UNIT – I: RELATIONS AND FUNCTIONS

CHAPTER-1

RELATIONS AND FUNCTIONS

Topic-1 Relations

Concepts Covered • *Types of relations and their identification* • *Equivalence class*

Revision Notes

1. Definition

A relation R, from a non-empty set A to another non-empty set B is mathematically as an subset of $A \times B$. Equivalently, any subset of $A \times B$ is a relation from A to B. Thus, R is a relation from A to B

$\Leftrightarrow R \subseteq A \times B$

$\Leftrightarrow R \subseteq \{(a, b) : a \in A, b \in B\}$

Illustrations:

(a) Let $A = \{1, 2, 4\}$, $B = \{4, 6\}$. Let $R = \{(1, 4), (1, 6), (2, 4), (2, 6), (4, 4) (4, 6)\}$. Here $R \subseteq A \times B$ and therefore R is a relation from A to B.

(b) Let $A = \{1, 2, 3\}$, $B = \{2, 3, 5, 7\}$, Let $R = \{(2, 3), (3, 5), (5, 7)\}$. Here $R \not\subset A \times B$ and therefore R is not a relation from A to B. Since $(5, 7) \in R$ but $(5, 7) \notin A \times B$.

(c) Let $A = \{-1, 1, 2\}$, $B = \{1, 4, 9, 10\}$ let $a \in A$ and $b \in B$ and $a\,R\,b$ means $a^2 = b$ then, $R = \{(-1, 1), (1, 1), (2, 4)\}$.

Note:
- A relation from A to B is also called a relation from A into B.
- $(a, b) \in R$ is also written as aRb (read as a is related to b).
- Let A and B be two non-empty finite sets having p and q elements respectively. Then $n(A \times B) = n(A)\cdot n(B) = pq$. Then total number of subsets of $A \times B = 2^{pq}$. Since each subset of $A \times B$ is a relation from A to B, therefore total number of relations from A to B will be 2^{pq}.

2. Domain & range of a relation

(a) **Domain of a relation:** Let R be a relation from A to B. The domain of relation R is the set of all those elements $a \in A$ such that $(a, b) \in R\ \forall\ b \in B$.

Thus, $\text{Dom.}(R) = \{a \in A : (a, b) \in R\ \forall\ b \in B\}$.

That is, the domain of R is the set of first components of all the ordered pairs which belong to R.

(b) Range of a relation: Let R be a relation from A to B. The range of relation R is the set of all those elements $b \in B$ such that $(a, b) \in R \ \forall \ a \in A$.

Thus, Range of $R = \{b \in B : (a, b) \in R \ \forall \ a \in A\}$.

That is, the range of R is the set of second components of all the ordered pairs which belong to R.

(c) Co-domain of a relation: Let R be a relation from A to B. Then B is called the co-domain of the relation R. So we can observe that co-domain of a relation R from A into B is the set B as a whole.

Illustrations: Let $a \in A$ and $b \in B$ and

(i) Let $A = \{1, 2, 3, 7\}, B = \{3, 6\}$. If aRb means $a < b$.

Then we have

$$R = \{(1, 3), (1, 6), (2, 3), (2, 6), (3, 6)\}.$$

Here, Dom.$(R) = \{1, 2, 3\}$,

Range of $R = \{3, 6\}$, Co-domain of $R = B = \{3, 6\}$

(ii) Let $A = \{1, 2, 3\}, B = \{2, 4, 6, 8\}$.

If $R_1 = \{(1, 2), (2, 4), (3, 6)\}$,

and $R_2 = \{(2, 4\}, (2, 6), (3, 8), (1, 6)\}$

Then both R_1 and R_2 are related from A to B because

$$R_1 \subseteq A \times B, R_2 \subseteq A \times B$$

Here, Dom

$(R_1) = \{1, 2, 3\}$, Range of $R_1 = \{2, 4, 6\}$;

Dom$(R_2) = \{2, 3, 1\}$, Range of $R_2 = \{4, 6, 8\}$

3. Types of relations from one set to another set

(a) Empty relation: A relation R from A to B is called an empty relation or a void relation from A to B if $R = \phi$.

For example, Let

$$A = \{2, 4, 6\}, B = \{7, 11\}$$

Let $R = \{(a, b) : a \in A, b \in B \text{ and } |a - b| \text{ is even}\}$.

Here R is an empty relation.

(b) Universal relation: A relation R from A to B is said to be the universal relation if $R = A \times B$.

For example, Let

$$A = \{1, 2\}, B = \{1, 3\}$$

Let $R = \{(1, 1), (1, 3), (2, 1), (2, 3)\}$.

Here, $R = A \times B$, so relation R is a universal relation.

Note:

- The void relation *i.e.*, ϕ and universal relation *i.e.*, $A \times A$ on A are respectively the smallest and largest relations defined on the set A. Also these are also called **Trivial Relations** and other relation is called a **Non-Trivial Relation**.
- The relations $R = \phi$ and $R = A \times A$ are two **extreme relations**.

(c) Identity relation: A relation R defined on a set A is said to be the identity relation on A if $R = \{(a, b) : a \in A, b \in A$ and $a = b\}$

Thus identity relation

$R = \{(a, a) : \forall \ a \in A\}$

The identity relation on set A is also denoted by I_A.
For example, Let $A = \{1, 2, 3, 4\}$,
Then $I_A = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$.
But the relation given by
$$R = \{(1, 1), (2, 2), (1, 3), (4, 4)\}$$
is not an identity relation because element of I_A is not related to elements 1 and 3.

Note:
- In an identity relation on A every element of A should be related to itself only.

(d) Reflexive relation: A relation R defined on a set A is said to be reflexive if $a\ R\ a\ \forall\ a \in A$ *i.e.*, $(a, a) \in R\ \forall\, a \in A$.
For example, Let $A = \{1, 2, 3\}$ and R_1, R_2, R_3 be the relations given as
$$R_1 = \{(1, 1), (2, 2), (3, 3)\},$$
$$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3)\} \text{ and}$$
$$R_3 = \{(2, 2), (2, 3), (3, 2), (1, 1)\}$$
Here R_1 and R_2 are reflexive relations on A but R_3 is not reflexive as $3 \in A$ but $(3, 3) \notin R_3$.

Note:
- The universal relation on a non-void set A is reflexive.
- The identity relation is always a reflexive relation but the converse may or may not be true. As shown in the example above, R_1 is both identity as well as reflexive relation on A but R_2 is only reflexive relation on A.

(e) Symmetric relation: A relation R defined on a set A is symmetric if
$(a, b) \in R \Rightarrow (b, a) \in R\ \forall\ a, b \in A$ *i.e.*, $aRb \Rightarrow bRa$ (*i.e.*, whenever aRb then bRa).
For example, Let $A = \{1, 2, 3\}$,
$$R_1 = \{(1,2),(2,1)\}, R_2 = \{(1,2),(2,1),(1,3),(3,1)\}.$$
$$R_3 = \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$$
$$R_4 = \{(1, 3), (3, 1), (2, 3)\}$$
Here R_1, R_2 and R_3 are symmetric relations on A. But R_4 is not symmetric because $(2, 3) \in R_4$ but $(3, 2) \notin R_4$.

(f) Transitive relation: A relation R on a set A is transitive if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$
i.e., aRb and $bRc \Rightarrow aRc$.
For example, Let $A = \{1, 2, 3\}$,
$$R_1 = \{(1, 2), (2, 3), (1, 3), (3, 2)\}$$
and
$$R_2 = \{(1, 3), (3, 2), (1, 2)\}$$
Here R_2 is transitive relation whereas R_1 is not transitive because $(2, 3) \in R_1$ and $(3, 2) \in R_1$ but $(2, 2) \notin R_1$.

(g) Equivalence relation: Let A be a non-empty set, then a relation R on A is said to be an equivalence relation if

(i) R is reflexive *i.e.*,
$$(a, a) \in R\ \forall\ a \in A \text{ i.e., } aRa.$$
(ii) R is symmetric *i.e.*,
$$(a, b) \in R$$
$$\Rightarrow \quad (b, a) \in R\ \forall\ a, b \in A \text{ i.e., } aRb \Rightarrow bRa.$$
(iii) R is transitive *i.e.*,
$$(a, b) \in R \text{ and } (b, c) \in R$$
$$\Rightarrow \quad (a, c) \in R\ \forall\ a, b, c \in A$$
i.e., aRb and $bRc \Rightarrow aRc$.

For example, Let $A = \{1, 2, 3\}$
$$R = \{(1,2),(1,1),(2,1),(2,2),(3,3)(1,3),(3,1),(3,2),(2,3)\}$$
Here R is reflexive, symmetric and transitive. So R is an equivalence relation on A.

Equivalence classes: Let A be an equivalence relation in a set A and let $a \in A$. Then, the set of all those elements of A which are related to a, is called equivalence class determined by a and it is denoted by $[a]$. Thus, $[a] = \{b \in A : (a, b) \in A\}$

Mnemonics

Types of relation
RIPE STRAWBERRY TO EAT
Interpretations
Ripe – reflexive
Strawberry – Symmetric
To – transitive
Eat - Equivalence

Note:

- Two equivalence classes are either disjoint or identical.
- An equivalence relation R on a set A partitions the set into mutually disjoint equivalence classes.
- An important property of an equivalence relation is that it divides the set into pair-wise disjoint subsets called **equivalence classes** whose collection is called **a partition of the set**.

  Note that the union of all equivalence classes give the whole set.

  e.g., Let R denotes the equivalence relation in the set Z of integers given by $R = \{(a, b) : 2 \text{ divides } a - b\}$. Then the equivalence class [0] is $[0] = [0, \pm 2, \pm 4, \pm 6, ...]$.

Key Words

Disjoint: These are sets which have no elements in common.

4. Tabular representation of a relation

In this form of representation of a relation R from set A to set B, elements of A and B are written in the first column and first row respectively. If $(a, b) \in R$ then we write '1' in the row containing a and column containing b and if $(a, b) \notin R$ then we write '0' in the same manner.

For example, Let $A = \{1, 2, 3\}$,

$B = \{2, 5\}$ and $R = \{(1, 2), (2, 5), (3, 2)\}$, then

R	**2**	**5**
1	1	0
2	0	1
3	1	0

5. Inverse relation

Let $R \subseteq A \times B$ be a relation from A to B. Then, the inverse relation of R, to be denoted by R^{-1}, is a relation from B to A defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$

Thus $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1} \ \forall\, a \in A, b \in B$.

Clearly, **Domain** (R^{-1}) = **Range** of R, Range of R^{-1} = **Domain** (R).

Key Words

Domain and Range: The set of x coordinate values is called domain and the set of y coordinate values is called range.

Also, $(R^{-1})^{-1} = R$.

For example, Let $A = \{1, 2, 4\}$, $B = \{3, 0\}$ and let $R = \{(1, 3), (4, 0), (2, 3)\}$ be a relation from A to B, then $R^{-1} = \{(3, 1), (0, 4), (3, 2)\}$.

Key Facts

1. **(i)** A relation R from A to B is an empty relation or void relation if $R = \phi$
   (ii) A relation R on a set A is an empty relation or void relation if $R = \phi$
2. **(i)** A relation R from A to B is a universal relation if $R = A \times B$.
   (ii) A relation R on a set A is an universal relation if $R = A \times A$.
3. A relation R on a set A is reflexive if aRa, $\forall\, a \in A$.
4. A relation R on a set A is symmetric if whenever aRb, then bRa for all $a, b \in A$.
5. A relation R on a set A is transitive if whenever aRb and bRc then aRc for all $a, b, c \in A$.
6. A relation R on A is identity relation if $R = \{(a, a) \ \forall\, a \in A\}$ *i.e.*, R contains only elements of the type $(a, a) \ \forall\, a \in A$ and it contains no other element.
7. A relation R on a non-empty set A is an equivalence relation if the following conditions are satisfied :
   (i) R **is reflexive** *i.e.*, for every $a \in A$, $(a, a) \in R$ *i.e.*, aRa.
   (ii) R **is symmetric** *i.e.*, for $a, b \in A$, $aRb \Rightarrow bRa$ *i.e.*, $(a, b) \in R \Rightarrow (b, a) \in R$.
   (iii) R **is transitive** *i.e.*, for all $a, b, c \in A$, we have, aRb and $bRc \Rightarrow aRc$ *i.e.*, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$.

TYPES OF INTERVALS

(i) **Open Intervals:** If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying strictly between a and b is called an open interval. It is denoted by $]a, b[$ or (a, b) *i.e.*, $\{x \in R : a < x < b\}$.

(ii) **Closed Intervals:** If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying between a and b such that it includes both a and b as well is known as a closed interval. It is denoted by $[a, b]$ *i.e.*, $\{x \in R : a \leq x \leq b\}$.

(iii) **Open Closed Interval:** If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying between a and b such that it excludes a and includes only b is known as an open closed interval. It is denoted by $]a, b]$ or $(a, b]$ *i.e.*, $\{x \in R : a < x \leq b\}$.

(iv) **Closed Open Interval:** If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying between a and b such that it includes only a and excludes b is known as a closed open interval. It is denoted by $[a, b[$ or $[a, b)$ *i.e.*, $\{x \in R : a \leq x < b\}$.

Example 1

Let N denote the set of all natural numbers and R be the relation on N × N defined by (a, b) R(c, d) if $ad(b + c) = bc(a + d)$. Show that R is an equivalence relation.

Sol.

