UNIT III: CALCULUS

CHAPTER-1
INTEGRALS

Revision Notes

Indefinite Integral

➢ **Meaning of Integral of Function**

If differentiation of a function $F(x)$ is $f(x)$ *i.e.*, if $\frac{d}{dx}[F(x)] = f(x)$, then we say that one integral or primitive or anti-derivative of $f(x)$ is $F(x)$ and in symbols, we write, $\int f(x)dx = F(x) + C$.

Therefore, we can say that integration is the inverse process of differentiation.

➢ Methods of Integration

(a) Integration by Substitution Method:

In this method, we change the integral $\int f(x)dx$, where independent variable is x, to another integral in which independent variable is t (say) different from x such that x and t are related by $x = g(t)$.

Let $$u = \int f(x)dx \text{ then, } \frac{du}{dx} = f(x)$$

Again as $$x = g(t) \text{ so we have } \frac{dx}{dt} = g'(t)$$

Now $$\frac{du}{dt} = \frac{du}{dx}\cdot\frac{dx}{dt} = f(x)\cdot g'(t)$$

On integrating both sides w.r.t. t, we get

$$\int\left(\frac{du}{dt}\right)dt = \int f(x)g'(t)dt$$

or $$u = \int f[g(t)]g'(t)dt$$

i.e., $$\int f(x)dx = \int f[g(t)]g'(t)dt \text{, where } x = g(t).$$

So, it is clear that substituting $x = g(t)$ in $\int f(x)$ will give us the same result as obtained by putting $g(t)$ in place of x and $g'(t)dt$ in place of dx.

(b) Integration by Partial Fractions:

Consider $\frac{f(x)}{g(x)}$ defines a rational polynomial function.

- If the degree of numerator *i.e.*, $f(x)$ is **greater than or equal to** the degree of denominator *i.e.*, $g(x)$ then, this type of rational function is called an **improper rational function**. And if degree of $f(x)$ is **smaller than** the degree of denominator *i.e.*, $g(x)$, then this type of rational function is called a **proper rational function**.
- In rational polynomial function if the degree (*i.e.*, highest power of the variable) of numerator (Nr.) is **greater than or equal to** the degree of denominator (Dr.), then (without any doubt) **always perform the division** *i.e.*, divide the **Nr.** by **Dr.** before doing anything and thereafter use the following:

$$\frac{\text{Numerator}}{\text{Denominator}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Denominator}}$$

Table Demonstrating Partial Fractions or Various Forms

Form of the Rational Function	Form of the Partial Fraction
$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ **where $x^2 + bx + c$ can't be factorized further.**	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

(c) Integration by Parts:

If U and V be two functions of x, then

$$\int \underset{(I)}{U} . \underset{(II)}{V} \, dx = U\int V dx - \int\left\{\frac{dU}{dx}\int V dx\right\}dx$$

Definite Integral

➢ **Meaning of Definite Integral of Function**

If $\int f(x)dx = F(x)$ *i.e.,* $F(x)$, be an integral of $f(x)$, then $F(b) - F(a)$ is called the definite integral of $f(x)$ between the limits a and b and in symbols it is written as $\int_a^b f(x)dx = [F(x)]_a^b$. Moreover, the definite integral gives a unique and definite value (numeric value) of anti-derivative of the function between the given intervals. It acts as a substitute for evaluating the area analytically.

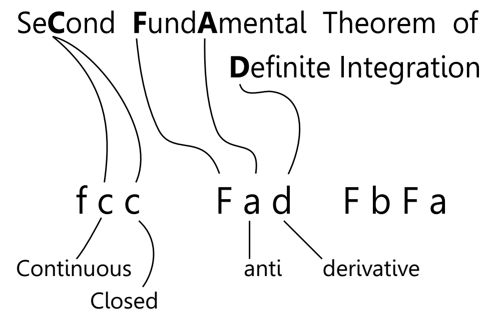
➢ **Definite Integral as The Limit of a Sum (First Principle of Integrals):**

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+...+f(a+(n-1)h)\right]$$

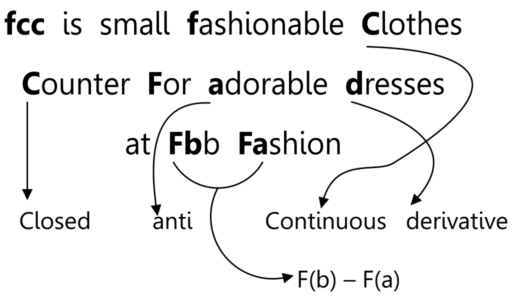
or, $$= (b-a)\lim_{h\to 0}\frac{1}{h}\left[f(a)+f(a+h)+........+f(a+(n-1)h)\right]$$

where, $$h = \frac{b-a}{n} \to 0 \text{ as } n \to \infty$$

Mnemonic



You can also remember



Interpretation:

Let f be a continuous function defined on a closed interval [a,b] and F be an anti derivative of f. Then $\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$, where a and b are called limit of Integration.

Know the Formulae

Formulae for Indefinite Integrals

(a) $\int x^n dx = \frac{x^{n+1}}{n+1} + k,\ n \neq -1$

(b) $\int \frac{1}{x} dx = \log|x| + k$

(c) $\int a^x dx = \frac{1}{\log a} a^x + k$

(d) $\int e^{ax} dx = \frac{1}{a} e^{ax} + k$

(e) $\int \sin(ax) dx = -\frac{1}{a}\cos(ax) + k$

(f) $\int \cos(ax) dx = \frac{1}{a}\sin(ax) + k$

(g) $\int \tan x\, dx = \log|\sec x| + k$ **or** $-\log|\cos x| + k$

(h) $\int \cot x\, dx = \log|\sin x| + k$ **or** $-\log|\text{cosec}\, x| + k$

(i) $\int \sec x\, dx = \log|\sec x + \tan x| + k$ **or** $\log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + k$

(j) $\int \text{cosec}\, x\, dx = \log|\text{cosec}\, x - \cot x| + k$ **or** $\log\left|\tan\frac{x}{2}\right| + k$

(k) $\int \sec^2 x\, dx = \tan x + k$

(l) $\int \text{cosec}^2 x\, dx = -\cot x + k$

(m) $\int \sec x . \tan x\, dx = \sec x + k$

(n) $\int \text{cosec}\, x . \cot x\, dx = -\text{cosec}\, x + k$

(o) $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + k$

(p) $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + k$

(q) $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + k$

(r) $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + k$

(s) $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left|x + \sqrt{x^2-a^2}\right| + k$

(t) $\int \frac{1}{\sqrt{x^2+a^2}} dx = \log\left|x + \sqrt{x^2+a^2}\right| + k$

(u) $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + k$

(v) $\int \frac{1}{ax+b} dx = \frac{1}{a}\log|ax+b| + k$

(w) $\int \lambda dx = \lambda x + k$, where '$\lambda$' is a constant.

