 3KCl + 3H_2O$$

Hinsberg's Test

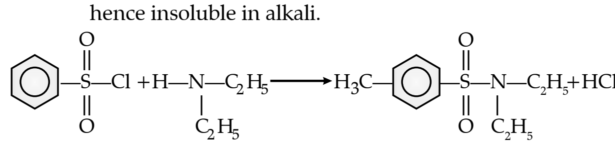
Benzenesulfonyl chloride ($C_6H_5SO_2Cl$) reacts with primary and secondary amines to produce sulphonamides.

(i) The reaction of benzene-sulfonyl chloride with primary amine yields N-ethyl benzene-sulfonyl amide. The hydrogen attached to the nitrogen in sulphonamide is strongly acidic due to the presence of strong electron withdrawing sulfonyl group. Hence, it is soluble in alkali.



N-Ethylbenzenesulphonamide
(Soluble in alkali)

(ii) In the reaction with a secondary amine, N, N-diethylbenzenesulfonamide is formed. Since N,N-diethyl benzene sulphonamide does not contain any hydrogen atom attached to a nitrogen atom, it is not acidic and hence insoluble in alkali.

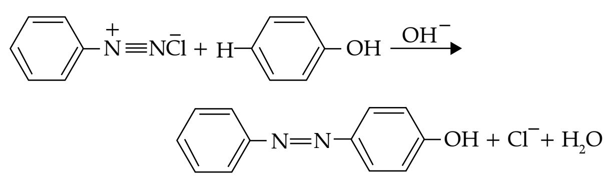


N.N-Diethylbenzenesulphonamide

(iii) Tertiary amines do not react with benzenesulphonyl chloride.

Coupling Reactions

Benzene diazonium chloride gets reacted with phenol in which the phenol molecule at its para position is mixed with the diazonium salt to give p-hydroxyazobenzene.



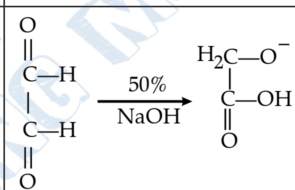
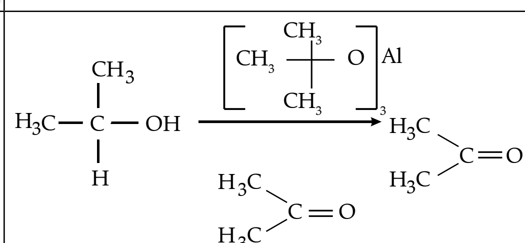
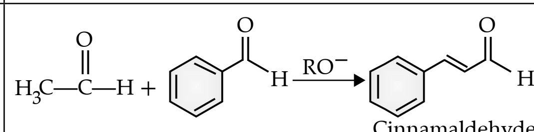
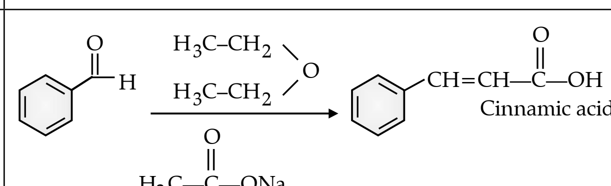
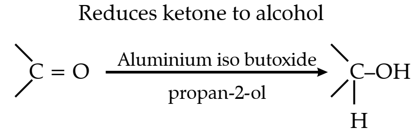
p – Hydroxyazobenzene (orange dye)

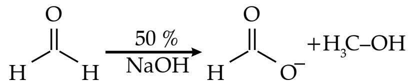
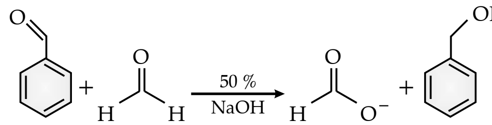
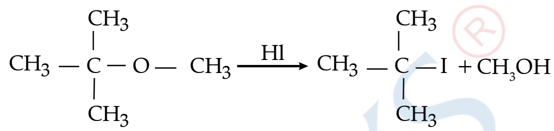
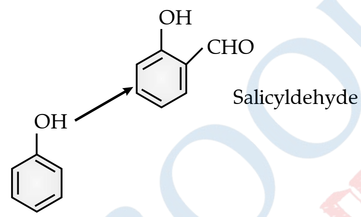
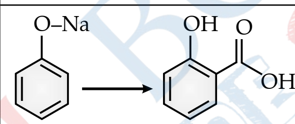
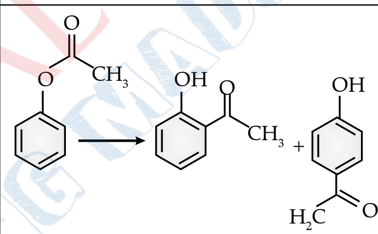
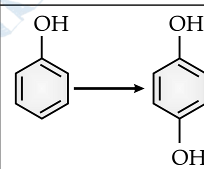
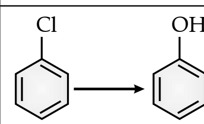
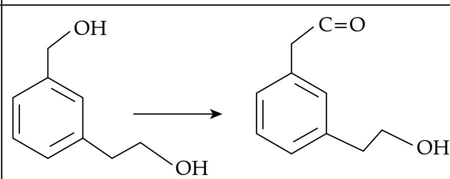
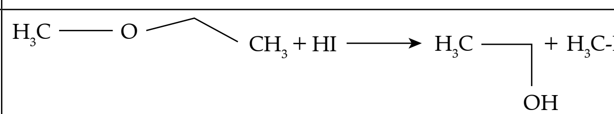
APPENDIX-F

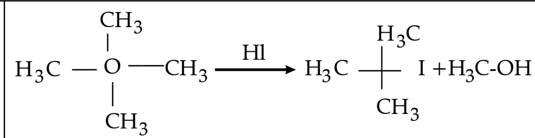
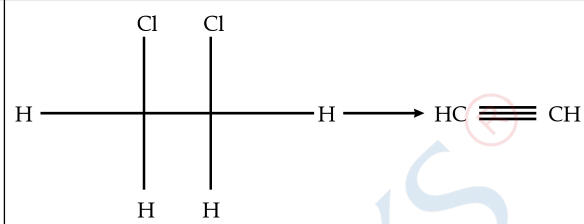
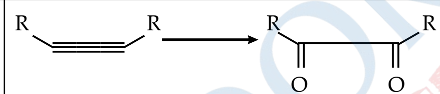
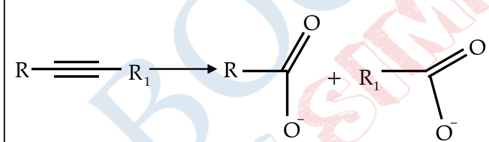
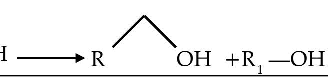
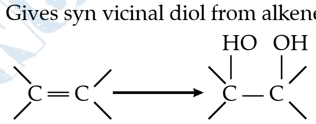
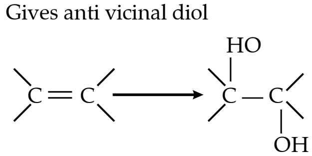
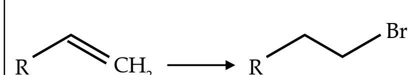
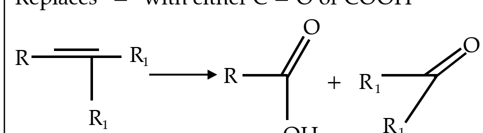
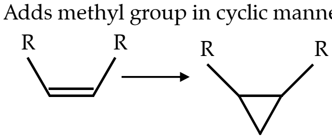
Important Reagents

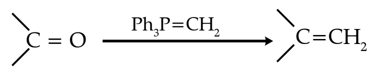
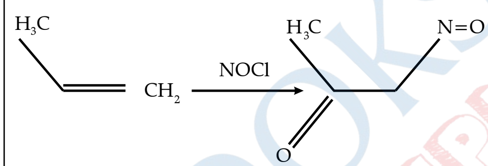
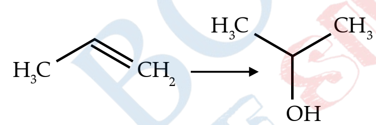
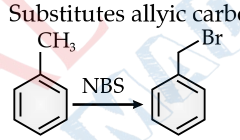
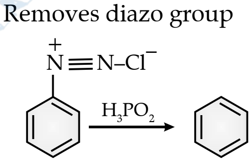
List of Organic Reagents

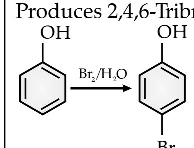
Aqueous NaOH	Reflux	Nucleophilic substitution, converts haloalkanes to alcohols.
Mg in dry ether	Reflux	Used to make Grignard reagents with haloalkanes.
PCl_5	Room temperature	Chlorinating agent, reacts with OH group in alcohols and carboxylic acids.
HNO_3 and H_2SO_4	55°C	Adds NO_2 group into benzene ring.
Cl_2 and $AlCl_3$	Warm gently	Adds Cl group into benzene ring.
CH_3CH_2Cl and $AlCl_3$	Warm gently	Adds CH_3CH_2 group into benzene ring.
HCl and $NaNO_2$	Below 5°C	Forms diazonium salts with phenylamine.

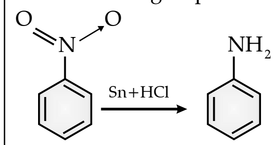
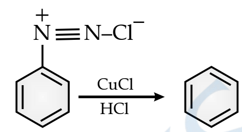
Name of Reagent	Conditions	Example of its Use
$K_2Cr_2O_7$ with conc. H_2SO_4	Warm gently	**Oxidising agent,** used commonly for oxidising secondary alcohols to ketones.
Excess conc. H_2SO_4	Heat to 170°C	Dehydrating agent, used to dehydrate alcohols to alkenes.
$Cl_2(g)$	Ultra violet light	Free radical reaction, used to convert alkanes to haloalkanes.
Br_2 in CCl_4	Room temperature, in the dark	Electrophilic addition, converts alkenes to dihaloalkanes.
$H_2(g)$	Nickel catalyst, 300°C and 30 atmospheric pressure	**Hydrogenating agent,** used to convert benzene to cyclohexane.
$H_2(g)$	Nickel catalyst, 150°C	**Reducing agent,** used to convert alkenes to alkanes
Tin in hydrochloric acid	Reflux	Reducing agent for converting nitrobenzene to phenylamine.
Acidified $KMnO_4$	Room temperature	**Oxidising agent,** converts alkenes to diols.
NaOH in ethanol	Reflux	Elimination reaction, converts haloalkanes to alkenes.
NaOH (Intramolecular Cannizaro) reaction		
(i) Aluminium isobutoxide **(ii)** Acetone (Oppenaur Oxidation)		
RO^-(Claisen Schmidt Reaction)		
(i) Acetic anhydride **(ii)** Sodium acetate (Perkin's reaction)		
(i) Aluminium isobutoxide **(ii)** Propan-2 ol (MPV Reduction)		Reduces ketone to alcohol 

Cannizzaro Reaction		This is reaction of compounds which don't have alpha hydrogen. 
Cross Cannizaro reaction		
Anhydrous HI contains no water		
(i) $CHCl_3$ - Alc KOH **(ii)** H_3O^+ (Riemann Tiemann reaction)		
(i) CO_2 **(ii)** H_3O^+ (Kolbe's reaction)	High Pressure	
$AlCl_3$ (Fries rearrangement)	Heat	
(i) $K_2S_2O_8$ **(ii)** H_3O^+ (Elb's persulphate oxidation)		
(i) Fused NaOH **(ii)** H_3O^+ (Dow's process)	High Pressure	
$KMnO_4$, H^+ (or) $K_2Cr_2O_7$, H^+ (or) H_2CrO_4		Oxidises alcohol to acid
PCC (Pyridinium chloro chromate)	Solvent CS_2	Restricted oxidation of alcohol. Forms aldehyde.
MnO_2 special oxidising agent for alcohol		 MnO_2 selectively oxidizes primary or secondary allylic or benzylic alcohols to carbonyl compounds.
HI		
For 1 degree carbon		

For 3 degree carbon		CH_3-O–C$(CH_3)_3$ + HI = $(CH_3)_3$C–I + CH_3OH
Conc HI contains very less water		
(i) Alc KOH **(ii)** $NaNH_2$		
X_2/CCl_4		Adds both X on compound having double or triple bond
Cold Dil $KMnO_4$		
Hot $KMnO_4/OH^-$		
$CF_3SO_3^-$		Super Leaving Group
$LiAlH_4$–ether		Reduces ester to alcohol
$NaBH_4$– ROH (protic solvent)		Reduces aldehydes and ketones to alcohol but cannot act on ester
Cr_2O_3-Cu_2O	Heat, High pressure	Causes cleavage and reduction of ester $RCOOR_1$ = RCH_2OH + R_1–OH 
OsO_4 + H_2O-$NaHSO_3$ or Cold Dilute $KMnO_4$ (Hydroxylation)		Gives syn vicinal diol from alkene 
Per-formic acid (Anti Hydroxylation)		Gives anti vicinal diol 
HBr in presence of H_2O_2 (Only for HBr)		Anti Markovnikov's 
Hot Alkaline $KMnO_4$		Replaces "=" with either C = O or COOH 
CH_2-I_2, ZnCu		Adds methyl group in cyclic manner 

Alcoholic KOH	Heat	Causes dehydrohalogenation of alkyl halides to form alkenes. HX from compound and adds double bond
Zn dust		Convert vicinal dihalides to alkenes.
H_2 Pd/$BaSO_4$ or S-Quinoline (Lindlar's catalyst) or BH_3-THF		Reduces alkyne to *cis*-alkene. It also reduces acid halides to aldehydes.
$Ph_3P = CH_2$		Converts aldehyde or ketones to alkene. The reaction is known as Wittig Reaction. 
NOX		 Check the reaction. The final product should be $CH_3CH(X)CH_2N{=}O$
OMDM (Oxymercuration demercuration) **(i)** Hg $(OAc)_2$ + THF-H_2O **(ii)** $NaBH_4$-OH^-		
NBS		Substitutes allyic carbon with aldehyde 
(i) NH_2-NH_2 **(ii)** RO^-	heat	Reduces carbonyl group to ketone or aldehyde
H_3PO_2	Heat	Removes diazo group 
$LiAH_4$ or $NaBH_4$ or Ph_3 SnH		Reduction
R_2CuLi (Lithium dialkylcuprate)		Removes halogen from RX and adds R. The reaction is known as Corey-House synthesis. The reaction is used to produce an alkane. The R- from R_2CuLi displaces -X from R-X to produce an alkane.
Mg-ether		Adds Mg between R & X
Red P + HI		Reduces alcohols, aldehydes, ketones and carboxylic acids to alkanes.

Name	Condition	Example
$Br_2 + CS_2$		Causes mono-bromination of phenol to produce p-bromophenol.
$Br_2 + H_2O$	Compound should be more activated than benzene	Produces 2,4,6-Tribromophenol from phenol. 

Sn + HCl or Fe + HCl		Reduces nitro group 
CuCl + HCl		Removes Diazo group 
NH_4SH or Na_2S		Special regents which reduce only $-NO_2$ group to $-NH_2$
$CrO_2Cl_2 + H_3O^+$		Converts toluene to benzaldehyde

APPENDIX-G

Important Minerals, Compositions & their Chemical Formula

Mineral	Composition		Remarks
Apatite	Calcium phosphate	$Ca_{10}(PO_4)_6X_2$ (X is F, Cl, or (OH).]	Main mineral in phosphate rock
Asbestos	Hydrated magnesium silicate	$Mg_6(Si_4O_{12})(OH)_3$	In form of long fibres
Baryte	Barium sulphate	$BaSO_4$	Filler for pigments
Betonies	A clay mineral	$(Al.Mg)_8(Si_4O_{10})_3(OH)_{10}\bullet 12H_2O$	Agglomeration additive
Borax	Sodium borate	$Na_2B_4O_7\bullet 10H_2O$	
Clay	Hydrated aluminium silicates		Used in paper making
Cryolite	Sodium aluminium fluoride	Na_3AlF_6	Low melting point
Diamond - industrial	Crystalline carbon	C	The hardest mineral
Diatomite	Hydrated silica	$SiO_2(H_2O)_n$	Marine fossils, large surface area
Feldspar	A mineral group	K, Al silicates	
Fluorspar	Calcium fluoride	CaF_2	Main source of fluorine
Garnet	A group of silicates that crystallize in the cubic system	$Mg_3Fe_2Si_3O_{12}$	Abrasives, gemstones
Graphite	Carbon (crystalline)	C	
Gypsum	Calcium sulphate	$CaSO_4\bullet 2H_2O$	
Kaolinite	A clay mineral	$Al_4(Si_4O_{10})(OH)_8$	
Limestone	Calcium carbonate	$CaCO_3$	
Magnetite	Magnesium carbonate	$MgCO_3$	
Marble	Calcium carbonate	$CaCO_3$ crystalline	
Mica		K, Al silicates	
Nepheline syenite	Sodium aluminium silicate		

Potash	Potassium chloride and carbonate	KCl, K_2CO_3	Fertilizer
Pumice	Silicate		Porous, light, volcanic rock, large surface area
Quartz	Silica	SiO_2	
Salt	Sodium chloride	NaCl	
Sand and gravel	Silica	SiO_2	
Sulfur	Sulfur	S	
Talc	Hydrated magnesium silicate	$Mg_3(Si_4O_{10})(OH)_2$	Also known as soapstone
Trona	Sodium carbonate	$Na_2CO_3 \cdot NaHCO_3 \cdot 2H_2O$	
Vermiculite	Hydrated silicates	$(Mg,Fe^{2+},Fe^{3+})_3[(Al,Si)_4O_{10}](OH)_2 \cdot 4H_2O$	Expands and swells on heating
Zeolite	Hydrated alkali alumino silicates	$Na_2(AlO_2)_x(SiO_2)_y \cdot nH_2O$	Ion exchanger

APPENDIX-H

Important Metals & their Ores

S.No	Metal	Ores
1	Aluminium(Al)	Bauxite, Corundum, Feldspar, Cryolite, Alunite, Kaolin
2	Antimony(Sb)	Stibnite
3	Barium(Ba)	Barytes
4	Bismuth(Bi)	Bismuthate
5	Cadmium(Cd)	Greenockite
6	Calcium(Ca)	Dolomite, Calcite, Gypsum, Fluorspar, Asbestos
7	Cobalt(Co)	Smelite
8	Copper(Cu)	Cuprite, Copper glance ,Copper pyrites
9	Gold(Au)	Calaverite, Sylvenites
10	Iron(Fe)	Hematite, Limonite, Magnetite, Siderite, Iron pyrite, Copper pyrites
11	Lead(Pb)	Galena
12	Magnesium(Mg)	Magnesite, Dolomite, Epsom salt, Kieserite, Carnalite
13	Manganese(Mn)	Pyrolusite, Magnate
14	Mercury(Hg)	Cinnabar
15	Nickel(Ni)	Millerite
16	Potassium(K)	Nitrate(saltpetre), Carnallite
17	Silver(Ag)	Ruby silver, Horn silver
18	Sodium(Na)	Chile saltpetre , Trona, Borax, Common salt
19	Strontium(Sr)	Strontianite, Silestone
20	Tin(Sn)	Cassiterite
21	Uranium(U)	Carnallite, Pitch blende
22	Zinc(Zn)	Zinc blende, Zincite, Calamine

APPENDIX-A

Conversion Factors

Conversion factors may be read directly from these tables. For example, 1 degree = 2.778×10^{-3} revolutions, so 16.7° = $16.7 \times 2.778 \times 10^{-3}$ rev. The SI units are fully capitalized.