Step I : Given (a, b) R (c, d) as $ad(b + c) = bc(a + d)$

$\therefore \quad \forall a, b \in N$

or $\quad ab(b + a) = ba(a + b)$

or $\quad (a, b)$ R (a, b)

$\therefore$ R is reflexive. ...(i)

Step II : Let (a, b) R (c, d) for $(a, b), (c, d) \in N \times N$

$\therefore \quad ad(b + c) = bc(a + d)$ ...(ii)

Also, $\quad (c, d)$ R (a, b)

$\because \quad cb(d + a) = da(c + b)$

[By commutation of addition and multiplication on N]

$\therefore$ R is symmetric. ...(iii)

Step III : Let (a, b) R (c, d) and (c, d) R (e, f) for $a, b, c, d, e, f \in N$

$\therefore \quad ad(b + c) = bc(a + d)$ ...(iv)

and $\quad cf(d + e) = de(c + f)$ ...(v)

Dividing eqn. (iv) by $abcd$ and eqn. (v) by $cdef$

i.e., $\quad \frac{1}{c}+\frac{1}{b} = \frac{1}{d}+\frac{1}{a}$

and $\quad \frac{1}{e}+\frac{1}{d} = \frac{1}{f}+\frac{1}{c}$

On adding, we get

$$\frac{1}{c}+\frac{1}{b}+\frac{1}{e}+\frac{1}{d} = \frac{1}{d}+\frac{1}{a}+\frac{1}{f}+\frac{1}{c}$$

or $\quad af(b + e) = be(a + f)$

Hence, (a, b) R (e, f)

$\therefore$ R is transitive. ...(vi)

From equations (i), (iii) and (vi), R is an equivalence relation.

Topic-2 Functions

Concepts Covered • *Types of functions and their identification*

Revision Notes

1. **Domain:** If a function is expressed in the form $y = f(x)$, then domain of f means set of all those real values of x for which y is real (*i.e.*, y is well-defined).

   Remember the following points:

   (a) Negative number should not occur under the square root (even root) *i.e.*, expression under the square root sign must be always ≥ 0.

   (b) Denominator should never be zero.

   (c) For $\log_b a$ to be defined, $a > 0$, $b > 0$ and $b \neq 1$. Also note that $\log_b 1$ is equal to zero *i.e.*, 0.

2. **Range:** If a function is expressed in the form $y = f(x)$, then range of f means set of all possible real values of y corresponding to every value of x in its domain.

Remember the following points:

(a) At first find the domain of the given function.

(b) If the domain does not contain an interval, then find the values of y putting these values of x from the domain. The set of all these values of y obtained will be the range.

(c) If domain is the set of all real numbers R or set of all real numbers except a few points, then express x in terms of y and from this find the real values of y for which x is real and belongs to the domain.

3. **Function as a special type of relation:** A relation f from a set A to another set B is said be a function (or mapping) from A to B if with every element (say x) of A, the relation f relates a unique element (say y) of B. This y is called f - image of x. Also x is called pre-image of y under f.

4. **Difference between relation and function:** A relation from a set A to another set B is any subset of $A \times B$; while a function f from A to B is a subset of $A \times B$ satisfying following conditions:

(a) For every $x \in A$, there exists $y \in B$ such that $(x, y) \in f$.

(b) If $(x, y) \in f$ and $(x, z) \in f$ then, $y = z$.

S. No.	Function	Relation
(i)	Each element of A must be related to some element of B.	There may be some elements of A which are not related to any element of B.
(ii)	An element of A should not be related to more than one element of B.	An element of A may be related to more than one element of B.

5. **Real valued function of a real variable:** If the domain and range of a function f are subsets of R (the set of real numbers), then f is said to be **a real valued function of a real variable** or **a real function.**

6. Some important real functions and their domain & range

S. No. Function	Representation	Domain	Range
(i) Identity function	$I(x) = x \ \forall \ x \in R$	R	R
(ii) Modulus function or Absolute value function	$f(x) = \lvert x \rvert = \begin{cases} -x, \text{if } x < 0 \\ x, \text{ if } x \geq 0 \end{cases}$	R	$[0, \infty)$
(iii) Greatest integer function or Integral function or Step function	$f(x) = [x] \ \forall x \in R$	R	Z
(iv) Smallest integer function	$f(x) = [x] \ \forall \ x \in R$	R	Z
(v) Signum function	$f(x) = \begin{cases} \frac{\lvert x \rvert}{x}, \text{if } x \neq 0 \\ 0, \text{ if } \ x = 0 \end{cases}$ *i.e.,* $f(x) = \begin{cases} 1, \ x > 0 \\ 0, \ x = 0 \\ -1, x < 0 \end{cases}$	R	$\{-1, 0, 1\}$
(vi) Exponential function	$f(x) = a^x, \forall \ a > 0, a \neq 1$	R	$(0, \infty)$
(vii) Logarithmic function	$f(x) = \log_a x, \forall \ a \neq 1, a > 0$ and $x > 0$	$(0, \infty)$	R

7. Types of Function

(a) **One-one function (Injective function or Injection):** A function $f : A \to B$ is one-one function or injective function if distinct elements of A have distinct images in B.

Thus, $f : A \to B$ is one-one $\Leftrightarrow f(a) = f(b) \quad \Rightarrow \quad a = b, \forall a, b \in A$

$\Leftrightarrow \quad a \neq b \Rightarrow f(a) \neq f(b) \ \forall \ a, b \in A.$

- If A and B are two sets having m and n elements respectively such that $m \leq n$, then total number of one-one functions from set A to set B is ${}^nC_m \times m!$ *i.e.,* ${}^nP_m$.
- If $n(A) = n$, then the number of injective functions defined from A onto itself is $n!$.

ALGORITHM TO CHECK THE INJECTIVITY OF A FUNCTION

STEP 1: Take any two arbitrary elements a, b in the domain of f.

STEP 2: Put $f(a) = f(b)$.

STEP 3: Solve $f(a) = f(b)$. If it gives $a = b$ only, then f is a one-one function.

(b) Onto function (Surjective function or Surjection): A function $f : A \to B$ is onto function or a surjective function if every element of B is the f - image of some element of A. That implies $f(A) = B$ or range of f is the co-domain of f.

Thus, $f : A \to B$ is onto $\Leftrightarrow f(A) = B$ *i.e.*, range of f = co-domain of f.

ALGORITHM TO CHECK THE SURJECTIVITY OF A FUNCTION

STEP 1: Take an element $b \in B$, where B is the co-domain of the function.

STEP 2: Put $f(x) = b$.

STEP 3: Solve the equation $f(x) = b$ for x and obtain x in terms of b. Let $x = g(b)$.

STEP 4: If for all values of $b \in B$, the values of x obtained from $x = g(b)$ are in A, then f is onto. If there are some $b \in B$ for which values of x, given by $x = g(b)$, is not in A. Then f is not onto.

Mnemonics

Types of functions	Indian – injective
Indian Syndicate Bank	Syndicate – surjective
Interpretations	Bank - Bijective

Also note that a bijective function is also called a one-to-one function or one-to-one correspondence.

If $f : A \to B$ is a function such that,

(i) f is one-one $\Rightarrow n(A) \le n(B)$.

(ii) f is onto $\Rightarrow n(B) \le n(A)$.

For an ordinary finite set A, a one-one function $f : A \to A$ is necessarily onto and an onto function $f : A \to A$ is necessarily one-one for every finite set A.

(d) Identity function: The function $I_A : A \to A; I_A(x) = x, \forall x \in A$ is called an identity function on A.

Note:
- Domain (I_A) = A and Range $(I_A) = A$.

(e) Equal function: Two functions f and g having the same domain D are said to be equal if $f(x) = g(x)$ for all $x \in D$.

8. Constant and Types of Variables

(a) Constant: A constant is a symbol which retains the same value throughout a set of operations. So, a symbol which denotes a particular number is a constant. Constants are usually denoted by the symbols a, b, c, k, l, m, ... etc.

(b) Variable: It is a symbol which takes a number of values *i.e.*, it can take any arbitrary values over the interval on which it has been defined. *For example*, if x is a variable over R (set of real numbers) then we mean that x can denote any arbitrary real number. Variables are usually denoted by the symbols $x, y, z, u, v, \ldots$ etc.

(i) Independent variable: The variable which can take an arbitrary value from a given set is termed as an independent variable.

(ii) Dependent variable: The variable whose value depends on the independent variable is called a dependent variable.

9. Defining a Function

Consider A and B be two non-empty sets, then a rule f which associates **each element of** A **with a unique element of** B is called a function or the mapping from A to B or f maps A to B. If f is a mapping from A to B, then we write $f : A \to B$ which is read as 'f is mapping from A to B' or 'f is a function from A to B'.

If f associates $a \in A$ to $b \in B$, then we say that 'b **is the image of the element** a **under the function** f' or 'b **is the** f **- image of** a' or '**the value of** f **at** a' and denotes it by $f(a)$ and we write $b = f(a)$. The element a is called the **pre-image** or **inverse-image of** b.

Thus for a bijective function from A to B,

(a) A and B should be non-empty.

(b) Each element of A should have image in B.

(c) No element of A should have more than one image in B.

(d) If A and B have respectively m and n number of elements then the **number of functions defined from** A **to** B **is** n^m.

10. Domain, Co-domain and Range of A function

The **set** A **is called the domain** of the function f and the **set** B **is called the co- domain.** The set of the images of all the elements of A under the function f is called the **range of the function** f and is denoted as $f(A)$.

Thus range of the function f is $f(A)=\{f(x) : x \in A\}$.

Clearly $f(A) = B$ for a bijective function.

Note:
- It is necessary that every f-image is in B; but there may be some elements in B which are not the f-images of any element of A *i.e.*, whose pre-image under f is not in A.
- Two or more elements of A may have same image in B.
- $f : x \to y$ means that under the function f from A to B, an element x of A has image y in B.
- Usually we denote the function f by writing $y = f(x)$ and read it as 'y **is a function of** x'.

Example 1

Determine whether the function $f : A \to B$ defined by $f(x) = 4x + 7$, $x \in$ is one-one.

Show that no two elements in domain have same image in codomain.

Solution :

Given, $f : A \to B$ defined by $f(x) = 4x + 7, x \in A$

Let, $x_1, x_2 \in A$, such that $f(x_1) = f(x_2)$

$\Rightarrow 4x_1 + 7 = 4x_2 + 7 \Rightarrow 4x_1 = 4x_2 \Rightarrow x_1 = x_2$

So, f is one-one function.

CHAPTER-2

INVERSE TRIGONOMETRIC FUNCTIONS

Revision Notes

As we have learnt in class XI, the domain and range of trigonometric functions are given below:

S. No.	Function	Domain	Range
(i)	sine	R	$[-1, 1]$
(ii)	cosine	R	$[-1, 1]$
(iii)	tangent	$R-\left\{x : x=(2n+1)\frac{\pi}{2}; n \in Z\right\}$	R
(iv)	cosecant	$R - \{x : x = n\pi, n \in Z\}$	$R - (-1, 1)$
(v)	secant	$R-\left\{x : x=(2n+1)\frac{\pi}{2}; n \in Z\right\}$	$R - (-1, 1)$
(vi)	cotangent	$R - \{x : x = n\pi, n \in Z\}$	R

1. Inverse function

We know that if function $f : X \to Y$ such that $y = f(x)$ is **one-one** and **onto**, then we define another function $g : Y \to X$ such that $x = g(y)$, where $x \in X$ and $y \in Y$, which is also one-one and onto.

Key Words

One-one function: One to one function or one to one mapping states that each element of one set, say set A is mapped with a unique element of another set, say set B, where A and B are two different sets.
In terms of function, it states as if $f(x) = f(y) \Rightarrow x = y$, then f is one to one.

Onto function: If A and B are two sets, if for every element of B, there is atleast one or more element matching with set A, it is called onto function.

