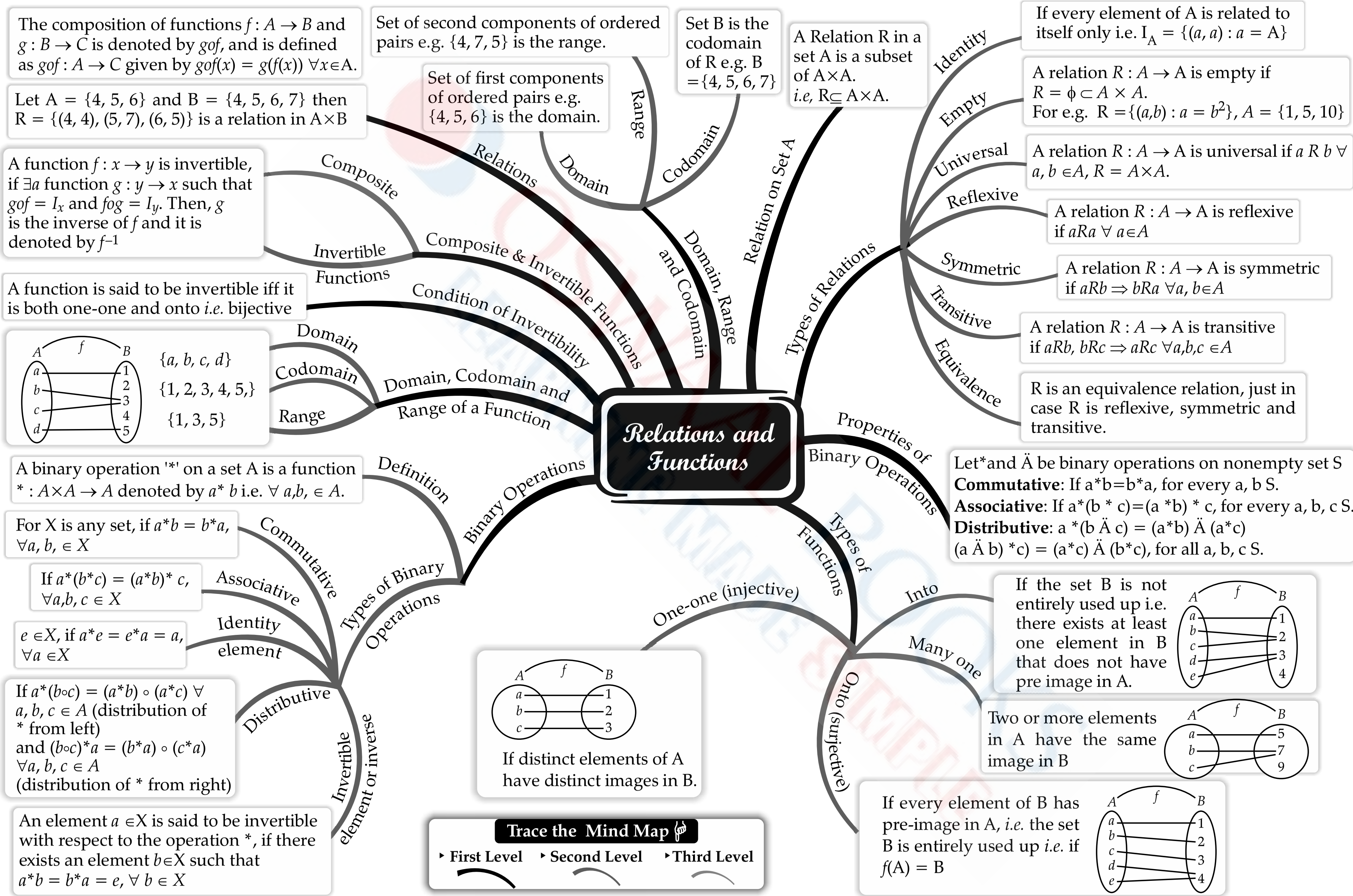
Relations and Functions
Relations
The composition of functions $f : A \to B$ and $g : B \to C$ is denoted by gof, and is defined as $gof : A \to C$ given by $gof(x) = g(f(x))\ \forall x \in A$.
Let A = {4, 5, 6} and B = {4, 5, 6, 7} then R = {(4, 4), (5, 7), (6, 5)} is a relation in A×B
Domain, Range and Codomain
Domain
Set of first components of ordered pairs e.g. {4, 5, 6} is the domain.
Range
Set of second components of ordered pairs e.g. {4, 7, 5} is the range.
Codomain
Set B is the codomain of R e.g. B ={4, 5, 6, 7}
Relation on Set A
A Relation R in a set A is a subset of A×A. i.e, R ⊆ A×A.
Types of Relations
Identity
If every element of A is related to itself only i.e. $I_A = \{(a, a) : a = A\}$
Empty
A relation $R : A \to A$ is empty if $R = \phi \subset A \times A$. For e.g. $R = \{(a,b) : a = b^2\}, A = \{1, 5, 10\}$
Universal
A relation $R : A \to A$ is universal if $a\ R\ b\ \forall\ a, b \in A, R = A \times A$.
Reflexive
A relation $R : A \to A$ is reflexive if $aRa\ \forall\ a \in A$
Symmetric
A relation $R : A \to A$ is symmetric if $aRb \Rightarrow bRa\ \forall a, b \in A$
Transitive
A relation $R : A \to A$ is transitive if $aRb, bRc \Rightarrow aRc\ \forall a,b,c \in A$
Equivalence
R is an equivalence relation, just in case R is reflexive, symmetric and transitive.
Composite & Invertible Functions
Composite
A function $f : x \to y$ is invertible, if $\exists a$ function $g : y \to x$ such that $gof = I_x$ and $fog = I_y$. Then, g is the inverse of f and it is denoted by f^{-1}
Invertible Functions
Condition of Invertibility
A function is said to be invertible iff it is both one-one and onto i.e. bijective
Domain, Codomain and Range of a Function
Domain
Codomain
Range
A f B
a b c d
1 2 3 4 5
{a, b, c, d}
{1, 2, 3, 4, 5,}
{1, 3, 5}
Properties of Binary Operations
Let*and Ä be binary operations on nonempty set S
Commutative: If a*b=b*a, for every a, b S.
Associative: If a*(b * c)=(a *b) * c, for every a, b, c S.
Distributive: a *(b Ä c) = (a*b) Ä (a*c)
(a Ä b) *c) = (a*c) Ä (b*c), for all a, b, c S.
Binary Operations
Definition
A binary operation '*' on a set A is a function $* : A \times A \to A$ denoted by $a * b$ i.e. $\forall\ a,b, \in A$.
Types of Binary Operations
Commutative
For X is any set, if $a*b = b*a$, $\forall a, b, \in X$
Associative
If $a*(b*c) = (a*b)*c$, $\forall a,b, c \in X$
Identity element
$e \in X$, if $a*e = e*a = a$, $\forall a \in X$
Distributive
If $a*(b \circ c) = (a*b) \circ (a*c)\ \forall\ a, b, c \in A$ (distribution of * from left)
and $(b \circ c)*a = (b*a) \circ (c*a)$ $\forall a, b, c \in A$ (distribution of * from right)
Invertible element or inverse
An element $a \in X$ is said to be invertible with respect to the operation *, if there exists an element $b \in X$ such that $a*b = b*a = e, \forall\ b \in X$
Types of Functions
Into
If the set B is not entirely used up i.e. there exists at least one element in B that does not have pre image in A.
A f B
a b c d e
1 2 3 4
Many one
Two or more elements in A have the same image in B
A f B
a b c
5 7 9
Onto (surjective)
If every element of B has pre-image in A, i.e. the set B is entirely used up i.e. if f(A) = B
A f B
a b c d e
1 2 3 4
One-one (injective)
A f B
a b c
1 2 3
If distinct elements of A have distinct images in B.
Trace the Mind Map
First Level
Second Level
Third Level

Inverse Trigonometric Function

Domain and Range of Inverse Trigonometric Function

- $y = \sin^{-1} x$
  - Domain: $-1 \le x \le 1$
  - Range: $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$, *i.e.*, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
- $y = \cos^{-1} x$
  - Domain: $-1 \le x \le 1$
  - Range: $0 \le y \le \pi$, *i.e.*, $[0, \pi]$
- $y = \tan^{-1} x$
  - Domain: All real numbers $(-\infty < x < \infty)$
  - Range: $\frac{-\pi}{2} < y < \frac{\pi}{2}$, *i.e.*, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
- $y = \cot^{-1} x$
  - Domain: All real numbers $(-\infty < x < \infty)$
  - Range: $0 < y < \pi$, *i.e.*, $[0, \pi]$
- $y = \sec^{-1} x$
  - Domain: $1 \le x, < \infty$ and $-\infty < x \le -1$ *i.e.*, $x \le -1$ or $x \ge 1$
  - Range: $0 \le y \le \frac{\pi}{2}$ and $\frac{\pi}{2} < y \le \pi$. Note that $y \ne \frac{\pi}{2}$, *i.e.* $[0, \pi] - \left\{\frac{\pi}{2}\right\}$
- $y = \operatorname{cosec}^{-1} x$
  - Domain: $-\infty < x \le -1$ and $1 \le x < \infty$ *i.e.*, $x \le -1$ or $x \ge -1$
  - Range: $\frac{-\pi}{2} \le y < 0$ and $0 < y \le \frac{\pi}{2}$. Note that $y \ne 0$, *i.e.*, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

What are Inverse Trigonometric Functions?

The expressions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$ are called inverse trigonometric function

Principal Value Branch and Principal Values

The range of an inverse trigonometric function is the principle value branch and those values which lie in the principal value branch is called the principal value of inverse trigonometric function

Important Formulae of Inverse Trigonometric Functions

Addition Formulae

(i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$, if $-1 \le x, y \le 1$ and $x^2 + y^2 \le 1$

(ii) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}]$, if $-1 \le x, y \le 1$ and $x + y \ge 0$

(iii) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$, if $xy < 1$

Subtraction Formulae

(i) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$, if $-1 \le x, y \le 1$ and $x^2 + y^2 \le 1$

(ii) $\cos^{-1} x - \cos^{-1} y = \cos^{-1}[xy + \sqrt{1-x^2}\sqrt{1-y^2}]$, if $-1 \le x, y \le 1$ and $x \le y$

(iii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}$, if $xy > -1$

Other Important Formulae

(i) $\sin^{-1}(-x) = -\sin^{-1} x, x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x, x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x, x \in R$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x, x \in R$

(v) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x, x \in R - (-1, 1)$

(vi) $\sec^{-1}(-x) = \pi - \sec^{-1} x, x \in R - (-1, 1)$

(vii) $\sin^{-1}x + \cos^{-1} x = \frac{\pi}{2}, x \in [-1, 1]$

(viii) $\sec^{-1}x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}, x \in R - (-1, 1)$

(ix) $\tan^{-1}x + \cot^{-1} x = \frac{\pi}{2}, x \in R$

(x) $2\sin^{-1} x = \sin^{-1}[2x\sqrt{1-x^2}], \frac{-1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

(xi) $\cos^{-1}x = \cos^{-1}[2x^2 - 1], 0 \le x \le 1$
$= 2\pi - \cos^{-1}(2x^2 - 1), -1 \le x \le 0$

(xii) $2\tan^{-1} x = \begin{cases} \sin^{-1}\frac{2x}{1+x^2}, & x \in [-1, 1] \\ \cos^{-1}\frac{1-x^2}{1+x^2}, & x \ge 0 \\ \tan^{-1}\frac{2x}{1-x^2}, & -1 < x < 1 \end{cases}$

(xiii) $\sin^{-1}x = \sin^{-1}[3x - 4x^3], \frac{-1}{2} \le x \le \frac{1}{2}$

(xiv) $3\cos^{-1}x = \cos^{-1}[4x^3 - 3x], \frac{-1}{2} \le x \le 1$

(xv) $\tan^{-1} x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$, if $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Some important Relations of Inverse Trigonometric Functions

(i) $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \operatorname{cosec}^{-1}\frac{1}{x}$

(ii) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \cot^{-1}\frac{x}{\sqrt{1-x^2}} = \sec^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}\frac{1}{\sqrt{1-x^2}}$

(iii) $\tan^{-1} x = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \cot^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{x} = \sec^{-1}\sqrt{1+x^2}$

Elementary Properties of Inverse Trigonometric Function

- $\sin^{-1} x$
  - $\sin^{-1}(\sin\theta) = \theta$, for all $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
  - $\sin(\sin^{-1} x) = x$, for all $x \in [-1, 1]$
- $\cos^{-1} x$
  - $\cos^{-1}(\cos x) = x$, for all $x \in [0, \pi]$
  - $\cos(\cos^{-1}x) = x$, for all $x \in [-1, 1]$
- $\tan^{-1} x$
  - $\tan^{-1}(\tan x) = x$, for all $x \in \left(\frac{-x}{2}, \frac{x}{2}\right)$
  - $\tan(\tan^{-1}x) = x$, for all $x \in R$
- $\cot^{-1} x$
  - $\cot^{-1}(\cot x) = x$, for all $x \in (0, \pi)$
  - $\cot(\cot^{-1}x) = x$, for all $x \in R$
- $\sec^{-1} x$
  - $\sec^{-1}(\sec x) = x$, for all $x \in [0, \pi], x \ne \frac{\pi}{2}$
  - $\sec(\sec^{-1}x) = x$, for all $x \in (-\infty, -1) \cup [1, \infty]$, i.e., $R - (-1, 1)$
- $\operatorname{cosec}^{-1} x$
  - $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$, for all $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, $\theta \ne 0$
  - $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x$, for all $x \in [-\infty, -1] \cup (1, \infty)$, *i.e.*, $R - (-1, 1)$

Trace the Mind Map
- First Level
- Second Level
- Third Level

Matrices Part- A

What is Matrix ?