(x) $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2-a^2}\right| + k$

(y) $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2+a^2}\right| + k$

(z) $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + k$

Formulae for definite Integrals

(a) $\int_a^b f(x)dx = F(b) - F(a)$

(b) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(c) $\int_a^b f(x)dx = \int_a^b f(t)dt \ (dx = dt)$

(d) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, a < c < b$

(e) $\int_o^a f(x)dx = \int_o^a f(a-x)dx$

(f) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(g) $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_o^a f(x)dx, \text{ if } f(x) \text{ is an even function } i.e., f(-x) = f(x) \\ 0, \text{ if } f(x) \text{ is an odd function } i.e., f(-x) = -f(x) \end{cases}$

(h) $\int_{-a}^{a} f(x)dx = \int_o^a \{f(x) + f(-x)\}dx$

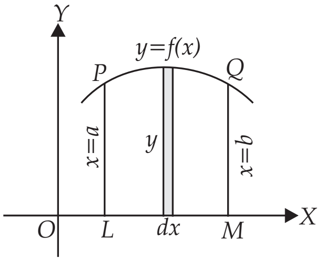
(i) $\int_o^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x) \\ 0 \quad ; \text{ if } f(2a-x) = -f(x) \end{cases}$

(j) $\int_0^{2a} f(x)dx = \int_o^{2a} \{f(x) + f(2a-x)\}dx$

❑❑

CHAPTER-2
APPLICATIONS OF THE INTEGRALS

Revision Notes



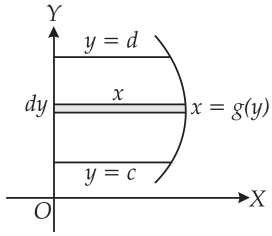
➢ **Area Under Simple Curves:**

(i) Let us find the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$. Consider the area under the curve as composed by large number of thin vertical strips.

Let there be an arbitrary strip of height y and width dx.

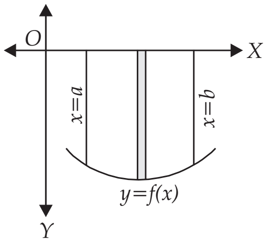
Area of elementary strip $dA = y\,dx$, where $y = f(x)$. Total area A of the region between x-axis ordinates $x = a$, $x = b$ and the curve $y = f(x)$ = sum of areas of elementary thin strips across the region $PQML$.

$$A = \int_a^b y\,dx = \int_a^b f(x)\,dx$$



(ii) The area A of the region bounded by the curve $x = g(y)$, y-axis and the lines $y = c$ and $y = d$ is given by

$$A = \int_c^d x\,dy = \int_c^d g(y)dy$$

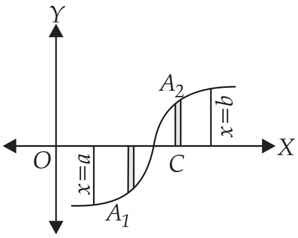


(iii) If the curve under consideration lies below x-axis, then $f(x) < 0$ from $x = a$ to $x = b$, the area bounded by the curve $y = f(x)$ and the ordinates $x = a$, $x = b$ and y-axis is negative. But, if the numerical value of the area is to be taken into consideration, then

$$\text{Area} = \left|\int_a^b f(x)dx\right|$$

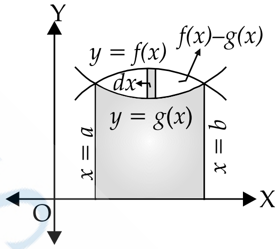
(iv) It may also happen that some portion of the curve is above x-axis and some portion is below x-axis as shown in the figure. Let A_1 be the area below x-axis and A_2 be the area above the x-axis. Therefore, area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$ is given by

$$A = |A_1| + |A_2|$$



➢ **Area Between Two Curves:**

(i) Let the two curves be $y = f(x)$ and $y = g(x)$, as shown in the figure. Suppose these curves intersect at $x = a$ and $x = b$ with width dx.



Consider the elementary strip of height y,

where $\qquad y = f(x)$

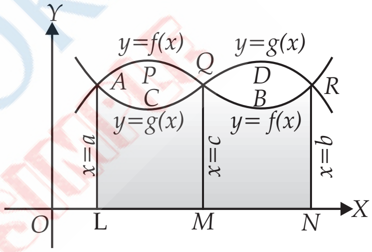
$\therefore \qquad dA = y\,dx$

$$\Rightarrow \qquad A = \int_a^b [f(x) - g(x)]dx$$

$$= \int_a^b f(x)dx - \int_a^b g(x)dx$$

= Area bounded by the curve $\{y = f(x)\}$ – Area bounded by the curve $\{y = g(x)\}$, where $f(x) > g(x)$.

(ii) If the two curves $y = f(x)$ and $y = g(x)$ intersect at $x = a$, $x = b$ and $x = c$, such that $a < c < b$.



If $f(x) > g(x)$ in $[a, c]$ and $g(x) \leq f(x)$ in $[c, b]$, then the area of the regions bounded by the curve = Area of region *PACQP* + Area of region *QDRBQ*.

$$= \int_a^c |f(x) - g(x)|\,dx + \int_c^b |g(x) - f(x)|\,dx.$$

❑❑

CHAPTER-3

DIFFERENTIAL EQUATIONS

Revision Notes

Basic Differential Equations

➢ **Orders and Degrees of Differential Equation:**

- We shall prefer to use the following notations for derivatives.
- $\frac{dy}{dx} = y', \frac{d^2y}{dx^2} = y'', \frac{d^3y}{dx^3} = y'''$
- For derivatives of higher order, it will be in convenient to use so many dashes as super suffix therefore, we use the notation y_n for nth order derivative $\frac{d^ny}{dx^n}$.
- Order and degree (if defined) of a differential equation are always positive integers.

Variable Separable Methods

➢ **Solutions of differential equations:**

(a) **General Solution:** The solution which contains as many as arbitrary constants as the order of the differential equations, *e.g.* $y = \alpha \cos x + \beta \sin x$ is the general solution of $\frac{d^2y}{dx^2} + y = 0$.

(b) **Particular Solution:** Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution *e.g.* $y = 3\cos x + 2\sin x$ is a particular solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

(c) **Solution of Differential Equation by Variable Separable Method:** A variable separable form of the differential equation is the one which can be expressed in the form of $f(x)dx = g(y)dy$. The solution is given by $\int f(x)dx = \int g(y)dy + k$, where k is the constant of integration.

Linear Differential Equations

➢ **Solutions of Differential Equations:**

➲ **Linear differential equation in *y*:** It is of the form $\frac{dy}{dx} + P(x)y = Q(x)$, where $P(x)$ and $Q(x)$ are either constant or function of x only.

➢ **Solving Linear Differential Equation in *y*:**

STEP 1: Write the given differential equation in the form $\frac{dy}{dx} + P(x)y = Q(x)$.

STEP 2: Find the Integration Factor $(I.F.) = e^{\int P(x)dx}$.

STEP 3: The solution is given by, $y.(I.F.) = \int Q(x).(I.F.)dx + k$, where k is the constant of integration.

➲ **Linear differential equation in *x*:** It is of the form $\frac{dx}{dy} + P(y)x = Q(y)$, where $P(y)$ and $Q(y)$ are either constant or function of y only.

Homogeneous Differential Equations

➢ **Homogeneous Differential Equations and their solution:**

➲ **Identifying a Homogeneous Differential equation:**

STEP 1: Write down the given differential equation in the form $\frac{dy}{dx} = f(x, y)$.

STEP 2: If $f(kx, ky) = k^n f(x, y)$, then the given differential equation is homogeneous of degree 'n'.