In such a case, Domain of g = Range of f

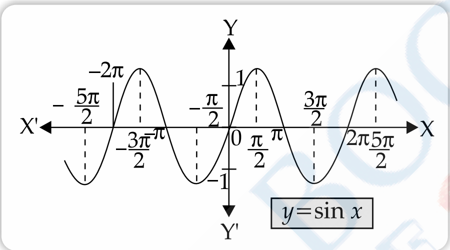
and Range of g = Domain of f

g is called the inverse of f

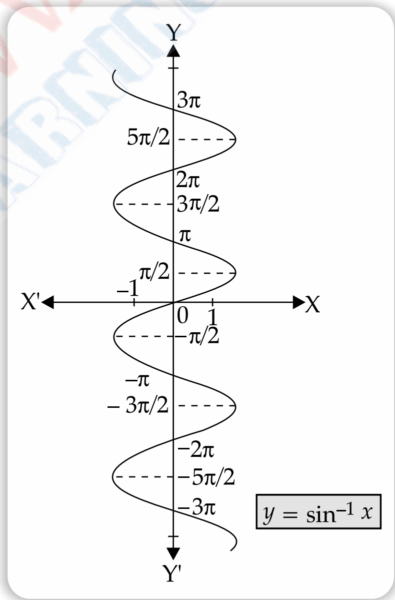
$$g = f^{-1}$$

or Inverse of $g = g^{-1} = (f^{-1})^{-1} = f$

The graph of sine function is shown here:

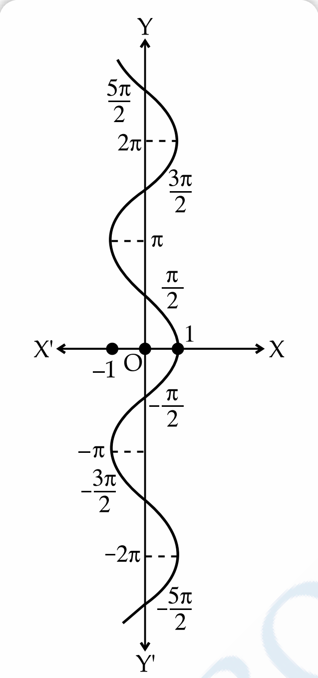


Principal value of branch function $\sin^{-1}$: It is a function with domain $[-1, 1]$ and range $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right], \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and so on corresponding to each interval, we get a branch of the function $\sin^{-1} x$. The branch with range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is called the principal value branch. Thus, $\sin^{-1} : [-1, 1] \rightarrow \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

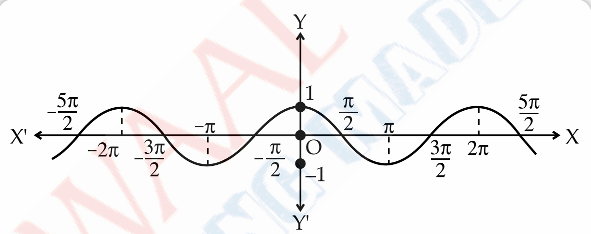


Principal value branch of function $\cos^{-1}$: The graph of the function $\cos^{-1}$ is as shown in figure. Domain of the function $\cos^{-1}$ is $[-1, 1]$. Its range in one of the intervals $(-\pi, 0)$, $(0, \pi)$, $(\pi, 2\pi)$, etc. is one-one and onto with the range $[-1, 1]$. The branch with range $(0, \pi)$ is called the principal value branch of the function $\cos^{-1}$.

Thus, $\cos^{-1} : [-1, 1] \rightarrow [0, \pi]$



$y = \cos^{-1} x$

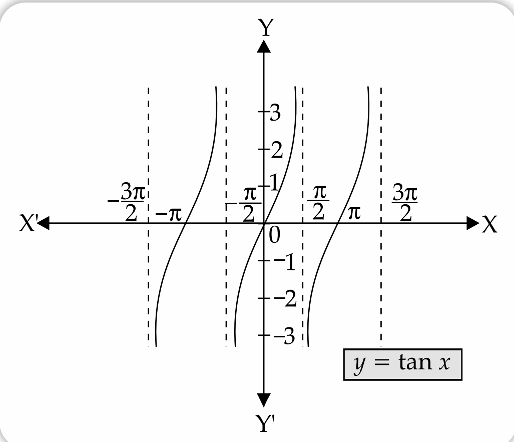


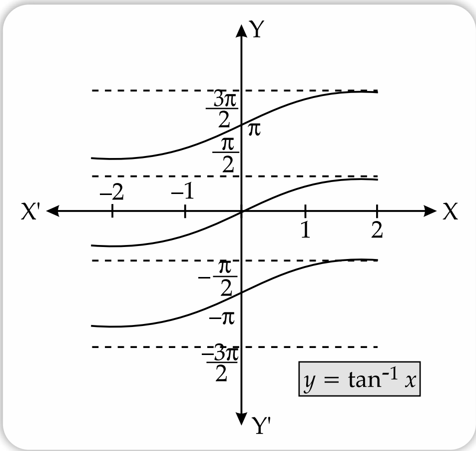
$y = \cos x$

Principal value branch function $\tan^{-1}$: The function $\tan^{-1}$ is defined whose domain is set of real numbers and range is one of the intervals,

$$\left(\frac{-3\pi}{2}, \frac{\pi}{2}\right), \left(\frac{-\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), ...$$

Graph of the function is as shown in the figure:

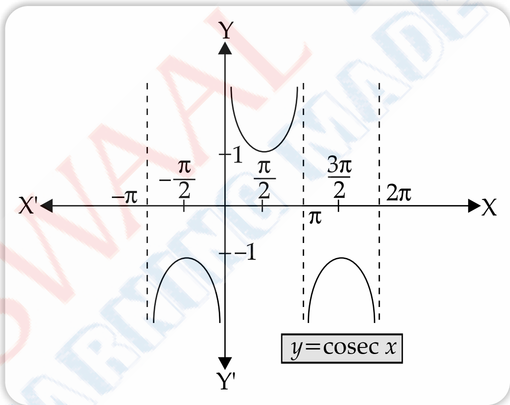


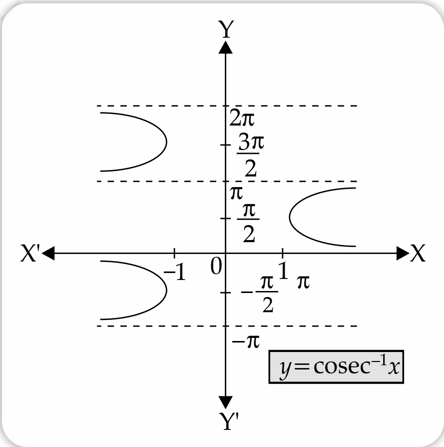


The branch with range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is called the principal value branch of function $\tan^{-1}$. Thus, $\tan^{-1} : R \rightarrow \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Principal value branch of function cosec^{-1} : The graph of function cosec^{-1} is shown in the figure. The cosec^{-1} is defined on a function whose domain is $R - (-1, 1)$ and the range is any one of the interval,

$$\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] - \{\pi\}, \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}, \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}, \ldots$$



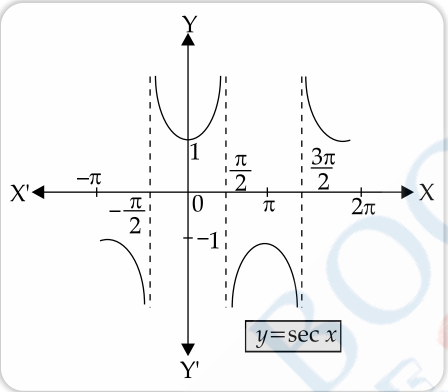


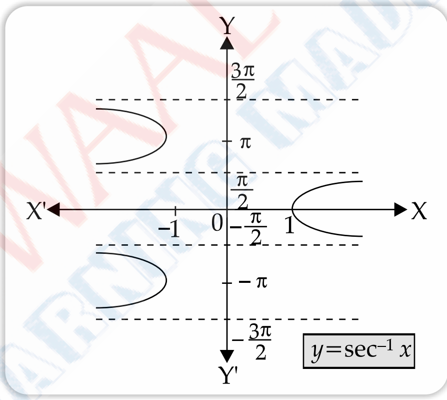
The function corresponding to the range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ is called the principal value branch of cosec^{-1}.

Thus, $\text{cosec}^{-1}: R-(-1,1) \to \left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$

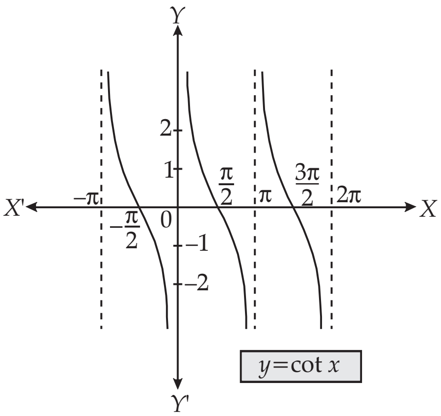
Principal value branch of function $\sec^{-1}$: The graph of function $\sec^{-1}$ is shown in figure. The $\sec^{-1}$ is defined as a function whose domain $R - (-1, 1)$ and range is $[-\pi, 0] - \left[\frac{-\pi}{2}\right]$, $[0, \pi]-\left\{\frac{\pi}{2}\right\}$, $[\pi, 2\pi]-\left\{\frac{3\pi}{2}\right\}$, etc. Function corresponding to range $[0, \pi]-\left\{\frac{\pi}{2}\right\}$ is known as the principal value branch of $\sec^{-1}$.

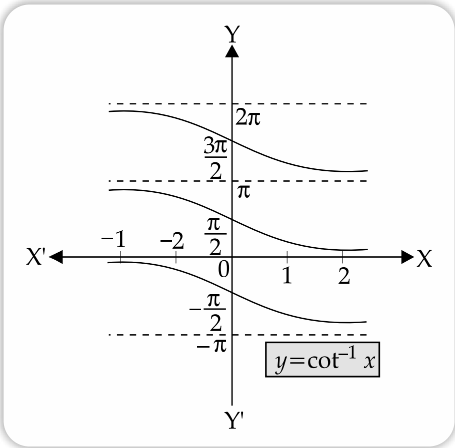
Thus, $\sec^{-1}: R-(-1,1) \to [0,\pi]-\left\{\frac{\pi}{2}\right\}$





The principal value branch of function $\cot^{-1}$:
The graph of function $\cot^{-1}$ is shown below:





The $\cot^{-1}$ function is defined on function whose domain is R and the range is any of the intervals, $(-\pi, 0)$, $(0, \pi)$, $(\pi, 2\pi)$, ...

The function corresponding to $(0, \pi)$ is called the principal value branch of the function $\cot^{-1}$.

Then, $\cot^{-1} : R \rightarrow (0, \pi)$

The principal value branch of trigonometric inverse functions is as follows:

Inverse Function	Domain	Principal Value Branch
$\sin^{-1}$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$
cosec^{-1}	$R - (-1, 1)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1}$	$R - (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\tan^{-1}$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}$	R	$(0, \pi)$

Key Facts

- Inverse trigonometric functions were considered early in the 1700s by Daniel Bernoulli, who use 'A.sin' for the inverse sine of a number, and in 1736. Euler wrote "At" for the inverse tangent.
- Inverse trigonometric functions one used to find the elevation of sun to the ground. The angle of tilt of the building can be found using inverse trigonometric functions.
- Inverse trigonometric functions help in identifying the angles of bridges to build scale models.
- Inverse trigonometric functions are often called 'arc functions', since given a value of a trigonometric function, they produce the length of arc needed to obtain that value.

(3) Principal Value:

Numerically smallest angle is known as the principal value.

Finding the principal value: For finding the principal value, following algorithm can followed :

STEP 1: First draw a trigonometric circle and mark the quadrant in which the angle may lie.

STEP 2: Select anti-clockwise direction for 1st and 2nd quadrants and clockwise direction for 3rd and 4th quadrants.

STEP 3: Find the angles in the first rotation.

STEP 4: Select the numerically least (magnitude wise) angle among these two values. The angle thus found will be the principal value.

STEP 5: In case, two angles one with positive sign and the other with the negative sign qualify for the numerically least angle then, it is the convention to select the angle with positive sign as principal value.

The principal value is never numerically greater than π.

(4) To simplify inverse trigonometric expressions, following substitutions can be considered:

Expression	Substitution
$a^2 + x^2$ or $\sqrt{a^2+x^2}$	$x = a\tan\theta$ or $x = a\cot\theta$
$a^2 - x^2$ or $\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$x^2 - a^2$ or $\sqrt{x^2-a^2}$	$x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ or $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$	$x^2 = a^2\cos 2\theta$
$\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$	$x = a\sin^2\theta$ or $x = a\cos^2\theta$
$\sqrt{\frac{x}{a+x}}$ or $\sqrt{\frac{a+x}{x}}$	$x = a\tan^2\theta$ or $x = a\cot^2\theta$

Note the following and keep them in mind:

- The symbol $\sin^{-1}x$ is used to denote the **smallest angle** whether positive or negative, such that the sine of this angle will give us x. Similarly $\cos^{-1}x$, $\tan^{-1}x$, $\text{cosec}^{-1}x$, $\sec^{-1}x$ and $\cot^{-1}x$ are defined.
- You should note that $\sin^{-1}x$ can be written as **arcsin** x. Similarly, other Inverse Trigonometric Functions can also be written as arccos x, arctan x, arcsec x etc.
- Also note that $\sin^{-1}x$ (and similarly other Inverse Trigonometric Functions) **is entirely different from** $(\sin x)^{-1}$. In fact, $\sin^{-1}x$ is the measure of an angle in Radians whose sine is x whereas $(\sin x)^{-1}$ is $\frac{1}{\sin x}$ (which is obvious as per the laws of exponents).
- Keep in mind that these inverse trigonometric relations are **true only in their domains** *i.e.*, they are valid only for some values of 'x' for which inverse trigonometric functions are well defined.

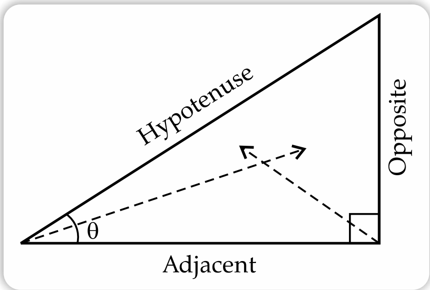
Mnemonics

Inverse trigonometric ratio can be used to find the angle of a right triangle when given two sides of the triangle.