A matrix of order $m \times n$ is an ordered rectangular array of numbers or functions having 'm' rows and 'n' columns. The matrix $A = [a_{ij}]_{m\times n}$ is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ a_{m1} & a_{m2} & \dots a_{mn} \end{bmatrix}_{m\times n}$$

Types of Matrices

- **Row matrix**: It is of the form $A = [a_{ij}]_{1\times n}$
  e.g. $\begin{bmatrix} 2 & 5 & -4 \end{bmatrix}_{1\times 3}, \begin{bmatrix} \sqrt{2} & 4 \end{bmatrix}_{1\times 2}$
- **Column matrix**: It is of the form $A = [a_{ij}]_{m\times 1}$
  e.g. $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}_{3\times 1}, \begin{bmatrix} 4 \\ 5 \end{bmatrix}_{2\times 1}$
- **Square matrix**: Here $m = n$ (no. of rows = no. of columns)
  $A = [a_{ij}]_{n\times n}$
  e.g. $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & -4 \\ 0 & -1 & -2 \end{bmatrix}_{3\times 3}$
- **Diagonal matrix**: All non-diagonal entries are zero i.e.
  $a_{ij} = 0 \ \forall \ i \neq j$
  e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3}$
- **Scalar matrix**: $a_{ij} = 0,\ i \neq j$ and $a_{ij} = k$ (scalar), $i = j$
  e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
- **Identity matrix**: $a_{ij} = 0$ when $i \neq j$ and $a_{ij} = 1$ when $i = j$
  e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$
- **Zero matrix**: All entries are zero
  e.g. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

Equality of two matrix

$A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are said to be equal, if $a_{ij} = b_{ij} \ \forall \ i, j$

e.g. $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 4/2 & 2-1 \\ \sqrt{9} & 0 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 0 \end{bmatrix}$

Transpose of matrix

A new matrix whose rows are columns of the original, it is denoted by A^T or A'

e.g. $A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 5 \\ 5 & 4 & 7 \end{bmatrix}$

Operation on Matrices

- **Addition**: Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, than $A+B = [a_{ij} + b_{ij}]$
  e.g. $A = \begin{bmatrix} 1 & 0 \\ 2 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}$
  $A+B = \begin{bmatrix} 2 & -2 \\ 2 & 10 \end{bmatrix}$
- **Multiplication**: Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$, then $C = AB = [C_{ik}]_{m\times p}$
  when $C_{ik} = \sum_{f=1}^{n} a_{ij} \cdot b_{jx}$
  e.g. $A = \begin{bmatrix} \boxed{a \quad b} \\ c \quad d \end{bmatrix}$ and $B = \begin{bmatrix} \boxed{\begin{matrix} e \\ g \end{matrix}} & \begin{matrix} f \\ h \end{matrix} \end{bmatrix}$
  then, $AB = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$
- **Multiplication of a Matrix by a Scalar Matrices**: If $A = [a_{ij}]_{m\times n}$, then $kA = [ka_{ij}]_{m\times n}$
  e.g. $A = \begin{bmatrix} 2 & -4 \\ 5 & 6 \end{bmatrix}, 3A = \begin{bmatrix} 6 & -12 \\ 15 & 18 \end{bmatrix}$
- **Existence of non - zero matrices whose product is the zero matrix**: Product of two matrices is zero matrix, as it is not necessary that one of the matrices is a zero matrix
  e.g. $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Symmetric and Skew-Symmetric Matrix

- **Symmetric**: If $A^T = A$, e.g. : $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$
  $A^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = A$
- **Skew-Symmetric**: If $A^T = -A$,
  e.g. : $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,
  $A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -A$

Invertible matrix

A square matrix $[A]_{n\times n}$ is said to be an invertible matrix if there exists another square matrix $[B]_{n\times n}$ such that $AB = BA = I$

Elementary Row and Column Operations

- Interchange of any two rows or two columns: $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$
- Multiplication of the elements of any row or column by a non -zero number: $R_i \leftrightarrow kR_j$ or $C_i \leftrightarrow kC_j$
- Addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non-zero number: $R_i \leftrightarrow R_i + kR_j$ or $C_i \leftrightarrow C_i + kC_j$

Trace the Mind Map
- First Level
- Second Level
- Third Level

Matrices (Part- B)

Determinant of Square Matrix

A unique number (real or complex) can be associated, to every square matrix A $=[a_{ij}]$ of order m, is called determinant of matrix A. It is denoted by $|A|$ or det (A)

Expansion of a Determinant

- **of Order 2**

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}.a_{22} - a_{21}.a_{22}$$

- **of Order 3**

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}.a_{33} - a_{32}.a_{23}) - a_{12}(a_{21}.a_{33} - a_{31}.a_{23}) + a_{13}(a_{21}.a_{32} - a_{31}.a_{22})$$

Minors and Cofactors

- **Minors**

If $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then minor of $a_{11} = M_{11}$

$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, Minor of $a_{12} = M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and so on.

- **Cofactos**

If the minor M_{ij} is multiplied by $(-1)^{i+j}$, then we obtain the Cofactor of a_{ij}. It is denoted by A_{ij} or C_{ij}, i.e., $A_{ij} = (-1)^{i+j}$ M_{ij}, cofactor of $a_{11} = (-1)^{1+1} M_{11}$ $A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

Adjoint of a Matrix

- **Statement**

Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order n and C_{ij} (or A_{ij}) be the cofactor of a_{ij} in the determinant $|A|$. Then, the adjoint of A is defined as the transpose of the cofactor matrix and it is denoted by adj (A).

- **Example**

If $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then adj $A = \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}$,

where A_{ij}'s are coafactors of A.

Singular and Non-Singular Matrix

- **Non-Singular**

If A is a square matrix and $|A| \neq 0$, then A is known as non-singular matrix.

- **Singular**

If A is a square matrix, and $|A| = 0$, then A is known as singular matrix.

Properties of Adjoint of a Matrix

If A is a square matrix of order $n\times n$, then
(i) $A(\text{adj } A) = (\text{adj } A)A = |A|I_n$
(ii) adj $(A^T) = (\text{adj } A)^T$
(iii) $|\text{adj } A| = |A|^{n-1}$, provided $|A| \neq 0$

Inverse of Square Matrix

- **Statement**

Let A be a non-singular (*i.e.* $|A| \neq 0$) square matrix. Then, a square matrix B, such that $AB = BA = I$ is called of A inverse of A and it is denoted by A^{-1}

i.e. $$A^{-1} = \frac{1}{|A|} \text{adj}(A).$$

- **Properties**

Let A and B be square matrices of same order, then
(i) A square matrix is invertible if and only if it is non-singular
(ii) $(A^{-1})^{-1} = A$
(iii) $(A')^{-1} = (A^{-1})'$
(iv) $(AB)^{-1} = B^{-1}A^{-1}$
(v) $|A^{-1}| = |A|^{-1}$
(vi) $AA^{-1} = A^{-1}A = I$
(vii) $(kA)^{-1} = \frac{1}{k}A^{-1}$, if $k \neq 0$

Solutions of System of Linear Equations (Martin's Rule)

Let system of linear equation in three variables be

$a_1x+b_1y+c_1z = d_1$, $a_2x+b_2y+c_2z = d_2$ and $a_3x+b_3y+c_3z = d_3$

It can be written in the matrix form as $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ or AX = B

Where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ If $|A| \neq 0$, then solution of system of linear equations is given by $X = A^{-1}B$. This method of solving a system of linear equations is also known as **matrix method**.

Consistency and Inconsistency of system of Linear Equations

A system of equations is consistent if it has one or more solutions while it is inconsistence if it has no solutions

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- First Level
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Determinants

Determinant of Square Matrics

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of Order 2

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of Order 3

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\,(a_{22}.a_{33} - a_{32}.a_{23}) - a_{12}\,(a_{21}.a_{33} - a_{31}.a_{23}) + a_{13}\,(a_{21}.a_{32} - a_{31}.a_{22})$$

Minors and Cofactors

Minors

If $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then minor of $a_{11} = M_{11}$

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Cofactos

If the minor M_{ij} is multiplied by $(-1)^{i+j}$, then we obtain the factor of a_{ij}. It is denoted by A_{ij} or C_{ij}, i.e., $A_{ij} = (-1)^{i+j}$ M_{ij}, cofactor of $a_{11} = (-1)^{1+1} M_{11}$ or $A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

Properties of Determinants

- If each element of a row or column is zero, then determinant becomes zero. e.g. $A = \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \therefore |A| = 0$
- The value of a determinant remains unchanged if its rows and columns are interchanged. e.g. $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
- If any two adjaceat rows or columns are interchanged then the determinant retains its absolute value, but its sign is changed. e.g. $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_1 = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_1 & c_1 \end{vmatrix}$ $\Rightarrow \Delta_1 = -1\Delta$
- If each element of a row or a column of a determinant is multiplied by a constant k, then the value of new determinant are k times the value of the original determinant. e.g. $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_1 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow \Delta_1 = k\Delta$
- If each element of a row (or column of determinant is the sum of the two more terms, then the determinants can be expressed as the sum of the two or more determinants. e.g. $\begin{vmatrix} a_1+a_2 & b & c \\ p_1+p_2 & q & r \\ u_1+u_2 & v & w \end{vmatrix} = \begin{vmatrix} a_1 & b & c \\ p_1 & q & r \\ u_1 & v & w \end{vmatrix} + \begin{vmatrix} a_2 & b & c \\ p_2 & q & r \\ u_2 & v & w \end{vmatrix}$

Cramer's Rule

The solution of the system $a_1x + b_1y + c_1z + d$, $a_2x + b_2y + c_2z + d_2$ and $a_3x + b_3y + c_3z + d_3$ is given by $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$ and $z = \frac{D_z}{D}$, where

If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

provided that $D \neq 0$

Area of triangle

If $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are the vertices of triangle,

Area of $\Delta = \frac{1}{2}\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$

For eg: if (1, 2), (3, 4) amd (–2, 5) are the vertices, then area of the triangle is

Area of $\Delta = \frac{1}{2}\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ -2 & 5 & 1 \end{bmatrix}$

$= \frac{1}{2}|1(4-5) - 2(3+2) + 1(15+8)| = 6$ sq. units.

we take positie value of the determinant because area in poisitve

Consistency and Inconsistency of system of Linear Equations

A system of equations is consistent if it has one or more solutions while it is inconsistence if it has no solutions

Applications of determinants and matrics

- If $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$ then we an write $AX + B$,

  Where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$
- Unique solution of $AX = B$ is $X = A^{-1}B$, $|A| \neq 0$
- $AX = B$ is consistent or inconsistent according as the solution existes or not.
- For a square matrix A in $AX = B$, if
  - (i) $|A| \neq 0$ then there exists unique solution.
  - (ii) $|A| = 0$ and $(adj. A)\,B \neq 0$, then no solution.
  - (iii) if $|A| = 0$ and $(adj. A).B = 0$ then system may or may not be consistent.

Trace the Mind Map

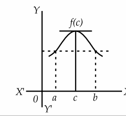
- First Level
- Second Level
- Third Level

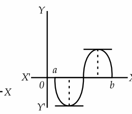
Continuity Differentiability and Differentiation

Trace the Mind Map: First Level, Second Level, Third Level

Rolle's Mean Value Theorems and its Geometric Representation

If a function $f(x)$
(i) is continuous in the closed interval $[a, b]$
(ii) is differentiable in an open interval (a, b)
(ii) $f(a) = f(b)$, then there will be atleast one point c in the interval (a, b), such that $f'(c)=0$

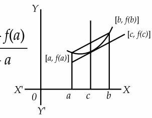




Lagrange's mean Value Theorem and its Geometrical Representation

If a function $f(x)$ is
(i) continuous in the closed interval $[a, b]$
(ii) differentiable in an open interval (a, b)
Then, there will be atleast one point c.
where $a<c<b$, such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



L' Hospital's Rule

If $f(x)$ and $g(x)$ are two function of x such that
(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$
(ii) both are continuous at $x=a$
(iii) both are differentiable at $x=a$
(iv) $f'(x)$ and $g'(x)$ are defined at the point $x=a$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ provided that $g'(a) \neq 0$
Above rule is also applicable, if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$

Continuity of a Function at a Point

A function $f(x)$ is said to be continuous at a point $x=a$, if $(\text{LHL})_{x=a}$, if $(\text{RHL})_{x=a} = f(a)$ or $\lim_{x\to a} f(x) = f(a)$ where, $(\text{LHL})_{x=a} = \lim_{x\to a^-} f(x)$ and $(\text{RHL})_{x=a} = \lim_{x\to a^+} f(x)$

Continuity of a Function in all interval

Statement

A function $f(x)$ is continuous in an internal (a, b) if $f(x)$ is continuous at every point in that internal and f is continuous in $[a, b]$ if f is continuous in the interval (a, b) and also at the point a from the right and at the point b than the right

Example

Let $f(x) = \begin{cases} x, & x>4 \\ \frac{x}{4}+2, & 4\leq x \end{cases}$

Is f continuous on the interval (2, 5) ?

Algebra of Continuous Function

Let f and g are two real functions, and also continuous at real number k, then,
(i) $f+g$ is continuous at $x=k$
(ii) $f-g$ is continuous at $x=k$
(iii) $f.g$ is continuous at $x=k$
(iv) mf is continuous, where m is any constant
(v) $\frac{f}{g}$ is continuous at $x=k$, where $g(k) \neq 0$

Removable Discontinuity

Statement

If the limit of the function exists but does not equal the value of the function at that point

Example

Let $g(x) = \frac{x-5}{\sqrt{x-4}-1}$ when $x \neq 5$

Differentiabity

A function $f(x)$ is said to be differentiable at a point $x=a$, if left hand derivative at $(x=a)$ equals to Right hand derivative at $(x=a)$ i.e. LHD at $(x=a)$ = RHD (at $x=a$).

where Right hand derivative, $\text{R}f'(a) = \lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}, h>0$

Left hand derivative, $\text{L}f'(a) = \lim_{h\to 0^+}\frac{f(a-h)-f(a)}{-h}, h>0$

Differentiation (The process of finding derivative of a function)

Fundamental Rules for Derivatives

(i) $\frac{d}{dx}(k) = 0$
where, 'k' is constant
(ii) $\frac{d}{dx}\{k.f(x)\} = k\frac{d}{dx}\{f(x)\}$
where, 'k' is a constant
(iii) $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}\{g(x)\}$
(iv) Product Rule : –
$\frac{d}{dx}\{f(x).g(x)\} = f(x)\frac{d}{dx}\{g(x)\} + g(x)\frac{d}{dx}\{f(x)\}$
(v) Quotient Rule : –
$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx}\{f(x)\} - f(x)\frac{d}{dx}\{g(x)\}}{\{g(x)\}^2}$

Derivatives of Some Standard Functions

Trigonometric Functions

(i) $\frac{d}{dx}(\sin x) = \cos x$
(ii) $\frac{d}{dx}(\cos x) = -\sin x$
(iii) $\frac{d}{dx}(\tan x) = \sec^2 x$
(iv) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$
(v) $\frac{d}{dx}(\sec x) = \sec x \tan x$
(vi) $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

Algebraic Functions

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$
(ii) $\frac{d}{dx}\left(\frac{1}{x^n}\right) = -nx^{-n-1}$
(iii) $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

Exponential Functions

(i) $\frac{d}{dx}(e^x) = e^x$
(ii) $\frac{d}{dx}(a^x) = a^x \log_e a, a>0$

Logarithmic Functions

(i) $\frac{d}{dx}(\log_e x) = \frac{1}{x}, x>0$
(ii) $\frac{d}{dx}(\log_a x) = \frac{1}{x\log_e a}$

Inverse Trigonometric Functions

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}; |x|<1$
(ii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}; |x|<1$
(iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}; x \in R$
(iv) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}; x \in R$
(v) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}; |x|>1$
(vi) $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}; |x|>1$

Differentiation of Implicit Functions

In such a case, to find $\frac{dy}{dx}$, differentiate both sides of the given relation w.r.t. x keeping in mind that the derivative of $f(y)$ w.r.t. x is $\frac{d\phi}{dx}\cdot\frac{dy}{dx}$
e.g. (i) $\frac{d}{dx}(\tan y) = \sec^2 y\frac{dy}{dx}$
(ii) $\frac{d}{dx}(y^4) = 4y^3\frac{dy}{dx}$

Differentiation of a Function with respect to another function

If $y = f(x)$ and $z = g(x)$ are two functions $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$

Differentiation of Parametric Function

$y = f(t)$ and $x = g(t)$,
$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f'(t)}{g'(t)}$

Differentiation of Composite Function

Statement

The derivative of the composite function is the product of their derivatives.