➲ **Solving a Homogeneous Differential Equation :**

CASE I: If $\frac{dy}{dx} = f(x, y)$

Put $y = vx$

or $\frac{dy}{dx} = v + x\frac{dv}{dx}$

CASE II: If $\frac{dx}{dy} = f(x, y)$

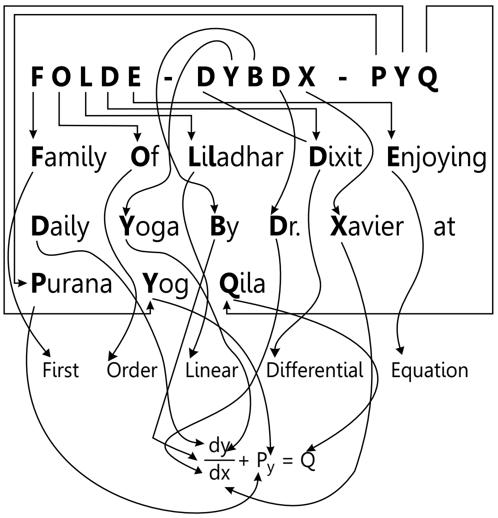
Put $x = vy$

or $\frac{dx}{dy} = v + y\frac{dv}{dy}$

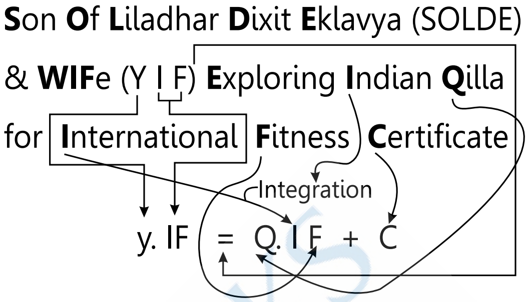
Then, we separate the variables to get the required solution.

Mnemonic

Linear Differential Equations



SOLDE-YIF-EIQ-IFC



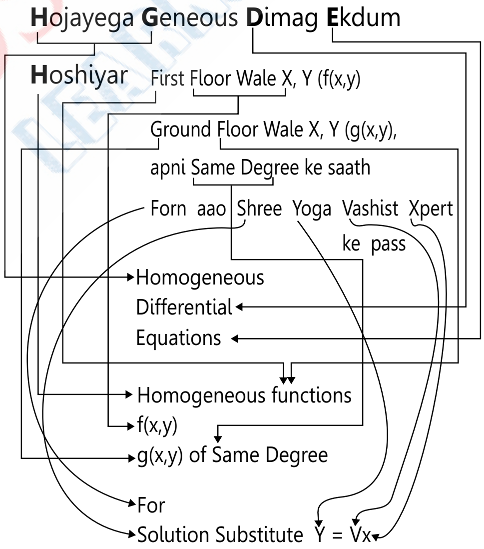
S — Solution
O — Of
L — Linear
D — Differential
E — Equation

Interpretation:

Differential equation is of the form $\frac{dy}{dx} + py = Q$, where P and Q are constants or the function of 'x' is called a first order linear differential equations. Its solution is given as

$$Y.IF = \int Q.IF + C$$

Homogeneous Differential Equation



Interpretation:

Differential equation can be expressed in the form $\frac{dy}{dx} = f(x,y)$ or $\frac{dx}{dy} = g(x,y)$ where $f(x, y)$ and $g(x, y)$ are homogeneous functions of sum is called a homogeneous Differential equation. These equations can be solved by substituting $y = vx$ so that dependent variable y is changed to another variable v, where v is some unknown function.

Know the Terms

(a) Order of a differential equation: It is the order of the highest order derivative appearing in the differential equation.

(b) Degree of a differential equation: It is the degree (power) of the highest order derivative, when the differential coefficients are made free from the radicals and the fractions.

❑❑

UNIT IV: VECTORS & THREE-DIMENSIONAL GEOMETRY

CHAPTER-4
VECTORS

Revision Notes

Basic Algebra of Vectors

1. Vector : Basic Introduction:

- A quantity having magnitude as well as the direction is called a vector. It is denoted as $\overrightarrow{AB}$ or $\vec{a}$. Its magnitude (or modulus) is $|\overrightarrow{AB}|$ or $|\vec{a}|$ otherwise, simply AB or a.
- Vectors are denoted by symbols such as $\vec{a}$. [Pictorial representation of vector]

2. Initial and Terminal Points:

The initial and terminal points, means, that point from which the vector originates and terminates respectively.

3. Position Vector:

The position vector of a point say $P(x, y, z)$ is $\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and the magnitude is $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.

The vector $\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is said to be in its **component form**. Here x, y, z are called the scalar components or rectangular components of $\vec{r}$ and $x\hat{i}, y\hat{j}, z\hat{k}$ are the vector components of $\vec{r}$ along x, y, z-axis respectively.

- Also, $\overrightarrow{AB}$ = (Position Vector of B) – (Position Vector of A). For example, let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$. Then, $\overrightarrow{AB}$ = $(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$.
- Here $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors along the axes OX, OY and OZ respectively (The discussion about unit vectors is given later under 'types of vectors').

4. Direction Ratios and Direction Cosines:

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then coefficient of $\hat{i}, \hat{j}, \hat{k}$ in $\vec{r}$ *i.e.*, x, y, z are called the direction ratios (abbreviated as DRs) of vector $\vec{r}$. These are denoted by a, b, c (*i.e.*, $a = x$, $b = y$, $c = z$; in a manner we can say that scalar components of vector $\vec{r}$ and its DRs both are the same).

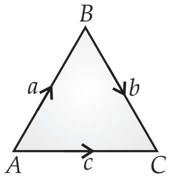
Also, the coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\vec{r}$ (which is the unit vector of $\vec{r}$) *i.e.*, $\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}}$ are called direction cosines (which is abbreviated as DCs) of vector $\vec{r}$.

- These direction cosines are denoted by l, m, n such that $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$ and $l^2 + m^2 + n^2 = 1 \Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.
- It can be easily concluded that $\frac{x}{r} = l = \cos\alpha$, $\frac{y}{r} = m = \cos\beta$, $\frac{z}{r} = n = \cos\gamma$.

  Therefore, $\vec{r} = lr\hat{i} + mr\hat{j} + nr\hat{k} = r(\cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k})$. [Here $r = |\vec{r}|$].

5. Addition of vectors

(a) Triangular law: If two adjacent sides (say sides AB and BC) of a triangle ABC are represented by $\vec{a}$ and $\vec{b}$ taken in same order, then the third side of the triangle taken in the reverse order gives the sum of vectors $\vec{a}$ and $\vec{b}$ *i.e.,* $\vec{AC} = \vec{AB} + \vec{BC} \Rightarrow \vec{AC} = \vec{a} + \vec{b}$.



- Also since $\vec{AC} = -\vec{CA} \Rightarrow \vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$.
- And $\vec{AB} + \vec{BC} - \vec{AC} = \vec{0} \Rightarrow \vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$.

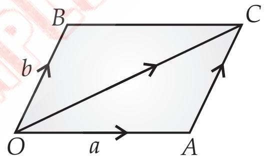
(b) Parallelogram law: If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and the direction by the two adjacent sides (say AB and AD) of a parallelogram $ABCD$, then their sum is given by that diagonal of parallelogram which is co-initial with $\vec{a}$ and $\vec{b}$ *i.e.,* $\vec{OC} = \vec{OA} + \vec{OB}$.