Plane Angle

°	′	″	RADIAN	rev
1 degree = 1	60	3600	1.745×10^{-2}	2.778×10^{-3}
1 minute = 1.667×10^{-2}	1	60	2.909×10^{-4}	4.630×10^{-5}
1 second = 2.778×10^{-4}	1.667×10^{-2}	1	4.848×10^{-6}	7.716×10^{-7}
1 RADIAN = 57.30	3438	2.063×10^{5}	1	0.1592
1 revolution = 360	2.16×10^{4}	1.296×10^{6}	6.283	1

Solid Angle

1 sphere = 4π steradians = 12.57 steradians

Length

cm	m	km	In.	ft	mi
1 centimeter = 1	10^{-2}	10^{-5}	0.3937	3.281×10^{-2}	6.214×10^{-6}
1 meter = 100	1	10^{-3}	39.37^{4}	3.281	6.214×10^{-4}
1 kilometer = 10^{5}	1000	1	3.937×10^{4}	3281	0.6214
1 inch = 2.540	2.540×10^{-2}	2.540×10^{-25}	1	8.333×10^{-2}	1.578×10^{-5}
1 foot = 30.48	0.3048	3.048×10^{-4}	12	1	1.894×10^{-4}
1 mile = 1.609×10^{5}	1609	1.609	6.336×10^{4}	5280	1

1 angstrom = 10^{-10} m	1 fermi = 10^{-15} m	1 fathom = 6 ft	1 rod = 16.5 ft
1 nautical mile = 1852 m	1 light-year = 9.460×10^{12} km	1 Bohr radius = 5.292×10^{-11} m	1 mil = 10^{-3} in.
= 1.151 miles = 6076 ft	1 parsec = 3.084×10^{13} km	1 yard = 3 ft	1 nm = 10^{-9} m

Area

m^2	cm^2	ft^2	$In.^2$
1 SQUARE METER = 1	10^{4}	10.76	1550
1 square centimetre = 10^{-4}	1	1.076×10^{-3}	0.1550
1 square foot = 9.290×10^{-2}	929.0	1	144
1 square inch = 6.452×10^{-4}	6.452	6.944×10^{-3}	1

1 square mile = 2.788×10^{7} ft^2 = 640 acres	1 acre = 43 560 ft^2
1 barn = 10^{-28} m^2	1 hectare = 10^{4} m^2 = 2.471 acres

Volume

m^3	cm^3	L	ft^3	in^3
1 CUBIC METER = 1	10^6	1000	35.31	6.102×10^4
1 cubic centimeter $=10^{-6}$	1	1.000×10^{-3}	3.531×10^{-5}	6.102×10^{-2}
1 liter = 1.000×10^{-3}	1000	1	3.531×10^{-2}	61.02
1 cubic foot = 2.832×10^{-2}	2.832×10^{-4}	28.32	1	1728
1 cubic inch = 1.639×10^{-5}	16.39	1.639×10^{-2}	5.787×10^{-4}	1

1 U.S. fluid gallon = 4 U.S. fluid quarts = 8 U.S. pints = 128 U.S. fluid ounces = 231 $in.^3$

1 British imperial gallon = 277.4 $in.^3$ = 1.201 U.S. fluid gallons

Mass

Quantities in the colored ares are not mass units but are often used as such. For example, when we write 1 kg "=" 2.205 lb, this means that a kilogram is a *mass* that *weighs* 2.205 pounds at a location where g has the standard value of 9.80665 m/s^2.

g	Kg	slug	u	oz	lb	ton
1 gram = 1	0.001	6.852×10^{-5}	6.022×10^{23}	3.527×10^{-2}	2.205×10^{-3}	1.102×10^{-6}
1 KILOGRAM = 1000	1	6.852×10^{-2}	6.022×10^{26}	35.27	2.205	1.102×10^{-3}
1 slug = 1.459×10^4	14.59	1	8.786×10^{27}	514.8	32.17	1.609×10^{-2}
1 atomic mass unit = 1.661×10^{-24}	1.661×10^{-27}	1.138×10^{-28}	1	5.857×10^{-26}	3.662×10^{-27}	1.830×10^{-30}
1 ounce = 28.35	2.835×10^{-2}	1.943×10^{-3}	1.718×10^{25}	1	6.250×10^{-2}	3.125×10^{-5}
1 pound=453.6	0.4536	3.108×10^{-2}	2.732×10^{26}	16	1	0.0005
1 ton = 9.072×10^5	907.2	62.16	5.463×10^{29}	3.2×10^4	2000	1

1 metric ton = 1000 kg

Density

Quantities in the colored areas are weight densities and, as such, are dimensionally different from mass densities. See the note for the mass table.

$slug/ft^3$	KILOGRAM/METER^3	g/cm^3	lb/ft^3	$lb/in.^3$
1 slug per foot^3 = 1	515.4	0.5154	32.17	1.862×10^{-2}
1 KILOGRAM Per METER^3 = 1.940×10^{-3}	1	0.001	6.243×10^{-2}	3.613×10^{-5}
1 gram per centimeter^3 = 1.940	1000	1	62.43	3.613×10^{-2}
1 pound per foot^3=3.108×10^{-2}	16.02	16.02×10^{-2}	1	5.787×10^{-4}
1 pound per inch^3=53.71	2.768×10^4	27.68	1728	1

Time

y	d	h	min	s
1 year = 1	365.25	8.766×10^3	5.259×10^5	3.156×10^7
1 day = 2.738×10^{-3}	1	24	1440	8.640×10^4
1 hour = 1.141×10^{-4}	4.167×10^{-2}	1	60	3600
1 minute = 1.901×10^{-6}	6.944×10^{-4}	1.667×10^{-2}	1	60
1 SECOND =3.169×10^{-8}	1.157×10^{-5}	2.778×10^{-4}	1.667×10^{-2}	1

Speed

ft/s	km/h	m/s	mi/h	cm/s
1 foot per second = 1	1.097	0.3048	0.6818	30.48
1 kilometer per hour = 0.9113	1	0.2278	0.6214	27. 78
1 METER per SECOND = 3.281	3.6	1	2.237	100
1 mile per hour = 1.467	1.609	0.4470	1	44.70
1 centimeter per second = 3.281×10^{-2}	3.6×10^{-2}	0.01	2.237×10^{-2}	1

APPENDIX-B

The Greek Alphabet

Alpha	A	α	Iota	I	ι	Rho	P	ρ
Beta	B	β	Kappa	K	κ	Sigma	Σ	σ
Gamma	Γ	γ	Lambda	Λ	λ	Tau	T	τ
Delta	Δ	δ	Mu	M	μ	Upsilon	Υ	υ
Epsilon	E	ε	Nu	N	ν	Phi	ϕ	φ
Zeta	Z	ζ	Xi	Ξ	ξ	Chi	X	χ
Eta	H	η	Omicron	O	o	Psi	Ψ	ψ
Theta	Θ	θ	Pi	Π	π	Omega	Ω	ω

APPENDIX-C

Concept Base Mathematical Formula

ALGEBRA – 1

1.1 Set Identities

Sets : A, B, C, Universal set : U, Complement : A', Proper subset : $A \subset B$, Empty set : $\varnothing$, Union of sets : $A \cup B$, Intersection of sets : $A \cap B$, Difference of sets : $A - B$

1. $A \subset U$
2. $A \subset A$
3. $A = B$ if $A \subset B$ and $B \subset A$.
4. **Empty set** $\varnothing \subset A$
5. **Union of Sets**
   $C = A \cup B \{x \mid x \in A \text{ or } x \in B\}$
6. **Commutativity** $A \cup B = B \cup A$
7. **Associativity** $A \cup (B \cup C) = (A \cup B) \cup C$
8. **Intersection of Sets**
   $C = A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
9. **Commutativity** $A \cap B = B \cap A$
10. **Associativity** $A \cap (B \cap C) = (A \cap B) \cap C$
11. **Distributivity**
    $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$
    $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$
12. **Idempotency** $A \cap A = A, \quad A \cup A = A$
13. **Domination** $A \cap \varnothing = \varnothing \quad A \cup U = U$
14. **Identity** $A \cup \varnothing = A, \quad A \cap U = A$
15. **Complement** $A' = \{x \in U \mid x \notin A\}$
16. **Complement of intersection and Union**
    $A \cup A' = U, \quad A \cap A' = \varnothing$
17. **De - Morgan's Laws**
    $(A \cup B)' = A' \cap B', \ (A \cap B)' = A' \cup B'$
18. **Difference of Sets**
    $C = B - A = \{x \mid x \in B \text{ and } x \notin A\}$
19. $B - A = B - (A \cap B)$
20. $B - A = B \cap A'$
21. $A - A = \varnothing$
22. $A - B = A$ *iff* $A \cap B = \varnothing$.
23. $(A - B) \cap C = (A \cap C) - (B \cap C)$
24. $A' = U - A$
25. **Cartesian Product**
    $C = A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$

1.2 Sets of Numbers

Natural numbers : N, Whole numbers : W, Integers : Z, Positive integers : Z^+, Negative integers : Z^-, Rational numbers : Q, Real numbers : R, Complex numbers : C

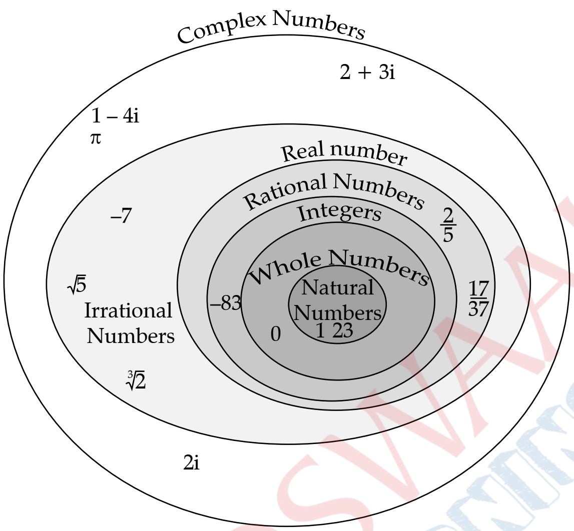
1. **Natural Numbers**
   Counting numbers : $N = \{1, 2, 3, \ldots\}$.
2. **Whole Numbers**
   Counting numbers and zero : $W = \{0, 1, 2, 3, \ldots\}$.
3. **Integers**
   Whole number and their opposites and zero :

$Z^+ = N = \{1, 2, 3, \ldots\}$,

$Z^- = \{\ldots, -3, -2, -1\}$,

$Z = Z^- \cup \{0\} \cup Z^+ = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

4. **Rational numbers**
   Repeating or terminating decimals:
   $$Q = \left\{ x \,\middle|\, x = \frac{a}{b} \text{ and } a \in Z \text{ and } b \in Z \text{ and } b \neq 0 \right\}.$$
5. **Irrational Numbers**
   Nonrepeating and nonterminating decimals.
6. **Real Numbers**
   Union of rational and irrational numbers : R.
7. **Complex Numbers**
   $C = \{x + iy \mid x \in R \text{ and } y \in R\}$,
   where i is the imaginary unit.
8. $N \subset W \subset Z \subset Q \subset R \subset C$



1.3 Basic Identities

Real numbers: a, b, c

1. **Additive Identity :** $a + 0 = a$
2. **Additive Inverse :** $a + (-a) = 0$
3. **Commutative of Addition :** $a + b = b + a$
4. **Associative of Addition :** $(a + b) + c = a + (b + c)$
5. **Definition of Subtraction :** $a - b = a + (-b)$
6. **Multiplicative Identity :** $a .1 = a$
7. **Multiplicative Inverse** $a . \frac{1}{a} = 1, \ a \neq 0$
8. **Multiplication times 0** $a.0 = 0$.
9. **Commutative of Multiplication :** $a.b = b.a$
10. **Associative of Multiplication :** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
11. **Distributive Law :** $a(b + c) = ab + ac$
12. **Definition of Division :** $\frac{a}{b} = a \cdot \frac{1}{b}$

1.4 Complex Numbers

Natural number : n, Imaginary unit : i, Complex number : z, Real part : a, c, Imaginary part : bi, di, Modulus of a complex number : r, r_1, r_2, Argument of a complex number: $\varphi, \varphi_1, \varphi_2$

1.

$i^1 = i$	$i^5 = i$	$i^{4n+1} = i$
$i^2 = -1$	$i^6 = -1$	$i^{4n+2} = -1$
$i^3 = -i$	$i^7 = -i$	$i^{4n+3} = -i$
$i^4 = -1$	$i^8 = 1$	$i^{4n} = 1$

2. $z = a + bi$
3. **Complex Plane**
4. $(a + bi) + (c + di) = (a + c) + (b + d)i$
5. $(a + bi) - (c + di) = (a - c) + (b - d)i$
6. $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$
7. $\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} \cdot i$
8. **Conjugate Complex Numbers :** $\overline{a + bi} = a - bi$
9. $a = r \cos \varphi, b = r \sin \varphi$
10. **Polar Presentation of Complex Numbers**
    $a + bi = r(\cos \varphi + i \sin \varphi)$
11. **Modulus and Argument of a Complex Number**
    If $a + bi$ is a complex number, then
    $r = \sqrt{a^2 + b^2}$ (modulus), and $\varphi = \arctan \frac{b}{a}$ (argument).
12. **Product in Polar Representation**
    $z_1 \cdot z_2 = r_1 (\cos \varphi_1 + i \sin \varphi_1) \cdot r_2 (\cos \varphi_2 + i \sin \varphi_2)$
    $= r_1 r_2 [\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)]$
13. **Conjugate Numbers in Polar Representation**
    $$\overline{r(\cos \varphi + i \sin \varphi)} = r[\cos(-\varphi) + i \sin(-\varphi)]$$
14. **Inverse of a Complex Number in Polar Representation**
    $$\frac{1}{r(\cos \varphi + i \sin \varphi)} = \frac{1}{r}[\cos(-\varphi) + i \sin(-\varphi)]$$
15. **Quotient in Polar Representation**
    $$\frac{z_1}{z_2} = \frac{r_1(\cos \varphi_1 + i \sin \varphi_1)}{r_2(\cos \varphi_2 + i \sin \varphi_2)}$$
    $$= \frac{r_1}{r_2}[\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2)]$$
16. **Power of a Complex Number**
    $$z^n = [r(\cos \varphi + i \sin \varphi)]^n = r^n[\cos(n\varphi) + i \sin(n\varphi)]$$
17. **Formula "De Moivre"**
    $$(\cos \varphi + i \sin \varphi)^n = \cos(n\varphi) + i \sin(n\varphi)$$
18. **n^{th} Root of a Complex Number**
    $$\sqrt[n]{z} = \sqrt[n]{r(\cos \varphi + i \sin \varphi)}$$
    $$= \sqrt[n]{r}\left(\cos \frac{\varphi + 2\pi k}{n} + i \sin \frac{\varphi + 2\pi k}{n}\right),$$
    where $k = 0, 1, 2, \ldots, n - 1$.
19. **Euler's Formula** $e^{ix} = \cos x + i \sin x$

1.5 Basic Algebra

Real numbers: a, b, c, Natural number: n

1. $a^3 - b^3 = (a-b)(a^2+ab+b^2)$
2. $a^3 + b^3 = (a+b)(a^2-ab+b^2)$
3. $a^5 - b^5 = (a-b)(a^2+a^3b+a^2b^2+ab^3+b^4)$
4. $a^5 + b^5 = (a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)$
5. **If n is odd, then**

   $a^n + b^n = (a+b)(a^{n-1}-a^{n-2}b+a^{n-3}b^2-\ldots-ab^{n-2}+b^{n-1})$.
6. **If n is even, then**

   $a^n - b^n = (a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+ab^{n-2}+b^{n-1})$,

   $a^n + b^n = (a+b)(a^{n-1}-a^{n-2}b+a^{n-3}b^2-\ldots-ab^{n-2}-b^{n-1})$.
7. **Binomial Formula**

   $$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n$$

   where ${}^nC_k = \frac{n!}{k!(n-k)!}$ are the binomial coefficients.
8. $(a+b+c+\ldots+u+v)^2 = a^2+b^2+c^2+\ldots+u^2+v^2+ +2(ab+ac+\ldots+au+av+bc+\ldots+bu+bv+\ldots+uv)$

1.6 Properties of Powers and Roots

Base (positive real numbers): a, b, Powers (rational numbers): n, m

Powers

1. $a^m a^n = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$
3. $(ab)^m = a^m b^m$
4. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
5. $(a^m)^n = a^{mn}$
6. $a^0 = 1, a \neq 0$
7. $a^1 = a$
8. $a^{-m} = \frac{1}{a^m}$
9. $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Roots

Bases: a, b, Powers (rational numbers): n, m, $a, b \geq 0$ for even roots ($n = 2k, k \in N$)

1. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$
2. $\sqrt[n]{a}\sqrt[m]{b} = \sqrt[nm]{a^m b^n}$
3. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}, b \neq 0$
4. $\frac{\sqrt[n]{a}}{\sqrt[m]{b}} = \frac{\sqrt[nm]{a^m}}{\sqrt[nm]{b^n}} = \sqrt[nm]{\frac{a^m}{b^n}}, b \neq 0$
5. $\left(\sqrt[n]{a^m}\right)^p = \sqrt[n]{a^{mp}}$
6. $\left(\sqrt[n]{a}\right)^n = a$
7. $\sqrt[n]{a^m} = \sqrt[np]{a^{mp}}$
8. $\sqrt[n]{a^m} = a^{\frac{m}{n}}$
9. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$
10. $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$
11. $\frac{1}{\sqrt[n]{a}} = \frac{\sqrt[n]{a^{n-1}}}{a}, a \neq 0$.

12. $\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} \pm \sqrt{\frac{a-\sqrt{a^2-b}}{2}}$
13. $\frac{1}{\sqrt{a} \pm \sqrt{b}} = \frac{\sqrt{a} \mp \sqrt{b}}{a-b}$

1.7 Concept of Logarithms

Positive real numbers: x, y, a, c, k, Natural number: n

1. Definition of Logarithm

   $y = \log_a x$ if and only if $x = a^y$, $a > 0$, $a \neq 1$.
2. $\log_a 1 = 0$
3. $\log_a a = 1$
4. $\log_a 0 = \begin{cases} -\infty & \text{if } a > 1 \\ +\infty & \text{if } a < 1 \end{cases}$
5. $\log_a(xy) = \log_a x + \log_a y$
6. $\log_a \frac{x}{y} = \log_a x - \log_a y$
7. $\log_a(x^n) = n\log_a x$
8. $\log_a \sqrt[n]{x} = \frac{1}{n}\log_a x$
9. $\log_a x = \frac{\log_c x}{\log_c a} = \log_c x \cdot \log_a c, c > 0, c \neq 1, a \neq 1$
10. $\log_a c = \frac{1}{\log_c a}$
11. $x = a^{\log_a x}$
12. **Logarithm to Base 10** $\log_{10} x = \log x$
13. **Natural Logarithm**

    $\log_e x = \ln x$, where $e = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = 2.718281828\ldots$
14. $\log x = \frac{1}{\ln 10}\ln x = 0.434294 \ln x$
15. $\ln x = \frac{1}{\log e}\log x = 2.302585 \log x$

1.8 Concept of Equations

Real numbers: a, b, c, p, q, u, v, Solutions : x_1, x_2, y_1, y_2, y_3

1. **Linear Equation in One Variable**

   $ax + b = 0, x = -\frac{b}{a}$.
2. **Quadratic Equation**

   $ax^2 + bx + c = 0,\ x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.
3. **Discriminant** $D = b^2 - 4ac$
4. **Viete's Formulas**

   If $x^2 + px + q = 0$, then

   $\begin{cases} x_1 + x_2 = -p \\ x_1x_2 = q \end{cases}$.
5. $ax^2 + bx = 0,\ x_1 = 0,\ x_2 = -\frac{b}{a}$.