Chain Rule

Let y be a real valued function which is a composite of two function say $y = f(u)$, where $u = g(x)$
Then, $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = f'(a).g'(x)$
i.e. $\frac{d}{dx}[f\{g(x)\}] = f'\{g(x)\}.g'(x)$

Logarithmic Differentiation

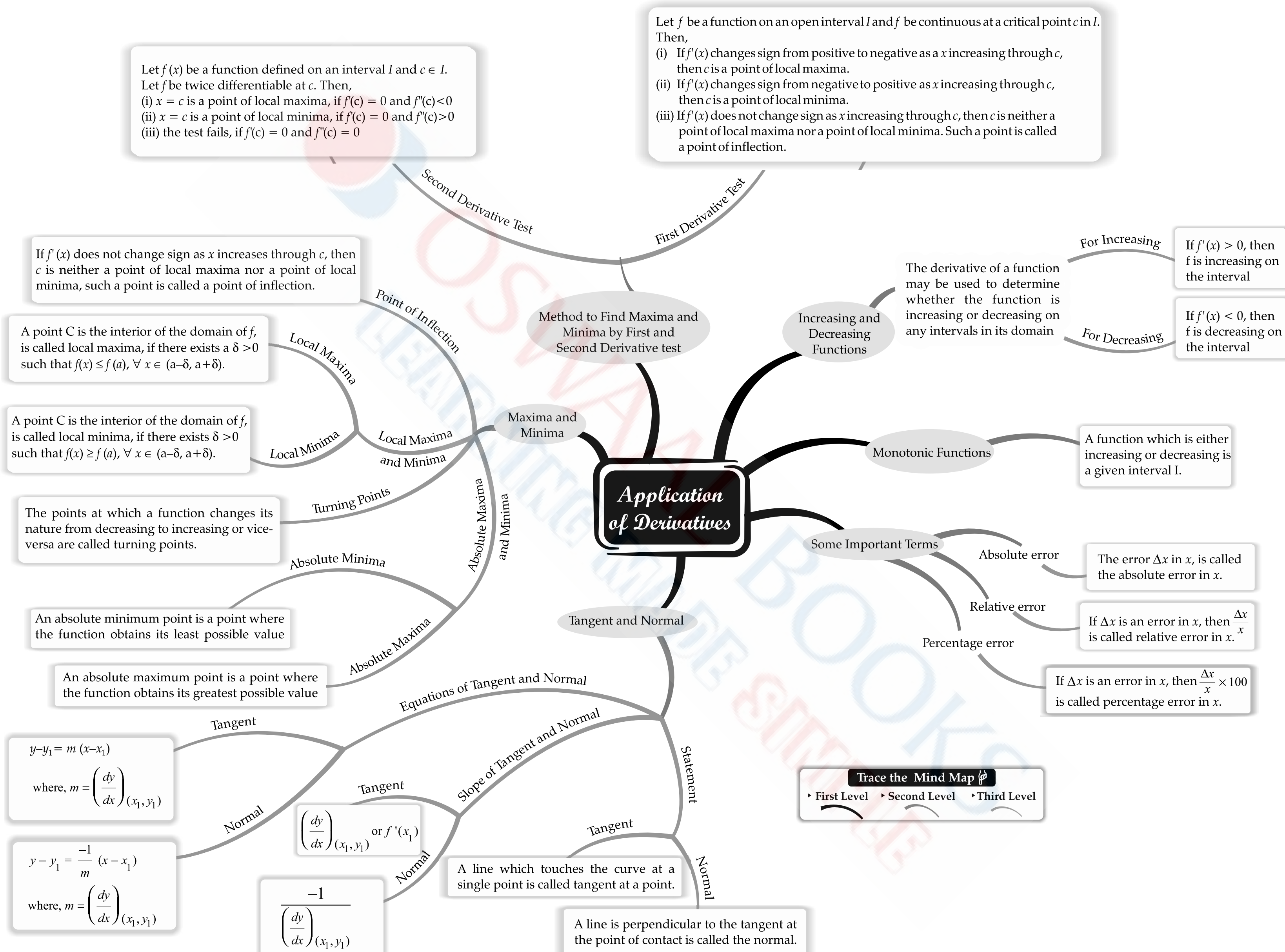
The functions, which can be evaluated by using logarithmic differentiation, are of type $y = [f(x)]^{g(x)}$ or $y = f_1(x)f_2(x)f_3(x)$.....or

$$y = \frac{f_1(x)f_2(x)f_3(x).....}{g_1(x)g_2(x)g_3(x).....}$$

(i) If $y = u(x)v(x)w(x)$, then $\log y = \log u(x) + \log v(x) + \log w(x)$
(ii)If $y = \frac{u(x)}{v(x)}$, then $\log y = \log u(x) - \log v(x)$
(iii)If $y = [f(x)]^{g(x)}$, then $\log y = g(x)\log[f(x)]$

Successive Differentiation

$\frac{dy}{dx}$ can be differentiated with respect to x.
It is writen as $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ of $f''(x)$.
Again $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$ or $f'''(x)$

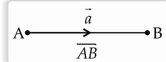

Application of Derivatives
Method to Find Maxima and Minima by First and Second Derivative test
First Derivative Test
Let f be a function on an open interval I and f be continuous at a critical point c in I. Then,
(i) If $f'(x)$ changes sign from positive to negative as a x increasing through c, then c is a point of local maxima.
(ii) If $f'(x)$ changes sign from negative to positive as x increasing through c, then c is a point of local minima.
(iii) If $f'(x)$ does not change sign as x increasing through c, then c is neither a point of local maxima nor a point of local minima. Such a point is called a point of inflection.
Second Derivative Test
Let $f(x)$ be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then,
(i) $x = c$ is a point of local maxima, if $f'(c) = 0$ and $f''(c)<0$
(ii) $x = c$ is a point of local minima, if $f'(c) = 0$ and $f''(c)>0$
(iii) the test fails, if $f'(c) = 0$ and $f''(c) = 0$
Increasing and Decreasing Functions
The derivative of a function may be used to determine whether the function is increasing or decreasing on any intervals in its domain
For Increasing
If $f'(x) > 0$, then f is increasing on the interval
For Decreasing
If $f'(x) < 0$, then f is decreasing on the interval
Monotonic Functions
A function which is either increasing or decreasing is a given interval I.
Some Important Terms
Absolute error
The error Δx in x, is called the absolute error in x.
Relative error
If Δx is an error in x, then $\frac{\Delta x}{x}$ is called relative error in x.
Percentage error
If Δx is an error in x, then $\frac{\Delta x}{x} \times 100$ is called percentage error in x.
Maxima and Minima
Point of Inflection
If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima, such a point is called a point of inflection.
Local Maxima and Minima
Local Maxima
A point C is the interior of the domain of f, is called local maxima, if there exists a $\delta > 0$ such that $f(x) \le f(a)$, $\forall x \in (a-\delta, a+\delta)$.
Local Minima
A point C is the interior of the domain of f, is called local minima, if there exists $\delta > 0$ such that $f(x) \ge f(a)$, $\forall x \in (a-\delta, a+\delta)$.
Turning Points
The points at which a function changes its nature from decreasing to increasing or vice-versa are called turning points.
Absolute Maxima and Minima
Absolute Minima
An absolute minimum point is a point where the function obtains its least possible value
Absolute Maxima
An absolute maximum point is a point where the function obtains its greatest possible value
Tangent and Normal
Equations of Tangent and Normal
Tangent
$y-y_1 = m(x-x_1)$
where, $m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$
Normal
$y - y_1 = \frac{-1}{m}(x - x_1)$
where, $m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$
Slope of Tangent and Normal
Tangent
$\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ or $f'(x_1)$
Normal
$\frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$
Statement
Tangent
A line which touches the curve at a single point is called tangent at a point.
Normal
A line is perpendicular to the tangent at the point of contact is called the normal.
Trace the Mind Map
First Level
Second Level
Third Level

Vectors

Scalars and vectors

- Quantities which have only magnitude and no direction is called scaler
- Quantities which have magnitude as well as direction is called vector.

Representation



Magnitude and direction of vector

- Magnitude of vector $\overline{AB}$ is $|\overline{AB}|$
- Any line which can be described by a vector called its direction vector.

Position vector

Position vector of a point P(x,y,z) is $x\hat{i}+y\hat{j}+z\hat{k}$ and its magnitude is $OP(r)=\sqrt{x^2+y^2+z^2}$, where O(0,0,0) be the origin. Position vector of P (2, 3, 5) is $2\hat{i}+3\hat{j}+5\hat{k}$ and its magnitude is $\sqrt{2^2+3^2+5^2}=\sqrt{38}$

Direction cosines and direction ratios

- Direction cosines are generally donated by letters l, m and n *i.e.*, $l=\cos\alpha$, $m=\cos\beta$, $n=\cos\gamma$.
- $\cos\alpha=\frac{x}{|\vec{r}|}$, $\cos\beta=\frac{y}{|\vec{r}|}$ and $\cos\gamma=\frac{z}{|\vec{r}|}$

$\Rightarrow l=\frac{x}{|\vec{r}|}$, $m=\frac{y}{|\vec{r}|}$, and $n=\frac{z}{|\vec{r}|}$

$\Rightarrow x=l|\vec{r}|$, $y=m|\vec{r}|$ and $z=n|\vec{r}|$

$|\vec{r}|=\sqrt{x^2+y^2+z^2}$

- Numbers $l|\vec{r}|$, $m|\vec{r}|$ and $n|\vec{r}|$ are proportional to the direction cosines are called direction ratios of vector $\vec{r}$
- For e.g: If $\overline{AB}=\hat{i}+2\hat{j}+3k$,

then $|\vec{r}|=\sqrt{1+4+9}=\sqrt{14}$

Direction ratios are (1, 2, 3)

and direction cosines are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

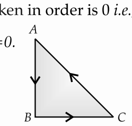
Types of Vectors

(i) Zero vector (initial and terminal points coincide)
(ii) Unit vector (magnitude is unity)
(iii) Coinitial vectors (same initial points)
(iv) Collinear vectors (parallel to the same line)
(v) Equal vectors (same magnitude and direction)
(vi) Negative of a vector (same magnitude and opposite direction)

Addition of vector

Triangle law of vector addition

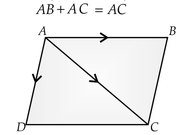
- In ΔABC, we have $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{CA}$, where $\overrightarrow{CA}$ is resultant vector.
- The vector sum of the three sides of a triangle taken in order is 0 *i.e.*, $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0.$



Parallelogram law of vector addition

The vector sum of two coinitials vector is given by diagonal of the parallelogram whose adjacent sides are given vectors, if $\overrightarrow{AB}$, $\overrightarrow{AD}$ are given vectors, then

$\overrightarrow{AB}+\overrightarrow{AC}=\overrightarrow{AC}$



Components of vectors

In $\overrightarrow{OP}(\text{or } \vec{r})=x\hat{i}+y\hat{j}+z\hat{k}$

- $\hat{i}, \hat{j}$ and $\hat{k}$ are unit vectors along the *X*-axis, *Y*-axis and *Z*-axis
- x, y and z are called scalar components of $\vec{r}$
- $x\hat{i}, y\hat{j}$ and $z\hat{k}$ are called vector components of $\vec{r}$

Vector joining of two points

$\overline{P_1P_2}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$

$\overline{P_1P_2}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$,

the distance between two points.

Dot (or scalar) product of two vectors

$\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

where $0\le\theta\le\pi$

Cross (or vector) product of two vectors

$\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\hat{n}$,

such that $0\le\theta\le\pi$

where, $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}$ $\vec{b}$ and $\hat{n}$ form a right handed system.

Properties of cross product of two vectors

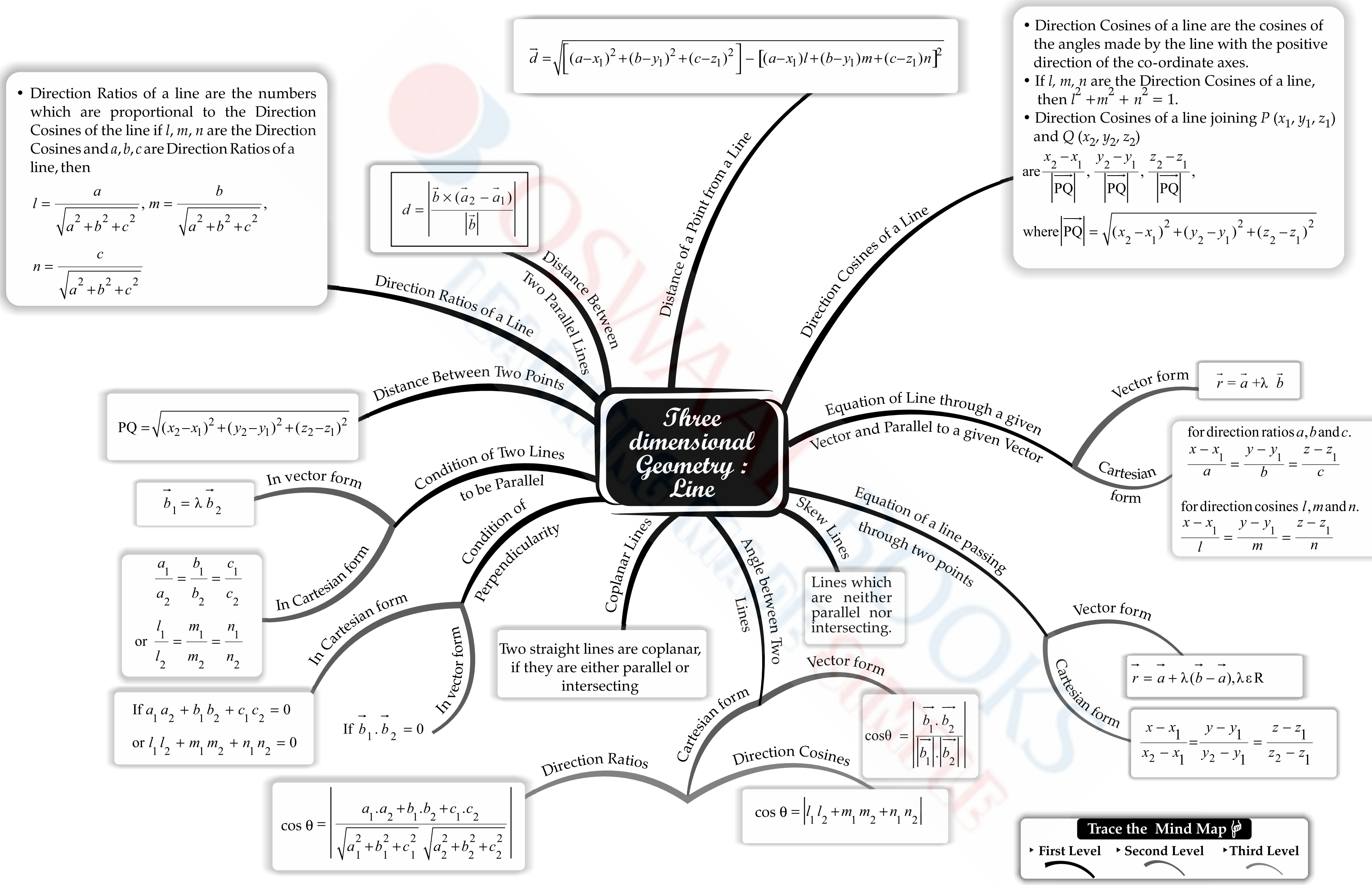
- If $\vec{a}$ and $\vec{b}$ are two vectors

then $\vec{a}\times\vec{b}=\begin{bmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{bmatrix}$