6. Properties of Vector Addition

(a) Commutative property: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Consider $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ be any two given vectors,

then $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k} = \vec{b} + \vec{a}$.



(b) Associative property: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

(c) Additive identity property: $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$.

(d) Additive inverse property: $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$.

Note : Multiplication of a vector by a scalar

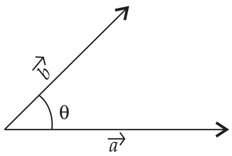
Let $\vec{a}$ be any vector and k be any non-zero scalar. Then the product $k\vec{a}$ is defined as a vector whose magnitude is $|k|$ times that of $\vec{a}$ and the direction is

(i) same as that of $\vec{a}$ if k is positive, and **(ii)** opposite as that of $\vec{a}$ if k is negative.

Dot Product of Vectors

1. Products of Two Vectors and Projection of Vectors

(a) Scalar Product or Dot Product : The dot product of two vectors $\vec{a}$ and $\vec{b}$ is defined by,



$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$.

Consider $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

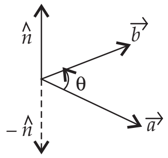
➲ **Projection of a vector :** $\vec{a}$ on the other vector say $\vec{b}$ is given as $\left(\dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}\right)$.

➲ **Projection of a vector :** $\vec{b}$ on the other vector say $\vec{a}$ is given as $\left(\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\right)$.

Cross Product

1. The cross product or vector product of two vectors $\vec{a}$ and $\vec{b}$ is defined by,

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$, where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. For better illustration, see figure.



Consider $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

then, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$.

- **Properties/Observations of Cross Product**
  - ➲ $\hat{i} \times \hat{i} = |\hat{i}||\hat{i}| \sin 0 = \vec{0}$ or $\hat{i} \times \hat{i} = \vec{0} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k}$.
  - ➲ $\hat{i} \times \hat{j} = |\hat{i}||\hat{j}| \sin\frac{\pi}{2} . \hat{k} = \hat{k}$ or $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$.
  - ➲ $\vec{a} \times \vec{b}$ is a vector $\vec{c}$ (say) then this vector $\vec{c}$ is perpendicular to both the vectors $\vec{a}$ and $\vec{b}$.
  - ➲ $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \,||\, \vec{b}$ or, $\vec{a} = \vec{0}$, $\vec{b} = \vec{0}$.
  - ➲ $\vec{a} \times \vec{a} = \vec{0}$.
  - ➲ $\vec{a} \times \vec{b} \ne \vec{b} \times \vec{a}$ (Commutative property does not hold for cross product).
  - ➲ $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ (Left distributive).
  - ➲ $(\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}$ (Right distributive).

  (Distributive property of the vector product or cross product)

2. Relationship between Vector product and Scalar product [Lagrange's Identity]

or $$|\vec{a} \times \vec{b}|^2 + \left(\vec{a} \cdot \vec{b}\right)^2 = |\vec{a}|^2 \cdot |\vec{b}|^2$$

3. Cauchy-Schwarz inequality:

For any two vectors $\vec{a}$ and $\vec{b}$, always have $|\vec{a} \cdot \vec{b}| \le |\vec{a}||\vec{b}|$.

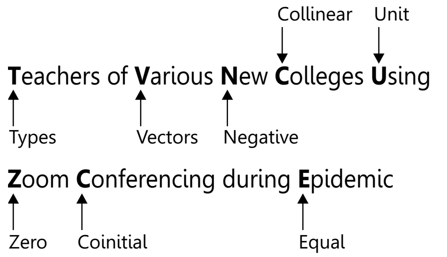
Note:

- If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle, then the area of triangle can be obtained by evaluating $\frac{1}{2}|\vec{a} \times \vec{b}|$.
- If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then the area of parallelogram can be obtained by evaluating $|\vec{a} \times \vec{b}|$.

- The area of the parallelogram with diagonals $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|\vec{a}\times\vec{b}|$.

Mnemonic

(i) Types Of Vectors (A)

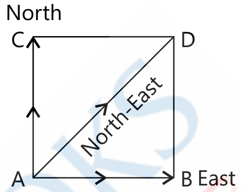


Interpretation:

1. **Zero Vector -** Initial and terminal points coincide
2. **Unit Vector -** Magnitude is unity
3. **Co-initial Vectors -** Same initial points
4. **Collinear vectors -** Parallel to the same Line
5. **Equal Vectors -** Same magnitude and same direction
6. **Negative of a vector -** Same magnitude, opposite direction

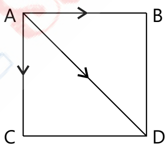
(ii) Properties Of Vectors (B)

"Neither choose East nor north, always choose North-East and save your time".



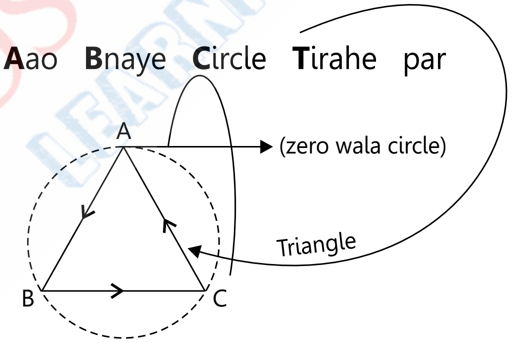
Interpretation:

The vector sum of two co-initial vectors is given by the diagonal of the parallelogram whose adjacent sides are given vectors.



$\vec{AB} + \vec{AC} = \vec{AD}$

(iii) Properties Of Vectors (C)



Interpretation:

The vector sum of the three sides of a triangle taken in order is $\vec{0}$ *i.e* $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$

Know the Terms

Types of Vectors:

(a) **Zero or Null vector:** It is that vector whose initial and terminal points are coincident. It is denoted by $\vec{0}$. Its magnitude is 0 (zero).

- Any non-zero vector is called a **proper vector**.

(b) **Co-initial vectors:** Those vectors (two or more) having the same starting point are called the co-initial vectors.

(c) **Co-terminus vectors:** Those vectors (two or more) having the same terminal point are called the co-terminus vectors.

(d) Negative of a vector: The vector which has the same magnitude as the $\vec{r}$ but opposite direction. It is denoted by $-\vec{r}$. Hence if, $\vec{AB} = \vec{r}$ or $\vec{BA} = -\vec{r}$ *i.e.*, $\vec{AB} = -\vec{BA}$, $\vec{PQ} = -\vec{QP}$ etc.

(e) Unit vector: It is a vector with the unit magnitude. The unit vector in the direction of vector $\vec{r}$ is given by $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$ such that $|\hat{r}| = 1$, so, if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then its unit vector is :

$$\hat{r} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}.$$

- Unit vector perpendicular to the plane $\vec{a}$ and $\vec{b}$ is : $\pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.

(f) Reciprocal of a vector: It is a vector which has the same direction as the vector $\vec{r}$ but magnitude equal to the reciprocal of the magnitude of $\vec{r}$. It is denoted as $\vec{r}^{\,-1}$. Hence $\left|\vec{r}^{\,-1}\right| = \frac{1}{|\vec{r}|}$.

(g) Equal vectors: Two vectors are said to be equal if they have the same magnitude as well as direction, regardless of the position of their initial points.