6. $ax^2 + c = 0,\ x_{1,2} = \pm\sqrt{-\frac{c}{a}}.$

7. **Cubic Equation. Cardano's Formula.**

$y^3 + py + q = 0,$

$y_1 = u + v, y_{2,3} = -\frac{1}{2}(u+v) \pm \frac{\sqrt{3}}{2}(u+v)i,$

where

$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^2}},\ v = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^2}}.$

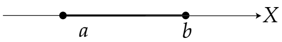
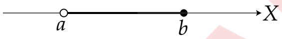
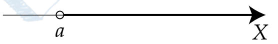
1.9 Inequalities

Variables: x, y, z

Real numbers : $\begin{cases} a, b, c, d \\ a_1, a_2, a_3, \ldots, a_n \end{cases}, m, n$

Determinants : D, D_x, D_y, D_z

1. Inequalities, Interval Notations and Graphs

Inequality	Interval Notation	Graph
$a \le x \le b$	$[a, b]$	
$a < x \le b$	$(a, b]$	
$a \le x < b$	$[a, b)$	
$a < x < b$	(a, b)	
$-\infty < x \le b$, $x \le b$	$(-\infty, b)$	
$-\infty < x < b$, $x < b$	$(-\infty, b)$	
$a \le x < \infty$, $x \ge a$	$[a, \infty)$	
$a < x < \infty$, $x > a$	(a, ∞)	

2. If $a > b$ and $m > 0$, then $ma > mb$.

3. If $a > b$ and $m > 0$, then $\frac{a}{m} > \frac{b}{m}$.

4. If $a > b$ and $m < 0$, then $ma < mb$.

5. If $a > b$ and $m < 0$, then $\frac{a}{m} < \frac{b}{m}$.

6. If $0 < a < b$ and $n > 0$, then $a^n < b^n$.

7. If $0 < a < b$ and $n < 0$, then $a^n > b^n$.

8. If $0 < a < b$, then $\sqrt[n]{a} < \sqrt[n]{b}$.

9. $\sqrt{ab} \le \frac{a+b}{2}$, where $a > 0, b > 0$; an equality is valid only if $a = b$.

10. $a + \frac{1}{a} \ge 2$, where $a > 0$; an equality takes place only at $a = 1$.

11. $\sqrt[n]{a_1 a_2 \ldots a_n} \le \frac{a_1 + a_2 + \ldots + a_n}{n}$, where $a_1, a_2, \ldots, a_n > 0$.

12. If $ax + b > 0$ and $a > 0$, then $x > -\frac{b}{a}$.

13. If $ax + b > 0$ and $a < 0$, then $x < \frac{b}{a}$.

14. If $x^2 < a$, then $|x| < \sqrt{a}$, where $a > 0$.

15. If $x^2 > a$, then $|x| > \sqrt{a}$, where $a > 0$.

16. If $\frac{f(x)}{g(x)} > 0$, then $\begin{cases} f(x).g(x) > 0 \\ g(x) \ne 0 \end{cases}$

17. If $\frac{f(x)}{g(x)} < 0$, then $\begin{cases} f(x).g(x) < 0 \\ g(x) \ne 0 \end{cases}$

SERIES – 2

2.1 Arithmetic Series

Initial term: a_1, n^{th} term : a_n, Difference between successive terms : d, Number of terms in the series : n, Sum of the first n terms : S_n

1. $a_n = a_{n-1} + d = a_{n-2} + 2d = \ldots = a_1 + (n-1)d$

2. $a_1 + a_n = a_2 + a_{n-1} = \ldots = a_i + a_{n+1-i}$

3. $a_i = \frac{a_{i-1} + a_{i+1}}{2}$

4. $S_n = \frac{a_1 + a_n}{2}.n = \frac{(2a_1 + (n-1)d)}{2}.n$

2.2 Geometric Series

Initial term: a_1, n^{th} term: a_n, Common ratio: r, Number of terms in the series: n, Sum of the first n terms: S_n, Sum to infinity: S

1. $a_n = ra_{n-1} = a_1 r^{n-1}$

2. $a_1.a_n = a_2.a_{n-1} = \ldots = a_i.a_{n+1-i}$

3. $a_i = \sqrt{a_{i-1}.a_{i+1}}$

4. $S_n = \frac{a_n r - a_1}{r-1} = \frac{a_1(r^n - 1)}{r-1}$

5. $S = \lim_{n\to\infty} S_n = \frac{a_1}{1-r}$

For $|r| < 1$, the sum S converges as $n \to \infty$.

2.3 Some Finite Series

Number of terms in the series : n

1. $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

2. $2 + 4 + 6 + \ldots + 2n = n(n+1)$

3. $1+3+5+...+(2n-1)=n^2$

4. $k+(k+1)+(k+2)+...+(k+n-1)=\frac{n(2k+n-1)}{2}$

5. $1^2+2^2+3^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$

6. $1^3+2^3+3^3+...+n^3=\left[\frac{n(n+1)}{2}\right]^2$

7. $1^2+2^2+3^2+...+(2n-1)^2=\frac{n(4n^2-1)}{3}$

8. $1^3+3^3+5^3+...+(2n-1)^3=n^2(2n^2-1)$

9. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....+\frac{1}{2^n}+...=2$

10. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{n(n+1)}+...=\frac{n}{(n+1)}$

11. $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{(n-1)!}+...=e$

2.4 Infinite Series

Sequence : $\{a_n\}$, First term : a_1, N^{th} term : a_n

1. **Infinite Series**

$$\sum_{n=1}^{\infty} a_n = a_1+a_2+...+a_n+...$$

2. **n^{th} Partial Sum**

$$S_n=\sum_{n=1}^{\infty} a_n = a_1+a_2+...+a_n$$

3. **Convergence of Infinite Series**

$$\sum_{n=1}^{\infty} a_n = L \text{, if } \lim_{n\to\infty} S_n = L$$

4. **n^{th} Term Test**

- If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.
- If $\lim_{n\to\infty} a_n \neq 0$, then the series is divergent.

2.5 Properties of Convergent Series

Convergent Series : $\sum_{n=1}^{\infty} a_n = A, \sum_{n=1}^{\infty} b_n = B$, Real number : c

1. $\sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty} a_n+\sum_{n=1}^{\infty} b_n = A+B$

2. $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n = cA$.

2.6 Convergence Tests

1. **The Comparison Test**

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series such that $0 < a_n \leq b_n$ for all n.

- If $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is also convergent.
- If $\sum_{n=1}^{\infty} a_n$ is divergent then $\sum_{n=1}^{\infty} b_n$ is also divergent.

2. **The Limit Comparison Test**

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series such that a_n and b_n are positive for all n.

- If $0 < \lim_{n\to\infty}\frac{a_n}{b_n} < \infty$ then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are either both convergent or both divergent.
- If $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$ then $\sum_{n=1}^{\infty} b_n$ convergent implies that $\sum_{n=1}^{\infty} a_n$ is also convergent.
- If $\lim_{n\to\infty}\frac{a_n}{b_n} = \infty$ then $\sum_{n=1}^{\infty} b_n$ divergent implies that $\sum_{n=1}^{\infty} a_n$ is also divergent.

3. **p – series**

p – series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converge s for $p > 1$ and diverges for $0 < p \leq 1$.

4. **The Integral Test**

Let $f(x)$ be a function which is continuous, positive, and decreasing for all $x \geq 1$. The series

$$\sum_{n=1}^{\infty} f(n) = f(1)+f(2)+f(3)+...+f(n)+...$$

Converges if $\int_1^{\infty} f(x)\,dx$ converges, and diverges if

$\int_1^{n} f(x)\,dx \to \infty$ as $n \to \infty$.

5. **The Ratio Test**

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms.

- If $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} < 1$ then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.
- $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1$ then $\sum_{n=1}^{\infty} a_n$ may converge or diverge and the ratio test is inconclusive; some other test must be used.

6. **The Root Test**

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms.

- If $\lim_{n\to\infty}\sqrt[n]{a_n} < 1$ then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\lim_{n\to\infty}\sqrt[n]{a_n} > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

- If $\lim_{n\to\infty}\sqrt[n]{a_n}=1$ then $\sum_{n=1}^{\infty}a_n$ may converge or diverge, but no conclusion can be drawn from this test.

2.7 Power Series

Real numbers: x, x_0, Power series : $\sum_{n=0}^{\infty}a_nx^n, \sum_{n=0}^{\infty}a_n(x-x_0)^n$,
Whole number : n, Radius of Convergence : R.

1. **Power Series in x**

$$\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots$$

2. **Power Series in $(x-x_0)$**

$$\sum_{n=0}^{\infty}a_n(x-x_0)^n=a_0+a_1(x-x_0)+a_2(x-x_0)^2+\ldots+a_n(x-x_0)^n+\ldots$$

3. **Interval of Convergence**
The set of those value of x for which the function $f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$ is convergent is called the interval of convergence.

4. **Radius of Convergence**
If the interval of convergence is (x_0-R, x_0+R) for some $R\geq 0$, the R is called the radius of convergence. It is given as

$$R=\lim_{n\to\infty}\frac{1}{\sqrt[n]{a_n}} \text{ or } R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

2.8 Power Series Expansions for Some Functions

Whole number: n, Real number : x

1. $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots$

2. $a^x=1+\frac{x\ln a}{1!}+\frac{(x\ln a)^2}{2!}+\frac{(x\ln a)^3}{3!}+\ldots+\frac{(x\ln a)^n}{n!}+\ldots$

3. $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+\frac{(-1)^nx^{n+1}}{n+1}\pm\ldots, -1<x\leq 1.$

4. $\ln\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\ldots\right), |x|<1.$

5. $\ln x=2\left[\frac{x-1}{x+1}+\frac{1}{3}\left(\frac{x-1}{x+1}\right)^3+\frac{1}{5}\left(\frac{x-1}{x+1}\right)^5\ldots\right], x>0.$

6. $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+\frac{(-1)^nx^{2n}}{(2n)!}\pm\ldots$

7. $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots+\frac{(-1)^nx^{2n+1}}{(2n+1)!}\pm\ldots$

8. $\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\frac{62x^9}{2835}+\ldots, |x|<\frac{\pi}{2}.$

9. $\cot x=\frac{1}{x}-\left(\frac{x}{3}+\frac{x^3}{45}+\frac{2x^5}{945}+\frac{2x^7}{4725}+\ldots\right), 0<|x|<\pi.$

10. $\sin^{-1}x=x+\frac{x^3}{2.3}+\frac{1.3x^5}{2.4.5}+\ldots+\frac{1.3.5\ldots(2n-1)x^{2n+1}}{2.4.6\ldots(2n)(2n+1)}+\ldots, |x|<1.$

11. $\cos^{-1}x=\frac{\pi}{2}-\left(x+\frac{x^3}{2.3}+\frac{1.3x^5}{2.4.5}+\ldots+\frac{1.3.5\ldots(2n-1)x^{2n+1}}{2.4.6\ldots(2n)(2n+1)}+\ldots\right), |x|<1$

12. $\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^3}{5}-\frac{x^7}{7}+\ldots+\frac{(-1)^nx^{2n+1}}{2n+1}\ldots, |x|<1.$

13. $\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots+\frac{x^{2n}}{(2n)!}+\ldots$

14. $\sinh x=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\ldots+\frac{x^{2n+1}}{(2n+1)!}+\ldots$

2.9 Binomial Series

Whole numbers: n, m, Real number : x Combinations : ${}^nC_m$

1. $(1+x)^n=1+{}^nC_1x+{}^nC_2x^2+\ldots+{}^mC_nx^m+\ldots+x^n$

2. ${}^nC_m=\frac{n(n-1)\ldots[n-(m-1)]}{m!}, |x|<1.$

3. $\frac{1}{1+x}=1-x+x^2-x^3+\ldots, |x|<1.$

4. $\frac{1}{1-x}=1+x+x^2+x^3+\ldots, |x|<1.$

5. $\sqrt{1+x}=1+\frac{x}{2}-\frac{x^2}{2\cdot 4}+\frac{1\cdot 3x^3}{2\cdot 4\cdot 6}-\cdot\frac{1\cdot 3\cdot 5x^4}{2\cdot 4\cdot 6\cdot 8}+\ldots, |x|\leq 1.$

6. $\sqrt[3]{1+x}=1+\frac{x}{3}-\frac{1.2x^2}{3\cdot 6}+\frac{1\cdot 2\cdot 5x^3}{3\cdot 6\cdot 9}-\frac{1\cdot 2\cdot 5\cdot 8x^4}{3\cdot 6\cdot 9\cdot 12}+\ldots, |x|\leq 1.$

TRIGONOMETRY – 3

3.1 Periodicity of Trigonometric Functions

1. $\sin(\alpha\pm 2\pi n)=\sin\alpha$, period 2π or 360°.
2. $\cos(\alpha\pm 2\pi n)=\cos\alpha$, period 2π or 360°.
3. $\tan(\alpha\pm\pi n)=\tan\alpha$, period π or 180°.
4. $\cot(\alpha\pm\pi n)=\cot\alpha$, period π or 180°.

3.2 Relations between Trigonometric Functions

1. $\sin\alpha=\pm\sqrt{1-\cos^2\alpha}=\pm\sqrt{\frac{1}{2}(1-\cos 2\alpha)}$

$$=2\cos^2\left(\frac{\alpha}{2}-\frac{\pi}{4}\right)-1=\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$

2. $\cos\alpha = \pm\sqrt{1-\sin^2\alpha} = \pm\sqrt{\frac{1}{2}(1+\cos 2\alpha)}$

$= 2\cos^2\frac{\alpha}{2} - 1 = \dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$

3. $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \pm\sqrt{\sec^2\alpha - 1} = \dfrac{\sin 2\alpha}{1+\cos 2\alpha} = \dfrac{1-\cos 2\alpha}{\sin 2\alpha}$

$= \pm\sqrt{\dfrac{1-\cos 2\alpha}{1+\cos 2\alpha}} = \dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$

4. $\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha} = \pm\sqrt{\csc^2\alpha - 1} = \dfrac{1+\cos 2\alpha}{\sin 2\alpha} = \dfrac{\sin 2\alpha}{1-\cos 2\alpha}$

$= \pm\sqrt{\dfrac{1+\cos 2\alpha}{1-\cos 2\alpha}} = \dfrac{1+\tan^2\frac{\alpha}{2}}{2\tan\frac{\alpha}{2}}$

5. $\sec\alpha = \dfrac{1}{\cos\alpha} = \pm\sqrt{1+\tan^2\alpha} = \dfrac{1+\tan^2\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$

6. $\operatorname{cosec}\alpha = \dfrac{1}{\sin\alpha} = \pm\sqrt{1+\cot^2\alpha} = \dfrac{1+\tan^2\frac{\alpha}{2}}{2\tan^2\frac{\alpha}{2}}$

3.3 Addition and Substitution Formulas

1. $\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$

2. $\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

3. $\tan(\alpha\pm\beta) = \dfrac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$

4. $\cot(\alpha\pm\beta) = \dfrac{1\mp\tan\alpha\tan\beta}{\tan\alpha\pm\tan\beta} = \dfrac{\cot\alpha\cot\beta\mp 1}{\cot\beta\pm\cot\alpha}$

3.4 Multiple Angle Formulas

1. $\sin 2\alpha = 2\sin\alpha\cdot\cos\alpha$

2. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1-2\sin^2\alpha = 2\cos^2\alpha - 1$

3. $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha} = \dfrac{2}{\cot\alpha - \tan\alpha}$

4. $\cot 2\alpha = \dfrac{\cot^2\alpha - 1}{2\cot\alpha} = \dfrac{\cot\alpha - \tan\alpha}{2}$

5. $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha = 3\cos^2\alpha\cdot\sin\alpha - \sin^3\alpha$

6. $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha = \cos^3\alpha - 3\cos\alpha\cdot\sin^2\alpha$

7. $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}$

8. $\cot 3\alpha = \dfrac{\cot^3\alpha - 3\cot\alpha}{3\cot^2\alpha - 1}$

9. $\sin 4\alpha = 4\sin\alpha\cos\alpha(1-2\sin^2\alpha)$

10. $\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$

11. $\tan 4\alpha = \dfrac{4\tan\alpha - 4\tan^3\alpha}{1-6\tan^2\alpha+\tan^4\alpha}$

12. $\cot 4\alpha = \dfrac{1-6\tan^2\alpha+\tan^4\alpha}{4\tan\alpha - 4\tan^3\alpha}$

13. $\sin 5\alpha = 5\sin\alpha - 20\sin^3\alpha + 16\sin^5\alpha$

14. $\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$

15. $\tan 5\alpha = \dfrac{\tan^5\alpha - 10\tan^3\alpha + 5\tan\alpha}{1-10\tan^2\alpha + 5\tan^4\alpha}$

16. $\cot 5\alpha = \dfrac{1-10\tan^2\alpha+5\tan^4\alpha}{\tan^5\alpha - 10\tan^3\alpha + 5\tan\alpha}$

3.5 Half Angle Formulas and Identifiers.

1. $\sin\frac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$

2. $\cos\frac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$

3. $\tan\frac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}} = \dfrac{\sin\alpha}{1+\cos\alpha} = \dfrac{1-\cos\alpha}{\sin\alpha}$

$= \operatorname{cosec}\alpha - \cot\alpha$

4. $\cot\frac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{1-\cos\alpha}} = \dfrac{\sin\alpha}{1-\cos\alpha} = \dfrac{1+\cos\alpha}{\sin\alpha}$

$= \operatorname{cosec}\alpha + \cot\alpha$

5. $\sin\alpha = \dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$

6. $\cos\alpha = \dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$

7. $\tan\alpha = \dfrac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$

8. $\cot\alpha = \dfrac{1-\tan^2\frac{\alpha}{2}}{2\tan\frac{\alpha}{2}} = \dfrac{\cot^2\frac{\alpha}{2}-1}{2\cot\frac{\alpha}{2}}$

3.6 Transforming of Trigonometric Expressions to Product

1. $\sin\alpha + \sin\beta = 2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$

2. $\sin\alpha - \sin\beta = 2\cos\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$

3. $\cos\alpha + \cos\beta = 2\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$

4. $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

5. $\tan\alpha \pm \tan\beta = \frac{\sin(\alpha\pm\beta)}{\cos\alpha\cdot\cos\beta}$

6. $\cot\alpha \pm \cot\beta = \frac{\sin(\beta\pm\alpha)}{\sin\alpha\cdot\sin\beta}$

7. $\cos\alpha + \sin\alpha = \sqrt{2}\cos\left(\frac{\pi}{4}-\alpha\right) = \sqrt{2}\sin\left(\frac{\pi}{4}+\alpha\right)$

8. $\cos\alpha - \sin\alpha = \sqrt{2}\sin\left(\frac{\pi}{4}-\alpha\right) = \sqrt{2}\cos\left(\frac{\pi}{4}+\alpha\right)$

9. $\tan\alpha + \cot\beta = \frac{\cos(\alpha-\beta)}{\cos\alpha\cdot\sin\beta}$

10. $1+\cos\alpha = 2\cos^2\frac{\alpha}{2}$

11. $1-\cos\alpha = 2\sin^2\frac{\alpha}{2}$

12. $1+\sin\alpha = 2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$

13. $1-\sin\alpha = 2\sin^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$

3.7 Transformation of Trigonometric Expression to Sum

1. $\sin\alpha\cdot\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$

2. $\cos\alpha\cdot\cos\beta = \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}$

3. $\sin\alpha\cdot\cos\beta = \frac{\sin(\alpha-\beta)+\sin(\alpha+\beta)}{2}$

4. $\tan\alpha\cdot\tan\beta = \frac{\tan\alpha+\tan\beta}{\cot\alpha+\cot\beta}$

5. $\cot\alpha\cdot\cot\beta = \frac{\cot\alpha+\cot\beta}{\tan\alpha+\tan\beta}$

6. $\tan\alpha\cdot\cot\beta = \frac{\tan\alpha+\cot\beta}{\cot\alpha+\tan\beta}$