SOH $\theta = \sin^{-1}\frac{\text{opposite}}{\text{hypotenuse}}$

CAH $\theta = \cos^{-1}\frac{\text{adjacent}}{\text{hypotenuse}}$

TOA $\theta = \tan^{-1}\frac{\text{opposite}}{\text{adjacent}}$



Key Formulae

TRIGONOMETRIC FORMULAE (ONLY FOR REFERENCE):

➢ **Relation between trigonometric ratios:**

(a) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (b) $\tan\theta = \dfrac{1}{\cot\theta}$ (c) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

(d) $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$ (e) $\sec\theta = \dfrac{1}{\cos\theta}$

➢ **Trigonometric Identities:**

(a) $\sin^2\theta + \cos^2\theta = 1$ (b) $\sec^2\theta = 1 + \tan^2\theta$ (c) $\text{cosec}^2\,\theta = 1 + \cot^2\theta$

➢ **Addition/subtraction/ formulae & some related results:**

(a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ (b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(c) $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(d) $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(e) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ (f) $\cot(A \pm B) = \dfrac{\cot B \cot A \mp 1}{\cot B \pm \cot A}$

➢ **Multiple angle formulae involving *A*, 2 *A* & 3 *A*:**

(a) $\sin 2A = 2\sin A\cos A$ (b) $\sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2}$

(c) $\cos 2A = \cos^2 A - \sin^2 A$ (d) $\cos A = \cos^2\dfrac{A}{2} - \sin^2\dfrac{A}{2}$

(e) $\cos 2A = 2\cos^2 A - 1$ (f) $2\cos^2 A = 1 + \cos 2A$

(g) $\cos 2A = 1 - 2\sin^2 A$ (h) $2\sin^2 A = 1 - \cos 2A$

(i) $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$ (j) $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(k) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ (l) $\sin 3A = 3\sin A - 4\sin^3 A$

(m) $\cos 3A = 4\cos^3 A - 3\cos A$ (n) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

➢ **Transformation of sums/differences into products & vice-versa:**

(a) $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ (b) $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

(c) $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$ (d) $\cos C - \cos D = -2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

(e) $2\sin A\cos B = \sin(A + B) + \sin(A - B)$ (f) $2\cos A\sin B = \sin(A + B) - \sin(A - B)$

(g) $2\cos A\cos B = \cos(A + B) + \cos(A - B)$ (h) $2\sin A\sin B = \cos(A - B) - \cos(A + B)$

➢ **Relations in different measures of Angle:**

(a) Angle in Radian Measure = (Angle in degree measure) × $\dfrac{\pi}{180'}$rad

(b) Angle in Degree Measure = (Angle in radian measure) × $\dfrac{180°}{\pi}$

(c) θ (in radian measure) = $\dfrac{l}{r} = \dfrac{\text{arc}}{\text{radius}}$

Also following are of importance as well:

(a) 1 right angle = 90° (b) 1° = 60′, 1′ = 60″

(c) $1° = \dfrac{\pi}{180°}$ = 0·01745 radians (Approx.) (d) 1 radian = 57°17′45″ or 206265 seconds.

➤ **General Solutions:**

(a) $\sin x = \sin y$ Or, $x = n\pi + (-1)^n y$, where $n \in Z$.

(b) $\cos x = \cos y$ Or, $x = 2n\pi \pm y$, where $n \in Z$.

(c) $\tan x = \tan y$ Or, $x = n\pi + y$, where $n \in Z$.

➤ **Relation in Degree & Radian Measures:**

Angles in Degree	0°	30°	45°	60°	90°	180°	270°	360°
Angles in Radian	0°	$\left(\frac{\pi}{6}\right)$	$\left(\frac{\pi}{4}\right)$	$\left(\frac{\pi}{3}\right)$	$\left(\frac{\pi}{2}\right)$	(π)	$\left(\frac{3\pi}{2}\right)$	(2π)

➤ **Trigonometric Ratio of Standard Angles:**

Degree	**0°**	**30°**	**45°**	**60°**	**90°**
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
$\cot x$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\operatorname{cosec} x$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec x$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

➤ **Trigonometric Ratios of Allied Angles:**

Angles (→) / **T – Ratios (↓)**	$\frac{\pi}{2}-\theta$	$\frac{\pi}{2}+\theta$	$\pi-\theta$	$\pi+\theta$	$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$	$2\pi-\theta$ **or** $-\theta$	$2\pi+\theta$
sin	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
cos	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$	$\cos\theta$
tan	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$
cot	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$
sec	$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta$	$-\sec\theta$	$-\sec\theta$	$-\operatorname{cosec}\theta$	$\operatorname{cosec}\theta$	$\sec\theta$	$\sec\theta$
cosec	$\sec\theta$	$\sec\theta$	$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta$	$-\sec\theta$	$-\sec\theta$	$-\operatorname{cosec}\theta$	$\operatorname{cosec}\theta$

UNIT – II : ALGEBRA

CHAPTER-3

MATRICES

Topic-1 Matrices and Operations

Concepts Covered • *Basic concept of matrices,* • *Types of matrices,* • *Operations on matrices*

Revision Notes

1. MATRIX - BASIC INTRODUCTION:

A matrix is an ordered rectangular **array** of numbers (real or complex) or functions which are known as elements or the entries of the matrix. It is denoted by the uppercase letters *i.e.* A, B, C etc.

Key Words

Array: An array is a rectangular arrangement of objects in equal rows (horizontal) and equal columns (vertical). Everyday example of arrays include a muffin tray and an egg carton.

Consider a matrix A given as,

Here in matrix A the horizontal lines of elements are said to constitute **rows** and vertical lines of elements are said to constitute **columns** of the matrix. Thus, matrix A has m **rows** and n **columns**. The array is enclosed by square brackets [ ], the parentheses ( ) or the double vertical bars || ||.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

- A matrix having m rows and n columns is called a matrix of order $m\times n$ (read as 'm by n' matrix). A matrix A of order $m \times n$ is depicted as $A = [a_{ij}]_{m \times n}$; $i, j \in N$.
- Also in general, a_{ij} means an element lying in the i^{th} row and j^{th} column.
- Number of elements in the matrix $A = [a_{ij}]_{m \times n}$ is given as mn.

2. TYPES OF MATRICES:

(i) Column matrix: A matrix having only one column is called a **column matrix** or **column vector**.

e.g : $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}_{3\times 1}, \begin{bmatrix} 4 \\ 5 \end{bmatrix}_{2\times 1}$

General notation : $A = [a_{ij}]_{m\times 1}$

Key Facts

- The term matrix was introduced by the 19^{th} century English Mathematician James Sylvester, but it was his friend the Mathematics Arthur Cayley who developed the algebraic aspect of matrices in two papers in the 1850s.
- The English Mathematician Cuthbert Edmund Cullis was the first to use modern bracket notation for matrices in 1913.

(ii) Row matrix: A matrix having only one row is called a **row matrix** or **row vector**.

e.g : $\begin{bmatrix} 2 & 5 & -4 \end{bmatrix}_{1\times 3}, \begin{bmatrix} \sqrt{2} & 4 \end{bmatrix}_{1\times 2}$

General notation : $A = [a_{ij}]_{1\times n}$

(iii) Square matrix: It is a matrix in which the number of rows is equal to the number of columns *i.e.*, an $n\times n$ matrix is said to constitute a square matrix of order $n\times n$ and is known as a **square matrix of order 'n'.**

e.g : $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & -4 \\ 0 & -1 & -2 \end{bmatrix}_{3\times 3}$ is a square matrix of order.

General notation : $A = [a_{ij}]_{n\times n}$

(iv) Diagonal matrix: A square matrix $A = [a_{ij}]_{m\times m}$ is said to be a **diagonal matrix** if all the elements, except those in the leading diagonal are zero *i.e.*, $a_{ij} = 0$, for all $i \neq j$.

e.g : $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3}$ is a diagonal matrix of order 3.

- Also there are **more notations** specifically used for the diagonal matrices. For instance, consider the matrix given above, it can also be written as diag (2, 5, 4) or diag [2, 5, 4].
- Note that the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{mm}$ of a square matrix $A = [a_{ij}]_{m\times m}$ of **order** m are said to constitute the **principal diagonal** or simply **the diagonal of the square matrix** A. These elements are known as **diagonal elements of matrix** A.

(v) Scalar matrix: A diagonal matrix $A = [a_{ij}]_{m\times m}$ is said to be a **scalar matrix** if its diagonal elements are equal.

i.e., $a_{ij} = \begin{cases} 0, & \text{when } i \neq j \\ k, & \text{when } i = j \text{ for some constant } k \end{cases}$

$e.g: \begin{bmatrix} 17 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 17 \end{bmatrix}_{3\times3}$ is a scalar matrix of order **3.**

(vi) Unit or Identity matrix: A square matrix $A = [a_{ij}]_{m\times m}$ is said to be an **identity matrix** if $a_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$

A **unit matrix** can also be defined as the **scalar matrix** in which all diagonal elements are equal to **unity**. We denote the identity matrix of order m by I_m or I.

$e.g: I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}.$

(vii) Zero matrix or Null matrix: A matrix is said to be a **zero matrix** or **null matrix** if each of its elements is '0'.

$e.g.,: \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}, [0 \quad 0]_{1\times2.}$

(viii) Horizontal matrix: A $m\times n$ matrix is said to be a **horizontal matrix** if $m < n$.

$e.g: \begin{bmatrix} 1 & 2 & 5 \\ 4 & 8 & -9 \end{bmatrix}_{2\times3}$

(ix) Vertical matrix: A $m\times n$ matrix is said to be a **vertical matrix** if $m > n$.

$e.g: \begin{bmatrix} -5 & -1 \\ 8 & -9 \\ 4 & 0 \end{bmatrix}_{3\times2}$

3. EQUALITY OF MATRICES:

Two matrices A and B are said to be equal and written as $A = B$, if they are of the **same order** and their **corresponding elements are identical** i.e. $a_{ij} = b_{ij}$ i.e., $a_{11} = b_{11}, a_{22} = b_{22}, a_{32} = b_{32}$ etc.

4. ADDITION OF MATRICES:

If A and B are two $m \times n$ matrices, then another $m \times n$ matrix obtained by adding the corresponding elements of the matrices A and B is called the sum of the matrices A and B and is denoted by '$A + B$'.

Thus if $A = [a_{ij}], B = [b_{ij}]$, or $A + B = [a_{ij} + b_{ij}]$.

Properties of matrix addition:

- Commutative property:

$$A + B = B + A$$

- Associative property:

$$A + (B + C) = (A + B) + C$$

- Cancellation laws:

(i) Left cancellation: $A + B = A + C \Rightarrow B = C$

(ii) Right cancellation: $B + A = C + A \Rightarrow B = C$.

5. MULTIPLICATION OF A MATRIX BY A SCALAR:

If a $m \times n$ matrix A is multiplied by a scalar k (say), then the new kA matrix is obtained by multiplying each element of matrix A by scalar k. Thus, if $A = [a_{ij}]$ and it is multiplied by a scalar k, then $kA = [ka_{ij}]$, i.e. $A = [a_{ij}]$ or $kA = [ka_{ij}]$.

e.g :

$$A = \begin{bmatrix} 2 & -4 \\ 5 & 6 \end{bmatrix} \text{ or } 3A = \begin{bmatrix} 6 & -12 \\ 15 & 18 \end{bmatrix}$$

6. MULTIPLICATION OF TWO MATRICES:

Let $A = [a_{ij}]$ be a $m \times n$ matrix and $B = [b_{jk}]$ be a $n \times p$ matrix such that the number of columns in A is equal to the number of rows in B, then the $m \times p$ matrix $C = [c_{ik}]$ such that $[c_{ik}] = \Sigma_{j=1}^{n} a_{ij} b_{jk}$ is said to be the product of the matrices A and B in that order and it is denoted by AB *i.e.* "$C = AB$".

Properties of matrix multiplication:

- Note that the product AB is defined only when the number of columns in matrix A is equal to the number of rows in matrix B.
- If A and B are $m \times n$ and $n \times p$ matrices, respectively, then the matrix AB will be an $m \times p$ matrix *i.e.*, order of matrix AB will be $m \times p$.
- In the product AB, A is called the **pre-factor** and B is called the **post-factor**.
- If two matrices A and B are such that AB is possible then it is not necessary that the product BA is also possible.
- If A is a $m \times n$ matrix and both AB as well as BA are defined, then B will be a $n \times m$ matrix.
- If A is a $n \times n$ matrix and I_n be the unit matrix of order n, then $A I_n = I_n A = A$.
- Matrix multiplication is **associative** *i.e.*, $A(BC) = (AB)C$.
- Matrix multiplication is **distributive** over the **addition** *i.e.*, $A.(B + C) = AB + AC$.
- Matrix multiplication is not commutative.

7. IDEMPOTENT MATRIX:

A square matrix A is said to be an idempotent matrix if $A^2 = A$.

For example,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \text{ are idempotent matrices.}$$

8. TRANSPOSE OF A MATRIX:

If $A = [a_{ij}]_{m \times n}$ be a $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of matrix A is said to be a **transpose of matrix** A. The transpose of A is denoted by A' or A^T *i.e.*, if $A^T = [a_{ji}]_{n \times m}$.