Thus $\vec{a} = \vec{b} \Leftrightarrow \begin{cases} |\vec{a}| = |\vec{b}| \\ \vec{a} \text{ and } \vec{b} \text{ have same direction} \end{cases}$

Also, if $\vec{a} = \vec{b} \Rightarrow a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \Rightarrow a_1 = b_1, a_2 = b_2, a_3 = b_3$.

(h) Collinear or Parallel vector: Two vectors $\vec{a}$ and $\vec{b}$ are collinear or parallel if there exists a non-zero scalar λ such that $\vec{a} = \lambda\vec{b}$.

- It is important to note that the respective coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\vec{a}$ and $\vec{b}$ are proportional provided they are parallel or collinear to each other.
- The DRs of parallel vectors are same (or are in proportion).
- The vectors $\vec{a}$ and $\vec{b}$ will have same or opposite direction as λ is positive or negative respectively.
- The vectors $\vec{a}$ and $\vec{b}$ are collinear if $\vec{a}\times\vec{b} = \vec{0}$.

(i) Free vectors: The vectors which can undergo parallel displacement without changing its magnitude and direction are called free vectors.

Know the Formulae

The position vector of a point say *P* dividing a line segment joining the points *A* and *B* whose position vectors are $\vec{a}$ and $\vec{b}$ respectively, in the ratio *m* : *n*.

(a) Internally, $\vec{OP} = \frac{m\vec{b} + n\vec{a}}{m+n}$ **(b)** Externally, $\vec{OP} = \frac{m\vec{b} - n\vec{a}}{m-n}$

- Also if point *P* is the mid-point of line segment *AB*, then $\vec{OP} = \frac{\vec{a}+\vec{b}}{2}$.

➲ Angle between two vectors $\vec{a}$ and $\vec{b}$ can be found by the expression given below :

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \text{ or, } \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right).$$

➲ Angle between two vectors $\vec{a}$ and $\vec{b}$ in terms of cross-product can be found by the expression given here :

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} \text{ or } \theta = \sin^{-1}\left(\frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}\right).$$

Know the Properties (Dot Product)

- **Properties/Observations of Dot product**

➲ $\hat{i}.\hat{i} = |\hat{i}||\hat{i}|\cos 0 = 1$ or $\hat{i}.\hat{i} = 1 = \hat{j}.\hat{j} = \hat{k}\cdot\hat{k}$

➲ $\hat{i}.\hat{j} = |\hat{i}||\hat{j}|\cos\frac{\pi}{2} = 0$ or $\hat{i}.\hat{j} = 0 = \hat{j}.\hat{k} = \hat{k}\cdot\hat{i}$

➲ $\vec{a}\,.\,\vec{b} \in R$, where R is real number *i.e.*, any scalar.

➲ $\vec{a}\,.\,\vec{b} = \vec{b}\,.\,\vec{a}$ (Commutative property of dot product).

➲ $\vec{a}\,.\,\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ or $|\vec{a}| = 0$ or $|\vec{b}| = 0$.

➲ If $\theta = 0$, then $\vec{a}\,.\,\vec{b} = |\vec{a}||\vec{b}|$. Also $\vec{a}\,.\,\vec{a} = |\vec{a}|^2 = a^2$; as θ in this case is 0.

Moreover if $\theta = \pi$, then $\vec{a}\,.\,\vec{b} = -|\vec{a}||\vec{b}|\cdot$

➲ $\vec{a}.\left(\vec{b} + \vec{c}\right) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ (Distributive property of dot product).

➲ $\vec{a}.\left(-\vec{b}\right) = -\left(\vec{a}.\vec{b}\right) = \left(-\vec{a}\right).\vec{b}$.

Theorem

Triangle Inequality

For any two vectors $\vec{a}$ and $\vec{b}$, we always have $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$.

Proof: The given inequality holds trivially when either $\vec{a} = 0$ or $\vec{b} = 0$ *i.e.*, in such a case $|\vec{a} + \vec{b}| = 0 = |\vec{a}| + |\vec{b}|$.

So, let us check it for $|\vec{a}| \neq 0 \neq |\vec{b}|$.

Then consider $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a}.\vec{b}$

or $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta$

For $\cos\theta \leq 1$, we have : $2|\vec{a}||\vec{b}|\cos\theta \leq 2|\vec{a}||\vec{b}|$

or $|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta \leq |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|$

or $|\vec{a} + \vec{b}|^2 \leq \left(|\vec{a}| + |\vec{b}|\right)^2$

or $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$ **Hence proved**

❑❑

CHAPTER-5
THREE DIMENSIONAL GEOMETRY

Revision Notes

Direction Ratios and Direction Cosines

1. Direction Cosines of a Line:

- If A and B are two points on a given line L, then direction cosines of vectors $\vec{AB}$ and $\vec{BA}$ are the direction cosines (d.c.'s) of line L. Thus if α, β, γ are the direction-angles which the line L makes with the positive direction of x, y, z-axis respectively, then its d.c.'s are $\cos\alpha, \cos\beta, \cos\gamma$.
- If direction of line L is reversed, the direction angles are replaced by their supplements *i.e.*, $\pi-\alpha, \pi-\beta, \pi-\gamma$ and so are the d.c.'s *i.e.*, the direction cosines become $-\cos\alpha, -\cos\beta, -\cos\gamma$.
- So, a line in space has two set of d.c.'s *viz* $\pm\cos\alpha, \pm\cos\beta, \pm\cos\gamma$.
- The d.c.'s are generally denoted by l, m, n. Also $l^2+m^2+n^2=1$ and so we can deduce that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$. Also $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$.
- The d.c.'s of a line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are $\pm\frac{x_2-x_1}{AB}, \pm\frac{y_2-y_1}{AB}, \pm\frac{z_2-z_1}{AB}$; where AB is the distance between the points A and B *i.e.*, $AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.

2. Direction Ratios of a Line:

Any three numbers a, b, c (say) which are proportional to DCs *i.e.*, l, m, n of a line are called the **direction ratios** (d.r.'s) of the line. Thus, $a=\lambda l, b=\lambda m, c=\lambda n$ for any $\lambda \in R-\{0\}$.

Consider, $$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\frac{1}{\lambda} \quad \text{(say)}$$

or $$l=\frac{a}{\lambda}, m=\frac{b}{\lambda}, n=\frac{c}{\lambda}$$

or $$\left(\frac{a}{\lambda}\right)^2+\left(\frac{b}{\lambda}\right)^2+\left(\frac{c}{\lambda}\right)^2=1 \quad [\text{Using } l^2+m^2+n^2=1]$$

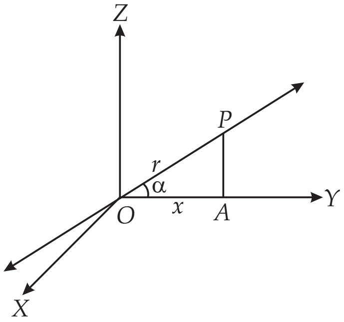
or $$\lambda=\pm\sqrt{a^2+b^2+c^2}$$

Therefore, $$l=\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m=\pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n=\pm\frac{c}{\sqrt{a^2+b^2+c^2}}.$$

- The DRs of a line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are $x_2-x_1, y_2-y_1, z_2-z_1$ or $x_1-x_2, y_1-y_2, z_1-z_2$.
- Direction ratios are sometimes called as **Direction Numbers.**

3. Relation Between the Direction Cosines of a Line:

Consider a line L with d.c's l, m, n. Draw a line passing through the origin and $P(x, y, z)$ and parallel to the given line L. From P draw a perpendicular PA on the X-axis, suppose $OP=r$



Now in ΔOAP, $\angle PAO=90°$

we have, $$\cos\alpha=\frac{OA}{OP}=\frac{x}{r} \text{ or } x=lr.$$

Similarly, we can obtain $y=mr$ and $z=nr$.