3.8 Powers of Trigonometric Functions

1. $\sin^2\alpha = \frac{1-\cos2\alpha}{2}$

2. $\cos^2\alpha = \frac{1+\cos2\alpha}{2}$

3. $\sin^3\alpha = \frac{3\sin\alpha-\sin3\alpha}{4}$

4. $\cos^3\alpha = \frac{3\cos\alpha+\cos3\alpha}{4}$

5. $\sin^4\alpha = \frac{\cos4\alpha-4\cos2\alpha+3}{8}$

6. $\cos^4\alpha = \frac{\cos4\alpha+4\cos2\alpha+3}{8}$

7. $\sin^5\alpha = \frac{10\sin\alpha-5\sin3\alpha+\sin5\alpha}{16}$

8. $\cos^5\alpha = \frac{10\cos\alpha+5\cos3\alpha+\cos5\alpha}{16}$

9. $\sin^6\alpha = \frac{10-15\cos2\alpha+6\cos4\alpha-\cos6\alpha}{32}$

10. $\cos^6\alpha = \frac{10+15\cos2\alpha+6\cos4\alpha+\cos6\alpha}{32}$

3.9 Relations between Inverse Trigonometric Functions

1. $\sin^{-1}(-x) = -\sin^{-1}x$

2. $\sin^{-1}x = \frac{\pi}{2}-\cos^{-1}x$

3. $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}, 0\le x\le1.$

4. $\sin^{-1}x = -\cos^{-1}\sqrt{1-x^2}, -1\le x\le0.$

5. $\sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}}, x^2<1.$

6. $\sin^{-1}x = \cot^{-1}\frac{\sqrt{1-x^2}}{x}, 0<x\le1.$

7. $\sin^{-1}x = \cot^{-1}\frac{\sqrt{1-x^2}}{x}-\pi, -1\le x\le0.$

8. $\cos^{-1}(-x) = \pi-\cos^{-1}x$

9. $\cos^{-1}x = \frac{\pi}{2}-\sin^{-1}x$

10. $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}, 0\le x\le1.$

11. $\cos^{-1}x = \pi-\sin^{-1}\sqrt{1-x^2}, -1\le x\le0.$

12. $\cos^{-1}x = \tan^{-1}\frac{\sqrt{1-x^2}}{x}, 0<x\le1.$

13. $\cos^{-1}x = \pi+\tan^{-1}\frac{\sqrt{1-x^2}}{x}, -1\le x<0$

14. $\cos^{-1}x = \cot^{-1}x\frac{x}{\sqrt{1-x^2}}, -1\le x<1.$

15. $\tan^{-1}(-x) = -\tan^{-1}x$

16. $\tan^{-1}x = \frac{\pi}{2}-\cot^{-1}x$

17. $\tan^{-1}x = \sin^{-1}\frac{x}{\sqrt{1+x^2}}$

18. $\tan^{-1}x = \cos^{-1}\frac{1}{\sqrt{1+x^2}}, x\ge0.$

19. $\tan^{-1}x = \cos^{-1}\frac{1}{\sqrt{1+x^2}}, x\le0.$

20. $\tan^{-1} x = \frac{\pi}{2} - \tan^{-1}\frac{1}{x}, x > 0.$

21. $\tan^{-1} x = -\frac{\pi}{2} - \tan^{-1}\frac{1}{x}, x < 0.$

22. $\tan^{-1} x = \cot^{-1}\frac{1}{x}, x > 0.$

23. $\tan^{-1} x = \cot^{-1}\frac{1}{x} - \pi, x < 0.$

24. $\cot^{-1}(-x) = \pi - \cot^{-1} x$

25. $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$

26. $\cot^{-1} x = \sin^{-1}\frac{1}{\sqrt{1+x^2}}, x > 0.$

27. $\cot^{-1} x = \pi - \sin^{-1}\frac{1}{\sqrt{1+x^2}}, x < 0.$

28. $\cot^{-1} x = \cos^{-1}\frac{x}{\sqrt{1+x^2}}$

29. $\cot^{-1} x = \tan^{-1}\frac{1}{x}, x > 0.$

30. $\cot^{-1} x = \pi + \tan^{-1}\frac{1}{x}, x < 0.$

3.10 Trigonometric Equations

Whole number : *n*

1. $\sin x = a, x = (-1)^n \sin^{-1} a + \pi n$
2. $\cos x = a, x = \pm\cos^{-1} a + 2\pi n$
3. $\tan x = a, x = \tan^{-1} a + \pi n$
4. $\cot x = a, x = \cot^{-1} a + \pi n$

3.11 Relations to Hyperbolic Functions

Imaginary unit : *i*

1. $\sin(ix) = i \sinh x$
2. $\tan(ix) = i \tanh x$
3. $\cot(ix) = -i \coth x$
4. $\sec(ix) = \operatorname{sech} x$
5. $\operatorname{cosec}(ix) = -i \operatorname{cosech} x$

MATRICES AND DETERMINANTS – 4

Matrices: A, B, C

Elements of a matrix : $a_i, b_i, a_{ij}, b_{ij}, c_{ij}$

Determinants of a matrix : det A

Minor of an element a_{ij} : M_{ij}

Cofactor of an element a_{ij} : C_{ij}

Transpose of a matrix : A^T, A

Adjoint of a matrix : adj A

Trace of a matrix : tr A

Inverse of a matrix: A^{-1}

Real number: k

Real variable: x_i

Natural numbers: m, n

4.1 Determinants

1. **3rd Order Determinant**

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

2. **n^{th} Order Determinant**

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nm} \end{vmatrix}$$

3. **Minor :** The minor M_{ij} associated with the element a_{ij} of n^{th} order matrix A is the $(n-1)^{th}$ order determinant derived from the matrix A by deletion of its i^{th} row and j^{th} column.
4. **Cofactor** $C_{ij} = (-1)^{i+j} M_{ij}$
5. **Laplace Expansion of n^{th} Order Determinant**

Laplace expansion by elements of the i^{th} row

$$\det A = \sum_{j=1}^{n} a_{ij}C_{ij}, i = 1, 2, \dots, n.$$

Laplace expansion by elements of the j^{th} column

$$\det A = \sum_{i=1}^{n} a_{ij}C_{ij}, j = 1, 2, \dots, n.$$

4.2 Properties of Determinants

1. The value of a determinant remains unchanged if rows are changed to column and columns to rows.

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

2. If two rows (or two columns) are interchanged, the sign of the determinant is changed.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}$$

3. If two rows (or two columns) are identical, the value of the determinant is zero.

$$\begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} = 0$$

4. If the elements of any row (or column) are multiplied by a common factor, the determinant is multiplied by that factor.

$$\begin{vmatrix} ka_1 & kb_1 \\ a_2 & b_2 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

5. If the elements of any row (or column) are increased (or decreased) by equal multiples of the corresponding elements of any other row (or column), the value of the determinant is uncharged.

$$\begin{vmatrix} a_1 + kb_1 & b_1 \\ a_2 + kb_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

4.3 Matrices

1. **Definition :** An $m \times n$ matrix A is a rectangular array of elements (numbers or functions) with m rows and n columns.

$$A = [a_{ij}] = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}$$

2. **Square matrix** is a matrix of order $n \times n$.

   A square matrix $\lfloor a_{ij} \rfloor$ is symmetric if $a_{ij} = a_{ji}$, *i.e.* it is symmetric about the leading diagonal.

   A square matrix $\lfloor a_{ij} \rfloor$ is skew-symmetric if $a_{ij} = -a_{ji}$.
3. **Diagonal matrix** is a square matrix with all elements zero except those on the leading diagonal.
4. **Unit matrix** is a diagonal matrix in which the elements on the leading diagonal are all unity. The unit matrix is denoted by I.
5. **Null matrix** A null matrix is one whose elements are all zero.

4.4 Operations with Matrices

1. Two matrices A and B are equal if, and only if, they are both of the same shape $m \times n$ and corresponding elements are equal.
2. Two matrices A and B can be added (or subtracted) if, and only if, they have the same shape $m \times n$. If

$$A = [a_{ij}] = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}, \text{ and}$$

$$B = [b_{ij}] = \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{vmatrix},$$

$$\text{then } A + B = \begin{vmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{1n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{vmatrix}.$$

3. If k is a scalar, and $A = \lfloor a_{ij} \rfloor$ is a matrix, then

$$kA = \lfloor ka_{ij} \rfloor = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}.$$

4. **Multiplication of Two Matrices :** Two matrices can be multiplied together only when the number of columns in the first is equal to the number of rows in the second.

$$\text{If } A = [a_{ij}] = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}, \text{ and}$$

$$B = [b_{ij}] = \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nk} \end{vmatrix},$$

$$\text{Then } AB = C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1k} \\ c_{21} & c_{22} & \cdots & c_{2k} \\ \vdots & \vdots & & \vdots \\ b_{m1} & c_{m2} & \cdots & c_{mk} \end{bmatrix},$$

where, $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{\lambda=1}^{n} a_{i\lambda}b_{\lambda j}$

$(i = 1, 2, \ldots, m ; j = 1, 2, \ldots, k)$.

$$\text{Thus if } A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, B = [b_i] = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

$$\text{Then } AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \end{bmatrix}.$$

5. **Transpose of a Matrix**
   If the rows and columns of a matrix are interchanged, then the new matrix is called the transpose of the original matrix. If A is the original matrix, its transpose is denoted A^T or A'.
   - The matrix A is orthogonal if $AA^T = I$.
   - If the matrix product AB is defined, then $(AB)^T = B^T A^T$.
6. **Adjoint of Matrix :** If A is a square $n \times n$ matrix, its adjoint, denoted by adj A, is the transpose of the matrix of cofactors C_{ij} of A : $\text{adj}A = [C_{ij}]^T$.
7. **Trace of Matrix :** If A is a square $n \times n$ matrix, its trace, denoted by tr A, is defined to be the sum of the terms on the leading diagonal : $\text{tr } A = a_{11} + a_{22} + \ldots + a_{nn}$.
8. **Inverse of a Matrix :** If A is a square $n \times n$ matrix with a nonsingular determinant det A, then its inverse A^{-1} is given by
   $$A^{-1} = \frac{\text{adj} A}{\det A}.$$
   - If the matrix product AB is defined, then
   $$(AB)^{-1} = B^{-1}A^{-1}.$$
9. If A is square $n \times n$ matrix, the eigen vectors X satisfy the equation $AX = \lambda X$, While the eigen values λ satisfy the characteristic equation $|A - \lambda I| = 0$.

4.5 Systems of Linear Equations

Variables: $x, y, z, x_1, x_2, \ldots$, Real numbers: $a_1, a_2, a_3, b_1, a_{11}, a_{12}, \ldots$

Determinants: D, D_x, D_y, D_z, Matrices: A, B, X

1. $$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 , \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

2. $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, $z = \frac{D_z}{D}$ (Cramer's rule),

   $$\text{where } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

   $$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

3. If $D \neq 0$, then the system has a single solution:

   $$x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}.$$

If $D = 0$ and $D_x \neq 0$ (or $D_y \neq 0$ or $D_z \neq 0$), then the system has no solution.
If $D = D_x = D_y = D_z = 0$, then the system has infinitely many solutions.

4. Matrix Form of a System of n Linear Equations in n Unknowns. The set of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

Can be written in matrix form

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix},$$

i.e. $A.X = B$,

Where $A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$

5. Solution of a Set of Linear Equations $n \times n$
$X = A^{-1}.B$, where A^{-1} is the inverse of A.

VECTORS – 5

Vectors : $\vec{u}, \vec{v}, \vec{w}, \vec{r}, \overrightarrow{AB}, \ldots$

Vector length : $|\vec{u}|, |\vec{v}|, \ldots$

Unit vectors : $\vec{i}, \vec{j}, \vec{k}$

Null vector : $\vec{0}$

Coordinates of vector $\vec{u}$: X_1, Y_1, Z_1

Coordinates of vector $\vec{v}$: X_2, Y_2, Z_2

Scalars : λ, μ

Direction cosines: $\cos\alpha$, $\cos\beta$, $\cos\gamma$.

Angle between two vectors : θ

5.1 Vector Coordinates

1. Unit Vectors
$\vec{i} = (1,0,0)$, $\vec{k} = (0, 0,1)$,
$\vec{j} = (0, 1,0)$, $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$.

2. $\vec{r} = \overrightarrow{AB} = (x_1 - x_0)\vec{i} + (y_1 - y_0)\vec{j} + (z_1 - z_0)\vec{k}$

3. $|\vec{r}| = |\overrightarrow{AB}| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$

4. If $\overrightarrow{AB} = \vec{r}$, then $\overrightarrow{BA} = -\vec{r}$.

5. $X = |\vec{r}|\cos\alpha$, $Y = |\vec{r}|\cos\beta$, $Z = |\vec{r}|\cos\gamma$,

6. If $\vec{r}$ $(X, Y, Z) = \vec{r_1}$ (X_1, Y_1, Z_1) then $X = X_1$, $Y = Y_1$, $Z = Z_1$

5.2 Scalar Product

1. **Scalar Product of Vectors** $\vec{u}$ and $\vec{v}$ $\vec{u}\cdot\vec{v} = |\vec{u}|\cdot|\vec{v}|\cdot\cos\theta$,
Where θ is the angle between vectors $\vec{u}$ and $\vec{v}$.

2. **Scalar product in Coordinate Form**
If $\vec{u} = (X_1, Y_1, Z_1)$, $\vec{v} = (X_2, Y_2, Z_2)$, then
$\vec{u}\cdot\vec{v} = X_1X_2 + Y_1Y_2 + Z_1Z_2$.

3. **Angle Between Two Vectors**
If $\vec{u} = (X_1, Y_1, Z_1)$, $\vec{v} = (X_2, Y_2, Z_2)$, then
$$\cos\theta = \frac{X_1X_2 + Y_1Y_2 + Z_1Z_2}{\sqrt{X_1^2 + Y_1^2 + Z_1^2}\sqrt{X_2^2 + Y_2^2 + Z_2^2}}.$$

4. **Commutative Property** $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

5. **Associative Property** $(\lambda\vec{u})\cdot(\mu\vec{v}) = \lambda\mu\vec{u}\cdot\vec{v}$

6. **Distributive Property** $\vec{u}\cdot(\vec{v} + \vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$

7. $\vec{u}\cdot\vec{v} = 0$ if $\vec{u}$, $\vec{v}$ are orthogonal $\left(\theta = \frac{\pi}{2}\right)$.

8. $\vec{u}\cdot\vec{v} > 0$ if $0 < \theta < \frac{\pi}{2}$ and $\vec{u}\cdot\vec{v} < 0$ if $\frac{\pi}{2} < \theta < \pi$.

9. $\vec{u}\cdot\vec{v} \leq |\vec{u}|\cdot|\vec{v}|$ and $\vec{u}\cdot\vec{v} = |\vec{u}|\cdot|\vec{v}|$ if $\vec{u}$, $\vec{v}$ are parallel ($\theta = 0$).

10. If $\vec{u} = (X_1, Y_1, Z_1)$, then $\vec{u}\cdot\vec{u} = \vec{u}^2 = |\vec{u}|^2 = X_1^2 + Y_1^2 + Z_1^2$.

11. $\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$ and $\vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{k} = \vec{k}\cdot\vec{i} = 0$

5.3 Vector Product

1. Vector Product of Vectors $\vec{u}$ and $\vec{v}$ $\vec{u}\times\vec{v} = \vec{w}$, where
   - $|\vec{w}| = |\vec{u}|\cdot|\vec{v}|\cdot\sin\theta$, where $0 \leq \theta \leq \frac{\pi}{2}$;
   - $\vec{w} \perp \vec{u}$ and $\vec{w} \perp \vec{v}$;
   - Vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ form a right – handed screw.

2. $\vec{w} = \vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}$

$$\vec{w} = \vec{u}\times\vec{v} = \left(\begin{vmatrix} Y_1 & Z_1 \\ Y_2 & Z_2 \end{vmatrix}, -\begin{vmatrix} X_1 & Z_1 \\ X_2 & Z_2 \end{vmatrix}, \begin{vmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{vmatrix}\right)$$

3. $S = |\vec{u}\times\vec{v}| = |\vec{u}|\cdot|\vec{v}|\cdot\sin\theta$

4. **Angle Between Two Vectors** $\sin\theta = \frac{\vec{u}\times\vec{v}}{|\vec{u}|\cdot|\vec{v}|}$

5. **Non commutative Property** $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$

6. **Associative Property** $(\lambda\vec{u})\times(\mu\vec{v}) = \lambda\mu\vec{u}\times\vec{v}$

7. **Distributive Property** $\vec{u}\times(\vec{v} + \vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$

8. $\vec{u}\times\vec{v} = \vec{0}$ if $\vec{u}$ and $\vec{v}$ are parallel ($\theta = 0$).

9. $\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = \vec{0}$ and $\vec{i}\times\vec{j} = \vec{k}$, $\vec{j}\times\vec{k} = \vec{i}$, $\vec{k}\times\vec{i} = \vec{j}$

5.4. Triple Product

1. **Scalar Triple Product**
$[\vec{u}\vec{v}\vec{w}] = \vec{u}\cdot(\vec{v}\times\vec{w}) = \vec{v}\cdot(\vec{w}\times\vec{u}) = \vec{w}\cdot(\vec{u}\times\vec{v})$

2. $\left[\vec{u}\vec{v}\vec{w}\right]=\left[\vec{w}\vec{u}\vec{v}\right]=\left[\vec{v}\vec{w}\vec{u}\right]=-\left[\vec{v}\vec{u}\vec{w}\right]=-\left[\vec{w}\vec{v}\vec{u}\right]=-\left[\vec{u}\vec{w}\vec{v}\right]$

3. $k\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)=k\left[\vec{u}\vec{v}\vec{w}\right]$

4. **Scalar Triple Product Coordination Form**

$$\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)=\begin{vmatrix}X_1 & Y_1 & Z_1\\ X_2 & Y_2 & Z_2\\ X_3 & Y_3 & Z_3\end{vmatrix},$$

Where $\vec{u}=\left(X_1,Y_1,Z_1\right),\vec{v}=\left(X_2,Y_2,Z_2\right),\vec{w}=\left(X_3,Y_3,Z_3\right)$.

5. **Volume of Parallelepiped** $V=\left|\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)\right|$

6. **Volume of Pyramid** $V=\dfrac{1}{6}\left|\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)\right|$

7. If $\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)=0$, then the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are **linearly dependent**, so $\vec{w}=\lambda\vec{u}+\mu\vec{v}$ for some scalars λ and μ.

8. If $\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)\neq 0$, then the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are **linearly independent.**

9. **Vector Triple Product** $\vec{u}\times\left(\vec{v}\times\vec{w}\right)=\left(\vec{u}\cdot\vec{w}\right)\vec{v}-\left(\vec{u}\cdot\vec{v}\right)\vec{w}$

COORDINATE SYSTEM – 6

6.1 Two – Dimensional Coordinate System

Point coordinates : x_0, x_1, x_2, y_0, y_1, y_2

Polar coordinates: r, φ

Real number : λ

Positive real number: a, b, c,

Distance between two points : d

Area : A

1. **Distance Between Two Points**

$$d=AB=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$$

2. **Dividing a Line Segment in the Ratio** $\lambda : 1$

$$x_0=\frac{x_1+\lambda x_2}{1+\lambda},\quad y_0=\frac{y_1+\lambda y_2}{1+\lambda}.$$

3. **Midpoint of a Line Segment**

$$x_0=\frac{x_1+x_2}{2},\quad y_0=\frac{y_1+y_2}{2},\lambda=1.$$

4. **Centroid (Intersection of Medians) of a Triangle**

$$x_0=\frac{x_1+x_2+x_3}{3},\quad y_0=\frac{y_1+y_2+y_3}{3},$$

Where $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ are vertices of the triangles ABC.