For example,

$$\begin{bmatrix} 5 & -4 & 1 \\ 0 & \sqrt{5} & 3 \end{bmatrix}^T = \begin{bmatrix} 5 & 0 \\ -4 & \sqrt{5} \\ 1 & 3 \end{bmatrix}$$

PROPERTIES OF TRANSPOSE OF MATRICES:

(i) $(A + B)^T = A^T + B^T$

(ii) $(A^T)^T = A$

(iii) $(kA)^T = kA^T$, where k is any constant

(iv) $(AB)^T = B^T A^T$

(v) $(ABC)^T = C^T B^T A^T$

Mnemonics

Types of Matrices

Ram → **R**ow Matrix

Charan → **C**olumn Matrix

Says → **S**quare Matrix

Drink → **D**iagonal Matrix

Sprite → **S**calar Matrix

and **N**escafe → **N**ull Matrix

Ice → **I**dentity Matrix

Tea → **T**riangular Matrix

Matrix Multiplication

No. of columns of first matrix = No. of rows of second matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix}$$

R × **C**   **R** × C

Class **R**epresentative

Topic-2 Symmetric, Skew Symmetric and Invertible Matrices

Concepts Covered • *Symmetric Matrix,* • *Skew Symmetric Matrix,* • *Invertible Matrix* • *Uniqueness Theorem*

Revision Notes

Symmetric matrix: A square matrix $A = [a_{ij}]$ is said to be a **symmetric matrix** if $A^T = A$. *i.e.*, if $A = [a_{ij}]$, then $A^T = [a_{ji}] = [a_{ij}]$ or $A^T = A$.

For example :

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \begin{bmatrix} 2+i & 1 & 3 \\ 1 & 2 & 3+2i \\ 3 & 3+2i & 4 \end{bmatrix}$$

Skew symmetric matrix: A square matrix $A = [a_{ij}]$ is said to be a **skew symmetric matrix** if

$A^T = -[A]$ *i.e.*, if $A = [a_{ij}]$, then $A^T = [a_{ji}] = -[a_{ij}]$ or $A^T = -A$.

For example : $\begin{bmatrix} 0 & 1 & -5 \\ -1 & 0 & 5 \\ 5 & -5 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

Orthogonal matrix: A matrix A is said to be **orthogonal** if $A.A^T = I$, where A^T is transpose of A.

Invertible Matrix: An invertible matrix is a matrix for which matrix inversion operation exists, given that it satisfies the requisite conditions. Any given square matrix A of order $n \times n$ is called invertible if there exists another $n \times n$ square matrix B such that, AB = BA = I_n, where I_n is on identity matrix of order $n \times n$.

Example: Let matrix A = $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and matrix B = $\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

Now, AB = $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and BA = $\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Hence, A^{-1} = B and B is called the inverse of A.

So, A can also be the inverse of B or B^{-1} = A.

Uniqueness of Inverse of Matrix

If there exists an inverse of a square matrix, it is always unique.

Proof: Let A be a square matrix of order $n \times n$. Let us assume matrices B and C be inverses of matrix A.

Now, AB = BA = I, since B is the inverse of matrix Δ.

Similarly, AC = CA = I

But, B = BI = B(AC) = (BA)C = IC = C

This proves B = C, or B and C are the same matrices.

Key Fact

- Note that $[a_{ji}] = -[a_{ij}]$ or $[a_{ii}] = -[a_{ii}]$ or $2[a_{ii}] = 0$ (Replacing j by i). *i.e.*, all the diagonal elements in a skew symmetric matrix are zero.
- For any matrices, AA^T and A^TA are symmetric matrices.
- For a square matrix A, the matrix $A + A^T$ is a symmetric matrix and $A - A^T$ is always a skew-symmetric matrix.
- Also note that any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix *i.e.*, $A = P + Q$ where $P = \frac{A+A^T}{2}$ is a symmetric matrix and $Q = \frac{A-A^T}{2}$ is a skew symmetric matrix.

CHAPTER-4
DETERMINANTS

Topic-1 Determinants, Minors & Co-factors

Concepts Covered • *Determinant value of a matrix,* • *Co-factor and Minor of a matrix* • *Inverse of matrix using Adjoint method,* • *Area of triangle with the help of determinant*

Revision Notes

Determinants, Minors & Co-factors

(a) Determinant: A unique number (real or complex) can be associated to every square matrix $A = [a_{ij}]$ of order m. This number is called the determinant of the square matrix A, where $a_{ij} = (i, j)^{th}$ element of A.

For instance, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then, determinant of matrix A is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A)$ and its value is given by $ad - bc$.

(b) Minors: Minors of an element a_{ij} of a determinant *(or a determinant corresponding to matrix A)* is the determinant obtained by deleting its i^{th} row and j^{th} column in which a_{ij} lies. Minor of a_{ij} is denoted by M_{ij}. Hence, we can get 9 minors corresponding to the 9 elements of a third order (*i.e.*, 3 × 3) determinant.

(c) Co-factors : Cofactor of an element a_{ij}, denoted by A_{ij}, is defined by $A_{ij} = (-1)^{(i+j)} M_{ij}$, where M_{ij} is minor of a_{ij}. Sometimes C_{ij} is used in place of A_{ij} to denote the co-factor of element a_{ij}.

1. ADJOINT OF A SQUARE MATRIX:

Let $A = [a_{ij}]$ be a square matrix. Also, assume $B = [A_{ij}]$, where A_{ij} is the cofactor of the elements a_{ij} in matrix A. Then the transpose B^T of matrix B is called the **adjoint of matrix** A and it is denoted by "*adj* (A)".

To find adjoint of a 2 × 2 matrix: Follow this, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ or $adj\ A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

For example, consider a square matrix of order 3 as $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$, then in order to find theadjoint matrix A, we find a matrix B (formed by the co-factors of elements of matrix A as mentioned above in the definition)

i.e., $B = \begin{bmatrix} 15 & -2 & -6 \\ -10 & -1 & 4 \\ -1 & 2 & -1 \end{bmatrix}$. Hence, $adj\ A = B^T = \begin{bmatrix} 15 & -10 & -1 \\ -2 & -1 & 2 \\ -6 & 4 & -1 \end{bmatrix}$

2. SINGULAR MATRIX AND NON-SINGULAR MATRIX:

(a) Singular matrix: A square matrix A is said to be singular if $|A| = 0$ *i.e.,* its determinant is zero.

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 12 \\ 1 & 1 & 3 \end{bmatrix}$

$= 1(15 - 12) - 2(12 - 12) + 3(4 - 5) = 3 - 0 - 3 = 0$

$\therefore$ A is singular matrix.

$B = \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix} = 12 - 12 = 0$

$\therefore$ B is singular matrix.

(b) Non-singular matrix: A square matrix A is said to be non-singular if $|A| \neq 0$.

e.g. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$= 0(0 - 1) - 1(0 - 1) + 1(1 - 0)$

$= 0 + 1 + 1 = 2 \neq 0$

$\therefore$ A is non-singular matrix.

- A square matrix A is **invertible** if and only if A is **non-singular**.

3. **ALGORITHM TO FIND A^{-1} BY DETERMINANT METHOD:**

STEP 1: Find $|A|$.

STEP 2: If $|A| = 0$, then, write "A is a singular matrix and hence not invertible". Else write "A is a non-singular matrix and hence invertible".

STEP 3: Calculate the co-factors of elements of matrix A.

STEP 4: Write the matrix of co-factors of elements of A and then obtain its transpose to get $adj.A$ (*i.e.*, adjoint A).

STEP 5: Find the inverse of A by using the relation $A^{-1} = \frac{1}{|A|}(adj\,A)$.

4. **AREA OF TRIANGLE:**

Area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \text{ sq. units}$$

- Since area is a positive quantity, we take absolute value of the determinant.
- If the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear, then $\Delta = 0$.
- The equation of a line passing through the points (x_1, y_1) and (x_2, y_2) can be obtained by the expression given here:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Key Facts

- In mathematics, the determinant is a scalar value that is a function of the entries of a square matrix.
- There are 10 main properties of determinants which include reflection property, all-zero property, proportionality or repetition property, switching property, scalar multiple property, sum properly, invariance properly, factor properly, triangle properly and w-factor properly.

Topic-2 Solutions of System of Linear Equations

Concepts Covered • *Unique Solution,* • *Consistent System,* • *Inconsistent System*

Revision Notes

SOLVING SYSTEM OF EQUATIONS BY MATRIX METHOD [INVERSE MATRIX METHOD]

(a) Homogeneous and Non-homogeneous system : A system of equations $AX = B$ is said to be a homogeneous system if $B = O$. Otherwise it is called a non-homogeneous system of equations.

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3$$

STEP 1 : Assume

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \text{ and } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

STEP 2 : Find $|A|$. Now there may be following situations :

(i) $|A| \neq 0 \Rightarrow A^{-1}$ exists. It implies that the given system of equations is consistent and therefore, the system has **unique solution**. In that case, write

$$AX = B$$

$$\Rightarrow \quad X = A^{-1}B \qquad \left[\text{where } A^{-1} = \frac{1}{|A|}(adj\,A)\right]$$

Then by using the definition of equality of matrices, we can get the values of x, y and z.

(ii) $|A| = 0 \Rightarrow A^{-1}$ does not exist. It implies that the given system of equations may be **consistent** or **inconsistent**.

Key Words

Consistent System: A system is considered to be consistent if it has atleast one solution.
Inconsistent System: If a system has no solution, it is said to be inconsistent.

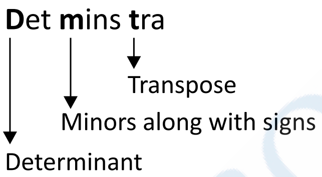
In order to check proceed as follow:

$\Rightarrow$ Find $(adj\ A)\ B$. Now, we may have either $(adj\ A)\ B \neq O$ or $(adj\ A)\ B = O$.

- If $(adj\ A)B = O$, then the given system may be consistent or inconsistent.
  To check, put $z = k$ in the given equations and proceed in the same manner in the new two variables system of equations assuming $d_i - c_i k$, $1 \le i \le 3$ as constant.
- And if $(adj\ A)\ B \neq O$, then the given system is inconsistent with no solutions.

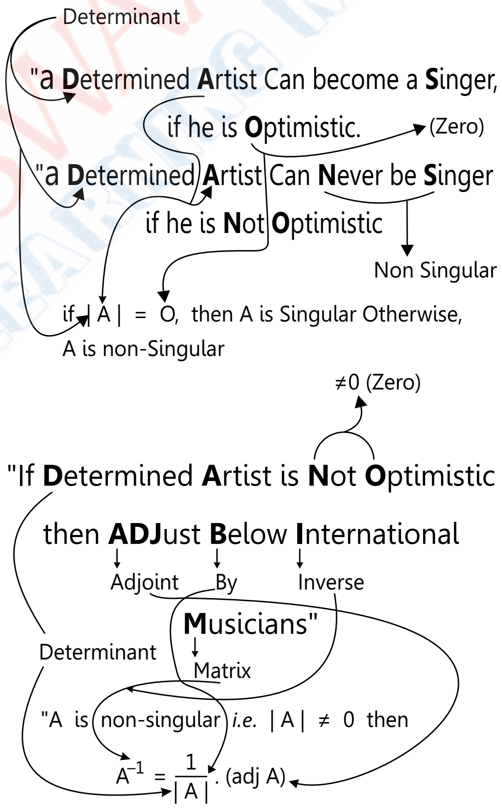
Mnemonics-1

Inverse of a Square Matrix



Mnemonics-2

Singular Matrix
A square matrix is said to be singular matrix if determinant of matrix denoted by |A| is zero otherwise it is non pingular matrix
Inverse Of a Matrix



Interpretation :
Singular & Non Singular Matrix -
if $|A| = 0$, then A is singular. Otherwise A is non-singular
Inverse of a Matrix–

Inverse of a Matrix exists if A is non- singular *i.e* $|A| \neq 0$,and is given by $A^{-1} = \frac{1}{|A|} \text{adj } A$

UNIT – III: CALCULUS

CHAPTER-5
CONTINUITY & DIFFERENTIABILITY

Topic-1 Continuity

Concepts Covered • *Left hand Limit,* • *Right Hand Limit*

Revision Notes

FORMULAE FOR LIMITS:

(a) $\lim_{x\to 0} \cos x = 1$

(b) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

(c) $\lim_{x\to 0} \frac{\tan x}{x} = 1$

(d) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1$

(e) $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$

(f) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a,\ a > 0$

(g) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(h) $\lim_{x\to 0} \frac{\log_e(1+x)}{x} = 1$

(i) $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

- For a function $f(x)$, $\lim_{x\to m} f(x)$ exists if $\lim_{x\to m^-} f(x) = \lim_{x\to m^+} f(x)$.

- A function $f(x)$ is continuous at a point $x = m$ if, $\lim_{x\to m^-} f(x) = \lim_{x\to m^+} f(x) = f(m)$, where $\lim_{x\to m^-} f(x)$ is **Left Hand Limit** of $f(x)$ at $x = m$ and $\lim_{x\to m^+} f(x)$ is **Right Hand Limit** of $f(x)$ at $x = m$. Also $f(m)$ is the value of function $f(x)$ at $x = m$.

- A function $f(x)$ is continuous at $x = m$ (say) if, $f(m) = \lim_{x\to m} f(x)$ *i.e.*, a function is continuous at a point in its domain if the **limit value of the function** at that point **equals** the value of the function at the same point.

- For a continuous function $f(x)$ at $x = m$, $\lim_{x\to m} f(x)$ can be directly obtained by evaluating $f(m)$.

- Indeterminate forms or meaningless forms: $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 1^{\infty}, 0^0, \infty^0$.