Therefore, $x^2+y^2+z^2=r^2(l^2+m^2+n^2)$

But we know that $x^2+y^2+z^2=r^2$

Hence, $l^2+m^2+n^2=1$.

Lines & Its Equations in Different forms

1. Equation of a line passing through two given points:

Consider the two given points as $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ with position vectors $\vec{a}$ and $\vec{b}$ respectively. Also, assume $\vec{r}$ as the position vector of any arbitrary point $P(x, y, z)$ on the line L passing through A and B.

Thus $\vec{OA} = \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k},\ \vec{OB} = \vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k},\ \vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

(a) Vector equation of a line: Since the points A, B and P all lie on the same line which means that they are all collinear points.

Further it means, $\vec{AP} = \vec{r} - \vec{a}$ and $\vec{AB} = \vec{b} - \vec{a}$ are collinear vectors, *i.e.*,

$$\vec{AP} = \lambda \vec{AB}$$

or $$\vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a})$$

or $$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}), \text{ where } \lambda \in R.$$

this is the vector equation of the line.

(b) Cartesian equation of a line: By using the vector equation of the line $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, we get

$$x\hat{i} + y\hat{j} + z\hat{k} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + \lambda\left[(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}\right]$$

On equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$, we get

$$x = x_1 + \lambda(x_2 - x_1),\ y = y_1 + \lambda(y_2 - y_1),\ z = z_1 + \lambda(z_2 - z_1) \quad \text{...(i)}$$

On eliminating λ, we have

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

2. Angle between two lines:

(a) When DRs or DCs of the two lines are given:

Consider two lines L_1 and L_2 with d.r.'s in proportion to a_1, b_1, c_1 and a_2, b_2, c_2 respectively; d.c.'s as l_1, m_1, n_1 and l_2, m_2, n_2. Consider $\vec{b}_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b}_2 = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$. These vectors $\vec{b}_1$ and $\vec{b}_2$ are parallel to the given lines L_1 and L_2. So in order to find the angle between the lines L_1 and L_2, we need to get the angle between the vectors $\vec{b}_1$ and $\vec{b}_2$.

So the acute angle θ between the vectors $\vec{b}_1$ and $\vec{b}_2$ (and hence lines L_1 and L_2) can be obtained as,

$$\vec{b}_1 . \vec{b}_2 = |\vec{b}_1||\vec{b}_2|\cos\theta$$

Thus, $$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

- Also, in terms of DCs: $\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$.
- Sine of angle is given as: $$\sin\theta = \left|\frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|.$$

(b) When vector equations of two lines are given:

Consider vector equations of lines L_1 and L_2 as $\vec{r}_1 = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r}_2 = \vec{a}_2 + \mu\vec{b}_2$ respectively.

Then, the acute angle θ between the two lines is given by the relation

$$\cos\theta = \left|\frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|.$$

(c) When Cartesian equation of two lines are given:

Consider the lines L_1 and L_2 in Cartesian form as,

$$L_1: \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

$$L_2: \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

Then the acute angle θ between the lines L_1 and L_2 can be obtained by,

$$\cos\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

Note:

- For two perpendicular lines: $a_1a_2 + b_1b_2 + c_1c_2 = 0$, $l_1l_2 + m_1m_2 + n_1n_2 = 0$.
- For two parallel lines: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$; $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

3. Shortest Distance between two Lines:

If two lines are in the same plane, *i.e.*, they are coplanar, they will intersect each other if they are non-parallel. Hence, the shortest distance between them is zero. If the lines are parallel then the shortest distance between them will be the perpendicular distance between the lines, *i.e.*, the length of the perpendicular drawn from a point on one line onto the other line. Adding to this discussion, in space, there are lines which are neither intersecting nor parallel. In fact, such pairs of lines are non-coplanar and are called the skew lines.

Plane & Its Equations in Various Forms

1. Plane and its Equation:

A plane is a surface such that if any two points are taken on it, the line segment joining them lies completely on the surface.

(a) Equation of plane in Normal unit vector form:

Consider a plane at distance d from the origin such that $\vec{ON}$ is the normal from the origin to plane and $\hat{n}$ is a unit vector along $\vec{ON}$. Then $\vec{ON} = d\hat{n}$, if $ON = d$. Consider $\vec{r}$ be the position vector of any arbitrary point $P(x, y, z)$ on the plane.

➲ **Vector form of the Equation of plane:** Since P lies on the plane, so $\vec{NP}$ is perpendicular to the vector $\vec{ON}$.

That implies $\vec{NP}.\vec{ON} = 0$

or $(\vec{r} - d\hat{n}).d\hat{n} = 0$

or $(\vec{r} - d\hat{n}).\hat{n} = 0$ [∵ $d \neq 0$]

or $\vec{r}.\hat{n} - d\hat{n}.\hat{n} = 0$

or $\vec{r}.\hat{n} = d$ [∵ $\hat{n}.\hat{n} = 1$]

This is the vector equation of the plane.

➲ **Cartesian form of the equation of plane:** If l, m, n are the DCs of the normal $\hat{n}$ to the given plane. Then by using $\vec{r}.\hat{n} = d$

we get, $$(x\hat{i} + y\hat{j} + z\hat{k}).(l\hat{i} + m\hat{j} + n\hat{k}) = d$$

or $$lx + my + nz = d.$$

This is the cartesian equation of the plane.

- Also if a, b, c are the DRs of the normal $\hat{n}$ to the plane, then the cartesian equation of plane becomes $ax + by + cz = d$.

(b) Equation of plane perpendicular to a given vector and passing through a given point:

Assume that the plane passes through a point $A(x_1, y_1, z_1)$ with the position vector $\vec{a}$ and is perpendicular to the vector $\vec{m}$ with DRs as A, B, C ($\therefore \vec{m} = A\hat{i} + B\hat{j} + C\hat{k}$).

Also consider $P(x, y, z)$ as any arbitrary point on the plane with position vector as $\vec{r}$.

➲ **Vector form of the equation of plane:** As $\vec{AP}$ lies in the plane and $\vec{m}$ is perpendicular to the plane. So $\vec{AP}$ is perpendicular to $\vec{m}$.

or $$\vec{AP}.\vec{m} = 0$$

or $$(\vec{r} - \vec{a}).\vec{m} = 0$$

or $$(\vec{r} - \vec{a}).[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$$

This is the vector equation of the plane.

- The above obtained equation of plane can also be expressed as $\vec{r}.\vec{m} = \vec{a}.\vec{m}$.

➲ **Cartesian form of the equation of plane:** As $\overrightarrow{AP} = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$, so by using $(\vec{r} - \vec{a}).\vec{m} = 0$, we get

$$[(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}].(A\hat{i} + B\hat{j} + C\hat{k}) = 0$$

or $$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

or $$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

This is the cartesian equation of the plane.