- Area of parallelogram $=|\vec{a}\times\vec{b}|$
- Area of triangle $=\frac{1}{2}|\vec{a}\times\vec{b}|$

Trace the Mind Map

First Level · Second Level · Third Level


Three dimensional Geometry : Line
Direction Cosines of a Line
• Direction Cosines of a line are the cosines of the angles made by the line with the positive direction of the co-ordinate axes.
• If l, m, n are the Direction Cosines of a line, then $l^2 + m^2 + n^2 = 1$.
• Direction Cosines of a line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\frac{x_2-x_1}{|\overrightarrow{PQ}|}, \frac{y_2-y_1}{|\overrightarrow{PQ}|}, \frac{z_2-z_1}{|\overrightarrow{PQ}|}$, where $|\overrightarrow{PQ}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$
Distance of a Point from a Line
$\vec{d} = \sqrt{\left[(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\right] - \left[(a-x_1)l+(b-y_1)m+(c-z_1)n\right]^2}$
Distance Between Two Parallel Lines
$d = \left|\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$
Direction Ratios of a Line
• Direction Ratios of a line are the numbers which are proportional to the Direction Cosines of the line if l, m, n are the Direction Cosines and a, b, c are Direction Ratios of a line, then
$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \frac{b}{\sqrt{a^2+b^2+c^2}}, n = \frac{c}{\sqrt{a^2+b^2+c^2}}$
Distance Between Two Points
$PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$
Condition of Two Lines to be Parallel
In vector form
$\vec{b}_1 = \lambda \vec{b}_2$
In Cartesian form
$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ or $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$
Condition of Perpendicularity
In Cartesian form
If $a_1a_2 + b_1b_2 + c_1c_2 = 0$ or $l_1l_2 + m_1m_2 + n_1n_2 = 0$
In vector form
If $\vec{b}_1 \cdot \vec{b}_2 = 0$
Coplanar Lines
Two straight lines are coplanar, if they are either parallel or intersecting
Angle between Two Lines
Cartesian form
Direction Ratios
$\cos\theta = \left|\frac{a_1.a_2+b_1.b_2+c_1.c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$
Direction Cosines
$\cos\theta = |l_1l_2+m_1m_2+n_1n_2|$
Vector form
$\cos\theta = \left|\frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$
Skew Lines
Lines which are neither parallel nor intersecting.
Equation of a line passing through two points
Vector form
$\vec{r} = \vec{a} + \lambda(\vec{b}-\vec{a}), \lambda \varepsilon R$
Cartesian form
$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$
Equation of Line through a given Vector and Parallel to a given Vector
Vector form
$\vec{r} = \vec{a} + \lambda\vec{b}$
Cartesian form
for direction ratios a, b and c.
$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$
for direction cosines l, m and n.
$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$
Trace the Mind Map
First Level
Second Level
Third Level

Applications of Calculus

Cost Function
The total cost function is expressed as C = c(x), explicitly and f (c, x) = 0 is the implicit form where x denotes the quantity produced of a certain commodity at total cost c.

Total Cost (Broken into Two Parts)

Fixed Cost
Total fixed cost,

$\boxed{\text{TFC} = \text{TC, when } x = 0}$

Variable Cost
Total cost = total fixed cost + total variable cost

$\boxed{\text{TC} = \text{TFC} + \text{TVC}}$

Average Cost
It represents the cost per unit,

$\boxed{\text{AC} = \frac{\text{C}}{x}}$

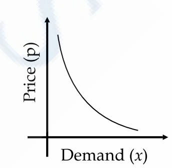
Where, C = total cost,

x = number of commodities

Demand Function
- It represents the functional relationship between demand and price of a commodity
- If p denotes the price per unit and x is the number of units demanded by a consumer at that price, then

(i) demand function in explicit form, $x = f$ (p)

(ii) demand function in implicit form, $f(x, p) = 0$



Revenue Function
Total revenue = selling price per unit of the commodity × quantity sold.

$\boxed{\text{R or R}(x) = p.x}$

Profit Function
Difference between revenue function and cost function.

$\boxed{\text{P}(x) = \text{R}(x) - \text{C}(x)}$

Breakeven Point
When total revenue equals to total cost incurred

$\boxed{\text{R}(x) = \text{C}(x)}$

Marginal Cost
It is the rate of change of total cost with respect to x (output).

$\boxed{\text{MC} = \frac{d}{dx}(\text{C}) = \frac{d\text{C}}{dx}}$

Relation between Average Cost and Marginal Cost
If C is the total cost of producing and marketing x units of commodity, then

$\boxed{\frac{d}{dx}(\text{AC}) = \frac{1}{2}(\text{MC} - \text{AC})}$

Average Revenue
It is the revenue generated per unit of output sold

$\boxed{\text{AR} = \frac{\text{Total Revenue}}{\text{Number of units sold}} = \frac{\text{R}}{x} = \frac{p.x}{x} = p}$

Marginal Revenue
It is the rate of change of total revenue with respect to quantity sold.

$\boxed{\text{MR} = \frac{d\text{R}}{dx} = \frac{d}{dx}(px)}$

Trace the Mind Map
- First Level
- Second Level
- Third Level

ON TIPS NOTES

Note making is a skill that we use in many walks of life: at school, university and in the world of work. However, accurate note making requires a thorough understanding of concepts. We, at Oswaal, have tried to encapsulate all the chapters from the given syllabus into the following **ON TIPS NOTES**. These notes will not only facilitate better understanding of concepts, but will also ensure that each and every concept is taken up and every chapter is covered in totality. So, go ahead and use these to your advantage… go get the **OSWAAL ADVANTAGE!!**

Section A

Chapter 1 - Relations and Functions

Relation Let A be a non-empty set and $R \in A \times A$. Then, R is called a relation on A. If $(a, b) \in R$, then say that 'a' is related to 'b' and write $\boldsymbol{aRb}$. If $(a, b) \notin R$, then we write $\boldsymbol{a \not R b}$.

Types of Relation

- **Empty Relation:** A relation R on a set A is called an empty or null or void relation, if no element of A is related to any element of A *i.e.* $R = \phi \subset A \times A$.
- **Identity Relation:** A relation I on a set A is called an identity relation, if every element of a set A is related to itself only. The identity relation on set A is defined as $I_A = \{(x, y) \in A \times A \mid x = y\}$
- **Universal Relation:** A relation R on a set A is called universal relation, if each element of A is related to every element of A *i.e.* $R = A \times A$.
- **Reflexive Relation:** A relation R defined on set A is said to be reflexive, if $(x, x) \in R, \forall x \in A$ *i.e.* $x R x, \forall x \in A$.
- **Symmetric Relation:** A relation R defined on set A is said to be symmetric, if $(x, y) \in R \Rightarrow (y, x) \in R, \forall x, y \in A$ *i.e.* $x R y \Rightarrow y R x \; \forall x, y \in A$.
- **Transitive Relation:** A relation R defined on set A is said to be transitive, if $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R, \forall x, y, z \in A$ *i.e.* $x R y$ and $y R z \Rightarrow x R z \; \forall x, y, z \in A$.
- **Equivalence Relation:** A relation R on a set A is called an equivalence relation, if it is reflexive, symmetric and transitive.
- **Equivalence Classes:** Consider, an arbitrary equivalence relation on R on an arbitrary set X, R divides X into mutually disjoint subsets A_i is called partitions or subdivisions of X, satisfying
  (i) all elements of A_i are related to each other for all i
  (ii) no element of A_i is related to any element of A_j; $i \neq j$.
  (iii) $A_i \cup A_j = X$ and $A_i \cap A_j = \phi$, $i \neq j$. Then, subset A_i are called equivalence classes.

Function

Let A and B be two non-empty sets. Then, a relation f from A to B which associates each element $x \in A$, to a unique element of $f(x) \in B$ is called a function from A to B and we write $f : A \to B$. Here, A is called the **domain** of f. *i.e.*, dom $(f) = A$ and B is called the **codomain** of f. Also, $\{f(x) : x \in A\} \subseteq B$ is called the **range** of f.
Note *Every function is a relation but every relation is not a function.*

Types of Function

1. **One-one (Injective) Function:** A function $f: A \rightarrow B$ is said to be one-one, if distinct element of A have distinct images in B, *i.e.* if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ or $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ $\forall$, $x_1, x_2 \in A$.
2. **Many-one Function:** A function $f: A \rightarrow B$ is said to be many-one, if two or more than two elements in A have the same image in B, *i.e.* if $x_1 \neq x_2$, then $f(x_1) = f(x_2)$.
3. **Onto (Surjective) Function:** A function $f: A \rightarrow B$ is said to be onto, if every element in B has its pre-image in A, *i.e.* if for each $y \in B$, there exists an element $x \in A$, such that $f(x) = y$.
4. **Into Function:** A function $f: A \rightarrow B$ is said to be into, if atleast one element of B do not have a pre-image in A.
5. **One-one and Onto (Bijective) Function:** A function $f: X \rightarrow Y$ is said to be one-one and onto, if f is both one-one and onto.

Composition of Function

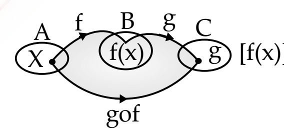
Let $f: A \rightarrow B$ and $g: B \rightarrow C$, be any two functions, then the composition of f and g denoted by the function gof and defined as $gof: A \rightarrow C$, such that $(gof)(x) = g\{f(x)\}, \forall\, x \in A$.

Note (i) *In generally, $gof \neq fog$.*

(ii) *In generally, if gof is one-one, then f is one-one; and if gof is onto, then g is onto.*

Invertible Function

A function $f: X \rightarrow Y$ is defined to be invertible, if there exists a function $g: Y \rightarrow X$, such that $gof = I_x$ and $fog = I_y$. The function g is called the inverse of f and it is denoted by f^{-1}. Thus, f is invertible, then f must be one-one and onto and *vice-versa.*



Note: (i) *If $f: X \rightarrow Y$, $g: Y \rightarrow Z$ and $h: Z \rightarrow S$ are three functions, then ho(gof) = (hog)of.*

(ii) *Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be two invertible functions. Then, gof is also invertible with $(gof)^{-1} = f^{-1} o g^{-1}$.*

Binary Operation

Let S be a non-empty set and $*$ be an operation on S such that

$a \in S,\ b \in S$

$\Rightarrow \quad a * b \in S, \forall\ a, b \in S$

Then, $*$ is called a binary operation on S.

Types of Binary Operation

(i) A binary operation $*$ on set S is said to be commutative, if $a * b = b * a, \forall\ a, b \in S$.

(ii) A binary operation $*$ on S is said to be associative, if $(a * b) * c = a * (b * c); \forall\ a, b, c \in S$.

(iii) An element $e \in S$ is said to be the identity element of a binary operation $*$ on set S, if

$a * e = e * a = a, \forall\ a \in S$

(iv) An element $a \in S$ is said to be invertible with respect to the operation $*$, if there exists an element $b \in S$, such that

$a * b = b * a = e, \forall\ b \in S$.

Then, element b is called inverse of element a.

Chapter 2 - Inverse Trigonometric Function

Trigonometric functions are not one-one and onto on their natural domains and ranges, so their inverse does not exist in all values but their inverse may exists in some interval of their restricted domains and ranges. Thus, we can say that, inverse of trigonometric functions are defined within restricted domains of corresponding trigonometric functions. Inverse of f is denoted by f^{-1}.

Note: **(i)** $\sin^{-1} x \neq (\sin x)^{-1}$ **(ii)** $\sin^{-1} x \neq \sin^{-1}\left(\frac{1}{x}\right)$ **(iii)** $\sin^{-1} x \neq \left(\frac{1}{\sin x}\right)$

Domain and Principal Value branch (Range) of Inverse Trigonometric Functions

Function	Domain	Principal value branch (Range)
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\cot^{-1} x$	R	$(0, \pi)$

T-ratios of Some Standard Angles

Angle (θ) / Ratio	$0° = 0$	$30° = \frac{\pi}{6}$	$45° = \frac{\pi}{4}$	$60° = \frac{\pi}{3}$	$90° = \frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Properties of Inverse Trigonometric Functions

Property I

(i) $\sin^{-1}(\sin\theta) = \theta,\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos\theta) = \theta,\ \theta \in [0, \pi]$

(iii) $\tan^{-1}(\tan\theta) = \theta,\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iv) $\cot^{-1}(\cot\theta) = \theta,\ \theta \in (0, \pi)$

(v) $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta,\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

(vi) $\sec^{-1}(\sec\theta) = \theta,\ \theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

(vii) $\sin(\sin^{-1} x) = x,\ x \in [-1, 1]$

(viii) $\cos(\cos^{-1} x) = x,\ x \in [-1, 1]$

(ix) $\tan(\tan^{-1} x) = x,\ x \in R$

(x) $\cot(\cot^{-1} x) = x,\ x \in R$

(xi) $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x,\ x \in (-\infty, -1] \cup [1, \infty)$

(xii) $\sec(\sec^{-1} x) = x,\ x \in (-\infty; -1] \cup [1, \infty)$

Property II

(i) $\sin^{-1}(-x) = -\sin^{-1} x,\ x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x,\ x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x,\ x \in R$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x,\ x \in R$

(v) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x,\ x \in (-\infty; 1] \cup [1, \infty)$

(vi) $\sec^{-1}(-x) = \pi - \sec^{-1} x,\ x \in (-\infty; 1] \cup [1, \infty)$

Property III

(i) $\sin^{-1}(1/x) = \operatorname{cosec}^{-1} x,\ x \geq 1 \text{ or } x \leq -1$