Topic-2 Differentiability

Concepts Covered • *Left Hand Derivative,* • *Right Hand Derivative,* • *Relation between Continuity and Differentiability*

Derivative of Some Standard Functions:

(a) $\frac{d}{dx}(x^n) = nx^{n-1}$

(b) $\frac{d}{dx}(k) = 0$, where k is any constant

(c) $\frac{d}{dx}(a^x) = a^x \log_e a,\ a > 0$

(d) $\frac{d}{dx}(e^x) = e^x$

(e) $\frac{d}{dx}(\log_a x) = \frac{1}{x\log_e a} = \frac{1}{x}\log_a e$

(f) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(g) $\frac{d}{dx}(\sin x) = \cos x$

(h) $\frac{d}{dx}(\cos x) = -\sin x$

(i) $\frac{d}{dx}(\tan x) = \sec^2 x$

(j) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(k) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(l) $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

(m) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, x \in (-1, 1)$

(n) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, x \in (-1, 1)$

(o) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, x \in R$

(p) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, x \in R$

(q) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$, where $x \in (-\infty, -1) \cup (1, \infty)$

(r) $\frac{d}{dx}(\text{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$, where $x \in (-\infty, -1) \cup (1, \infty)$

Following derivatives should also be memorized by you for quick use:

(i) $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$

(ii) $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

➲ **Left Hand Derivative of $f(x)$ at $x = m$,**

$$Lf'(m) = \lim_{x \to m^-} \frac{f(x) - f(m)}{x - m} \text{ and,}$$

Right Hand Derivative of $f(x)$ at $x = m$,

$$Rf'(m) = \lim_{x \to m^+} \frac{f(x) - f(m)}{x - m}$$

For a **function to be differentiable** at a point, **LHD and RHD at that point should be equal**.

➲ **Derivative of y *w.r.t.* x:** $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$.

Also, for very-very small value h, $f'(x) = \frac{f(x+h) - f(x)}{h}$, (as $h \to 0$)

Relation between Continuity and Differentiability:

(i) If a function is differentiable at a point, it is continuous at that point as well.

(ii) If a function is not differentiable at a point, it may or may not be continuous at that point.

(iii) If a function is continuous at a point, it may or may not differentiable at that point.

(iv) If a function is **discontinuous** at a point, it is not be differentiable at that point.

Rules of Derivatives:

- Product or Leibniz's rule of derivatives:

$$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$$

- Quotient Rule of derivatives:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2} = \frac{vu' - uv'}{v^2}.$$

Key Word

Discontinuous Function: A discontinuous function is a function in algebra that has a point where either the function is not defined at that point or the LHL and RHL of the function are equal but not equal to the value of the function at that point or the limit of the function does not exist at the given point.

Key Facts

- All differentiable functions happen to be continuous but not all continuous functions can said to be differentiable.
- A function is said to be continuously differentiable if the derivative exists and is itself a continuous function.
- $f(x) = 0$ is a continuous function because it is an unbroken line, without holes or jumps.
- If $f(0) = \infty$, then function is continuous at 0.
- All polynomial functions are continuous functions.

Mnemonics

Quotient Rule of Derivative

Ho D Hi Minus Hi D Ho Over Ho Ho

In mathematical notation, $\dfrac{\text{Ho D Hi} - \text{Hi D Ho}}{\text{ho ho}}$

where, Ho → function in numerator

Hi → function in denominator

D → derivative of

CHAPTER-6
APPLICATIONS OF DERIVATIVES

Topic-1 Rate of Change of Bodies

Revision Notes

Interpretation of $\frac{dy}{dx}$ as a rate measure:

If two variables x and y are varying with respect to another variables say t, *i.e.*, if $x = f(t)$, then by the Chain Rule, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \frac{dx}{dt} \neq 0$$

Thus, the rate of change of y with respect to x can be calculated by using the rate of change of y and that of x both with respect to t.

Also, if y is a function of x and they are related as $y = f(x)$ then, $f'(\alpha)$, *i.e.*, represents the rate of change of y with respect to x at the instant when $x = \alpha$.

Topic-2 Increasing/Decreasing Functions

Concepts Covered • *Increasing function,* • *Decreasing function,* • *Constant function* • *Monotonic function*

Revision Notes

1. A function $f(x)$ is said to be an increasing function in $[a, b]$, if as x increases, $f(x)$ also increases *i.e.*, if $\alpha, \beta \in [a, b]$ and $\alpha > \beta$, $f(\alpha) > f(\beta)$.

   If $f'(x) \geq 0$ lies in (a, b), then $f(x)$ is an increasing function in $[a, b]$, provided $f(x)$ is continuous at $x = a$ and $x = b$.

2. A function $f(x)$ is said to be a **decreasing function** in $[a, b]$, if, as x increases, $f(x)$ decreases *i.e.*, if $\alpha, \beta \in [a, b]$ and $\alpha > \beta \Rightarrow f(\alpha) < f(\beta)$.

   If $f'(x) \leq 0$ lies in (a, b), then $f(x)$ is a decreasing function in $[a, b]$ provided $f(x)$ is continuous at $x = a$ and $x = b$.

➲ A function $f(x)$ is a **constant function** in $[a, b]$ if $f'(x) = 0$ for each $x \in (a, b)$.

➲ By **monotonic function** $f(x)$ in interval I, we mean that f is either **only increasing** in I or **only decreasing** in I.

3. **Finding the intervals of increasing and/or decreasing of a function:**

ALGORITHM

STEP 1: Consider the function $y = f(x)$.

STEP 2: Find $f'(x)$.

STEP 3: Put $f'(x) = 0$ and solve to get the critical point(s).

STEP 4: The value(s) of x for which $f'(x) > 0$, $f(x)$ is increasing; and the value(s) of x for which $f'(x) < 0$, $f(x)$ is decreasing.

Topic-3 Maxima and Minima

Concepts Covered • *Local Maxima,* • *Local Minima,* • *Absolute Maxima,* • *Absolute Minima,* • *First derivative test,* • *Second derivative test*

Revision Notes

1. **Understanding maxima and minima:**

Consider $y = f(x)$ be a well defined function on an **interval** I, then

Key Word

Interval: In mathematics, an interval is a set of real numbers between two given numbers called end points of the interval.

(a) f is said to have a **maximum value** in I, if there exists a point c in I such that $f(c) > f(x)$, for all $x \in I$.

The value corresponding to $f(c)$ is called the maximum value of f in I and the point c is called the **point of maximum value of f in I.**

(b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) < f(x)$, for all $x \in I$.

The value corresponding to $f(c)$ is called the minimum value of f in I and the point c is called the **point of minimum value of f in I.**

(c) f is said to have an **extreme value** in I, if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I.
The value $f(c)$ in this case, is called an extreme value of f in I and the point c called an **extreme point**.

Know the Terms

1. Let f be a real valued function and also take a point c from its **domain**, then

Key Word

Domain: The domain refers to the set of possible input values, the domain of a graph consists of all input values shown on the x-axis.

(i) c is called a point of **local maxima** if there exists a number $h > 0$ such that $f(c) > f(x)$, for all x in $(c - h, c + h)$. The value $f(c)$ is called the **local maximum value of** f.

(ii) c is called a point of **local minima** if there exists a number $h > 0$ such that $f(c) < f(x)$, for all x in $(c - h, c + h)$. The value $f(c)$ is called the **local minimum value of** f.

2. Critical points

It is a point c (say) in the domain of a function $f(x)$ at which either $f'(x)$ vanishes *i.e.*, $f'(c) = 0$ or f is not differentiable.

3. First Derivative Test:

Consider $y = f(x)$ be a well defined function on an open interval I. Now proceed as have been mentioned in the following algorithm:

STEP 1: Find $\frac{dy}{dx}$.

STEP 2: Find the critical point(s) by putting $\frac{dy}{dx} = 0$. Suppose $c \in I$ (where I is the interval) be any critical point point and f be continuous at this point c. Then we may have following situations :

- $\frac{dy}{dx}$ changes sign from **positive to negative** as x increases through c, then the function attains a **local maximum** at $x = c$.
- $\frac{dy}{dx}$ changes sign from **negative to positive** as x increases through c, then the function attains a **local minimum** at $x = c$.
- $\frac{dy}{dx}$ **does not change sign** as x increases through c, then $x = c$ is **neither** a point of **local maximum nor** a point of **local minimum**. Rather in this case, the point $x = c$ is called the **point of inflection**.

4. Second Derivative Test:

Consider $y = f(x)$ be a well defined function on an open interval I and twice differentiable at a point c in the interval. Then we observe that:

- $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$.
  The value $f(c)$ is called the local maximum value of f.
- $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$
  The value $f(c)$ is called the local minimum value of f.
  This test fails if $f'(c) = 0$ and $f''(c) = 0$. In such a case, we use **first derivative test** as discussed above.

5. Absolute maxima and absolute minima:

If f is a continuous function on a **closed interval** I, then f has the absolute maximum value and f attains it atleast once in I. Also f has the absolute minimum value and the function attains it atleast once in I.

ALGORITHM

STEP 1: Find all the critical points of f in the given interval, *i.e.*, find all the points x where either $f'(x) = 0$ or f is not differentiable.
STEP 2: Take the end points of the given interval.
STEP 3: At all these points (*i.e.*, the points found in (STEP 1 and STEP 2) calculate the values of f.
STEP 4: Identify the maximum and minimum value of f out of the values calculated in STEP 3. This maximum value will be the **absolute maximum value** of f and the minimum value will be the **absolute minimum value** of the function f.

Absolute maximum value is also called as **globalmaximum value** or **greatest value**. Similarly absolute minimum value is called as **global minimum value** or the **least value.**

Key Facts

- Rate of change itself is something that we use daily, like comparing one's salary, the weather or even how long it takes for a car to arrive some place.
- When a cycle moves along a road, then the road becomes the tangent at each point when the wheel rolls on it.
- Maxima and minina is used to solve optimization problems such as maximizing profit, minimizing the amount of material used in manufacturing or finding the maximum height a rocket can reach.

CHAPTER-7

INTEGRALS

Topic-1 Indefinite Integral

Concepts Covered • *Meaning of Integral of function* • *Integration by Substitution* • *Integration by partial fraction* • *Integration by parts* • *Formulae for indefinite Integral*

Revision Notes

➢ **Meaning of Integral of Function**

If differentiation of a function $F(x)$ is $f(x)$ *i.e.*, if $\frac{d}{dx}[F(x)] = f(x)$, then we say that one integral or primitive or anti-derivative of $f(x)$ is $F(x)$ and in symbols, we write, $\int f(x)dx = F(x) + C$.

Therefore, we can say that integration is the inverse process of differentiation.

➢ **Methods of Integration**

(a) Integration by Substitution Method :

In this method, we change the integral $\int f(x)dx$, where independent variable is x, to another integral in which independent variable is t (say) different from x such that x and t are related by $x = g(t)$.

Let $\quad u = \int f(x)dx$ then, $\frac{du}{dx} = f(x)$

Again as $\quad x = g(t)$ so we have $\frac{dx}{dt} = g'(t)$

Now $\quad \frac{du}{dt} = \frac{du}{dx} \cdot \frac{dx}{dt} = f(x) \cdot g'(t)$

On integrating both sides w.r.t. t, we get

$$\int\left(\frac{du}{dt}\right)dt = \int f(x)g'(t)dt$$

or $\quad u = \int f[g(t)]g'(t)dt$

i.e., $\quad \int f(x)dx = \int f[g(t)]g'(t)dt$, where $x = g(t)$.

So, it is clear that substituting $x = g(t)$ in $\int f(x)$ will give us the same result as obtained by putting $g(t)$ in place of x and $g'(t)dt$ in place of dx.

(b) Integration by Partial Fractions:

Consider $\frac{f(x)}{g(x)}$ defines a rational polynomial function.

➲ If the degree of numerator *i.e.*, $f(x)$ is **greater than or equal to** the degree of denominator *i.e.*, $g(x)$ then, this type of rational function is called an **improper rational function**. And if degree of $f(x)$ is **smaller than** the degree of denominator *i.e.*, $g(x)$, then this type of rational function is called a **proper rational function**.

- In rational polynomial function if the degree (*i.e.*, highest power of the variable) of numerator (Nr.) is **greater than or equal to** the degree of denominator (Dr.), then (without any doubt) **always perform the division** *i.e.*, divide the **Nr.** by **Dr.** before doing anything and thereafter use the following:

$$\frac{\text{Numerator}}{\text{Denominator}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Denominator}}$$

Table Demonstrating Partial Fractions or Various Forms

Form of the Rational Function	Form of the Partial Fraction
$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where $x^2 + bx + c$ can't be factorized further.	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

(c) Integration by Parts :
If U and V be two functions of x, then

$$\int \underset{(I)}{U} . \underset{(II)}{V}\, dx = U\int V dx - \int\left\{\frac{dU}{dx}\int V dx\right\}dx$$

Key Formulae

Formulae for Indefinite Integrals

(a) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

(b) $\int \frac{1}{x} dx = \log|x| + C$

(c) $\int a^x dx = \frac{1}{\log a} a^x + C$

(d) $\int e^{ax} dx = \frac{1}{a} e^{ax} + C$

(e) $\int \sin(ax) dx = -\frac{1}{a}\cos(ax) + C$

(f) $\int \cos(ax) dx = \frac{1}{a}\sin(ax) + C$

(g) $\int \tan x dx = \log|\sec x| + C$ **or** $-\log|\cos x| + C$

(h) $\int \cot x dx = \log|\sin x| + C$ **or** $-\log|\text{cosec } x| + C$

(i) $\int \sec x dx = \log|\sec x + \tan x| + C$ **or** $\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C$

(j) $\int \text{cosec } x dx = \log|\text{cosec } x - \cot x| + C$ **or** $\log\left|\tan\frac{x}{2}\right| + C$

(k) $\int \sec^2 x dx = \tan x + C$

(l) $\int \text{cosec}^2 x dx = -\cot x + C$

(m) $\int \sec x . \tan x dx = \sec x + C$

(n) $\int \text{cosec } x . \cot x dx = -\text{cosec } x + C$

(o) $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$

(p) $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(q) $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(r) $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(s) $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left|x+\sqrt{x^2-a^2}\right| + C$

(t) $\int \frac{1}{\sqrt{x^2+a^2}} dx = \log\left|x+\sqrt{x^2+a^2}\right| + C$

(u) $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)+C$

(v) $\int \frac{1}{ax+b}dx = \frac{1}{a}\log|ax+b|+C$

(w) $\int \lambda dx = \lambda x + C$, where '$\lambda$' is a constant.