5. **Incenter (Intersection of Angle Bisectors) of a Triangle**

$$x_0=\frac{ax_1+bx_2+cx_3}{a+b+c},\quad y_0=\frac{ay_1+by_2+cy_3}{a+b+c},$$

where $a = BC$, $b = CA$, $c = AB$ are the sides of ΔABC

6. **Circumcentre (Intersection of the Side Perpendicular Bisectors) of a Triangle**

$$x_0=\frac{\begin{vmatrix}x_1^2+y_1^2 & y_1 & 1\\ x_2^2+y_2^2 & y_2 & 1\\ x_3^2+y_3^2 & y_3 & 1\end{vmatrix}}{2\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}},\quad y_0=\frac{\begin{vmatrix}x_1 & x_1^2+y_1^2 & 1\\ x_2 & x_2^2+y_2^2 & 1\\ x_3 & x_3^2+y_3^2 & 1\end{vmatrix}}{2\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}}$$

7. **Orthocentre (Intersection of Altitudes) of a Triangle**

$$x_0=\frac{\begin{vmatrix}y_1 & x_2x_3+y_1^2 & 1\\ y_2 & x_3x_1+y_2^2 & 1\\ y_3 & x_1x_2+y_3^2 & 1\end{vmatrix}}{\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}},\quad y_0=\frac{\begin{vmatrix}x_1^2+y_2y_3 & x_1 & 1\\ x_2^2+y_3y_1 & x_2 & 1\\ x_3^2+y_1y_2 & x_3 & 1\end{vmatrix}}{\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}}$$

8. **Area of a Triangle**

$$A=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=\frac{1}{2}\begin{vmatrix}x_2-x_1 & y_2-y_1\\ x_3-x_1 & y_3-y_1\end{vmatrix}$$

9. **Area of a Quadrilateral**

$$A=\frac{1}{2}\left[\begin{matrix}\left(x_1-x_2\right)\left(y_1+y_2\right)+\left(x_2-x_3\right)\left(y_2+y_3\right)+\\ \left(x_3-x_4\right)\left(y_3+y_4\right)\ +\left(x_4-x_1\right)\left(y_4+y_1\right)\end{matrix}\right]$$

10. **Distance Between Two Points in Polar Coordinates**

$$d=AB=\sqrt{r_1^2+r_2^2-2r_1r_2\ \cos\left(\varphi_2-\varphi_1\right)}$$

11. Converting Rectangular Coordinates to polar Coordinates $x = r\cos\varphi$, $y = r\sin\varphi$.

12. Converting Polar Coordinates to Rectangular Coordinates $r=\sqrt{x^2+y^2}$, $\tan\ \varphi=\dfrac{y}{x}$.

6.2 Straight Line in Plane

Point coordinates : $X, Y, x, x_0, x_1, y_0, y_1, a_1, a_2, \ldots$

Real numbers: $k, a, b, p, t, A, B, C, A_1, A_2, \ldots$

Angles : α, β

Angle between two lines : φ

Normal vector : $\vec{n}$

Position vectors : $\vec{r}$, $\vec{a}$, $\vec{b}$

1. **General Equation of a Straight Line**
$Ax + By + C = 0$

2. **Normal vector to a Straight Line**

The vector $\vec{n}\left(A,B\right)$ is normal to the line $Ax+By+C=0$.

3. **Explicit Equation of a Straight Line** (Slope – Intercept Form) $y = kx + b$.

4. **Gradient of a Line** $k=\tan\alpha=\dfrac{y_2-y_1}{x_2-x_1}$

5. **Equation of a Line** Given a Point and the Gradient $y=y_0+k\left(x-x_0\right)$,

Where k is the gradient, $P(x_0, y_0)$ is a point of the line.

6. **Equation of a Line That Passes Through Two Points**

$$(x_2-x_1)(y-y_1)=(y_2-y_1)(x-x_1)\ \text{ or, }\begin{vmatrix}x & y & 1\\ x_1 & y_1 & 1\\ x_2 & y_2 & 1\end{vmatrix}=0.$$

7. **Intercept Form** $\frac{x}{a}+\frac{y}{b}=1$

8. **Normal Form** $x\cos\beta+y\sin\beta-p=0$

9. **Point Direction Form** $\frac{x-x_1}{X}=\frac{y-y_1}{Y}$,

   where (X, Y) is the direction of the line and $P_1(x_1, y_1)$ lies on the line.

10. **Vertical Line** $x = a$

11. **Horizontal Line** $y = b$

12. **Vector Equation of a Straight Line** $\vec{r}=\vec{a}+t\vec{b}$,

    where

    O is the origin of the coordinates, X is any variable point on the line, $\vec{a}$ is the position vector of a known point A on the line, $\vec{b}$ is a known vector of direction, parallel to the line, t is a parameter, $\vec{r}=\overrightarrow{OX}$ is the position vector of any point X on the line.

13. **Straight Line in Parametric Form**

    $x=a_1+tb_1$ and $y=a_2+tb_2$

    where

    (x, y) are the coordinated of any unknown point on the line,

    (a_1, a_2) are the coordinates of a known point on the line,

    (b_1, b_2) are the coordinates of a vector parallel to the line, t is a parameter.

14. **Distance Form a Point To a Line**

    The distance from the point $P(a, b)$ to the line

    $Ax + By + C = 0$ is $d=\frac{|Aa+Bb+C|}{\sqrt{A^2+B^2}}$.

15. **Parallel Lines**

    Two lines $y = k_1x + b_1$ and $y = k_2x + b_2$ are parallel,

    If $k_1 = k_2$.

    Two lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$

    are parallel if $\frac{A_1}{A_2}=\frac{B_1}{B_2}$.

16. **Perpendicular Lines**

    Two lines $y = k_1x + b_1$ and $y = k_2x + b_2$ are perpendicular if $k_2=-\frac{1}{k_1}$ or, equivalently, $k_1k_2 = -1$.

    Two lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are perpendicular if $A_1A_2 + B_1B_2 = 0$.

17. **Angle Between Two Lines**

    $$\tan\varphi=\frac{k_2-k_1}{1+k_1k_2},\ \text{and}\ \cos\varphi=\frac{A_1A_2+B_1B_2}{\sqrt{A_1^2+B_1^2}\cdot\sqrt{A_2^2+B_2^2}}.$$

18. **Intersection of Two Lines**

    If two lines $A_1x+B_1y+C_1=0$ and $A_2x+B_2y+C_2=0$ intersect, the intersection point has coordinates

    $$x_0=\frac{-C_1B_2+C_2B_1}{A_1B_2-A_2B_1},\ y_0=\frac{-A_1C_2+A_2C_1}{A_1B_2-A_2B_1}$$

6.3 Circle

Radius : R, Centre of circle: (a,b), Point coordinates : $x, y, x_1, y_1, \ldots$, Real numbers: A, B, C, D, E, F, t.

1. **Equation of a circle Centred at the Origin** (Standard Form) $x^2 + y^2= R^2$

2. **Equation of a Circle Centred at Any Point** (a, b)

   $(x-a)^2+(y-b)^2 = R^2$

3. **Three Point Form**

   $$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix}=0$$

4. **Parametric Form**

   $$\begin{cases} x=R\cos t \\ y=R\sin t \end{cases},\ 0\le t\le 2\pi.$$

5. **General Form**

   $Ax^2+Ay^2+Dx+Ey+F = 0$ (A nonzero, $D^2 + E^2 > 4AF$).

   The centre of the circle has coordinates (a, b), where

   $$a=-\frac{D}{2A},\ b=-\frac{E}{2A}.$$

   The radius of the circle is $R=\sqrt{\frac{D^2+E^2-4AF}{2|A|}}$.

6.4 Ellipse

Semimajor axis : a, Semiminor axis: b, Foci : $F_1(-c, 0)$, $F_2(c, 0)$, Distance between the foci : $2c$, Eccentricity : e, Real numbers : A, B, C, D, E, F, t, Perimeter : L, Area : A.

1. **Equation of an Ellipse (Standard Form)** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

2. $r_1 + r_2 = 2a$,

   where r_1, r_2 are distances from any point $P(x, y)$ on the ellipse to the two foci.

3. $a^2 = b^2 + c^2$

4. **Eccentricity** $e=\frac{c}{a}<I$

5. **Equations of Directrices** $x=\pm\frac{a}{e}=\pm\frac{a^2}{c}$

6. **Parametric Form**

   $$\begin{cases} x=a\cos t \\ y=b\sin t \end{cases},0\le t\le 2\pi.$$

7. **General Form**

   $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$,

   where $B^2-4AC<0$.

8. **General Form with Axes Parallel to the Coordinate Axes**

   $Ax^2+Cy^2+Dx+Ey+F=0$, Where $AC > 0$.

9. **Circumference**

   $L = 4aE(e)$,

   where the function E is the complete elliptic integral of the second kind.

10. **Approximate Formulas of the Circumference**

    $$L=\pi\left(1.5(a+b)-\sqrt{ab}\right),$$

    $$L=\pi\sqrt{2\left(a^2+b^2\right)}.$$

11. **Area of Ellipse** $A=\pi ab$

6.5 Hyperbola

Transverse axis : a, Conjugate axis : b, Foci : $F_1(-c, 0)$, $F_2(c, 0)$, Distance between the foci : $2c$, Eccentricity : e,

Asymptotes : s, t, Real numbers : A, B, C, D, E, F, t, k.

1. **Equation of a Hyperbola** (Standard Form)

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

2. $|r_1-r_2|=2a$, where r_1, r_2 are distances from any point $P(x, y)$ on the hyperbola to the two foci.

3. **Equations of Asymptotes** $y=\pm\frac{b}{a}x$

4. $c^2 = a^2 + b^2$

5. **Eccentricity** $e=\frac{c}{a}>1$

6. **Equations of Directrices** $x=\pm\frac{a}{e}=\pm\frac{a^2}{c}$

7. **Parametric Equations** of the Right Branch of a Hyperbola

$$\begin{cases} x=a\cosh t \\ y=b\sinh t \end{cases}, 0\le t\le 2\pi.$$

8. **General Form**

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0, \text{ where } B^2-4AC>0$$

9. **General Form with Axes Parallel to the Coordinate Axes**

$$Ax^2+Cy^2+Dx+Ey+F=0,$$

where $AC < 0$.

10. **Asymptotic Form**

$$xy=\frac{e^2}{4}, \text{ or } y=\frac{k}{x}, \text{ where } k=\frac{e^2}{4}.$$

In this case, the asymptotes have equations $x = 0$ and $y = 0$.

6.6 Parabola

Focal parameter : p, Focus : F, Vertex : $M(x_0, y_0)$,

Real numbers : $A, B, C, D, E, F, p, a, b, c$.

1. **Equation of a Parabola** (Standard Form)

$y^2=2px$

- Equation of the directrix $x=-\frac{p}{2}$,
- Coordinates of the focus $F\left(\frac{p}{2},0\right)$,
- Coordinates of the vertex $M(0,0)$.

2. **General Form**

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

Where $B^2 - 4AC = 0$.

3. $y=ax^2, p=\frac{1}{2a}$.

- Equation of the directrix $y=-\frac{p}{2}$
- Coordinates of the focus $F\left(0,\frac{p}{2}\right)$,
- Coordinates of the vertex $M(0,0)$.

4. **General Form, Axis Parallel to the y – axis**

$Ax^2 + Dx + Ey + F = 0$ (A, E nonzero),

$$y=ax^2+bx+c, p=\frac{1}{2a}.$$

- Equation of the directrix $y=y_0-\frac{p}{2}$,
- Coordinates of the focus $F\left(x_0, y_0+\frac{p}{2}\right)$
- Coordinates of the vertex

$$x_0=-\frac{b}{2a},\quad y_0=ax_0^2+bx_0+c=\frac{4ac-b^2}{4a}.$$

6.7 Three – Dimensional Coordinate System

Point coordinates: $x_0, y_0, z_0, x_1, y_1, z_1,\ldots$, Real number : λ, Distance between two points : d, Area : S, Volume: V

1. **Distance Between Two Points**

$$d=AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

2. **Dividing a Line Segment in the Ratio λ : 1**

$$x_0=\frac{x_1+\lambda x_2}{1+\lambda},\quad y_0=\frac{y_1+\lambda y_2}{1+\lambda},\quad z_0=\frac{z_1+\lambda z_2}{1+\lambda},$$

3. **Midpoint of a Line Segment**

$$x_0=\frac{x_1+x_2}{2},\quad y_0=\frac{y_1+y_2}{2},\quad z_0=\frac{z_1+z_2}{2},\quad \lambda=1$$

4. **Area of a Triangle**

The area of a triangle with vertices P_1 (x_1, y_1, z_1), P_2 (x_2, y_2, z_2), and P_3 (x_3, y_3, z_3), is given by

$$A=\frac{1}{2}\sqrt{\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2+\begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}^2+\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2}.$$

5. **Volume of a Tetrahedron**

The volume of a tetrahedron with vertices P_1 (x_1, y_1, z_1), P_2 (x_2, y_2, z_2), P_3 (x_3, y_3, z_3), and P_4 (x_4, y_4, z_4) is given by

$$V=\pm\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}, \text{ or}$$

$$V=\pm\frac{1}{6}\begin{vmatrix} x_1-x_4 & y_1-y_4 & z_1-z_4 \\ x_2-x_4 & y_2-y_4 & z_2-z_4 \\ x_3-x_4 & y_3-y_4 & z_3-z_4 \end{vmatrix}$$

Note : We choose the sign (+) or (–) so that to get a positive answer for volume.

6.8. Plane

Point coordinates: $x, y, z, x_0, y_0, z_0, x_1, y_1, z_1, \ldots$

Real numbers: $A, B, C, D, A_1, A_2, a, b, c, a_1, a_2, \lambda, p, t, \ldots$

Normal vectors : $\vec{n}$, $\vec{n}_1$, $\vec{n}_2$, Direction cosines: cosα, cosβ, cosγ, Distance from point to plane: d

1. **General Equation of a Plane**

   $Ax + By + Cz + D = 0$

2. **Normal Vector to a Plane**

   The vector $\vec{n}$ (A, B, C) is normal to the plane

   $Ax + By + Cz + D = 0.$

3. **Particular cases of the Equation of a Plane**

   $Ax + By + Cz + D = 0$

   - If $A = 0$, the plane is parallel to the x-axis.
   - If $B = 0$, the plane is parallel to the y-axis.
   - If $C = 0$, the plane is parallel to the z-axis.
   - If $D = 0$, the plane lies on the origin.
   - If $A = B = 0$, the plane is parallel to the xy-axis.
   - If $B = C = 0$, the plane is parallel to the yz-axis.
   - If $A = C = 0$, the plane is parallel to the xz-axis.

4. **Point Direction Form**

   $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$

   Where the point $P(x_0, y_0, z_0)$ lies in the plane, and the vector (A, B, C) is normal to the plane.

5. **Intercept Form**

   $$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

6. **Three Point Form**

   $$\begin{vmatrix} x-x_3 & y-y_3 & z-z_3 \\ x_1-x_3 & y_1-y_3 & z_1-z_3 \\ x_2-x_3 & y_2-y_3 & z_2-z_3 \end{vmatrix}=0, \text{ or, } \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0.$$

7. **Normal Form**

   $x \cos\alpha + y \cos\beta + z \cos\gamma - p = 0,$

   where p is the perpendicular distance from the origin to the plane, and cosα, cos β, cos γ are the direction cosines of any line normal to the plane.

8. **Parametric Form**

   $$\begin{cases} x = x_1 + a_1 s + a_2 t \\ y = y_1 + b_1 s + b_2 t, \\ z = z_1 + c_1 s + c_2 t \end{cases}$$

   where (x, y, z) are the coordinates of any unknown point on the line, the point $P(x_1, y_1, z_1)$ lies in the plane, the vectors (a_1, b_1, c_1) and (a_2, b_2, c_2) are parallel to the plane.

9. **Dihedral Angle Between Two Planes**

   If the planes are given by

   $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$,

   then the dihedral angle between them is

   $$\cos\varphi=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|\cdot|\vec{n}_2|}=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\times\sqrt{A_2^2+B_2^2+C_2^2}}.$$

10. **Parallel Planes**

    Two planes $A_1x+B_1y+C_1z+D_1=0$ and

    $A_2x+B_2y+C_2z+D_2=0$ are parallel if

    $$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}.$$

11. **Perpendicular Planes**

    Two planes $A_1x+B_1y+C_1z+D_1=0$ and

    $A_2x+B_2y+C_2z+D_2=0$ are perpendicular if

    $A_1A_2 + B_1B_2 + C_1C_2 = 0.$

12. Equation of a Plane Through $P(x_1, y_1, z_1)$ and Parallel to the Vectors (a_1, b_1, c_1) and (a_2, b_2, c_2)

    $$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

13. Equation of a Plane Through $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, and Parallel to the Vector (a, b, c).

    $$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a & b & c \end{vmatrix}=0$$

14. **Distance From a Point To a Plane**

    The distance from the point $P_1(x_1, y_1, z_1)$ to the plane $Ax + By + Cz + D = 0$ is

    $$d=\left|\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}\right|.$$

15. **Intersection of Two Planes**

    If two planes $A_1x+B_1y+C_1z+D_1=0$ and

    $A_2x+B_2y+C_2z+D_2=0$ intersect, the intersection straight line is given by

    $$\begin{cases} x = x_1 + at \\ y = y_1 + bt, \\ z = z_1 + ct \end{cases} \text{ or, } \frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c},$$

    where

    $$a=\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, b=\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}, c=\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}$$

    $$x_1=\frac{b\begin{vmatrix} D_1 & C_1 \\ D_2 & C_2 \end{vmatrix}-c\begin{vmatrix} D_1 & B_1 \\ D_2 & B_2 \end{vmatrix}}{a^2+b^2+c^2},$$

    $$y_1=\frac{c\begin{vmatrix} D_1 & A_1 \\ D_2 & A_2 \end{vmatrix}-a\begin{vmatrix} D_1 & C_1 \\ D_2 & C_2 \end{vmatrix}}{a^2+b^2+c^2},$$

    $$z_1=\frac{a\begin{vmatrix} D_1 & B_1 \\ D_2 & B_2 \end{vmatrix}-b\begin{vmatrix} D_1 & A_1 \\ D_2 & A_2 \end{vmatrix}}{a^2+b^2+c^2}.$$

6.9 Straight Line in Space

Point coordinates: $x, y, z, x_1, y_1, z_1, \ldots$

Direction cosines: cos α, cos β, cos γ

Real numbers : $A, B, C, D, a, b, c, a_1, a_2, t, \ldots$

Direction vectors of a line : $\vec{s}$, $\vec{s}_1$, $\vec{s}_2$

Normal vector to a plane : $\vec{n}$

Angle between two lines: φ

1. **Point Direction Form of the Equation of a Line**

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c},$$

where the point $P_1(x_1, y_1, z_1)$ lies on the line, and (a, b, c) is the direction vector of the line.

2. **Two Point Form**

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

3. **Parametric Form**

$$\begin{cases} x = x_1 + t\cot\alpha \\ y = y_1 + t\cos\beta, \\ z = z_1 + t\cos\gamma \end{cases}$$

where the point $P_1(x_1, y_1, z_1)$ lies on the straight line, $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the direction vector of the line, the parameter t is any real number.

4. **Angle Between Two Straight Lines**

$$\cos\varphi=\frac{\vec{s}_1\cdot\vec{s}_2}{|\vec{s}_1|\cdot|\vec{s}_2|}=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}.$$

5. **Parallel Lines**

Two lines are parallel if $\vec{s}_1 \,||\, \vec{s}_2$, or, $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.

6. **Perpendicular Lines**

Two lines are perpendicular if $\vec{s}_1\cdot\vec{s}_2=0$, or,

$a_1a_2+b_1b_2+c_1c_2=0.$

7. **Intersection of Two Lines**

Two lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and

$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ intersect if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0.$$

8. **Parallel Line and Plane**

The straight line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ and the plane

$Ax + By + Cz + D = 0$ are parallel if $\vec{n}\cdot\vec{s}=0$,

or, $Aa + Bb + Cc = 0$.