(c) Equation of plane passing through three non-collinear points:

Assume that the plane contains three non-collinear points $R(x_1, y_1, z_1)$, $S(x_2, y_2, z_2)$ and $T(x_3, y_3, z_3)$ with the position vectors as $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Let $P(x, y, z)$ be any arbitrary point in the plane whose position vector is $\vec{r}$.

2. Co-planarity of two Lines:

Assume that the given lines are $L_1 : \vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $L_2 : \vec{r} = \vec{a}_2 + \mu\vec{b}_2$ such that L_1 passes through $A(x_1, y_1, z_1)$ with position vector $\vec{a}_1$ and is parallel to $\vec{b}_1$ with DRs a_1, b_1, c_1. Also L_2 passes through $B(x_2, y_2, z_2)$ with position vector $\vec{a}_2$ and is parallel to $\vec{b}_2$ with the DRs a_2, b_2, c_2.

(a) Vector form of co-planarity of lines:

We know that $\overrightarrow{AB} = \vec{a}_2 - \vec{a}_1$. Now the lines L_1 and L_2 are coplanar if $\overrightarrow{AB}$ is perpendicular to $\vec{b}_1 \times \vec{b}_2$. That implies,

$\overrightarrow{AB}.(\vec{b}_1 \times \vec{b}_2) = 0$

or $$(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2) = 0$$

(b) Cartesian form of co-planarity of lines:

We know that $\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$, $\vec{b}_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b}_2 = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$. So by using $(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2) = 0$, we get

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

- Note that only coplanar lines can intersect each other in the plane they exist.

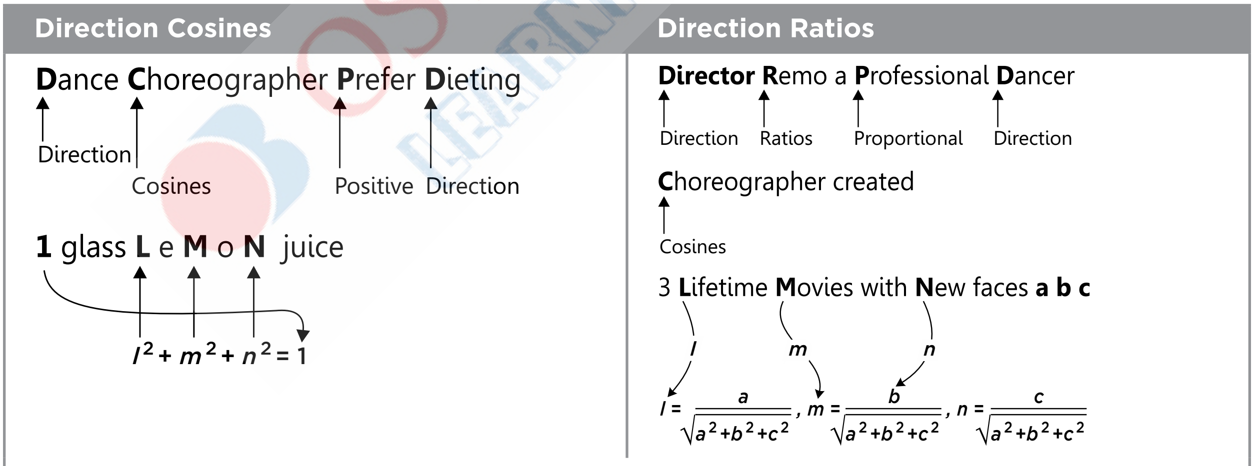
(c) Distance between two parallel planes : Assume that the two planes are $\vec{r}.\vec{m} = d_1$ *i.e.*, $Ax + By + Cz + D_1 = 0$ and $\vec{r}.\vec{m} = d_2$ *i.e.*, $Ax + By + Cz + D_2 = 0$.

Then the distance p(say) between them is given as

(i) Vector form: $$p = \frac{|d_1 - d_2|}{|\vec{m}|}.$$

(ii) Cartesian form: $$p = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}.$$

Mnemonic



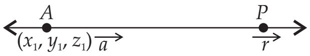
Interpretation:

Direction cosines of a line are the cosines of the angles made by the line with the positive directions of the co-ordinate axes. If l, m, n are the *D*. cs of a line, then $l^2+m^2+n^2 = 1$

Know the Terms

1. Equation of a line in space passing through a given point and parallel to a given vector:

Consider the line L is passing through the given point $A(x_1, y_1, z_1)$ with the position vector $\vec{a}$, where $\vec{d}$ is the given vector with d.r.'s a, b, c and $\vec{r}$ is the position vector of any arbitrary point $P(x, y, z)$ on the line.



Thus, $\vec{OA} = \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, \overline{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \vec{d} = a\hat{i} + b\hat{j} + c\hat{k}$.

(a) Vector equation of a line: As the line L is parallel to given vector $\vec{d}$ and points A and P are lying on the line so, $\overline{AP}$ is parallel to the $\vec{d}$.

or $\overline{AP} = \lambda\vec{d}$, where $\lambda \in R$ *i.e.,* set of real numbers

or $\vec{r} - \vec{a} = \lambda\vec{d}$

or $\vec{r} = \vec{a} + \lambda\vec{d}$.

This is the vector equation of line.

(b) Parametric equations: If DRs of the line are a, b, c, then by using $\vec{r} = \vec{a} + \lambda\vec{d}$, we get

$$x\hat{i} + y\hat{j} + z\hat{k} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + \lambda\left(a\hat{i} + b\hat{j} + c\hat{k}\right)$$

Now, as we equate the coefficients of $\hat{i}, \hat{j}, \hat{k}$, we get the parametric equations of line given as,

$$x = x_1 + \lambda a, y = y_1 + \lambda b, z = z_1 + \lambda c.$$

- Co-ordinates of any point on the line considered here are $(x_1 + \lambda a, y_1 + \lambda b, z_1 + \lambda c)$.

(c) Cartesian equation of a line: If we eliminate the parameter λ from the parametric equations of a line, we get the Cartesian equation of line as

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

- If l, m, n are the DCs of the line, then Cartesian equation of line becomes

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

- **Skew Lines:** Two straight lines in space which are neither parallel nor intersecting are known as the skew lines. They lie in different planes and are non-coplanar.
- **Line of Shortest distance:** There exists unique line perpendicular to each of the skew lines L_1 and L_2, and this line is known as the line of shortest distance (S.D.).

Know the Formulae

1. Distance Formula:

The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by the expression

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \text{ units.}$$

2. Section Formula:

The co-ordinates of a point Q which divides the line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the ratio $m : n$

(a) internally, are $\left(\frac{(mx_2 + nx_1)}{m+n}, \frac{(my_2 + ny_1)}{m+n}, \frac{(mz_2 + nz_1)}{m+n}\right)$

(b) externally, are $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$

3. Vector equation of the plane:

$$(\vec{r} - \vec{a}).\left[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\right] = 0$$

4. Cartesian form of the equation of plane:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

5. Equation of the plane in intercept form:

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

Note :

Equation of XY-plane : $z = 0$,

Equation of YZ-plane : $x = 0$,

Equation of ZX-plane : $y = 0$

6. Cartesian form for the angle between two planes:

$$\theta = \cos^{-1}\left|\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}\right|$$

Note:

- **For the parallel planes, we have :** $\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}$.
- **For the perpendicular planes, we have:** $A_1A_2 + B_1B_2 + C_1C_2 = 0$.