(ii) $\cos^{-1}(1/x) = \sec^{-1} x,\ x \geq 1 \text{ or } x \leq -1$

(iii) $\tan^{-1}\dfrac{1}{x} = \begin{Bmatrix} \cot^{-1} x &, & x > 0 \\ -\pi + \cot^{-1} x, & & x < 0 \end{Bmatrix}$

Property IV

(i) $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2},\ x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2},\ x \in R$

(iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \dfrac{\pi}{2},\ x \in (-\infty, -1] \cup [1, \infty)$

Property V

(i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right),\ xy < 1$

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right),\ xy > -1$

Property VI

(i) $2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2}), -\dfrac{1}{\sqrt{2}} \leq x \leq \dfrac{1}{\sqrt{2}}$

(ii) $2\cos^{-1} x = \cos^{-1}(2x^2 - 1),\ 0 \leq x \leq 1$

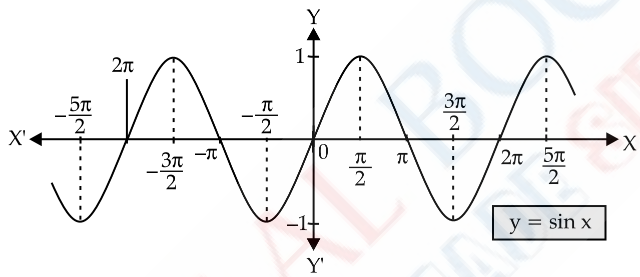
(iii) $2\tan^{-1} x = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right),\ |x| \leq 1 \text{ or } -1 \leq x \leq 1$

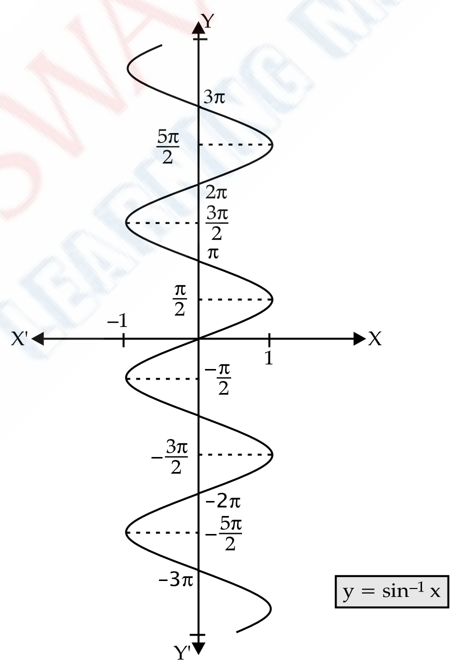
(iv) $2\tan^{-1}x = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), x \geq 0$

(v) $2\tan^{-1}x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), -1 < x \leq 1$

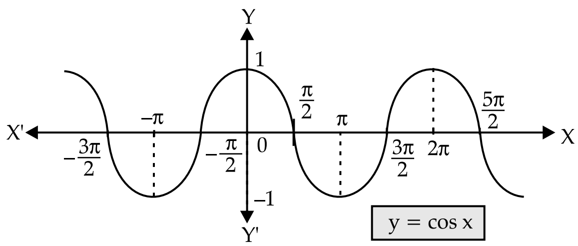
Graph of Functions and their Inverses

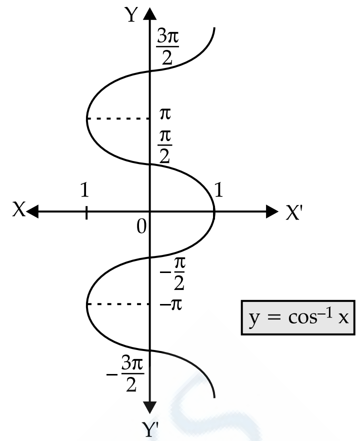
- **Principal value of branch function $\sin^{-1}$:** It is a function with domain $[-1, 1]$ and range $\left[\dfrac{-3\pi}{2}, \dfrac{-\pi}{2}\right], \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$ or $\left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$ and so on corresponding to each interval, we get a branch of the function $\sin^{-1} x$. The branch with range $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$ is called the principal value branch. Thus, $\sin^{-1}[-1,1] \rightarrow \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$.



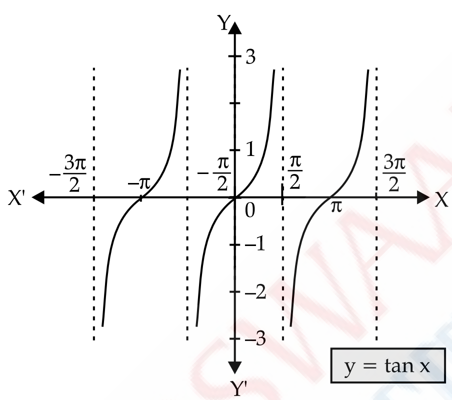


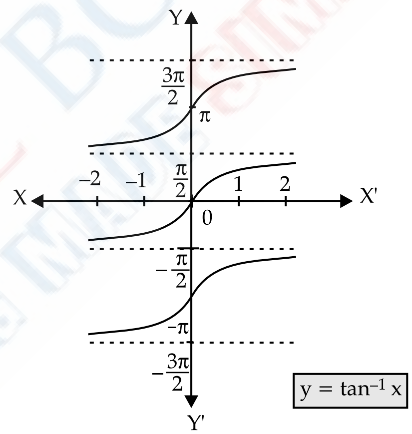
- **Principal value branch of function $\cos^{-1}$:** The graph of the function $\cos^{-1}$ is as shown in figure. Domain of the function $\cos^{-1}$ is $[-1, 1]$. Its range in one of the intervals $(-\pi, 0)$, $(0, \pi)$, $(\pi, 2\pi)$, etc. is one-one and onto with the range $[-1, 1]$. The branch with range $(0, \pi)$ is called the principal value branch of the function $\cos^{-1}$.
  Thus, $\cos^{-1}[-1,1] \rightarrow [0,\pi]$





- **Principal value branch function of tan^{-1} :** The function tan^{-1} is defined whose domain is set of real numbers and range is one of the intervals
  $\left(\frac{-3\pi}{2},\frac{-\pi}{2}\right),\left(\frac{-\pi}{2},\frac{\pi}{2}\right),\left(\frac{\pi}{2},\frac{3\pi}{2}\right),\ldots$
- **Graph of the function is as shown in the figure:**

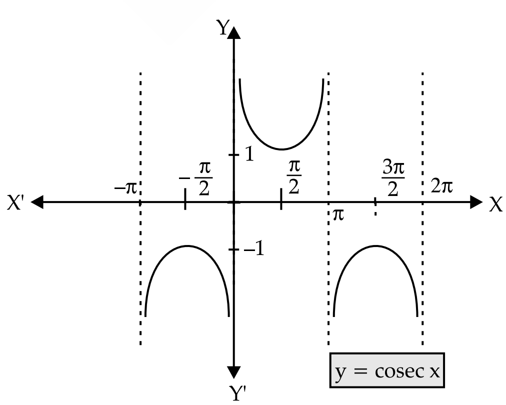


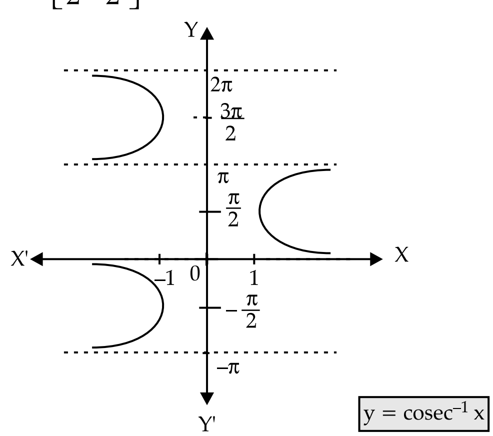


The branch with range $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ is called the principal value branch of function tan^{-1}. Thus, $\tan^{-1} : R \to \left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

- **Principal value branch of function cosec^{-1} :** The graph of function cosec^{-1} is shown in figure. The cosec^{-1} is defined on a function whose domain is $R - (-1, 1)$ and the range is any one of the interval,

$$\left[\frac{-3\pi}{2},\frac{-\pi}{2}\right]-\{\pi\},\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\},\left[\frac{\pi}{2},\frac{3\pi}{2}\right]-\{\pi\},\ldots$$



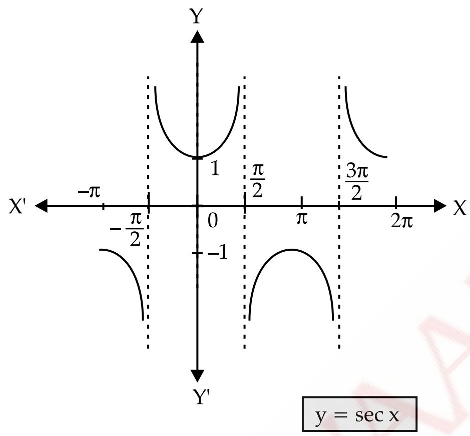


The function corresponding to the range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ is called the principal value branch of cosec^{-1}.

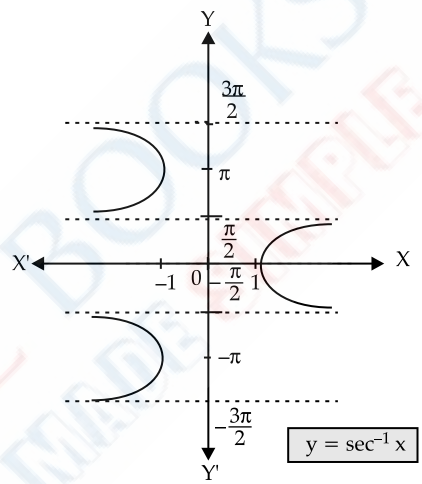
Thus, $\text{cosec}^{-1} : R(-1,1) \rightarrow \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

➢ **Principal value branch of function $\sec^{-1}$:** The graph of function $\sec^{-1}$ is shown in figure. The $\sec^{-1}$ is defined as a function whose domain $R-(-1,1)$ and range is $[-\pi, 0] - \left[\frac{-\pi}{2}\right], [0-\pi] - \left\{\frac{\pi}{2}\right\}, [\pi - 2\pi] - \left\{\frac{3\pi}{2}\right\}$, etc. Function corresponding to range $[0-\pi] - \left\{\frac{\pi}{2}\right\}$ is known as the principal value branch of $\sec^{-1}$.

Thus, $\sec^{-1} : R(-1,1) \rightarrow [0, \pi] - \left\{\frac{\pi}{2}\right\}$

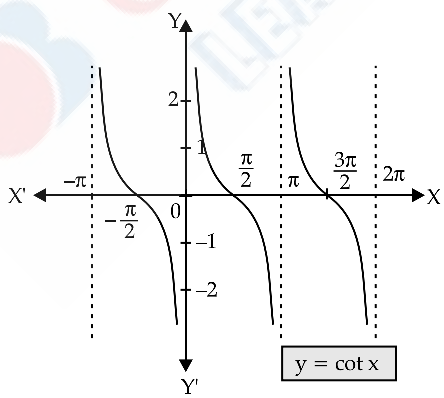


y = sec x

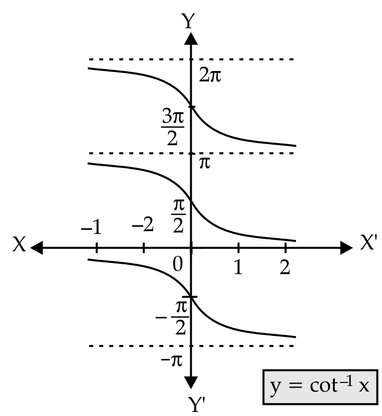


y = sec⁻¹ x

➢ **The principal value branch of function $\cot^{-1}$:** The graph of function $\cot^{-1}$ is shown below. The $\cot^{-1}$ function is defined on function whose domain is R and the range is any of the intervals, $(-\pi, 0)$, $(0, \pi)$, $(\pi, 2\pi)$, …. The function corresponding to $(0, \pi)$ is called the principal value branch of the function $\cot^{-1}$. Then, $\cot^{-1}: R \rightarrow (0, \pi)$



y = cot x



y = cot⁻¹ x

Some Useful Trigonometric Formulae

(i) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(ii) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(iii) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

(iv) $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

(v) $2\sin A \cos B = \sin(A+B) + \sin(A-B)$

(vi) $2\cos A \cos B = \sin(A+B) - \sin(A-B)$

(vii) $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

(viii) $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

(ix) $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

(x) $\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

(xi) $\cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

(xii) $\cos C - \cos D = -2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

or $2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{D-C}{2}\right)$

(xiii) $\sin 2x = 2\sin x \cos x$

(xiv) $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

(xv) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

(xvi) $1+\cos 2x = 2\cos^2 x$; $1-\cos 2x = 2\sin^2 x$

(xvii) $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$; $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

(xviii) $\sin 3x = 3\sin x - 4\sin^3 x$

(xix) $\cos 3x = 4\cos^3 x - 3\cos x$

(xx) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$

(xxi) $\sin^2\theta + \cos^2\theta = 1$

(xxii) $1+\tan^2\theta = \sec^2\theta$

(xxiii) $1+\cot^2\theta = \text{cosec}^2\theta$

➢ **To simplify inverse trigonometrical expressions, following substitutions can be considered:**

Expression	Substitution
a^2+x^2 or $\sqrt{a^2+x^2}$	$x = a\tan\theta$ or $x = a\cot\theta$
a^2-x^2 or $\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
x^2-a^2 or $\sqrt{x^2-a^2}$	$x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$
$\sqrt{\dfrac{a-x}{a+x}}$ or $\sqrt{\dfrac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$ or $\sqrt{\dfrac{a^2+x^2}{a^2-x^2}}$	$x^2 = a^2\cos 2\theta$

$\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$	$x = a \sin^2\theta$ or $x = a \cos^2\theta$
$\sqrt{\frac{x}{a+x}}$ or $\sqrt{\frac{a+x}{x}}$	$x = a \tan^2\theta$ $x = a \cot^2\theta$

Chapter 3 – Matrices

Matrix

A matrix is a rectangular array of numbers or functions. The number or functions are called the elements or the entries of the matrix.

Order of Matrix

A matrix of order $m \times n$ is of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix},$$

where m represents number of rows and n represents number of columns.
In notation form, it can be rewritten as

$$A = [a_{ij}]_{m \times n}$$

where, $1 \leq i \leq m$, $1 \leq j \leq n$ and $i, j \in N$. Here, a_{ij} is an element lying in the ith row and jth column.

Types of Matrices

(i) **Row matrix:** A matrix having only one row and many columns, is called a row matrix.

(ii) **Column matrix:** A matrix having only one column and many rows, is called a column matrix.

(iii) **Zero matrix or null matrix:** If all the elements of a matrix are zero, then it is called a zero matrix or null matrix. It is denoted by symbol O.

(iv) **Square matrix:** A matrix in which number of rows and number of columns are equal, is called a square matrix.

(v) **Diagonal matrix:** A square matrix is said to be a diagonal matrix, if all the elements lying outside the diagonal elements are zero.