(x) $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

(y) $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

(z) $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C$

Topic-2 Definite Integral

Concepts Covered • *Second fundamental theorem,* • *Properties of definite integral.*

Revision Notes

➤ **Meaning of Definite Integral of Function**

If $\int f(x)dx = F(x)$ *i.e.*, $F(x)$, be an integral of $f(x)$, then $F(b) - F(a)$ is called the definite integral of $f(x)$ between the limits a and b and in symbols it is written as $\int_a^b f(x)dx = [F(x)]_a^b$. Moreover, the definite integral gives a unique and definite value (numeric value) of **anti-derivative** of the function between the given intervals. It acts as a substitute for evaluating the area analytically.

Key Word

Anti-derivative: In calculus, an anti-derivative, inverse derivative, primitive function, primitive integral or indefinite integral of a function f is a differentiable function F whose derivative is equal to the original function f. This can be stated symbolically as $F' = f$.

Key Formulae

(a) $\int_a^b f(x)dx = F(b) - F(a)$

(b) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(c) $\int_a^b f(x)dx = \int_b^a f(t)dt \quad (dx = dt)$

(d) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, a < c < b$

(e) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

(f) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

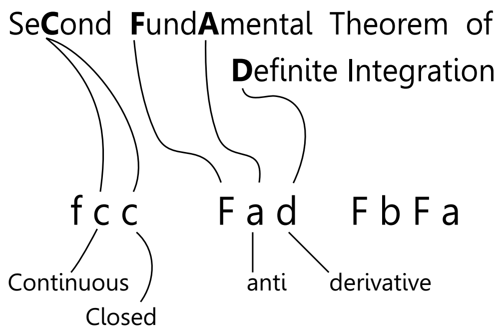
(g) $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \text{ if } f(x) \text{ is an even function } i.e., f(-x) = f(x) \\ 0, \text{ if } f(x) \text{ is an odd function } i.e., f(-x) = -f(x) \end{cases}$

(h) $\int_{-a}^a f(x)dx = \int_0^a \{f(x) + f(-x)\}dx$

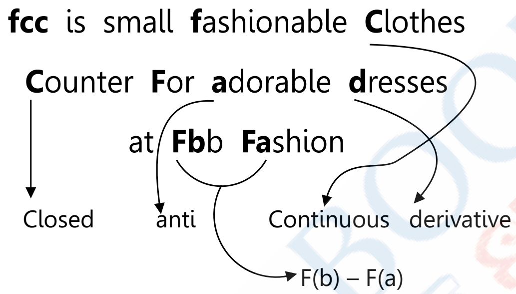
(i) $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x) \\ 0 \quad ; \text{ if } f(2a-x) = -f(x) \end{cases}$

(j) $\int_0^{2a} f(x)dx = \int_0^{2a} \{f(x) + f(2a-x)\}dx$

Mnemonics



You can also remember



Interpretation :

Let f be a continuous function defined on a closed interval $[a,b]$ and F be an anti derivative of f. Then $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$, where a and b are called limit of Integration.

Example 1

Evaluate the following definite integral :

$$\int_{-\pi}^{\pi} \frac{2x\,(1+\sin x)}{1+\cos^2 x}\,dx.$$

Solution:

Step I : Let

$$\begin{aligned} I &= \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx \\ &= \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\,dx + \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx \\ &= I_1 + I_2 \end{aligned}$$

Step II :

$$I_1 = 0 \qquad \text{(being an odd function)}$$

Step III :

$$I_2 = 2\int_0^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx$$

(being an even function)

$$\begin{aligned} \therefore \quad I &= I_2 \\ &= 2\int_0^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx \\ &= 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx \end{aligned}$$

Step IV :

Let $\qquad I_3 = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

Apply the property $\int_0^a f(x) = \int_0^a f(a-x)\,dx$,

$$\begin{aligned} &= \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx \\ &= \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx \\ &= \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}\,dx - \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx \\ &= \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx - I_3 \end{aligned}$$

Step V :

$$\therefore \qquad 2I_3 = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$

Putting $\cos x = t$ or $-\sin x\, dx = dt$

When $x = 0;\ t = 1\ \&\ x = \pi;\ t = -1$

$$2I_3 = -\pi\int_1^{-1} \frac{dt}{1+t^2}$$

$$= \pi\left[\tan^{-1} x\right]_{-1}^{1} \qquad \left[\because \int_0^b f(x)dx = \int_0^a f(x)dx\right]$$

$$= \frac{\pi^2}{2}$$

or $\qquad I_3 = \dfrac{\pi^2}{4}$

Hence $\qquad I = \pi^2 \qquad [\because I_3 = 4I]$

CHAPTER-8

APPLICATIONS OF THE INTEGRALS

Revision Notes

➤ **Area Under Simple Curves :**

(i) Let us find the area bounded by the **curve** $y = f(x)$, X-axis and the ordinates $x = a$ and $x = b$. Consider the area under the curve as composed by large number of thin vertical strips.

Key Words

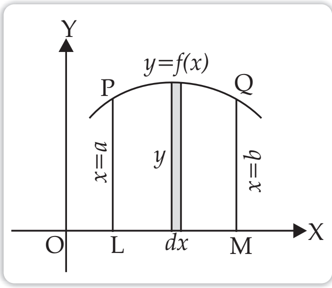
Curve: A **curve** is a continuous and smooth flowing line without any sharp turns. One way to recognize a curve is that it bends and changes its direction at least once.

- A **open curve** does not enclose any area within itself and it has two endpoints. Some of the open curves are given in the figure below.
- A **closed curve** has no end points and encloses an area (or a region). It is formed by joining the end points of an open curve together. *e.g.*: Circles, ellipses are formed from closed curves.
- A **simple curve** changes direction but does not cross itself while changing direction. A simple curve can be open and closed both.
- A **non-simple curve** crosses its own path.

Let there be an **arbitrary** strip of height y and width dx.

Area of elementary strip $dA = y\, dx$, where $y = f(x)$. Total area A of the region between X-axis ordinates $x = a$, $x = b$ and the curve $y = f(x)$ = sum of areas of elementary thin strips across the region $PQML$.

$$A = \int_a^b y\, dx = \int_a^b f(x)\, dx$$

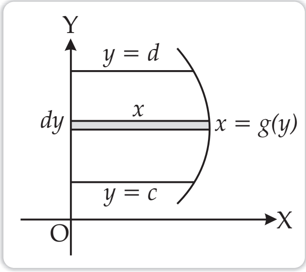


Key Words

Arbitrary: In mathematics, "arbitrary" just means "for all".
For example: "For all a, b, $a + b = b + a$". Another way to say this would be "$a + b = b + a$ for arbitrary a, b."

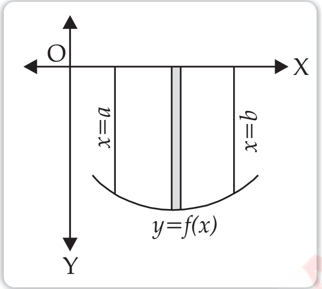
(ii) The area A of the region bounded by the curve $x = g(y)$, y-axis and the lines $y = c$ and $y = d$ is given by

$$A = \int_c^d x\,dy = \int_c^d g(y)\,dy$$



(iii) If the curve under consideration lies below X-axis, then $f(x) < 0$ from $x = a$ to $x = b$, the area bounded by the curve $y = f(x)$ and the ordinates $x = a$, $x = b$ and X-axis is negative. But, if the numerical value of the area is to be taken into consideration, then

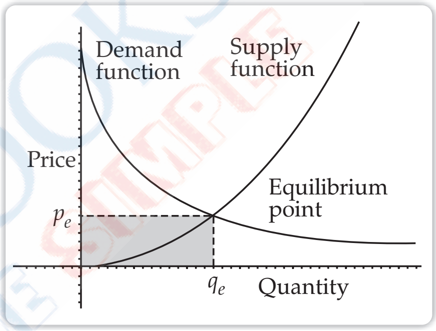
$$\text{Area} = \left|\int_a^b f(x)dx\right|$$



Amazing Fact

Application of Integration in Economics and Commerce

Integration helps us to find out the total cost function and total revenue function from the marginal cost. It is possible to find out consumer's surplus and producer's surplus from the demand and supply function. Cost and revenue functions are calculate through indefinite integral.



Key Words

Ordinate: The Cartesian coordinate obtained by measuring parallel to the Y-axis.

(iv) It may also happen that some portion of the curve is above X-axis and some portion is below X-axis as shown in the figure. Let A_1 be the area below x-axis and A_2 be the area above the X-axis. Therefore, area bounded by the curve $y = f(x)$, X-axis and the ordinates $x = a$ and $x = b$ is given by

$$A = |A_1| + |A_2|$$

CHAPTER-9

DIFFERENTIAL EQUATIONS

Topic-1 Basic Differential Equations

Concepts Covered • *Order of differential Equation*
• *Degree of differential Equation*

Revision Notes

➢ **Orders and Degrees of Differential Equation :**

- We shall prefer to use the following notations for derivatives.
- $\dfrac{dy}{dx} = y', \dfrac{d^2y}{dx^2} = y'', \dfrac{d^3y}{dx^3} = y'''$
- For derivatives of higher order, it will be in convenient to use so many dashes as super suffix therefore, we use the notation y_n for n^{th} order derivative $\dfrac{d^ny}{dx^n}$.
- Order and degree (if defined) of a **differential equation** are always positive integers.

Key Words

Differential Equation: In Mathematics, a differential equation is an equation with one or more derivatives of a function. The derivative of the function is given by dy/dx. In other words, it is defined as the equation that contains derivatives of one or more dependent variables with respect to one or more independent variables.

Know the terms

- **Order of a differential equation:** It is the order of the highest order derivative appearing in the differential equation.
- **Degree of a differential equation:** It is the degree (power) of the highest order derivative, when the differential coefficients are made free from the radicals and the fractions.

Topic-2 Variable Separable Methods

Concepts Covered • *General Solution,* • *Particular Solutions ,* • *Variable Separable Method*

Revision Notes

➢ **Solutions of differential equations :**

(a) General Solution : The solution which contains as many as arbitrary constants as the order of the differential equations, *e.g.* $y = \alpha \cos x + \beta \sin x$ is the general solution of $\frac{d^2y}{dx^2} + y = 0$.

(b) Particular Solution : Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution *e.g.* $y = 3 \cos x + 2 \sin x$ is a particular solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

(c) Solution of Differential by Variable Separable Method : A **variable** separable form of the differential equation is the one which can be expressed in the form of $f(x)dx = g(y)dy$. The solution is given by $\int f(x)dx = \int g(y)dy + k$, where k is the **constant** of integration.

Key Words

Variable: A value that keeps on changing is said to be variable. Variables are often represented by an alphabet like *a, b, c*, or *x, y, z*. Its value changes from time to time. ***e.g.:*** $3x + 5y = 7$ where x and y are variables that are changed according to the expression.

Constant: As the name implies, the constant is a value that remains constant ever. Constant has a fixed value and its value cannot be changed by any variable. Constants are represented by numbers.

e.g.: $3x + 5y = 7$, where 7 is the constant, we know its face value is 7 and it cannot be changed. But $3x$ and $5y$ are not constants because the variable x and y can change their value.

Topic-3 Linear Differential Equations

Concepts Covered • *Linear Differential Equations in x only and in y only*

Revision Notes

➢ **Solutions of Differential Equations:**

➲ **Linear differential equation in y :** It is of the form $\frac{dy}{dx} + P(x)y = Q(x)$, where $P(x)$ and $Q(x)$ are functions of x only.

- **Solving Linear Differential Equation in y :**

  STEP 1 : Write the given differential equation in the form $\frac{dy}{dx} + P(x)y = Q(x)$.

  STEP 2 : Find the **Integration Factor** (*I.F.*) = $e^{\int P(x)dx}$.

Key Word

Integrating Factor: An integrating factor is a function by which an ordinary differential equation can be multiplied in order to make it integrable.

STEP 3 : The solution is given by, $y.(I.F.) = \int Q(x).(I.F.)dx + k$, where k is the constant of integration.

- **Linear differential equation in x :** It is of the form $\frac{dx}{dy} + P(y)x = Q(y)$, where $P(y)$ and $Q(y)$ are functions of y only.

- **Solving Linear Differential Equation in x :**

  STEP 1 : Write the given differential equation in the form $\frac{dx}{dy} + P(y)x = Q(y)$.

  STEP 2 : Find the Integration Factor (*I.F.*) = $e^{\int P(y)dy}$.