9. **Perpendicular Line and Plane**

The straight line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ and the plane

$Ax + By + Cz + D = 0$ are perpendicular if $\vec{n}\,||\,\vec{s}$,

or $\frac{A}{a}=\frac{B}{b}=\frac{C}{c}$.

DIFFERENTIAL CALCULUS – 7

7.1 Limits of Functions

Function : $f(x)$, $g(x)$, Argument: x, Real constants: a, k

1. $\lim_{x\to a}[f(x)\pm g(x)]=\lim_{x\to a}f(x)\pm\lim_{x\to a}g(x)$

2. $\lim_{x\to a}[f(x)\cdot g(x)]=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x)$

3. $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}$, if $\lim_{x\to a}g(x)\neq 0$.

4. $\lim_{x\to a}[kf(x)]=k\lim_{x\to a}f(x)$

5. $\lim_{x\to a}f(g(x))=f\left(\lim_{x\to a}g(x)\right)$

6. $\lim_{x\to a}f(x)=f(a)$, if the function $f(x)$ is continuous at $x = a$.

7. $\lim_{x\to 0}\frac{\sin x}{x}=1$

8. $\lim_{x\to 0}\frac{\tan x}{x}=1$

9. $\lim_{x\to 0}\frac{\sin^{-1}x}{x}=1$

10. $\lim_{x\to 0}\frac{\tan^{-1}x}{x}=1$

11. $\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$

12. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$

13. $\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x=e^k$

14. $\lim_{x\to 0}a^x=1$

15. $\lim_{x\to 0}(1+x)=1$

16. $\lim_{x\to 0}e^x=1$

17. $\lim_{x\to 0}\frac{e^x-1}{x}$

18. $\lim_{x\to 0}\frac{a^x-1}{x}=l$

19. $\lim_{x\to 0}\frac{x^x-a^n}{x-a}=na$

7.2 Definition and Properties of the Derivative

Functions : f, g, y, u, v, Independent variable : x,
Real constant : k, Angle: α

1. $y'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{dy}{dx}$

2. $\frac{dy}{dx}=\tan\alpha$

3. $\frac{d(u+v)}{dx}=\frac{du}{dx}+\frac{dv}{dx}$

4. $\frac{d(u-v)}{dx}=\frac{du}{dx}-\frac{dv}{dx}$

5. $\frac{d(ku)}{dx}=k\frac{du}{dx}$

6. **Product Rule** $\frac{d(u\cdot v)}{dx}=\frac{du}{dx}\cdot v+u\cdot\frac{dv}{dx}$

7. **Quotient Rule** $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{\frac{du}{dx}\cdot v-u\cdot\frac{dv}{dx}}{v^2}$

8. **Chain Rule**

$y = f(g(x))$, $u = g(x)$,

$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$

9. **Derivative of Inverse Function**

$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$,

Where $x(y)$ is the inverse function of $y(x)$.

10. **Reciprocal Rule**

$$\frac{d}{dx}\left(\frac{1}{y}\right)=-\frac{\frac{dy}{dx}}{y^2}$$

11. **Logarithmic Differentiation** $y = f(x)$, $\ln y = \ln f(x)$,

$$\frac{dy}{dx} = f(x).\frac{d}{dx}[\ln f(x)].$$

7.3 Table of Derivatives

Independent variable : x, Real constants : C, a, b, c, Natural number : n

1. $\frac{d}{dx}(C) = 0$
2. $\frac{d}{dx}(x) = 1$
3. $\frac{d}{dx}(ax+b) = a$
4. $\frac{d}{dx}(ax^2+bx+c) = 2ax+b$
5. $\frac{d}{dx}(x^n) = nx^{n-1}$
6. $\frac{d}{dx}(x^{-n}) = -\frac{n}{x^{n+1}}$
7. $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$
8. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
9. $\frac{d}{dx}(\sqrt[n]{x}) = \frac{1}{n\sqrt[n]{x^{n-1}}}$
10. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
11. $\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$, $a > 0$, $a \neq 1$.
12. $\frac{d}{dx}(a^x) = a^x \ln a$, $a > 0$, $a \neq 1$.
13. $\frac{d}{dx}(e^x) = e^x$
14. $\frac{d}{dx}(\sin x) = \cos x$
15. $\frac{d}{dx}(\cos x) = -\sin x$
16. $\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x$
17. $\frac{d}{dx}(\cot x) = -\frac{1}{\sin^2 x} = -\text{cosec}^2 x$
18. $\frac{d}{dx}(\sec x) = \tan x \cdot \sec x$
19. $\frac{d}{dx}(\text{cosec}\, x) = -\cot x \cdot \text{cosec}\, x$
20. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
22. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
23. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
24. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
25. $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$
26. $\frac{d}{dx}(\sinh x) = \cosh x$
27. $\frac{d}{dx}(\cosh x) = \sinh x$
28. $\frac{d}{dx}(\tanh x) = \frac{1}{\cosh^2 x} = \text{sech}^2 x$
29. $\frac{d}{dx}(\coth x) = -\frac{1}{\sinh^2 x} = -\text{cosech}^2 x$
30. $\frac{d}{dx}(\text{sech}\, x) = -\text{sech}\, x \cdot \tanh x$
31. $\frac{d}{dx}(\text{cosech}\, x) = -\text{cosech}\, x \cdot \coth x$
32. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$
33. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$
34. $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, |x| < 1$.
35. $\frac{d}{dx}(\text{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$ $0 < x < 1$
36. $\frac{d}{dx}(\text{cosech}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}$, $x \neq 0$
37. $\frac{d}{dx}(\coth^{-1} x) = -\frac{1}{x^2-1}, |x| > 1$.
38. $\frac{d}{dx}(u^v) = vu^{v-1} \cdot \frac{du}{dx} + u^v \ln u . \frac{dv}{dx}$

7.4 Higher Order Derivatives

Functions: f, y, u, v, Independent variable : x, Natural number: n

1. **Second derivative**

$$f'' = (f')' = \left(\frac{dy}{dx}\right)' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

2. **Higher – Order derivative**

$$f^{(n)} = \frac{d^n y}{dx^n} = y^{(n)} = \left(f^{(n-1)}\right)'$$

3. $(u \pm v)^{(n)} = u^{(n)} \pm v^{(n)}$

4. **Leibnitz's Formulas**

$$(uv)'' = u''v + 2u'v' + uv''$$

$$(uv)''' = u'''v + 3u''v' + 3u'v'' + uv'''$$

$$(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v' + \frac{n(n-1)}{1.2}u^{(n-2)}v'' + \ldots + uv^{(n)}$$

5. $(x^m)^{(n)} = \frac{m!}{(m-n)!}x^{m-n}$

6. $\left(x^n\right)^{(n)} = n!$

7. $\left(\log_a x\right)^{(n)} = \dfrac{(-1)^{n-1}(n-1)!}{x^n \ln a}$

8. $\left(\ln x\right)^{(n)} = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$

9. $\left(a^x\right)^{(n)} = a^x \ln^n a$

10. $\left(e^x\right)^{(n)} = e^x$

11. $\left(a^{mx}\right)^{(n)} = m^n a^{mx} \ln^n a$

12. $\left(\sin x\right)^{(n)} = \sin\left(x + \dfrac{n\pi}{2}\right)$

13. $\left(\cos x\right)^{(n)} = \cos\left(x + \dfrac{n\pi}{2}\right)$

7.5 Applications of Derivative

Functions : f, g, y, Position of an object : s, Velocity : v, Acceleration : a, Independent variable : x, Time : t, Natural number: n

1. **Velocity and Acceleration** $s=f(t)$ is the position of an object relative to a fixed coordinate system at a time t,

   $v=s'=f'(t)$ is the instantaneous velocity of the object,

   $a=v'=s''=f''(t)$ is the instantaneous acceleration of the object.

2. **Tangent Line** $y - y_0 = f'(x_0)\,(x - x_0)$

3. **Normal Line** $y - y_0 = -\dfrac{1}{f'(x_0)}(x - x_0)$

4. **Increasing and Decreasing Functions.**

   If $f'(x_0)>0$, then $f(x)$ is increasing at x_0. ($x<x_1$, $x_2 < x$),

   If $f'(x_0)<0$, then $f(x)$ is decreasing at x_0. ($x_1 < x < x_2$),

   If $f'(x_0)$ does not exist or is zero, then the test fails.

5. **Local extrema**

   A function $f(x)$ has a local maximum at x_1 if and only if there exists some interval containing x_1 such that $f(x_1) \geq f(x)$ for all x in the interval.

   A function $f(x)$ has a local minimum at x_2 if and only if there exists some interval containing x_2 such that $f(x_2) \leq f(x)$ for all x in the interval.

6. **Critical Points**

   A critical point on $f(x)$ occurs at x_0 if and only if either $f'(x_0)$ is zero or the derivative doesn't exist.

7. **First Derivative Test for Local Extrema.**

   If $f(x)$ is increasing ($f'(x)>0$) for all x in some interval $(a, x_1]$ and $f(x)$ is decreasing ($f'(x)<0$) for all x in some interval $[x_1, b)$, then $f(x)$ has a local maximum at x_1.

8. If $f(x)$ is decreasing ($f'(x)<0$) for all x in some interval $(a, x_2]$ and $f(x)$ is decreasing ($f'(x)>0$) for all x in some interval $[x_2, b)$, then $f(x)$ has a local minimum at x_2.

9. **Second Derivative Test for Local Extrema.**

   If $f'(x_1) = 0$ and $f''(x_1)<0$, then $f(x)$ has a local maximum at x_1.

   If $f'(x_2) = 0$ and $f''(x_2)>0$, then $f(x)$ has a local minimum at x_2.

10. **Concavity.**

    If $f(x)$ is concave upward at x_0 if and only if $f'(x)$ is increasing at $x_0, x_3 < x$.

    If $f(x)$ is concave downward at x_0 if and only if $f'(x)$ is decreasing at $x_0, x < x_3$.

11. **Second derivative Test for Concavity.**

    If $f''(x_0)>0$, then $f(x)$ is concave upward at x_0.

    If $f''(x_0)<0$, then $f(x)$ is concave downward at x_0.

    If $f''(x)$ does not exist or is zero, then the test fails.

12. **Inflection Points**

    If $f'(x_3)$ exists and $f''(x)$ changes sign at $x = x_3$, then the point $(x_3, f(x_3))$ is an inflection point of the graph of $f(x)$. If $f''(x_3)$ exists at the inflection point, then $f''(x_3) = 0$.

13. **L 'Hopital's Rule**

    $$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)} \text{ if } \lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \left\{ {0 \atop \infty} \right..$$

7.6 Differential

Functions: f, u, v, Independent variable: x, Derivative of a function: $y'(x)$, $f'(x)$, Real constant: C, Differential of function $y=f(x)$: dy, Differential of x: dx, Small change in x: Δx, Small change in y: Δy

1. $dy = y'dx$
2. $f(x+\Delta x) = f(x) + f'(x)\Delta x$
3. Small Change in y

   $\Delta y = f(x+\Delta x) - f(x)$
4. $d(u \pm v) = du \pm dv$
5. $d(Cu) = Cdu$
6. $d(uv) = vdu + udv$
7. $d\left(\dfrac{u}{v}\right) = \dfrac{vdu - udv}{v^2}$

INTEGRAL CALCULUS – 8

Functions : f, g, u, v , Independent variables: x, t, ξ

Indefinite integral of a function : $\int f(x)dx,\ \int g(x)dx, \ldots$

Derivative of a function : $y'(x), f'(x), F'(x), \ldots$

Real constants: C, a, b, c, d, k, Natural numbers: m, n, i, j

8.1 Indefinite Integral

1. $\int f(ax)dx = \dfrac{1}{a}F(ax) + C$

2. $\int f(x)f'(x)dx = \dfrac{1}{2}f^2(x) + C$

3. $\int \dfrac{f'(x)}{f(x)}dx = \ln|f(x)| + C$

4. **Method of Substitution**

   $\int f(x)dx = \int f(u(t))u'(t)dt$ if $x = u(t)$.

5. $\int x^p dx = \dfrac{x^{p+1}}{p+1} + C,\ p \neq -1.$

6. $\int (ax+b)^n dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C,\ n \neq -1.$

7. $\int \dfrac{dx}{x} = \ln|x| + C$

8. $\int \dfrac{dx}{ax+b} = \dfrac{1}{a}\ln|ax+b| + C$

9. $\int \dfrac{ax+b}{cx+d}dx = \dfrac{a}{c}x + \dfrac{bc-ad}{c^2}\ln|cx+d| + C$

10. $\int\frac{dx}{(x+a)(x+b)}=\frac{1}{a-b}\ln\left|\frac{x+b}{x+a}\right|+C,\ a\neq b.$

11. $\int\frac{xdx}{a+bx}=\frac{1}{b^2}\left(a+bx-a\ln|a+bx|\right)+C$

12. $\int\frac{x^2dx}{a+bx}=\frac{1}{b^3}\left[\frac{1}{2}(a+bx)^2-2a(a+bx)+a^2\ln|a+bx|\right]+C$

13. $\int\frac{dx}{x(a+bx)}=-\frac{1}{a}\ln\left|\frac{a+bx}{x}\right|+C$

14. $\int\frac{dx}{x^2(a+bx)}=-\frac{1}{ax}+\frac{b}{a^2}\ln\left|\frac{a+bx}{x}\right|+C$

15. $\int\frac{xdx}{(a+bx)^2}=\frac{1}{b^2}\left(\ln|a+bx|+\frac{a}{a+bx}\right)+C$

16. $\int\frac{x^2dx}{(a+bx)^2}=\frac{1}{b^3}\left(a+bx-2a\ln|a+bx|-\frac{a^2}{a+bx}\right)+C$

17. $\int\frac{dx}{x(a+bx)^2}=\frac{1}{a(a+bx)}+\frac{1}{a^2}\ln\left|\frac{a+bx}{x}\right|+C$

18. $\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C$

19. $\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$

20. $\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+C$

21. $\int\frac{xdx}{x^2+a^2}=\frac{1}{2}\ln\left(x^2+a^2\right)+C$

22. $\int\frac{dx}{a+bx^2}=\frac{1}{\sqrt{ab}}\tan^{-1}\left(x\sqrt{\frac{b}{a}}\right)+C, ab>0.$

23. $\int\frac{xdx}{a+bx^2}=\frac{1}{2b}\ln\left|x^2+\frac{a}{b}\right|+C$

24. $\int\frac{dx}{x\left(a+bx^2\right)}=\frac{1}{2a}\ln\left|\frac{x^2}{a+bx^2}\right|+C$

25. $\int\frac{dx}{a^2-b^2x^2}=\frac{1}{2ab}\ln\left|\frac{a+bx}{a-bx}\right|+C$

26. $\int\frac{dx}{ax^2+bx+c}=\frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|+C,$
$b^2-4ac>0.$

27. $\int\frac{dx}{ax^2+bx+c}=\frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}+C,$
$b^2-4ac<0.$

8.2 Integral of Irrational Functions

1. $\int\frac{dx}{\sqrt{ax+b}}=\frac{2}{a}\sqrt{ax+b}+C$

2. $\int\sqrt{ax+b}\,dx=\frac{2}{3a}(ax+b)^{3/2}+C$

3. $\int\frac{xdx}{\sqrt{ax+b}}=\frac{2(ax-2b)}{3a^2}\sqrt{ax+b}+C$

4. $\int x\sqrt{ax+b}\,dx=\frac{2(3ax-2b)}{15a^2}(ax+b)^{3/2}+C$

5. $\int\frac{dx}{(x+c)\sqrt{ax+b}}=\frac{1}{\sqrt{b-ac}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b-ac}}{\sqrt{ax+b}-\sqrt{b-ac}}\right|+C,$
$b-ac>0.$

6. $\int\frac{dx}{(x+c)\sqrt{ax+b}}=\frac{1}{\sqrt{ac-b}}\tan^{-1}\sqrt{\frac{ax+b}{ac-b}}+C, b-ac<0.$

7. $\int\sqrt{\frac{ax+b}{cx+d}}dx=\frac{1}{c}\sqrt{(ax+b)(cx+d)}$
$-\frac{ad-bc}{c\sqrt{ac}}\ln\left|\sqrt{a(cx+d)}+\sqrt{c(ax+b)}\right|+C, a>0.$

8. $\int\sqrt{\frac{ax+b}{cx+d}}dx=\frac{1}{c}\sqrt{(ax+b)(cx+d)}$
$-\frac{ad-bc}{c\sqrt{ac}}\tan^{-1}\sqrt{\frac{a(cx+d)}{c(ax+b)}}+C,(a<0,c>0).$

9. $\int x^2\sqrt{ax+b}\,dx=\frac{2\left(8a^2-12abx+15b^2x^2\right)}{105b^3}\sqrt{(a+bx)^3}+C$

10. $\frac{\int x^2dx}{\sqrt{a+bx}}=\frac{2\left(8a^2-4abx+3b^2x^2\right)}{15b^3}\sqrt{a+bx}+C$

11. $\int\frac{dx}{x\sqrt{a+bx}}=\frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right|+C, a>0.$

12. $\int\frac{dx}{x\sqrt{a+bx}}=\frac{2}{\sqrt{-a}}\tan^{-1}\left|\frac{a+bx}{-a}\right|+C, a<0.$

13. $\int\sqrt{\frac{a-x}{b+x}}dx=\sqrt{(a-x)(b+x)}+(a+b)\sin^{-1}\sqrt{\frac{x+b}{a+b}}+C$

14. $\int\sqrt{\frac{a+x}{b-x}}dx=-\sqrt{(a+x)(b-x)}-(a+b)\sin^{-1}\sqrt{\frac{b-x}{a+b}}+C$

15. $\int\sqrt{\frac{1+x}{1-x}}dx=-\sqrt{1-x^2}+\sin^{-1}x+C$

16. $\int\frac{dx}{\sqrt{(x-a)(b-a)}}=2\sin^{-1}\sqrt{\frac{x-a}{b-a}}+C$

17. $\int\sqrt{a+bx-cx^2}\,dx=\frac{2cx-b}{4c}\sqrt{a+bx-cx^2}+$
$+\frac{b^2-4ac}{8\sqrt{c^3}}\sin^{-1}\frac{2cx-b}{\sqrt{b^2+4ac}}+C$

18. $\int\frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right|+C,$
$a>0.$

19. $\int\frac{dx}{\sqrt{ax^2+bx+c}}=-\frac{1}{\sqrt{a}}\sin^{-1}\frac{2ax+b}{4a}\sqrt{b^2-4ac}+C, a<0.$

20. $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|+C$

21. $\int x\sqrt{x^2+a^2}\,dx=\frac{1}{3}\left(x^2+a^2\right)^{3/2}+C$

22. $\int x^2\sqrt{x^2+a^2}\,dx=\frac{x}{8}\left(2x^2+a^2\right)\sqrt{x^2+a^2}$
$-\frac{a^4}{8}\ln\left|x+\sqrt{x^2+a^2}\right|+C$

23. $\int\frac{\sqrt{x^2+a^2}}{x^2}dx=-\frac{\sqrt{x^2+a^2}}{x}+\ln\left|x+\sqrt{x^2+a^2}\right|+C$

24. $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right|+C$

25. $\int \frac{\sqrt{x^2+a^2}}{x}dx = \sqrt{x^2+a^2}+a\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right|+C$

26. $\int \frac{xdx}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2}+C$

27. $\int \frac{x^2dx}{\sqrt{x^2+a^2}} = \frac{x}{2}\sqrt{x^2+a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|+C$

28. $\int \frac{dx}{x\sqrt{x^2+a^2}} = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right|+C$

29. $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C$

30. $\int x\sqrt{x^2-a^2}dx = \frac{1}{3}\left(x^2-a^2\right)^{3/2}+C$

31. $\int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2}+a\sin^{-1}\frac{a}{x}+C$

32. $\int \frac{\sqrt{x^2-a^2}}{x^2}dx = -\frac{\sqrt{x^2-a^2}}{x}+\ln\left|x+\sqrt{x^2-a^2}\right|+C$

33. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right|+C$

34. $\int \frac{xdx}{\sqrt{x^2-a^2}} = \sqrt{x^2-a^2}+C$

35. $\int \frac{x^2dx}{\sqrt{x^2-a^2}} = \frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C$

36. $\int \frac{dx}{x\sqrt{x^2-a^2}} = -\frac{1}{a}\sin^{-1}\frac{a}{x}+C$

37. $\int \frac{dx}{(x+a)\sqrt{x^2-a^2}} = \frac{1}{a}\sqrt{\frac{x-a}{x+a}}+C$

38. $\int \frac{dx}{(x-a)\sqrt{x^2-a^2}} = -\frac{1}{a}\sqrt{\frac{x+a}{x-a}}+C$

39. $\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x}+C$

40. $\int \frac{dx}{\left(x^2-a^2\right)^{3/2}} = -\frac{x}{a^2\sqrt{x^2-a^2}}+C$

41. $\int \left(x^2-a^2\right)^{3/2}dx = -\frac{x}{8}\left(2x^2-5a^2\right)\sqrt{x^2-a^2}+$

$$+\frac{3a^4}{8}\ln\left|x+\sqrt{x^2-a^2}\right|+C$$

42. $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

43. $\int x\sqrt{a^2-x^2}dx = -\frac{1}{3}\left(a^2-x^2\right)^{3/2}+C$

44. $\int x^2\sqrt{a^2-x^2}dx = \frac{x}{8}\left(2x^2-a^2\right)\sqrt{a^2-x^2}+\frac{a^4}{8}\sin^{-1}\frac{x}{a}+C$

45. $\int \frac{\sqrt{a^2-x^2}}{x}dx = \sqrt{a^2-x^2}+a\ln\left|\frac{x}{a+\sqrt{a^2-x^2}}\right|+C$

46. $\int \frac{\sqrt{a^2-x^2}}{x}dx = -\frac{\sqrt{a^2-x^2}}{x}-\sin^{-1}\frac{x}{a}+C$

47. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x+C$

48. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin\frac{x}{a}+C$

49. $\int \frac{xdx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}+C$

50. $\int \frac{x^2dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

51. $\int \frac{dx}{(x+a)\sqrt{a^2-x^2}} = -\frac{1}{2}\sqrt{\frac{a-x}{a+x}}+C$

52. $\int \frac{dx}{(x-a)\sqrt{a^2-x^2}} = -\frac{1}{2}\sqrt{\frac{a+x}{a-x}}+C$

53. $\int \frac{dx}{(x+b)\sqrt{a^2-x^2}} = \frac{1}{\sqrt{b^2-a^2}}\sin^{-1}\frac{bx+a^2}{a(x+b)}+C, b>a.$

54. $\int \frac{dx}{(x+b)\sqrt{a^2-x^2}} = \frac{1}{\sqrt{a^2-b^2}}$

$$\ln\left|\frac{x+b}{\sqrt{a^2-b^2}\sqrt{a^2-x^2}+a^2+bx}\right|+C,\ b<a.$$

55. $\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}+C$

56. $\int \left(a^2-x^2\right)^{3/2}dx = \frac{x}{8}\left(5a^2-2x^2\right)\sqrt{a^2-x^2}+\frac{3a^4}{8}\sin^{-1}\frac{x}{a}+C$

57. $\int \frac{dx}{\left(a^2-x^2\right)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}}+C$

8.3 Integrals of Trigonometric Functions

1. $\int \sin^2x\,dx = \frac{x}{2}-\frac{1}{4}\sin 2x+C$

2. $\int \cos^2x\,dx = \frac{x}{2}+\frac{1}{4}\sin 2x+C$

3. $\int \sin^3x\,dx = \frac{1}{3}\cos^3x-\cos x+C = \frac{1}{12}\cos 3x-\frac{3}{4}\cos x+C$

4. $\int \cos^3x\,dx = \sin x-\frac{1}{3}\sin^3x+C = \frac{1}{12}\sin 3x+\frac{3}{4}\sin x+C$

5. $\int \frac{dx}{\sin x} = \int \operatorname{cosec}x\,dx = \ln\left|\tan\frac{x}{2}\right|+C$

6. $\int \frac{dx}{\cos x} = \int \sec x\,dx = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{2}\right)\right|+C$

7. $\int \frac{dx}{\sin^2x} = \int \operatorname{cosec}^2x\,dx = -\cot x+C$

8. $\int \frac{dx}{\cos^2x} = \int \sec^2x\,dx = \tan x+C$

9. $\int \frac{dx}{\sin^3 x} = \int \csc^3 x\, dx = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C$

10. $\int \frac{dx}{\cos^3 x} = \int \sec^3 x\, dx = \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln\left|\tan x + \left(\frac{1}{\cos x}\right)\right| + C$

11. $\int \sin^2 x \cos^2 x\, dx = \frac{x}{8} - \frac{1}{32}\sin 4x + C$

12. $\int \tan x\, dx = -\ln|\cos x| + C$

13. $\int \frac{\sin x}{\cos^2 x} dx = \frac{1}{\cos x} + C = \sec x + C$

14. $\int \frac{\sin^2 x}{\cos x} dx = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| - \sin x + C$

15. $\int \tan^2 x\, dx = \tan x - x + C$

16. $\int \cot x\, dx = \ln|\sin x| + C$

17. $\int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{\sin x} + C = -\csc x + C$

18. $\int \frac{\cos^2 x}{\sin x} dx = \ln\left|\tan\frac{x}{2}\right| + \cos x + C$

19. $\int \cot^2 x\, dx = -\cot x - x + C$

20. $\int \frac{dx}{\cos x \sin x} = \ln|\tan x| + C$

21. $\int \frac{dx}{\sin^2 x \cos x} = -\frac{1}{\sin x} + \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{2}\right)\right| + C$

22. $\int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x} + \ln\left|\tan\frac{x}{2}\right| + C$

23. $\int \frac{dx}{\sin^2 x \cos^2 x} = \tan x - \cot x + C$

24. $\int \sin mx \sin nx\, dx = -\frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + C,\ m^2 \neq n^2.$

25. $\int \sin mx \cos nx\, dx = -\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)} + C,\ m^2 \neq n^2.$

26. $\int \cos mx \cos nx\, dx = \frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)}{2(m-n)} + C,\ m^2 \neq n^2.$

27. $\int \sec x \tan x\, dx = \sec x + C$

28. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$

29. $\int \sin x \cos^n x\, dx = \frac{\cos^{n+1} x}{n+1} + C$

30. $\int \sin^n x \cos x\, dx = \frac{\sin^{n+1} x}{n+1} + C$

31. $\int \sin^{-1} x\, dx = x\sin^{-1} x + \sqrt{1-x^2} + C$

32. $\int \cos^{-1} x\, dx = x\cos^{-1} x - \sqrt{1-x^2} + C$

33. $\int \tan^{-1} x\, dx = x\tan^{-1} x - \frac{1}{2}\ln(x^2+1) + C$

34. $\int \cot^{-1} x\, dx = x\cot^{-1} x + \frac{1}{2}\ln(x^2+1) + C$

8.4 Integrals of Hyperbolic Functions

1. $\int \sinh x\, dx = \cosh x + C$

2. $\int \cosh x\, dx = \sinh x + C$

3. $\int \tanh x\, dx = \ln \cosh x + C$

4. $\int \coth x\, dx = \ln|\sinh x| + C$

5. $\int \operatorname{sech}^2 x\, dx = \tanh x + C$

6. $\int \operatorname{cosech}^2 x\, dx = -\coth x + C$

7. $\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x + C$

8. $\int \operatorname{cosech} x \coth x\, dx = -\operatorname{cosech} x + C$

8.5 Integrals of Exponential and Logarithmic Functions

1. $\int e^x dx = e^x + C$

2. $\int a^x dx = \frac{a^x}{\ln a} + C$

3. $\int e^{ax} dx = \frac{e^{ax}}{a} + C$

4. $\int x e^{ax} dx = \frac{e^{ax}}{a^2}(ax-1) + C$

5. $\int \ln x\, dx = x\ln x - x + C$

6. $\int \frac{dx}{x\ln x} = \ln|\ln x| + C$

7. $\int x^n \ln x\, dx = x^{n+1}\left[\frac{\ln x}{n+1} - \frac{1}{(n+1)^2}\right] + C$

8 $\int e^{ax} \sin bx\, dx = \frac{a\sin bx - b\cos bx}{a^2+b^2} e^{ax} + C$

9. $\int e^{ax} \cos bx\, dx = \frac{a\cos bx + b\sin bx}{a^2+b^2} e^{ax} + C$

8.6 Reduction Formulas

1. $\int x^n e^{mx} dx = \frac{1}{m} x^n e^{mx} - \frac{n}{m}\int x^{n-1} e^{mx} dx$

2. $\int \frac{e^{mx}}{x^n} dx = -\frac{e^{mx}}{(n-1)x^{n-1}} + \frac{m}{n-1}\int \frac{e^{mx}}{x^{n-1}} dx, n \neq 1.$

3. $\int \sinh^n x\,dx = \frac{1}{n}\sinh^{n-1} x\cosh x - \frac{n-1}{n}\int \sinh^{n-2} x\,dx$

4. $\int \frac{dx}{\sinh^n x} = -\frac{\cosh x}{(n-1)\sinh^{n-1} x} - \frac{n-2}{n-1}\int \frac{dx}{\sinh^{n-2} x}, n \neq 1.$

5. $\int \cosh^n x\,dx = \frac{1}{n}\sinh x\cosh^{n-1} x\cosh x + \frac{n-1}{n}\int \cosh^{n-2} x\,dx$

6. $\int \frac{dx}{\cosh^n x} = -\frac{\sinh x}{(n-1)\cosh^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cosh^{n-2} x}, n \neq 1.$

7. $\int \sinh^n x\cosh^m x\,dx = \frac{\sinh^{n+1} x\cosh^{m-1} x}{n+m} + \frac{m-1}{n+m}\int \sinh^n x\cosh^{m-2} x\,dx$

8. $\int \sinh^n x\cosh^m x\,dx = \frac{\sinh^{n-1} x\cosh^{m+1} x}{n+m} - \frac{n-1}{n+m}\int \sinh^{n-2} x\cosh^m x\,dx$

9. $\int \tanh^n x\,dx = -\frac{1}{n-1}\tanh^{n-1} x + \int \tanh^{n-2} x\,dx, n \neq 1.$

10. $\int \coth^n x\,dx = -\frac{1}{n-1}\coth^{n-1} x + \int \coth^{n-2} x\,dx, n \neq 1.$

11. $\int \text{sech}^n x\,dx = \frac{\text{sech}^{n-2} x\tanh x}{n-1} + \frac{n-2}{n-1}\int \text{sech}^{n-2} x\,dx, n \neq 1.$

12. $\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$

13. $\int \frac{dx}{\sin^n x} = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} x}, n \neq 1.$

14. $\int \cos^n x\,dx = \frac{1}{n}\sin x\cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

15. $\int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} x}, n \neq 1.$

16. $\int \sin^n x\cos^m x\,dx = \frac{\sin^{n+1} x\cos^{m-1} x}{n+m} + \frac{m-1}{n+m}\int \sin^n x\cos^{m-2} x\,dx$

17. $\int \sin^n x\cos^m x\,dx = -\frac{\sin^{n-1} x\cos^{m+1} x}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} x\cos^m x\,dx$

18. $\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x' - \int \tan^{n-2} x\,dx, n \neq 1.$

19. $\int \cot^n x\,dx = -\frac{1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x\,dx, n \neq 1.$

20. $\int \sec^n x\,dx = \frac{\sec^{n-2} x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx, n \neq 1.$

21. $\int \text{cosec}^n x\,dx = \frac{\text{cosec}^{n-2} x\cot x}{n-1} + \frac{n-2}{n-1}\int \text{cosec}^{n-2} x\,dx, n \neq 1.$

22. $\int x^n \ln^m x\,dx = \frac{x^{n+1}\ln^m x}{n+1} - \frac{m}{n+1}\int x^n \ln^{m-1} x\,dx$

23. $\int \frac{\ln^m x}{x^n}dx = -\frac{\ln^m x}{(n-1)x^{n-1}} + \frac{m}{n-1}\int \frac{\ln^{m-1} x}{x^n}dx, n \neq 1.$

24. $\int \ln^n x\,dx = x\ln^n x - n\int \ln^{n-1} x\,dx$

25. $\int x^n \sinh x\,dx = x^n \cosh x - n\int x^{n-1}\cosh x\,dx$

26. $\int x^n \cosh x\,dx = x^n \sinh x - n\int x^{n-1}\sinh x\,dx$

27. $\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1}\cos x\,dx$

28. $\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1}\sin x\,dx$

29. $\int x^n \sin^{-1} x\,dx = \frac{x^{n+1}}{n+1}\sin^{-1} x - \frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{1-x^2}}dx$

30. $\int x^n \cos^{-1} x\,dx = \frac{x^{n+1}}{n+1}\cos^{-1} x + \frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{1-x^2}}dx$

31. $\int x^n \tan^{-1} x\,dx = \frac{x^{n+1}}{n+1}\tan^{-1} x - \frac{1}{n+1}\int \frac{x^{n+1}}{1+x^2}dx$

32. $\int \frac{x^n dx}{ax^n+b} = \frac{x}{a} - \frac{b}{a}\int \frac{dx}{ax^n+b}$

33. $\int \frac{dx}{(ax^2+bx+c)^n} = \frac{-2ax-b}{(n-1)(b^2-4ac)(ax^2+bx+c)^{n-1}} - \frac{2(2n-3)a}{(n-1)(b^2-4ac)}\int \frac{dx}{(ax^2+bx+c)^{n-1}}, n \neq 1.$

34. $\int \frac{dx}{(x^2+a^2)^n} = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2}\int \frac{dx}{(x^2+a^2)^{n-1}}, n \neq 1.$

35. $\int \frac{dx}{(x^2-a^2)^n} = -\frac{x}{2(n-1)a^2(x^2-a^2)^{n-1}} - \frac{2n-3}{2(n-1)a^2}\int \frac{dx}{(x^2-a^2)^{n-1}}, n \neq 1.$

8.7 Definite Integral – Properties

Definite integral of a function : $\int_a^b f(x)dx, \int_a^b g(x)dx, \ldots$

1. $\int_a^a f(x)dx = 0$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for $a < c < b$.

4. $\int_a^b f(x)dx \geq 0$ if $f(x) \geq 0$ on $[a, b]$.

5. $\int_a^b f(x)dx \leq 0$ if $f(x) \leq 0$ on $[a, b]$.

6. **Fundamental Theorem of Calculus**

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a) \text{ if } F'(x) = f(x).$$

7. **Method of Substitution**

If $x = g(t)$, then $\int_a^b f(x)dx = \int_c^d f(g(t))g'(t)dt$,

Where $c = g^{-1}(a)$, $d = g^{-1}(b)$.

8. **Trapezoidal Rule**

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f(x_i)\right]$$

9. **Simpson's Rule**

$$\int_a^b f(x)dx = \frac{b-a}{3n}\left[\begin{array}{l} f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) \\ +2f(x_4) + \ldots + 4f(x_{n-1}) + f(x_n)\end{array}\right],$$

Where $x_i = a + \frac{b-a}{n}i, i = 0, 1, 2, \ldots, n.$

10. **Area Between Two Curves**

$$A = \int_a^b [f(x) - g(x)]dx = F(b) - G(b) - F(a) + G(a),$$

Where $F'(x) = f(x)$, $G'(x) = g(x)$.

11. $\int xe^{-ax}dx = -\frac{1}{a^2}(ax+1)e^{-ax}$ $\boxed{\int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}}}$

12. $\boxed{\int x^2 e^{-ax}dx = -\frac{1}{a^3}(a^2x^2 + 2ax + 2)e^{-ax}}$

$$\int_0^\infty x^{2n} \cdot e^{-ax^2}dx = \frac{1\cdot3\cdot5\cdots\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$$

13. $\boxed{\int_0^\infty x^{2n+1} \cdot e^{-ax^2}dx = \frac{n!}{2a^{n+1}}}$

A $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots\ldots(x^2 < 1)$

B $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^2}{5!} - \ldots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots$$

$$\tan\theta = \theta + \frac{\theta^3}{3!} + \frac{2\theta^5}{5!} + \ldots$$

DIFFERENTIAL EQUATIONS - 9

9.1 First Order Ordinary Differential Equations

1. **Linear Equations**

$$\frac{dy}{dx} + p(x)y = q(x).$$

The general solution is

$$y = \frac{\int u(x)q(x)dx + C}{u(x)},$$

where $u(x) = \exp\left(\int p(x)dx\right)$.

2. **Separable Equations**

$$\frac{dy}{dx} = f(x,y) = g(x)h(y)$$

The general solution is given by

$$\int \frac{dy}{h(y)} = \int g(x)dx + C,$$

$H(y) = G(x) + C.$

3. **Homogeneous Equations**

The differential equation $\frac{dy}{dx} = f(x,y)$ is homogeneous, if the function $f(x, y)$ is homogeneous, that is $f(tx, ty) = f(x,y)$.

The substitution $z = \frac{y}{x}$ (then $y = zx$) leads to the separable equation

$$x\frac{dz}{dx} + z = f(1,z).$$

4. **Bernoulli Equation**

$$\frac{dy}{dx} + p(x)y = q(x)y^n.$$

The substitution $z = y^{1-n}$ leads to the linear equation

$$\frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x).$$

5. **Riccati Equation**

$$\frac{dy}{dx} = p(x) + q(x)y + r(x)y^2$$

If a particular solution y_1 is known, then the general solution can be obtained with the help of substitution

$z = \frac{1}{y - y_1}$, which leads to the first order linear equation

$$\frac{dz}{dx} = -[q(x) + 2y_1 r(x)]z - r(x.)$$

6. **Exact and Non exact Equations**

The equation $M(x,y)\,dx + N(x,y)dy = 0$

Is called exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, and non exact otherwise.

The general solution is $\int M(x,y)dx + \int N(x,y)dy = C$.

9.2 Second Order Ordinary Differential Equations

1. **Homogeneous Linear Equations with Constant Coefficients** $y''+py'+qy=0.$

   The characteristic equation is $\lambda^2+p\lambda+q=0.$

   If λ_1 and λ_2 are distinct real roots of the characteristic equation, then the general solution is

   $y=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x}$, where

   C_1 and C_2 are integration constants.

   If $\lambda_1=\lambda_2=-\frac{p}{2}$, then the general solution is

   $y=(C_1+C_2x)e^{-\frac{p}{2}x}.$

   If λ_1 and λ_2 are complex numbers:

   $\lambda_1=\alpha+\beta i$, $\lambda_2=\alpha-\beta i$, where

   $\alpha=-\frac{p}{2},\beta=\frac{\sqrt{4q-p^2}}{2},$

   then the general solution is $y=e^{ax}\left(C_1\cos\beta x+C_2\sin\beta x\right).$

2. **Inhomogeneous Linear Equation with Constant Coefficients**

   $y''+py'+qy=f(x).$

   The general solution is given by $y=y_p+y_h$, where

   y_p is a particular solution of the inhomogeneous equation and y_h is the general solution of the associated homogeneous equation.

   If the right side has the form

   $f(x)=e^{ax}(P_1(x)\cos\beta x+P_1(x)\sin\beta x),$

   then the particular solution y_p is given by

   $y_p=x^ke^{ax}\left(R_1(x)\cos\beta x+R_2(x)\sin\beta x\right),$

   Where the polynomials $R_1(x)$ and $R_2(x)$ have to be found by using the method of undetermined coefficients.

   - If $\alpha+\beta i$ is not a root of the characteristic equation, then the power $k=0$,
   - If $\alpha+\beta i$ is a simple root, then $k=1$,
   - If $\alpha+\beta i$ is a double root, then $k=2$,

3. **Differential Equations with y missing** $y''=f(x,y').$

   Set $u=y'$. Then the new equation satisfied by v is

   $u'=f(x,u),$

   Which is a first order differential equation.