(vi) **Scalar matrix:** A diagonal matrix in which all principal diagonal elements are equal, is called a scalar matrix.

(vii) **Unit matrix or identity matrix:** A square matrix having 1 (one) on its principal diagonal and 0 (zero) elsewhere, is called an identity matrix. It is denoted by symbol I.

(viii) **Equality of matrix:** Two matrices are said to be equal, if their order are same and their corresponding elements are also equal.

Addition of Matrices

Let A and B be two matrices each of order $m \times n$. Then, the sum of matrices $A+B$ is a matrix whose elements are obtained by adding the corresponding elements of A and B.

i.e. if $\quad A = [a_{ij}]_{m \times n}, B = [b_{ij}]_{m \times n}$

Then, $\quad A + B = [a_{ij} + b_{ij}]_{m \times n}$

Note: *If A and B are not of same order, then A + B is not defined.*

Difference of Matrices

If $A = [a_{ij}]$, and $B = [b_{ij}]$ are two matrices of the same order $m \times n$, then difference $A - B$ is defined as a matrix $D = [d_{ij}]$, where $d_{ij} = a_{ij} - b_{ij}, \forall i, j$.

Multiplication of a Matrix by a Scalar

Let $A = [a_{ij}]_{m \times n}$ be a matrix and k be any scalar. Then, KA is another matrix which is obtained by multiplying each element of A by k *i.e.* $kA = [ka_{ij}]_{m \times n}$.
(i) $k(A+B) = kA+kB$ **(ii)** $(k_1+k_2)A = k_1A+k_2A$

Multiplication of Matrices

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ be two matrices such that the number of columns of A is equal to the number of rows of B, then multiplication of A and B is denoted by AB, is given by $C_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$, where, C_{ij} is the element of matrix C and $C = AB$.
Note: *Generally, multiplication of matrices is not commutative i.e.* $AB \neq BA$.

Transpose of a Matrix

The matrix obtained by interchanging the rows and columns of a given matrix A, is called transpose of a matix. It is denoted by A' or A^T.

Properties of Transpose of Matrices

(i) $(A \pm B)' = A' \pm B'$
(ii) $(kA)' = kA'$, where k is any constant.
(iii) $(AB)' = B' A'$
(iv) $(A')' = A$

Symmetric and Skew-Symmetric Matrix

A square matrix A is called symmetric, if $A' = A$ and a square matrix A is called skew-symmetric, if $A' = -A$.

Properties of Symmetric and Skew-Symmetric Matrix

(i) For any square matrix A with real number entries, $A + A'$ is a symmetric matrix and $A - A'$ is a skew-symmetric matrix.
(ii) Any square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrices.

i.e. $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

(iii) The principal diagonal elements of a skew-symmetric matrix are always zero.

Elementary Operations of a Matrix

There are six operations (transformations) on a matrix, three of which are due to rows and three due to columns, which are known as elementary operations or transformations.

(i) **The interchange of any two rows or two columns** Symbolically, the interchange of *i*th and *j*th rows is denoted by $R_i \leftrightarrow R_j$ and interchange of *i*th and *j*th columns is denoted by $C_i \leftrightarrow C_j$.

(ii) **The multiplication of the elements of any row or column by a non-zero number** Symbolically, the multiplication of each element of the *i*th row by k, where $k \neq 0$ is denoted by $R_i \to kR_i$. The corresponding column operation is denoted by $C_i \to kC_i$.

(iii) **The addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non-zero number**. Symbolically, the addition to the elements of ith row, the corresponding elements of jth row multiplied by k is denoted by $R_i \to R_i + kR_j$. The corresponding column operation is denoted by $C_i \to C_i + kC_j$.

Invertible Matrices

A is a square matrix of order m is said to be invertible, if there exists another square matrix B of the same order m, such that $AB = BA = I$, where I is a unit matrix of same order m. The matrix B is called the inverse matrix of A and it is denoted by A^{-1}.

Note: **(i)** *A rectangular matrix does not posses inverse matrix.*
(ii) *If B is an inverse of A, then A is also the inverse of B.*

Properties of Invertible Matrices

Let A and B be two non-zero invertible matrices of same order.

(i) **Uniqueness of inverse** If inverse of a square matrix exists, then it is unique.
(ii) $AA^{-1} = A^{-1}A = I$
(iii) $(AB)^{-1} = B^{-1}A^{-1}$
(iv) $(A^{-1})^{-1} = A$
(v) $(A')^{-1} = (A^{-1})'$ or $(A^T)^{-1} = (A^{-1})^T$
where, A' or A^T is transpose of a matrix A.

Chapter 4 – Determinants

Determinant

Every square matrix A is associated to a number, which is called its determinant and it is denoted by det (A) or $|A|$.

Expansion of Determinant of Order (2×2)

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Expansion of Determinant of Order (3×3)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Note *Similarly, we can expand the above determinant corresponding to any row or column.*
If A is a $n \times n$ matrix, then $|kA| = k^n|A|$.

Properties of Determinants

(i) If the rows and columns of a determinant are interchanged, then the value of the determinant does not change.
(ii) If any two rows (columns) of a determinant are interchanged, then sign of determinant changes.
(iii) If any two rows (columns) of a determinant are identical, then the value of the determinant is zero.
(iv) If each element of a row (column) is multiplied by a non-zero constant k, then the value of the determinant is multiplied by k.
(v) If each element of any row (column) of a determinant is added k times the corresponding element of another row (column), then the value of the determinant remains unchanged.
(vi) If some or all elements of a row (column) of a determinant are expressed as sum of two (more) terms, then the determinant can be expressed as sum of two (more) determinants of the same order.

(vii) If all elements of any two rows (or column) of a determinant are proportional, then the value of such determinant becomes zero.

(viii) If all the elements of any row (or column) of a determinant are zero, then the value of such determinant becomes zero.

Minors and Cofactors

Minors: Minor of an element a_{ij} of a matrix is the determinant obtained by deleting ith row and jth column in which element a_{ij} lies. It is denoted by M_{ij}.

$$\text{If } A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then}$$

Minors of A are

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \text{ etc.}$$

Note: *The minor of an element of a determinant of order n ($n \geq 2$) is a determinant of order $n - 1$.*

Cofactors: If M_{ij} is the minor of an element a_{ij}, then the cofactor of a_{ij} is denoted by C_{ij} or A_{ij} and defined as follows

$$A_{ij} \text{ or } C_{ij} = (-1)^{i+j} M_{ij}$$

Note: *If elements of a row (column) are multiplied with cofactors of any other row (column), then their sum is zero.*

Area of Triangle

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a ΔABC. Then, its area is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \cdot \left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right|$$

(i) Since, area is positive quantity. So, we always take the absolute value of the determinant.

(ii) If area is given, then use both positive and negative values of the determinant for calculation.

Condition of Collinearity

Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear, when $\Delta = 0$.

i.e. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Adjoint of a Matrix

The adjoint of a square matrix A is defined as the transpose of the matrix formed by cofactors of A. Let $A = [a_{ij}]$ be a square matrix of order n, then adjoint of A, *i.e.* $adj\ A = C^T$, where $C = [C_{ij}]$ is the cofactor matrix of A.

Properties of Adjoint of Square Matrix

If A and B are two non-singular matrices of order n, then

(i) $A(adj\ A) = |A|\ I_n = (adj\ A)\ A$

(ii) $adj(A^T) = (adj\ A)^T$

(iii) $|adj\ A| = |A|^{n-1}$, if $|A| \neq 0$

(iv) $|adj[adj(A)]| = |A|^{(n-1)^2}$, if $|A| \neq 0$

(v) $(adj\ AB) = (adj\ B)\ (adj\ A)$

Singular and Non-singular Matrix

A square matrix A is said to be a singular matrix, if $|A| = 0$ and if $|A| \neq 0$, then matrix A is said to be non-singular matrix.

Inverse of a Matrix

A square matrix A has inverse, iff A is a non-singular ($|A| \neq 0$) matrix. The inverse of A is denoted by A^{-1}

i.e. $A^{-1} = \dfrac{1}{|A|}(adj\ A), |A| \neq 0$

Properties of Inverse of a Square Matrix

(i) $(A^{-1})^{-1} = A$

(ii) $(AB)^{-1} = B^{-1}A^{-1}$

(iii) $(A^T)^{-1} = (A^{-1})^T$

(iv) $(kA)^{-1} = kA^{-1}$

(v) $adj\ (A^{-1}) = (adj\ A)^{-1}$

System of Linear Equations

Let the system of linear equations be

$$a_1x+b_1y+c_1z = d_1,$$
$$a_2x+b_2y+c_2z = d_2$$
and $$a_3x+b_3y+c_3z = d_3$$

Then, in matrix form this system of equations can be written as $AX = B$

where, $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

A system of equations is consistent or inconsistent according as its solution exists or not.

(i) For a square matrix A in matrix equation $AX = B$

(a) If $|A| \neq 0$, then system of equations is **consistent** and has unique solution given by $X = A^{-1}B$.

(b) If $|A| = 0$ and $(adj\ A)\ B \neq 0$, then there exists no solution, *i.e.* system of equations is **inconsistent**.

(c) If $|A| = 0$ and $(adj\ A)\ B = 0$, then system of equations is consistent and has an infinite number of solutions.

(ii) When $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ in such cases, we have

(a) $|A| \neq 0 \Rightarrow$ System has only trivial solution
i.e. $x = 0, y = 0$ and $z = 0$

(b) $|A| = 0 \Rightarrow$ System has infinitely many solutions.

Chapter 5 – Continuity and Differentiability

Continuous Function

A real function f is said to be continuous, if it is continuous at every point in the domain f.

Continuity of a Function at a Point

Suppose, f is a real valued function on a subset of the real numbers and let c be a point in the domain of f. Then, f is continuous at c, if $\lim\limits_{x\to c} f(x) = f(c)$.

i.e. if $f(c) = \lim\limits_{x\to c^+} f(x) = \lim\limits_{x\to c^-} f(x),$ then $f(x)$ is continuous at $x = c$. Otherwise, $f(x)$ is discontinuous at $x = c$.

Note:
- **(i)** *Every constant function is continuous.*
- **(ii)** *Every identity function is continuous.*
- **(iii)** *Every rational function and polynomial function is continuous.*
- **(iv)** *All trigonometric functions are continuous in their domain.*

Algebra of Continuous Function

If f and g are two continuous functions in domain D, then
- **(i)** $(f + g)$ is continuous.
- **(ii)** $(f - g)$ is continuous.
- **(iii)** cf is continuous, where c is a constant.
- **(iv)** fg is continuous.
- **(v)** $\dfrac{f}{g}$ is continuous in domain provided that, $g(x) \neq 0$.

Differentiability or Derivability

A function f is said to be derivable or differentiable at $x = c$, if its left hand and right hand derivatives at c exist and are equal.
At $x = a$,
Right Hand Derivative

$Rf'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ and

Left Hand Derivative

$Lf'(a) = \lim\limits_{h\to 0} \dfrac{f(a-h) - f(a)}{-h}$. Then, $f(x)$ is differentiable at $x = a$, if $Rf'(a) = Lf'(a)$. Otherwise, $f(x)$ is not differentiable at $x = a$.

Differentiation

The process of finding derivative of a function is called differentiation.

Derivatives of Standard Functions

- **(i)** $\dfrac{d}{dx}(x^n) = nx^{n-1}$
- **(ii)** $\dfrac{d}{dx}(\text{constant}) = 0$
- **(iii)** $\dfrac{d}{dx}(cx^n) = cn\, x^{n-1}$, where c is a constant.

(iv) $\frac{d}{dx}(\sin x) = \cos x$

(v) $\frac{d}{dx}(\cos x) = -\sin x$

(vi) $\frac{d}{dx}(\tan x) = \sec^2 x$

(vii) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec}\, x \cot x$

(viii) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(ix) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(x) $\frac{d}{dx}(e^x) = e^x$

(xi) $\frac{d}{dx}(a^x) = a^x \log_e a,\ a > 0$

(xii) $\frac{d}{dx}(\log_e x) = \frac{1}{x},\ x > 0$

(xiii) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a},\ a > 0,\ a \neq 1$

(xiv) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}},\ -1 < x < 1$

(xv) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}},\ -1 < x < 1$

(xvi) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(xvii) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(xviii) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}},\ |x| > 1$

(xix) $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}},\ |x| > 1$

Algebra of Derivatives

(i) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$ [sum and difference rule]

(ii) $\frac{d}{dx}(u \cdot v) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$ [product rule]

(iii) $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{d}{dx}(u)-u\frac{d}{dx}(v)}{v^2}$ [quotient rule]

Derivative of Composite Function by Chain Rule

Let f be a real valued function which is a composite of two functions say $y = f(u)$ and $u = g(x)$.

Then, $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=f'(u)\cdot g'(x)$

i.e. $\frac{d}{dx}[f\{g(x)\}]=f'[g(x)]\cdot g'(x)$

Derivative of a Function with respect to Another Function

Let $y = f(x)$ and $z = g(x)$, we firstly differentiate both functions $f(x)$ and $g(x)$ with respect to x separately and then put these values in the following formulae.

$\frac{dy}{dz}=\frac{dy/dx}{dz/dx}$ or $\frac{dz}{dy}=\frac{dz/dx}{dy/dx}$

Logarithmic Differentiation

Suppose, given function is of the form $[u(x)]^{v(x)}$ or $[u(x)]^m$ or $u(x) \cdot v(x)$. Then, we take logarithm and use properties of logarithm to simplify it and then differentiate it.

Basic Properties of Logarithms

$$\left.\begin{array}{ll}\textbf{(i)} & \log_a mn=\log_a m+\log_a n\\ \textbf{(ii)} & \log_a \frac{m}{n}=\log_a m-\log_a n\\ \textbf{(iii)} & \log_a m^n = n\log_a m\end{array}\right\}\text{For } m>0, n>0, a>0 \text{ and } a\neq 1$$

(iv) $\log_a a=1; a>0$ and $a\neq 1$

(v) $\log_b a=\frac{\log_m a}{\log_m b}; a>0, b>0, b\neq 1, m>0$ and $m\neq 1$

(vi) $a^{\log_a m}=m;\ a>0, m>0,\ a\neq 1$

Second Order Derivative

Let $y=f(x)$, then $\frac{dy}{dx}=f'(x)$ is called the first derivative of y or $f(x)$, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called the second order derivative of y w.r.t x and it is denoted by $\frac{d^2y}{dx^2}$ or y'' or y^2.

L - Hospital rule states that for functions f and g which are differentiable on an open interval I except possibly at a point c contained in I, it $\lim_{x\to c} f(x)=\lim_{x\to c} g(x)=0$ or $\neq \infty$; $g'(x) \neq 0$ for all x in I with $x \neq c$ and

$$\lim_{x\to c}\frac{f'(x)}{g'(x)} \text{ exists, then}$$

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(c)}{g'(c)}$$

The differentiation of numerator and denominator often simplifies the quotient or converts it to a limit that can be evaluate directly.