❑❑

UNIT VI: PROBABILITY

CHAPTER-6
PROBABILITY

Revision Notes

Conditional Probability and Multiplication Theorem on Probability

1. Basic Definition of Probability:

Let S and E be the sample space and event in an experiment respectively.

Then,

$$\text{Probability} = \frac{\text{Number of Favourable Events}}{\text{Total number of Elementary Events}} = \frac{n(E)}{n(S)}$$

$$0 \le n(E) \le n(S)$$

$$0 \le P(E) \le 1$$

Hence, if $P(E)$ denotes the probability of occurrence of an event E, then $0 \le P(E) \le 1$ and $P(\bar{E}) = 1 - P(E)$ such that $P(\bar{E})$ denotes the probability of non-occurrence of the event E.

➲ **Note that $P(\bar{E})$ can also be represented as $P(E')$.**

2. Mutually Exclusive Or Disjoint Events:

Two events A and B are said to be mutually exclusive if the occurrence of one prevents the occurrence of the other *i.e.*, they can't occur simultaneously.

In this case, sets A and B are disjoint *i.e.*, $A \cap B = \phi$.

Consider an example of throwing a die. We have the sample space as, $S = \{1, 2, 3, 4, 5, 6\}$

Suppose A = the event of occurrence of a number greater than 4 = {5, 6}

B = the event of occurrence of an odd number = {1, 3, 5}

and C = the event of occurrence of an even number = {2, 4, 6}

In these events, events B and C are mutually exclusive events but A and B are not mutually exclusive events because they can occur together (when the number 5 comes up). Similarly A and C are not mutually exclusive events as they can also occur together (when the number 6 comes up).

3. Independent Events:

Two events are independent if the occurrence of one does not affect the occurrence of the other.

Consider an example of drawing two balls one by one with replacement from a bag containing 3 red and 2 black balls.

Suppose A = the event of occurrence of a red ball in the first draw

B = the event of occurrence of a black ball in the second draw

Then $$P(A) = \frac{3}{5},\ P(B) = \frac{2}{5}$$

Here probability of occurrence of event B is not affected by the occurrence or non-occurrence of event A.

Hence, events A and B are independent events.

4. Exhaustive Events:

Two or more events say A, B and C of an experiment are said to be exhaustive events, if

(a) their union is the total sample space *i.e.,* $A \cup B \cup C = S$

(b) the event A, B and C are disjoint in pairs *i.e.,* $A \cap B = \phi$, $B \cap C = \phi$ and $C \cap A = \phi$.

(c) $P(A) + P(B) + P(C) = 1$.

Consider an example of throwing a die. We have S = {1, 2, 3, 4, 5, 6}

Suppose A = the event of occurrence of an even number = {2, 4, 6}

B = the event of occurrence of an odd number = {1, 3, 5}

and C = the event of getting a number multiple of 3 = {3, 6}

In these events, the events A and B are exhaustive events as $A \cup B = S$ but the events A and C or the events B and C are not exhaustive events as $A \cup C \neq S$ and similarly $B \cup C \neq S$.

➲ If A and B are mutually exhaustive events, then we always have

$$P(A \cap B) = 0 \qquad [\text{As } n(A \cap B) = n(\phi) = 0]$$

$$\therefore \quad P(A \cup B) = P(A) + P(B).$$

➲ If A, B and C are mutually exhaustive events, then we always have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

5. Conditional Probability:

By the conditional probability, we mean the probability of occurrence of event A when B has already occurred.

The 'conditional probability of occurrence of event A when B has already occurred' is sometimes also called as probability of occurrence of event A w.r.t. B.

➲ $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, $B \neq \phi$ *i.e.,* $P(B) \neq 0$

➲ $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$, $A \neq \phi$ *i.e.,* $P(A) \neq 0$

➲ $P(\bar{A}|B) = \dfrac{P(\bar{A} \cap B)}{P(B)}$, $P(B) \neq 0$

➲ $P(A|\bar{B}) = \dfrac{P(A \cap \bar{B})}{P(\bar{B})}$, $P(\bar{B}) \neq 0$

➲ $P(\bar{A}|\bar{B}) = \dfrac{P(\bar{A} \cap \bar{B})}{P(\bar{B})}$, $P(\bar{B}) \neq 0$

➲ $P(A|B) + P(\bar{A}|B) = 1,\ B \neq \phi$.

Bayes' Theorem

BAYES' THEOREM:

If $E_1, E_2, E_3, \ldots\ldots, E_n$ are n non-empty events constituting a partition of sample space S *i.e.,* $E_1, E_2, E_3, \ldots, E_n$ are pair wise disjoint and $E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n = S$ and A is any event of non-zero probability, then

$$P(E_i|A) = \frac{P(E_i).P(A|E_i)}{\sum_{j=1}^{n} P(E_j)P(A|E_j)}, i = 1, 2, 3, ..., n$$

For example,

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2) + P(E_3).P(A|E_3)}$$

- Bayes' theorem is also known as the formula for the probability of causes.
- If $E_1, E_2, E_3, ... , E_n$ form a partition of S and A be any event, then

$$P(A) = P(E_1).P(A|E_1) + P(E_2).P(A|E_2) + ... + P(E_n).P(A|E_n)$$

$$[\because P(E_i \cap A) = P(E_i).P(A|E_i)]$$

- The probabilities $P(E_1), P(E_2), ..., P(E_n)$ which are known before the experiment takes place are called **prior probabilities** and $P(A|E_n)$ are called **posterior probabilities**.

Random Variable and its Probability Distributions

1. RANDOM VARIABLE:

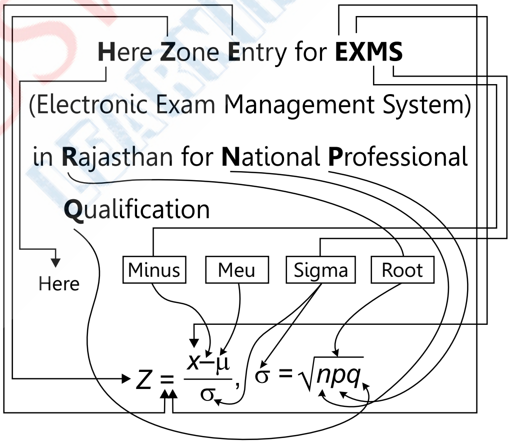
A random variable is a real-valued function defined over the sample space of an experiment. In other words, a random variable is a real-valued function whose domain is the sample space of a random experiment. A random variable is usually denoted by uppercase letters *X, Y, Z,* etc.

2. PROBABILITY DISTRIBUTION OF A RANDOM VARIABLE:

If the values of a random variable together with the corresponding probabilities are given, then this description is called a probability distribution of the random variable.

Mnemonic

Normal Distribution



Know the Terms

- **Discrete random variable:** It is a random variable that can take only finite or countable infinite number of values.
- **Continuous random variable:** A variable that can take any value between two given limits is called a continuous random variable.

Know the Formulae

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ *i.e.*, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

(b) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

(c) $P(\overline{A} \cap B) = P(\text{only } B) = P(B - A) = P(B \text{ but not } A) = P(B) - P(A \cap B)$

(d) $P(A \cap \overline{B}) = P(\text{only } A) = P(A - B) = P(A \text{ but not } B) = P(A) - P(A \cap B)$

❑❑