4. Differential Equations with x Missing $y''=f(y,y')$.

   Set $u=y'$. Since

   $y''=\frac{du}{dx}=\frac{du}{dy}\frac{dy}{dx}=u\frac{du}{dy},$

   We have $u\frac{du}{dy}=f(y,u),$

   Which is a first order differential equation.

9.3 Some Partial Differential Equations

1. **The Laplace Equation** $\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=0$

   Applies to potential energy function $u(x,y)$ for a conservative force field in the xy – plane. Partial differential equations of this type are called elliptic.

2. **The Heat Equation** $\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=\frac{\partial u}{\partial t}$

   Applies to the temperature distribution $u(x, y)$ in the xy plane when heat is allowed to flow from warm areas to cool ones. The equations of this type are called parabolic.

3. **The Wave Equation** $\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=\frac{\partial^2u}{\partial t^2}$

   Apples to the displacement $u(x, y)$ of vibrating membranes and other wave functions. The equations of this type are called hyperbolic.

PROBABILITY – 10

10.1 Permutations and Combinations

Permutations : ${}^nP_m$, Combinations : ${}^nC_m$, Whole numbers : n, m.

1. Factorial

   $n!=1\cdot 2\cdot 3\ldots(n-2)(n-1)n$

   $0!=1$

2. ${}^nP_n=n!$

3. ${}^nP_m=\frac{n!}{(n-m)!}$

4. Binomial Coefficient ${}^nC_m=\binom{n}{m}=\frac{n!}{(n-m)!\,m!}$

5. ${}^nC_m={}^nC_{n-m}$

6. ${}^nC_m+{}^nC_{m+1}={}^{n+1}C_{m+1}$

7. ${}^nC_0+{}^nC_1+{}^nC_2+\ldots+{}^nC_n=2^n$

8. **Pascal's Triangle**

Row 0							1						
Row 1						1		1					
Row 2					1		2		1				
Row 3				1		3		3		1			
Row 4			1		4		6		4		1		
Row 5		1		5		10		10		5		1	
Row 6	1		6		15		20		15		6		1

10.2 Probability Formulas

Events: A, B

Probability: P

Any positive real number : ε

Standard deviation : σ

Variance : σ^2

Density functions : $f(x)$, $f(t)$

Random variable :

X, Y, Z

Values of random variables : x, y, z

Expected value of X : μ

1. **Probability of an Event**

   $P(A)=\frac{m}{n},$

   where m is the number of possible positive outcomes, n is the total number of possible outcomes.

2. **Range of Probability Values** $0 \le P(A) \le 1$
3. **Certain Event** $P(A) = 1$
4. **Impossible Event** $P(A) = 0$
5. **Complement** $P(\overline{A}) = 1 - P(A)$
6. **Independent Events** $P(A/B) = P(A)$, $P(B/A) = P(B)$
7. **Addition Rule for Independent Events**
   $P(A \cup B) = P(A) + P(B)$
8. **Multiplication Rule for Independent Events**
   $P(A \cap B) = P(A) \cdot P(B)$
9. **General Addition Rule**
   $P(A \cup B) = P(A) + P(B) - P(A \cap B)$,
   Where $A \cup B$ is the union of events A and B,
   $A \cap B$ is the intersection of events A and B.
10. **Conditional Probability** $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$
11. $P(A \cap B) = P(B) \cdot P(A/B) = P(A) \cdot P(B/A)$
12. **Law of Total Probability** $P(A) = \sum_{i=1}^{m} P(B_i)P(A/B_i)$,

    Where B_i is a sequence of mutually exclusive events.
13. **Bayes' Theorem** $P(B/A) = \dfrac{P(A/B).P(B)}{P(A)}$
14. **Bayes' Formula**

    $$P(B_i/A) = \frac{P(B_i).P(A/B_i)}{\sum_{k=1}^{m} P(B_i).P(A/B_i)},$$

    Where B_i is a set of mutually exclusive events (hypotheses),

    A is the final event, $P(B_i)$ are the prior probabilities, $P(B_i/A)$ are the posterior probabilities,
15. **Law of Large Numbers**

    $$P\left(\left|\frac{S_n}{n} - \mu\right| \ge \varepsilon\right) \to 0 \text{ as } n \to \infty,$$

    $$P\left(\left|\frac{S_n}{n} - \mu\right| < \varepsilon\right) \to 1 \text{ as } n \to \infty,$$

    Where S_n is the sum of random variables, n is the number of possible outcomes.
16. **Chebyshev Inequality**

    $$P(|X - \mu| \ge \varepsilon) \le \frac{V(X)}{\varepsilon^2},$$

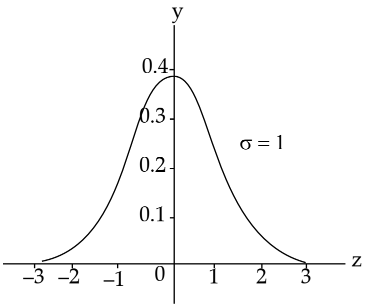
    Where $V(X)$ is the variance of X.
17. **Normal Density Function**

    $$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\alpha^2}},$$

    where x is a particular outcome.
18. **Standard Normal Density Function**

    $$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

    Average value $\mu=0$, deviation $\sigma=1$.



19. **Standard Z Value** $Z = \dfrac{X - \mu}{\sigma}$
20. **Cumulative Normal Distribution Function**

    $$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{(t-\mu)^2}{2\sigma^2}} dt,$$

    where x is a particular outcome, t is a variable of integration.
21. $P(\alpha < X < \beta) = F\left(\dfrac{\alpha - \mu}{\sigma}\right) - F\left(\dfrac{\beta - \mu}{\sigma}\right)$,

    where

    X is normally distributed random variable, F is cumulative normal distribution function,

    $P(\alpha < X < \beta)$ is interval probability.
22. $P(|X - \mu| < \varepsilon) = 2F\left(\dfrac{\varepsilon}{\sigma}\right)$

    where X is normally distributed random variable, F is cumulative normal distribution function.
23. **Cumulative Distribution Function**

    $$F(x) = P(X < x) = \int_{-\infty}^{x} f(t)dt,$$

    where t is a variable of integration.
24. **Bernoulli Trials Process**
    $\mu = np$, $\sigma^2 = npq$,
    where n is a sequence of experiments, p is the probability of success of each experiments, q is the probability of failure, $q = 1 - p$.
25. **Binomial Distribution Function**
    $b(n.p,q) = {}^{n}C_k p^k q^{n-k}$,
    $\mu = np$, $\sigma^2 = npq$,
    $f(x) = (q + pe^x)^n$,
    where n is the number of trials of selections, p is the probability of success, q is the probability of failure, $q = 1 - p$.
26. **Geometric Distribution**
    $P(T = j) = q^{j-1}p$,

    $$\mu = \frac{1}{p}, \sigma^2 = \frac{q}{p^2},$$

    where T is the first successful event is the series,

    j is the event number, p is the probability that any one event is successful, q is the probability of failure, $q = 1 - p$

    Poisson Distribution

$$P(X=k)\approx\frac{\lambda^k}{k!}e^{-\lambda},\lambda=np,\ \mu=\lambda,\sigma^2=\lambda,$$

where λ is the rate of occurrence, k is the number of positive outcomes.

27. Expected Value of Discrete Random Variables

$$\mu=E(X)=\sum_{i=1}^{n}x_ip_i,$$

where x_i is a particular outcome, p_i is its probability.

28. Expected Value of Continuous Random variables

$$\mu=E(X)=\int_{-\infty}^{\infty}xf(x)dx$$

29. Properties of Expectations

$E(X + Y) = E(X) + E(Y)$,

$E(X - Y) = E(X) - E(Y)$,

$E(cX) = cE(X)$,

$E(XY) = E(X)\cdot E(Y)$,

where c is a constant.

30. $E(X^2) = V(X) + \mu^2$,

where

$\mu = E(X)$ is the expected value

$V(X)$ is the variance.

31. Markov Inequality

$$\rho(X>k)\le\frac{E(X)}{k}$$

where k is some constant.

32. Variance of Discrete Random Variables

$$\sigma^2=V(X)=E\left[(X-\mu)^2\right]=\sum_{i=1}^{n}(x_i-\mu)^2P_i,$$

where

x_i is a particular outcome,

p_i is its probability.

33. Variance of Continuous Random Variables

$$\sigma^2=V(X)=E\left[(X-\mu)^2\right]=\int_{-\infty}^{\infty}(x_i-\mu)^2f(x)dx$$

34. Properties of Variance

$V(X + Y) = V(X) + V(Y)$,

$V(X - Y) = V(X) - V(Y)$,

$V(X + c) = V(X)$,

$V(cX) = c^2V(X)$

where c is a constant.

35. Standard Deviation $D(X)=\sqrt{V(X)}=\sqrt{E\left[(X-\mu)^2\right]}$

36. Covariance

$\mathrm{Cov}(X,Y) = E[(X-\mu(X))(Y-\mu(Y))] = E(XY)-\mu(X)\mu(Y)$,

where X is random variable, $V(X)$ is the variance of X, μ is the expected value of X or Y.

37. Correlation $\rho(X,Y)=\dfrac{\mathrm{cov}(X,Y)}{\sqrt{V(X)V(Y)}}$

Where

$V(X)$ is the variance of X, $V(Y)$ is the variance of Y.

APPENDIX-D

Famous Mathematicians and their contributions

THALES (Greek c. 600 B.C.)

The first Greek known to have used proof and strict logical reasoning to solve mathematical questions

PYTHAGORAS (Greek c. 500 B.C.)

Influential Greek philosopher and religious leader. He taught that numbers and ratios of numbers were the foundation of reality. He discovered many number patterns and the proof that the square root of two is not rational.

EUCLID (Greek c. 300 B.C.)

Organized Greek geometry into a mathematical system based on fundamental definitions, a few postulates and theorems that are logically deduced. This work, known as the Elements, had a profound influence on mathematics for thousands of years.

ARCHIMEDES (Greek c. 250 B.C.)

Discovered many fundamental properties of physics, such as the law of the lever; discovered a way to approximate pi as accurately as desired

APOLLONIUS (Greek c. 600 B.C.)

Discovered the family of curves known as the conic sections. He analyzed their properties using Greek geometry (not, however, with modern algebra equations or graphing techniques).

PTOLEMY (Greek c. 130 A.D.)

Invented a planetary system that was adopted as truth by the Christian church in Medieval Europe. In this system the Earth does not move and the planets, moon, stars and the Sun revolve around the Earth in circular paths with constant motion. This was described in his book the Almagest.

Al-KHWARIZMI (Hindu – Arabic c. 800 A.D.)

Wrote influential Arabic books on solving algebra problems and the Hindu – Arabic numberation system.

VIETE (Early Modern 1540 - 1603)

Introduced symbols into algebra.

DESCARTES (Early Modern 1596 - 1650)

Developed analytic geometry. He used a sophisticated symbolic algebra to show how algebra can be used to solve geometry problems and how algebra problems can be solved with geometry.

FERMAT (Early Modern 1601 - 1665)

Developed analytic geometry. He showed how a geometric curve, such as a conic section, could be drawn on a coordinate grid from an algebra equation. He also made important contributions to number theory, including the famous " Fermat's Last Theorem"

KEPLER (Early Modern 1571 - 1630)

Used real astronomical data to show that the planets orbit the sun in elliptical paths at varying rates of speed.

NEWTON (Early Modern 1643 - 1727)

Co – inventor of the calculus, He proved Kepler's laws mathematically in the style of Euclid in his book the Principal

LEIBNIZ (Early Modern 1646 - 1716)

Co-inventor of the calculus. His methods and symbolism is used today.

EULER (Early Modern 1707 - 1783)

A founding father to many branches of mathematics. He lived in the generation that followed Newton and Leibniz. Modern calculus for many modern symbols, such as $f(x)$, e , i, π .

GAUSS (Modern 1777 - 1855)

Discovered non-Euclidean geometry. He was a pioneer in many areas of modern mathematics.

CANTOR (Modern 1845 - 1918)

Invented the theory of infinite sets. He proved that the counting numbers and the real numbers have a different cardinality.

von NEUMANN (Modern 1903 - 1957)

Designed the fundamentals structure of modern computer design, known as the "von Neumann architecture". He also invented a branch of mathematics known as "game theory".

ARYABHATA(476 – 550AD)

1. Aryabhatta was born in 476 A.D. Kusumpur, India. He was the first in the line of great mathematicians from the classical age of Indian Mathematics and Astronomy.
2. His famous work are the "**Aryabhatiya**" and the "**Arya-siddhanta**". The Mathematical part of the Aryabhatiya covers arithmetic. algebra, plane and spherical trigonometry. The Arya-siddhanta, a lot work on astronomical computation.
3. **Approximation of Pi**: Aryabhata work on approximation for pi (π) and may have come to the conclusion that π is an irrational number. In the 2nd part of Aryabhatiya, he writes the ratio of circumference to diameter is 3.1416.
4. Aryabhata given the formula for area of a triangle. He also discussed the concept of sine in his work by the name of ardhajya. If we use Aryabhata's table and calculate the value of sin30° which is 1719/3438 = 0.5., the value is correct. His alphabetic code is commonly known as the Aryabhata cipher.
5. He was first person to say that Earth is spherical and it revolves around the sun.
6. He gave the formula $(a + b)^2 = a^2 + b^2 + 2ab$.
7. He taught the method of solving the following problems:

   $1 + 2 + 3 + \ldots\ldots\ldots\ldots + n = n(n + 1)/2$

   $1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + n^2 = n(n + 1)(2n + 1)/6$

   $1^3 + 2^3 + 3^3 + \ldots\ldots\ldots\ldots + n^3 = (n(n + 1)/2)^2$

BRAHMAGUPT (598 – 668 AD)

1. Brahmagupta was born in 598 A.D. in Bhinmal city in the state of Rajasthan. He was a mathematician and astronomer, who wrote many important works on mathematics and astronomy. His best known work is the "**Brahmasphuta-siddhanta**", written in 628 AD in Bhinmal.
2. He was the first to use zero as a number. He gave rules to compute with zero.
3. He gave four methods of multiplication.
4. He gave following formulae, used in G.P. series

   $a + ar + ar^2 + ar^3 + \ldots\ldots\ldots\ldots + ar^{n-1} = a(r^n - 1)/(r - 1)$
5. He gave the following formulae (Brahmagupta's formula):

   Area of a cyclic quadrilateral with side a, b, c, d

   $= \sqrt{(s-a)(s-b)(s-c)}$, where $2s = a + b + c + d$.

   Length of its diagonals

   $$= \sqrt{\frac{bc+ad}{ab+cd}}(ac+bd),\ \sqrt{\frac{ab+cd}{bc+ad}}(ac+bd)$$

BHASKARACHARYA (1114 – 1185 AD)

1. He was born in Bijapur in modern Karnataka. He and his work represent a significant contribution to mathematical and astronomical knowledge in the 12th century.
2. His main work "**Siddhanta Shiromani**" is divided into four parts called Lilawati, Bijaganit, Grahaganita and Goladhyaya. These four sections deal with arithmetic, algebra, mathematics of planets and spheres respectively.
3. He was the first to give that any number divided by zero gives infinity.
4. He was written a lot about zero, surds, permutation and combination.
5. He wrote," The hundredth part of the circumference of a circle seems to be straight. Our earth is a big sphere and that's why it appear to be flat."
6. He gave the formulae like :

   $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$.
7. He calculated derivatives of Trigonometric functions and formulae.
8. He developed spherical trigonometry along with a number of other trigonometric results.
9. He explained solution of quadratic, cubic and quartic indeterminate equations.
10. He developed a proof of Pythagoras Theorem by calculating the same area in two different ways and these cancel out terms to get $a^2 + b^2 = c^2$.
11. He gave first general method for finding the solution of the problem $x^2 - ny^2 = 1$ (so called Pell's equation).
12. He gave solution of Diophantine equations of second order such as $61x^2 + 1 = y^2$.

RAMANUJAN (1887 – 1920)

1. Ramanujan was born on 22nd of December 1887 in Erode, Madras Presidency. He made extraordinary contributions to mathematical analysis, number theory, infinite series, and continued fractions.
2. He demonstrated unusual mathematical skill at school, winning accolades and awards.
3. By 17, he had conducted his own mathematical research on Bernoulli numbers and the Euler-Mascheroni constant.
4. He discovered theorems of his own and rediscovered Euler's identity independently.
5. He sent a set 120 theorems to Professor Hardy of Cambridge. As result he invited Ramanujan to England.
6. He independently compiled nearly 3900 results (mostly identities and equations). Nearly all his claims have claims have now been proved correct.

7. Ramanujan Showed that any big number can be written as sum of not more then four prime numbers.
8. He showed that how to divide the numbers into two or more squares cubes.
9. **Ramanujan's Numbers :** When Mr.G.H. Hardy came to see Ramanujan in taxi number 1729, Ramanujan said that 1729 is the smallest number which can be written in the form of sum of cubes of two numbers in two ways, *i.e.*1729 = $9^3 + 10^3 = 1^3 + 12^3$ since than the number 1729 is calld Ramanujan's number.
10. In 1918, Ramanujan and Hardy studied the partition function P(n) extensively and gave a non-convergent asymptotic series that permits exact computation of the number of partition of an integer.
11. He discovered mock theta function in the last year of his life. For many years these functions where a mystry, but they are now known to be the holomorphic parts of harmonic weak mass forms.

SHAKUNTALA DEVI

1. She was born in 1939. She is an indian calculating prodigy.
2. By age 6, She demonstrated her calculation and memorization abilities at university of Mysore. At the age of 8, she had successes at Annamalai University by doing the same.
3. On June 18, 1980, She demonstrated the multiplication of two 13-digit numbers 7,686,369,774,870 × 2,465,099.745,779 picked at random by the Computer Department of Imperial College, London. She answered the question in 28 seconds. However, the time is more likely the time for dictating the answer (a 26-digit number) than the time for mental calculation(the time of 28 seconds was quoted on her website). Her answer was 18,947,668,177,995,426,773,730. This event is mentioned on page 26 of the 1995 Guinness Book of Records.
4. In Dallas, she competed with a computer to see who give the cube of 188138517 faster, she won. At University of USA she was asked to give the 23rd root of

   916748676920039158098660927585380162483106680144308622407126516427934657040867096593279205767480806790022783016354924852380335745316935111903596577547340075688186883056208210161291328455648957801588067 71.

   She answered in 50 seconds. The answer is 546372891. It took a Univac 1108 computer, full one minute (10 seconds more) to confirm that she was right after it was fed with 13000 instructions.
5. Now she is known to be Human Computer.

APPENDIX-E

ROMAN – NUMERALS

(A) Roman Numeral Symbols

Symbol	Number
I	1
V	5
X	10
L	50
C	100
D	500
M	1,000
$\overline{V}$	5,000
$\overline{X}$	10,000
$\overline{L}$	50,000
$\overline{C}$	100,000
$\overline{D}$	500,000
$\overline{M}$	1,000,000

(B) Roman Numerical Table

1	I	14	XIV	27	XXVII	150	CL
2	II	15	XV	28	XXVIII	200	CC
3	III	16	XVI	29	XXIX	300	CCC
4	IV	17	XVII	30	XXX	400	CD
5	V	18	XVIII	31	XXX1	500	D
6	VI	19	XIX	40	XL	600	DC
7	VII	20	XX	50	L	700	DCC
8	VIII	21	XXI	60	LX	800	DCCC
9	IX	22	XXII	70	LXX	900	CM
10	X	23	XXIII	80	LXXX	1000	M
11	XI	24	XXIV	90	XC	1600	MDC
12	XII	25	XXV	100	C	1700	MDCC
13	XIII	26	XXVI	101	CI	1900	MCM