Rolle's Theorem

If a function $y = f(x)$ is defined in $[a, b]$ and

(i) $f(x)$ is continuous in $[a, b]$.

(ii) $f(x)$ is differentiable in (a, b) and

(iii) $f(a) = f(b)$

Then, there exist atleast one value $c \in (a, b)$ such that $f'(c) = 0$.

Lagrange's Mean Value Theorem

If a function $f(x)$ is defined on $[a, b]$ and

(i) Continuous in $[a, b]$ and

(ii) Differentiable in (a, b), then there will be atleast one value

$c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Chapter 6 – Application of Derivatives

Rate of Change of Quantities

If $y = f(x)$ is a function, where y is dependent variable and x is independent variable. Then, $\dfrac{dy}{dx}$ [or $f'(x)$] represents the rate of change of y w.r.t. x and $\left(\dfrac{dy}{dx}\right)_{x = x_0}$ [or $f'(x_0)$] represents the rate of change of y w.r.t. x at $x = x_0$.

(i) Average rate of change of y w.r.t. $x = \dfrac{\Delta y}{\Delta x}$.

(ii) Instantaneous rate of change of y w.r.t. $x = \dfrac{dy}{dx}$.

Note *Here, $\dfrac{dy}{dx}$ is positive, if y increases as x increases and $\dfrac{dy}{dx}$ is negative, if y decreases as x increases.*

Increasing and Decreasing Functions

(i) **Increasing function** Let I be an open interval contained in the domain of a real valued function f. Then, f is said to be

(a) Increasing on I, if $x_1 < x_2$

$\Rightarrow \quad f(x_1) \le f(x_2), \forall x_1, x_2 \in I$

(b) Strictly increasing on I, if $x_1 < x_2$

$\Rightarrow \quad f(x_1) < f(x_2), \forall x_1, x_2 \in I$

(ii) **Decreasing functions** Let I be an open interval contained in the domain of a real valued function f. Then, f is said to be

(a) Decreasing on I, if $x_1 < x_2$

$\Rightarrow \quad f(x_1) \ge f(x_2), \forall x_1, x_2 \in I$

(b) Strictly decreasing on I, if $x_1 < x_2$

$\Rightarrow \quad f(x_1) > f(x_2), \forall x_1, x_2 \in I$

Theorem Let f be continuous on $[a, b]$ and differentiable on (a, b). Then,

(a) If $f'(x) > 0$ for each $x \in (a, b,)$, then $f(x)$ is said to be increasing in $[a, b]$ and strictly increasing in (a, b).
(b) If $f'(x) < 0$ for each $x \in (a, b,)$, then $f(x)$ is said to be decreasing in $[a, b]$ and strictly decreasing in (a, b).
(c) If $f'(x) = 0$ for each $x \in (a, b,)$, then f is said to be a constant function in $[a, b]$.

Tangents and Normals

A **tangent** is a straight line, which touches the curve $y = f(x)$ at a point. $\frac{dy}{dx}$ represents the gradient or slope of a curve $y = f(x)$.
A **normal** is a straight line perpendicular to a tangent to the curve $y = f(x)$ intersecting at the point of contact.

Slope of tangent and normal

If a tangent line to the curve $y = f(x)$ makes an angle θ with X-axis in the positive direction, then $\frac{dy}{dx}$ = Slope of tangent $= \tan\theta$ and slope of normal $= \frac{-1}{\text{slope of tangent}} = \frac{-1}{dy/dx}$

Equation of Tangent and Normal

Let $y = f(x)$ be a curve and $P(x_1, y_1)$ be a point on it. Then,

(a) equation of tangent at $P(x_1, y_1)$ is,

$$(y - y_1) = \left[\frac{dy}{dx}\right]_{(x_1, y_1)} (x - x_1)$$

(b) equation of normal at $P(x_1, y_1)$ is,

$$(y - y_1) = \frac{-1}{\left[\frac{dy}{dx}\right]_{(x_1, y_1)}} (x - x_1)$$

Angle of Intersection of Two Curves

Let $y = f_1(x)$ and $y = f_2(x)$ be the two curves and ϕ be the angle between their tangents at the point of their intersection $P(x_1, y_1)$.

Then, $$\tan\phi = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

where, $$m_1 = \left[\frac{dy}{dx}\right]_{(x_1, y_1)} \text{ for } y = f_1(x)$$

and $$m_2 = \left[\frac{dy}{dx}\right]_{(x_1, y_1)} \text{ for } y = f_2(x)$$

(a) If $m_1 m_2 = -1$, then tangents are perpendicular to each other. In this case, we say that the curves intersect each other orthogonally.
This also happens, when $m_1 = 0$ and $m_2 = \infty$.
(b) If $m_1 = m_2$, then tangents are parallel to each other.
(c) If $\phi = 0$, then curves touch each other.

Approximation

Let $y = f(x)$ be a function such that $f : D \to R, D \subset R$. Here, x is an independent variable and y is the dependent variable. Let Δx be a small change in x and Δy be the corresponding change in y and given by $\Delta y = f(x + \Delta x) - f(x)$. Then,

(i) The differential of x, denoted by dx, is defined by $dx = \Delta x$

(ii) The differential of y, denoted by dy, is defined by $dy = f'(x)dx$ or $dy = \left(\frac{dy}{dx}\right)\Delta x$.

If $dx = \Delta x$ is relatively small, when compared with x, dy is a good approximation of Δy and we denote it by $dy \simeq \Delta y$.

Maxima and Minima

Let f be a real valued function and c be an interior point in the domain of f. Then,

(i) Point c is called a **local maxima**, if there is a $h > 0$ such that $f(c) \geq f(x) \forall x$ in $(c-h,\ c+h)$. Here, value $f(c)$ is called the local maximum value of f.

(ii) Point c is called a point of **local minima**, if there is a $h > 0$ such that $f(c) \leq f(x) \forall x$ in $(c-h,\ c+h)$. Here, value $f(c)$ is called the local minimum value of f.

Critical Point

A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable, is called a critical point of f.

First Derivative Test

Let f be a function defined on an open interval I and let f be continuous at a critical point c in I. Then,

(i) If $f'(x)$ change sign from positive to negative as x increases through c, then c is a point of local maxima.

(ii) If $f'(x)$ change sign from negative to positive as x increases through point c, then c is a point of local minima.

(iii) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called **point of inflection**.

Second Derivative Test

Let f be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c, then

(i) $x = c$ is a point of local maxima, if $f'(c) = 0$ and $f''(c) < 0$. The value $f(c)$ is local maximum value of f.

(ii) $x = c$ is a point of local minima, if $f'(c) = 0$ and $f''(c) > 0$. Then, $f(c)$ is local minimum value of f.

(iii) If $f'(c) = 0$ and $f''(c) = 0$, then the test fails.

Absolute Maxima and Absolute Minima

Let f be a differentiable function on $[a, b]$ and c be a point in $[a, b]$ such that $f'(c) = 0$. Then, find $f(a)$, $f(b)$ and $f(c)$. The maximum of these values gives a maxima or absolute maxima and minimum of these values gives a minima or absolute minima.

Some Useful Results

(i) Area of a square $= x^2$ and Perimeter $= 4x$ where, x is the side of the square.

(ii) Area of a rectangle $= xy$ and Perimeter $= 2(x + y)$ where, x and y are the length and breadth of rectangle.

(iii) Area of a trapezium $= \frac{1}{2}$ (Sum of parallel sides) $\times$ (Perpendicular distance between them)

(iv) Area of a circle $= \pi r^2$ and Circumference of a circle $= 2\pi r$, where r is the radius of circle.

(v) Volume of sphere $=\frac{4}{3}\pi r^3$ and Surface area = $4\pi r^2$ where, r is the radius of sphere.

(vi) Total surface area of a right circular cylinder = $2\pi rh + 2\pi r^2$
Curved surface area of right circular cylinder = $2\pi rh$
and Volume = $\pi r^2 h$
where, r is the radius and h is the height of the cylinder.

(vii) Volume of a right circular cone $=\frac{1}{3}\pi r^2 h$, Curved surface area = πrl and Total surface area = $\pi r^2 + \pi rl$ where, r is the radius, h is the height and l is the slant height of the cone.

(viii) Volume of a parallelepiped = xyz and Surface area = $2(xy + yz + zx)$ where, x, y and z are the dimensions of parallelepiped.

(ix) Volume of a cube = x^3 and Surface area = $6x^2$ where, x is the side of the cube.

(x) Area of an equilateral triangle $=\frac{\sqrt{3}}{4}(\text{Side})^2$.

Chapter 7 – Vectors

Vector

A quantity that has magnitude as well as direction, is called a vector.

Scalar

A quantity that has magnitude only, is called scalar.

Magnitude of a Vector

The length of the vector $\vec{AB}$ or $\vec{a}$ is called the magnitude of $\vec{AB}$ or $\vec{a}$ and it is represented by $|\vec{AB}|$ or $|\vec{a}|$ or a.

If $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$, then $|\vec{a}| = \sqrt{a^2 + b^2 + c^2}$.

Note *Since, the length is never negative, so the notation* $|\vec{a}| < 0$ *is meaningless.*

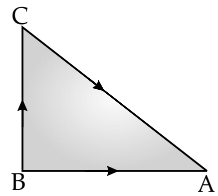
Types of Vector

1. **Zero or Null Vector:** A vector whose magnitude is zero *i.e.* whose initial and final points coincide, is called a null vector or zero vector.
2. **Unit Vector:** A vector whose magnitude is one unit. The unit vector in the direction of $\vec{a}$ is represented by $\hat{a}$. The unit vectors along X-axis, Y-axis and Z-axis are represented by $\hat{i}$, $\hat{j}$ and $\hat{k}$, respectively.
3. **Coinitial Vectors:** Two or more vectors having the same initial point are called co-initial vectors.
4. **Collinear Vectors:** The vectors which have same or parallel support are called collinear vectors.
5. **Like Vectors:** The vectors which have same direction are called like vectors.
6. **Unlike Vectors:** The vectors which have opposite directions are called unlike vectors.
7. **Equal Vectors:** Two vectors are equal, if they have same magnitude and direction.
8. **Negative of a Vector:** A vector whose magnitude is same as that of given vector but the direction is opposite is called negative vector of the given vector. e.g. Let $\overrightarrow{AB}$ be a vector, then $-\overrightarrow{AB}$ or $\overrightarrow{BA}$ is a negative vector.
9. **Coplanar Vectors:** A system of vectors is said to be coplanar, if their supports are parallel to same plane.

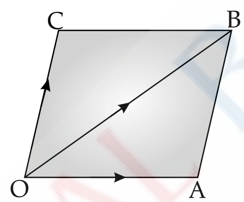
Addition of Vectors

(i) **Triangle Law of Vector Addition:** If two vectors are represented along two sides of a triangle taken in order, then their resultant is represented by the third side taken in opposite direction *i.e.* in ΔABC, by triangle law of vector addition, we have

$$\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$$



(ii) **Parallelogram Law of Vector Addition:** If two vectors are represented along the two adjacent sides of a parallelogram, then their resultant is represented by the diagonal of the parallelogram, which is co-initial with the given vector. If the sides OA and OC of parallelogram $OABC$ represents $\overrightarrow{OA}$ and $\overrightarrow{OC}$, respectively, then we get $\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB}$ or $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ $[\because \overrightarrow{AB} = \overrightarrow{OC}]$



Properties of Vector Addition

(i) For any two vectors $\vec{a}$ and $\vec{b}$,

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$ [commutative law]

(ii) For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$,

$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$ [associative law]

(iii) For any vector $\vec{a}$, we have $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$.

The zero vector $\vec{0}$ is called the additive identity for the vector addition.

(iv) For any vector $\vec{a}$, $\vec{a} + (-\vec{a}) = 0$

The vector $-\vec{a}$ is additive inverse of $\vec{a}$.

Difference of Vectors

If $\vec{a}$ and $\vec{b}$ are any two vectors, then their difference $\vec{a} - \vec{b}$ is defined as $\vec{a} + (-\vec{b})$.

Multiplication of a Vector by Scalar

Let $\vec{a}$ be a given vector and λ be a scalar. Then, the product of the vector $\vec{a}$ by the scalar λ, denoted by $\lambda \vec{a}$ is called the multiplication of vector $\vec{a}$ by the scalar λ.

Let $\vec{a}$ and $\vec{b}$ be any two vectors and k and m be any scalars.
Then,

(i) $k\vec{a}+m\vec{a}=(k+m)\vec{a}$

(ii) $k(m\vec{a})=(km)\vec{a}$

(iii) $k(\vec{a}+\vec{b})=k\vec{a}+k\vec{b}$

Section Formulae

Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$, respectively and P be a point which divides AB internally in the ratio m:n. Then, position vector of $\vec{P}$

$$=\frac{m\vec{b}+n\vec{a}}{m+n}.$$

If P divides AB externally in the ratio $m : n$. Then, position vector of $\vec{P}=\dfrac{m\vec{b}-n\vec{a}}{m-n}$.

If R is the mid-point of AB, then $\overrightarrow{OR}=\dfrac{\vec{a}+\vec{b}}{2}$.

Components of a Vector

If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, we say that the scalar components of $\vec{a}$ along X-axis, Y-axis and Z-axis are a_1, a_2 and a_3 respectively.

Important Results in Component Form

If $\vec{a}$ and $\vec{b}$ are any two vectors given in the component form such that $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$. Then, (a_1, a_2, a_3) and (b_1, b_2, b_3) are called direction ratios of $\vec{a}$ and $\vec{b}$, respectively.

(i) The sum (or resultant) of the vectors $\vec{a}$ and $\vec{b}$ is given by

$\vec{a}+\vec{b}=(a_1+b_1)\hat{i}+(a_2+b_2)\hat{j}+(a_3+b_3)\hat{k}$

(ii) The difference of the vectors $\vec{a}$ and $\vec{b}$ is given by

$\vec{a}-\vec{b}=(a_1-b_1)\hat{i}+(a_2-b_2)\hat{j}+(a_3-b_3)\hat{k}$

(iii) The vectors $\vec{a}$ and $\vec{b}$ are equal, iff

$$a_1=b_1,\ a_2=b_2$$

and $$a_3=b_3$$

(iv) The multiplication of vector $\vec{a}$ by any scalar λ is given by

$\lambda\vec{a}=(\lambda a_1)\hat{i}+(\lambda a_2)\hat{j}+(\lambda a_3)\hat{k}$

(v) If $\dfrac{b_1}{a_1}=\dfrac{b_2}{a_2}=\dfrac{b_3}{a_3}=k$ (constant)

Then, vectors $\vec{a}$ and $\vec{b}$ will be collinear.

(vi) If it is given that l, m and n are direction cosines of a vector, then $l\hat{i}+m\hat{j}+n\hat{k}=(\cos\alpha)\hat{i}+(\cos\beta)\hat{j}+(\cos\gamma)\hat{k}$ is the unit vector in the direction of that vector, where α, β and γ are the angles which the vector makes with X, Y and Z-axes, respectively.

Vector Joining Two Points

If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points, then vector joining P_1 and P_2 is

$$\begin{aligned}\overrightarrow{P_1P_2} &= \overrightarrow{OP_2}-\overrightarrow{OP_1}\\ &=\left(x_2\hat{i}+y_2\hat{j}+z_2\hat{k}\right)-\left(x_1\hat{i}+y_1\hat{j}+z_1\hat{k}\right)\\ &=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}\end{aligned}$$

Dot Product or Scalar Product

Let $\vec{a}$ and $\vec{b}$ be the two non-zero vectors inclined at an angle θ. Then, the scalar product or dot product of $\vec{a}$ and $\vec{b}$ is represented by $\vec{a}\cdot\vec{b}$, then $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$.

Important Results

(i) $\vec{a}\perp\vec{b}\Leftrightarrow\vec{a}\cdot\vec{b}=0$

(ii) $\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1$ and $\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=\vec{0}$

(iii) If $\theta=0$, then $\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|$

If $\theta=\pi$, then $\vec{a}\cdot\vec{b}=-|\vec{a}||\vec{b}|$

(iv) The angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \text{ or } \theta=\cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

(v) **Properties of scalar product**

(a) $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ [commutative]

(b) $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$

(c) $\vec{a}\cdot\vec{a}=|\vec{a}|^2=a^2$

where, a represents magnitude of vector $\vec{a}$.

(d) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=a^2-b^2$, where a and b represent the magnitude of vectors $\vec{a}$ and $\vec{b}$.

(e) $(\lambda\vec{a})\cdot\vec{b}=\lambda(\vec{a}\cdot\vec{b})$

(vi) Projection of $\vec{a}$ on $\vec{b}=\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)$

Note *If α, β and γ are the direction angles of vector*

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then its DC's is given as, $\cos\alpha = \frac{a_1}{|\vec{a}|}$, $\cos\beta = \frac{a_2}{|\vec{a}|}$, $\cos\gamma = \frac{a_3}{|\vec{a}|}$

Cross Product or Vector Product

Let θ be the angle between two non-zero vector $\vec{a}$ and $\vec{b}$, then the vector or cross product of $\vec{a}$ and $\vec{b}$ is represented by $\vec{a} \times \vec{b}$ and defined as $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$, where, $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

Important Results

(i) $\vec{a} \,||\, \vec{b} \Leftrightarrow \vec{a} \times \vec{b} = \vec{0}$

(ii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(iii) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

(iv) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

(v) $\vec{a} \times \vec{a} = \vec{0}$

(vi) $\hat{i} \times \hat{i} = \vec{0}$, $\hat{j} \times \hat{j} = \vec{0}$, $\hat{k} \times \hat{k} = \vec{0}$

(vii) $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$; $\hat{j} \times \hat{i} = -\hat{k}$, $\hat{k} \times \hat{j} = -\hat{i}$, $\hat{i} \times \hat{k} = -\hat{j}$

(viii) A unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is given by $\hat{n} = \frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$.

(ix) Area of a parallelogram with sides $\vec{a}$ and $\vec{b} = |\vec{a} \times \vec{b}|$

(x) Area of a parallelogram with diagonals $\vec{d_1}$ and $\vec{d_2} = \frac{1}{2}|\vec{d_1} \times \vec{d_2}|$

(xi) Area of a $\Delta ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}|\overrightarrow{BC} \times \overrightarrow{BA}| = \frac{1}{2}|\overrightarrow{CB} \times \overrightarrow{CA}|$

Chapter 8 – Three-Dimensional Geometry

Direction Cosines and Direction Ratios

If a line makes angles α, β and γ with X-axis, Y-axis and Z-axis, respectively, then $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$ are called the **direction cosines** of the line.

The relation between direction cosines of a line is, $l^2 + m^2 + n^2 = 1$.

Numbers proportional to the direction cosines of a line are called the **direction ratios** of the line.

If a, b and c are the direction ratios of a line, then

$$l = \frac{\pm a}{\sqrt{a^2+b^2+c^2}}, \; m = \frac{\pm b}{\sqrt{a^2+b^2+c^2}}$$

and $n = \dfrac{\pm c}{\sqrt{a^2+b^2+c^2}}$

For any line, direction cosines are unique but direction ratios are not unique.

Direction Cosines and Direction Ratios of a Line

The direction cosines and direction ratios of the line segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are respectively given by

$$\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_1}{PQ} \text{ and } (x_2-x_1, y_2-y_1, z_2-z_1)$$

where, $PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

Line

A line (or straight line) is a curve such that all points on the line segment joining two points of it lies on it.

Equation of a Line

(i) Equation of a line through a given point and parallel to a given vector.

In vector form: Equation of a line through a given point A with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is given by $\vec{r} = \vec{a} + t\vec{b}$, where t is a scalar.

In cartesian form:

The equation of a line passing through a point $A(x_1, y_1, z_1)$ and having direction ratios a, b and c is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

If l, m and n are the direction cosines of the line, the equation of the line is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

(ii) Equation of a line passing through two given points

In vector form: The vector equation of a line passing through two points with position vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$, where t is a scalar.

In cartesian form: The equation of a line passing through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Angle between two lines

In vector form: If θ is the angle between two lines $\vec{r} = \vec{a} + t\vec{b}$ and $\vec{r} = \vec{c} + s\vec{d}$, then $\cos\theta = \dfrac{\vec{b}\cdot\vec{d}}{|\vec{b}||\vec{d}|}$.

If two lines are perpendicular, then $\vec{b}\cdot\vec{d} = 0$ and if two lines are parallel, then $\vec{b} = \lambda\vec{d}$.

In cartesian form: If a_1, b_1, c_1 and a_2, b_2, c_2 are direction ratios of two lines respectively, then the angle θ between the lines is given by $\cos\theta = \left|\dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$

If θ is the angle between two lines with direction cosines l_1, m_1, n_1 and l_1, m_2, n_2, then $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$.

If lines are perpendicular, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$,

If lines are parallel, then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$,

Skew-lines

If two lines do not meet and not parallel, then they are known as skew-lines.

Shortest distance between two lines

The shortest distance between two skew-lines is the length of perpendicular to both the lines.

In vector form: Shortest distance between the skew-lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ is given by

$$SD = \frac{|(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$$

The shortest distance between the parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$ is given by

$$SD = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$$

In cartesian form: The shortest distance between the lines

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

$$\text{and } \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2} \text{ is}$$

$$\left|\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}\right|$$

Plane

A plane is a surface such that a line segment joining any two points on it lies wholly on it. A straight line which perpendicular to every line lying on a plane is called a normal to the plane.

Equation of Plane

(i) Equation of a plane in Normal form

In vector form: Let O be the origin and $\hat{n}$ be a unit vector in the direction of the normal ON to the plane and let $ON = p$. Then, the equation of the plane is $\vec{r} \cdot \hat{n} = p$.

In cartesian form: The general equation of a plane is $ax + by + cz + d = 0$.
The direction ratios of the normal to this plane are a, b and c.

(ii) Intercept form of the equation of a plane: If a plane cuts (intercepts) a, b and c with the coordinate axes, then the equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

(iii) Equation of a plane perpendicular to a given vector and passing through a given point

In vector form: The equation of a plane passing through the point $\vec{a}$ and perpendicular to the given vector $\vec{n}$ is $(\vec{r}-\vec{a})\cdot\vec{n} = \vec{0}$

In cartesian form: The equation of a plane passing through a point (x_1, y_1, z_1) is
$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$
where, a, b and c are direction ratios of perpendicular vector.

(iv) Equation of a plane passing through three non-collinear points

In vector form: The equation of plane passing through three non-collinear points $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $(\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})]=0$.

In cartesian form: The equation of a plane passing through three non-collinear points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

(v) Equation of a plane passing through the intersection of two planes

In vector form: The equation of plane passing through the intersection of two planes $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ is $\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2) = d_1+\lambda d_2$.

In cartesian form: The equation of a plane through the intersection of the planes

$$a_1x+b_1y+c_1z+d_1=0$$

and $\quad a_2x+b_2y+c_2z+d_2=0$ is

$$(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0$$

(vi) Equation of a plane parallel to a given plane

In vector form: The vector equation of a plane parallel to the given plane $\vec{r}\cdot\vec{n}=d_1$ is $\vec{r}\cdot\vec{n}=d_2$.

In cartesian form: The cartesian equation of a plane parallel to the given plane $ax + by + cz + d_1 = 0$ is $ax + by + cz + d_2 = 0$.

Angle between two planes

In vector form: The angle between two planes is the angle between their normals.

Angle between the planes $\vec{r}\cdot\vec{n}_1 = p_1$ and $\vec{r}\cdot\vec{n}_2 = p_2$ is given by $\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$.

In cartesian form: If θ is the angle between the planes

$a_1x + b_1y + c_1z + d_1 = 0$

and $a_2x + b_2y + c_2z + d_2 = 0$, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

If two planes are perpendicular, then $\vec{n}_1 \cdot \vec{n}_2 = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and if they are parallel, then $\vec{n}_1 \times \vec{n}_2 = 0$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Coplanarity of Two Lines

In cartesian form: Suppose, vector equations of two lines are $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$. Then, these lines are coplanar, iff $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 0$.

Two lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$

and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$

are coplanar, if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

The equation of plane containing the above lines is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

In vector form: Two lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$ are coplanar, if $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 0$.

Distance of a point from a plane

In cartesian form: Distance of a point $P(x_1, y_1, z_1)$ from a plane $ax + by + cz + d = 0$ is given by

$$p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

In vector form: The perpendicular distance of a point $\vec{a}$ from the plane $\vec{r} \cdot \vec{n} = d$, where $\vec{n}$ is normal to the plane, is

$$\frac{|\vec{a} \cdot \vec{n} - d|}{|\vec{n}|}$$

The length of the perpendicular from origin O to the plane $\vec{r} \cdot \vec{n} = d$ is $\frac{|d|}{|\vec{n}|}$.

Angle between a line and a plane

The angle θ between line $\vec{r} = \vec{a} + \lambda \vec{b}$ and plane $\vec{r} \cdot \vec{n} = d$

is $\cos\theta = \left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right|$ and $\sin\phi = \left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\cdot|\vec{n}|}\right|$, where $\phi = 90° - \theta$.

Chapter 9 – Application of Calculus

- **Cost function :**

  If x denotes the quantity produced of a certain commodity at total cost c, then the total cost function is given as $C = c(x)$, explicitly and $f(c, x) = 0$ is the implicit form. The total cost is divided into two parts, the fixed cost and the variable cost.

  (1) Fixed Cost :

  Fixed costs are those which incurred regardless of the level of production - like interest, rent, wages of permanent staff etc. Thus, Total Fixed Cost
  $(TFC) = TC$, when $x = 0$
  Fixed cost does not change whether there is any increase or decrease in level of production.

  (2) Variable Cost :

  Variable costs are those which vary with output.
  e.g., Raw materials and wages of casual labour etc. Thus,

  Total Cost = Total Fixed Cost + Total Variable Cost

  or $TC = TFC + TVC$

  (3) Average Cost :

  The average cost represents the cost per unit, *i.e.,* $AC = \frac{C}{x} = \frac{\text{Total cost}}{\text{number of commodities}}$

  (4) Demand Function :

  The demand function is the functional relationship between demand and price of a commodity. If P denotes the price per unit and x is the number of units demanded by a consumer at that price, then demand function in explicit form will be written as $x = f(P)$ and in implicit form as $f(x, P) = 0$.

  (5) Revenue Function :

  Revenue is the amount received by a company on selling a certain number of units of a commodity. Let P be the price per unit and x be the number of units sold. Then, total revenue

  $$\boxed{R \text{ or } R(x) = P \times x}$$

  Total revenue = Selling price per unit of the commodity × Quantity sold

  (6) Profit Function :

  Profit function is the difference of revenue function and cost function, *i.e.,*

  $$\boxed{P(x) = R(x) - C(x)}$$

  (7) Break even Point :

  The break even point is that point, where total revenue equals to the total cost incurred, *i.e.,* $P(x) = 0$ or $R(x) = C(x)$. At break even point, a company begins to earn profit.

  (8) Marginal Cost :

  The marginal cost is the rate of change of the total cost with respect to x (Output).

  $$\boxed{MC = \frac{d}{dx}(C) = \frac{dC}{dx},\ x > 0}$$

  (9) Relation between Average Cost (AC) and Marginal Cost (MC) :

  If C is the total cost of producing and marketing x units of commodity, then

  $$\boxed{\frac{d}{dx}(AC) = \frac{1}{x}(MC - AC)}$$

Here, three cases arise:

- For $MC > AC \Rightarrow AC$ increases with x and AC curve is rising.
- For $MC = AC \Rightarrow AC$ is constant at all levels of output.
- For $MC < AC \Rightarrow AC$ decreases with x and AC curve is falling.

(10) Average Revenue and Marginal Revenue :

(i) Average Revenue is the revenue generated per unit of output sold. It is denoted by AR, *i.e.*,

$$AR = \frac{\text{Total revenue}}{\text{Number of units sold}} = \frac{R}{x} = \frac{P \cdot x}{x} = P$$

(ii) Marginal Revenue is the rate of change of total revenue with respect to quantity sold

i.e., $$\boxed{MR = \frac{dR}{dx} = \frac{d}{dx}(Px)}$$

(11) Marginal Average Cost is called marginal average cost $\frac{d}{dx}(AC)$ may be denoted as (MAC). This is also called as slope of average cost curve.

➢ **Minimisation of *AC* and *MC* Functions :**

Using the concept of maxima and minima we can determine the level of output where per unit cost is minimum corresponding to a given total or average cost function. Following steps are used to find the optimal level of output:

Step 1:	From AC, determine $\frac{d}{dx}(AC)$.
Step 2:	Let $\frac{d}{dx}(AC) = 0$ and solve for x.
Step 3:	Find $\frac{d^2}{dx^2}(AC)$.
Step 4:	The value of x for which $\frac{d^2}{dx^2}(AC) > 0$ is the minimum per unit cost.
Step 5:	Minimum average cost can be determined by substituting this value of x in $AC(x)$. Similarly, we can find the minimum marginal cost or minimum cost.

➢ **Maximisation of Total Revenue :**

It is possible to determine the level of output at which the total revenue is maximum, if demand function is given. Total revenue, $R(x)$, is the maximum when marginal revenue is zero.

To maximise total revenue, following steps are to be followed:

Step 1:	Determine $\frac{dR}{dx}$.
Step 2:	Let $\frac{dR}{dx} = 0$ and solve for x.
Step 3:	Determine $\frac{d^2R}{dx^2}$.
Step 4:	The value for which $\frac{d^2R}{dx^2} < 0$ gives the maximum total revenue.
Step 5:	The maximum revenue can be obtained by putting this value of x in $R(x)$. Similarly, maximum profit can be determined.

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