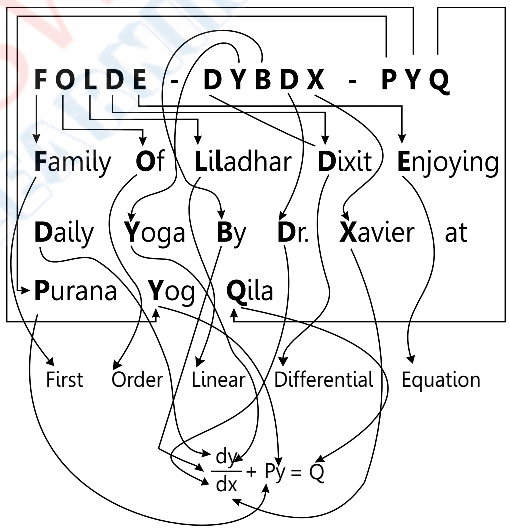
  STEP 3 : The solution is given by, $x.(I.F.) = \int Q(y).(I.F.)dy + \lambda$, where λ is the **constant of integration**.

Key Word

Constant of integration: A constant that is added to the function obtained by evaluating the indefinite integral of a given function, indicating that all indefinite integrals of the given function differ by, at most, a constant.

Mnemonics

Linear Differential Equations



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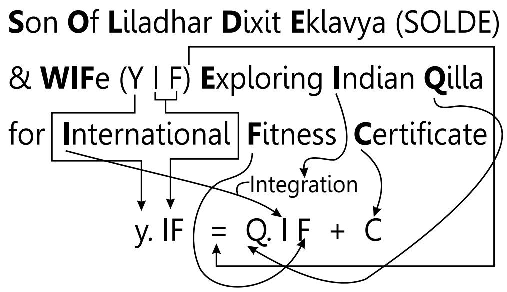


S — Solution
O — Of
L — Linear
D — Differential
E — Equation

Interpretation :

Differential equation is of the form $\frac{dy}{dx}+py=Q$, where P and Q are constants or the function of 'x' is called a first order linear differential equations. Its solution is given as

$$Y.IF=\int Q.IF+C$$

Topic-4 Homogeneous Differential Equations

Concepts Covered • *Solution of Homogenous Differential Equation of first order and first degree*

Revision Notes

➢ **Homogeneous Differential Equations and their solution :**

➲ **Identifying a Homogeneous Differential equation :**

STEP 1 : Write down the given differential equation in the form $\frac{dy}{dx} = f(x, y)$.

STEP 2 : If $f(kx, ky) = k^n f(x, y)$, then the given differential equation is **homogeneous** of degree 'n'.

Key Words

Homogeneous: To be **Homogeneous** a function must pass this test:
$f(zx, zy) = z^n f(x, y)$

Key Word

In other words,
Homogeneous is when we can take a function: $f(x, y)$ multiply each variable by z : $f(zx, zy)$
and then can rearrange it to get this: $z^n f(x, y)$
e.g.: $x + 3y$
Start with: $f(x, y) = x + 3y$
Multiply each variable by z : $f(zx, zy) = zx + 3zy$
Let's rearrange it by factoring out z : $f(zx, zy) = z(x + 3y)$
And $x + 3y$ is $f(x, y)$: $f(zx, zy) = z f(x, y)$
which is what we wanted, with $n = 1$: $f(zx, zy) = z^1 f(x, y)$

➲ **Solving a Homogeneous Differential Equation:**

CASE I : If $\dfrac{dy}{dx} = f(x, y)$

Put $y = vx$

or $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

CASE II : If $\dfrac{dx}{dy} = f(x, y)$

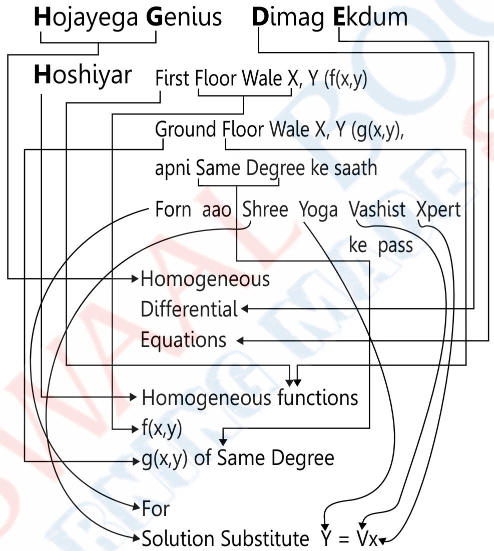
Put $x = vy$

or $\dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$

Then, we separate the variables to get the required solution.

Mnemonics

Homogeneous Differential Equation



Interpretation :

Differential equation can be expressed in the form $\dfrac{dy}{dx} = f(x, y)$ or $\dfrac{dx}{dy} = g(x, y)$ where $f(x,y)$ and $g(x,y)$ are homogeneous functions of sum is called a homogeneous Differential equation. These equations can be solved by substituting $y = vx$ so that dependent variable y is changed to another variable v, where v is some unknown function.

UNIT – IV: VECTORS & THREE-DIMENSIONAL GEOMETRY

CHAPTER-10

VECTORS

Topic-1 Basic Algebra of Vectors

Concepts Covered • *Basic concepts of vectors,* • *Operations on vectors* • *Different types of vectors,* • *Triangle Law,* • *Parallelogram Law*

Revision Notes

1. Vector : Basic Introduction :

- A physical quantity having **magnitude** as well as the direction is called a vector. It is denoted as $\vec{AB}$ or $\vec{a}$. Its magnitude (or modulus) is $|\vec{AB}|$ or $|\vec{a}|$ otherwise, simply AB or a.
- Vectors are denoted by symbols such as $\vec{a}$. [Pictorial representation of vector]

2. Initial and Terminal Points :

The initial and terminal points means that point from which the vector originates and terminates respectively.

Key Words

Magnitude: It is defined as the maximum extent of size and the direction of an object. Magnitude is used as a common factor in vector and scalar quantities.

3. Position Vector :

The position vector of a point say $P(x, y, z)$ is $\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and the magnitude is $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.

The vector $\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is said to be in its **component form**. Here x, y, z are called the scalar components or rectangular components of $\vec{r}$ and $x\hat{i}, y\hat{j}, z\hat{k}$ are the vector components of $\vec{r}$ along X, Y, Z-axis respectively.

- Also, $\vec{AB}$ = (Position Vector of B) – (Position Vector of A). For example, let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$. Then, $\vec{AB} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$.
- Here $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors along the axes OX, OY and OZ respectively (The discussion about unit vectors is given later under 'types of vectors').

4. Direction Ratios and Direction Cosines:

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then coefficient of $\hat{i}, \hat{j}, \hat{k}$ in $\vec{r}$ *i.e.*, x, y, z are called the direction ratios (abbreviated as d.r.'s) of vector $\vec{r}$. These are denoted by a, b, c (*i.e.*, $a = x$, $b = y$, $c = z$; in a manner we can say that scalar components of vector $\vec{r}$ and its d.r.'s both are the same).

Also, the coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\hat{r}$ (which is the unit vector of $\vec{r}$) *i.e.*, $\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}$ are called direction cosines (which is abbreviated as d.c.'s) of vector $\vec{r}$.

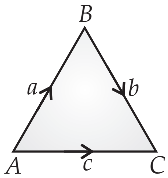
- These direction cosines are denoted by l, m, n such that $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$ and $l^2 + m^2 + n^2 = 1$ $\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.
- It can be easily concluded that $\frac{x}{r} = l = \cos\alpha$, $\frac{y}{r} = m = \cos\beta$, $\frac{z}{r} = n = \cos\gamma$.

  Therefore, $\vec{r} = lr\hat{i} + mr\hat{j} + nr\hat{k} = r(\cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k})$. [Here $r = |\vec{r}|$].

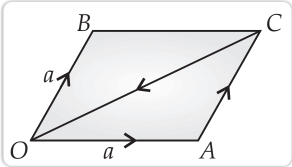
5. Addition of vectors

(a) Triangular law : If two adjacent sides (say sides AB and BC) of a triangle ABC are represented by $\vec{a}$ and $\vec{b}$ taken in same order, then the third side of the triangle taken in the reverse order gives the sum of vectors $\vec{a}$ and $\vec{b}$ *i.e.*, $\vec{AC} = \vec{AB} + \vec{BC} \Rightarrow \vec{AC} = \vec{a} + \vec{b}$.

- Also since $\vec{AC} = -\vec{CA} \Rightarrow \vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$.



(b) Parallelogram law : If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and the direction by the two adjacent sides (say *OA* and *OB*) of a parallelogram *OACB,* then their sum is given by that diagonal of parallelogram which is co-initial with $\vec{a}$ and $\vec{b}$ *i.e.,* $\vec{OC} = \vec{OA} + \vec{OB}$.



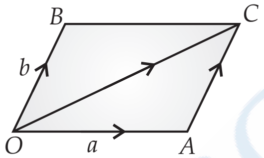
6. Properties of Vector Addition

(a) Commutative property : $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Consider $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ be any two given vectors, then

$\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k} = \vec{b} + \vec{a}.$

(b) Associative property : $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.



(c) Additive identity property : $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}.$

(d) Additive inverse property : $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}.$

Note : Multiplication of a vector by a scalar

Let $\vec{a}$ be any vector and k be any non-zero scalar. Then the product $k\vec{a}$ is defined as a vector whose magnitude is $|k|$ times that of $\vec{a}$ and the direction is

(i) same as that of $\vec{a}$ if k is positive, and **(ii)** opposite as that of $\vec{a}$ if k is negative.

Amazing Facts

- In biology, a vector is a living organism that transmits an infections agent from an infected animal a human or another animal. Vectors are frequency arthropods, such as mosquitoes, ticks, files, fleas an lice.
- In video games, we use vectors to represent the velocity of players, but also to control where they are aiming or what they can see (where they are facing).

Key Fact

- Vector calculus and its sub objective vector fields was invented by two men J. Willard Gibbs and Oliver Heaviside at the end of 19^{th} century.
- Vectors can be placed in a new position without rotating it. It still has the same magnitude and direction, and is identical to the vector at the beginning.

Know the Terms

Types of Vectors :

(a) Zero or Null vector : It is that vector whose initial and terminal points are coincident. It is denoted by $\vec{0}$. of course its magnitude is 0 (zero).

- Any non-zero vector is called a **proper vector**.

(b) Co-initial vectors : Those vectors (two or more) having the same starting point are called the co-initial vectors.

(c) **Co-terminus vectors :** Those vectors (two or more) having the same terminal point are called the co-terminus vectors.

(d) **Negative of a vector :** The vector which has the same magnitude as the $\vec{r}$ but opposite direction. It is denoted by $-\vec{r}$. Hence if, $\vec{AB} = \vec{r}$ or $\vec{BA} = -\vec{r}$ *i.e.,* $\vec{AB} = -\vec{BA}$, $\vec{PQ} = -\vec{QP}$ etc.

(e) **Unit vector :** It is a vector with the unit magnitude. The unit vector in the direction of vector $\vec{r}$ is given by $\hat{r} = \dfrac{\vec{r}}{|\vec{r}|}$ such that $|\hat{r}| = 1$, so, if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then its unit vector is :

$$\hat{r} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}.$$

- Unit vector perpendicular to the plane $\vec{a}$ and $\vec{b}$ is : $\pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.

(f) **Reciprocal of a vector :** It is a vector which has the same direction as the vector $\vec{r}$ but magnitude equal to the reciprocal of the magnitude of $\vec{r}$. It is denoted as $\vec{r}^{\,-1}$. Hence $\left|\vec{r}^{\,-1}\right| = \dfrac{1}{|\vec{r}|}$.

(g) **Equal vectors :** Two vectors are said to be equal if they have the same magnitude as well as direction, regardless of the position of their initial points.

Thus $\vec{a} = \vec{b} \Leftrightarrow \begin{cases} |\vec{a}| = |\vec{b}| \\ \vec{a} \text{ and } \vec{b} \text{ have same direction} \end{cases}$

Also, if $\vec{a} = \vec{b} \Rightarrow a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \Rightarrow a_1 = b_1,\ a_2 = b_2, a_3 = b_3$.

(h) **Collinear or Parallel vector :** Two vectors $\vec{a}$ and $\vec{b}$ are collinear or parallel if there exists a non-zero scalar λ such that $\vec{a} = \lambda\vec{b}$.

- It is important to note that the respective coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\vec{a}$ and $\vec{b}$ are proportional provided they are parallel or collinear to each other.
- The d.r's of parallel vectors are same (or are in proportion).
- The vectors $\vec{a}$ and $\vec{b}$ will have same or opposite direction as λ is positive or negative respectively.
- The vectors $\vec{a}$ and $\vec{b}$ are collinear if $\vec{a}\times\vec{b} = \vec{0}$.

(i) **Free vectors :** The vectors which can undergo parallel displacement without changing its magnitude and direction are called free vectors.

Key Formulae

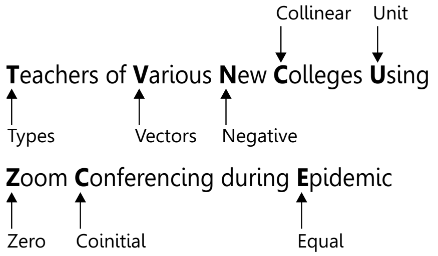
The position vector of a point say P dividing a line segment joining the points A and B whose position vectors are $\vec{a}$ and $\vec{b}$ respectively, in the ratio $m : n$.

(a) Internally, $\vec{OP} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$ **(b)** Externally, $\vec{OP} = \dfrac{m\vec{b} - n\vec{a}}{m-n}$

- Also if point P is the mid-point of line segment AB, then $\vec{OP} = \dfrac{\vec{a}+\vec{b}}{2}$.

Mnemonics

Types Of Vectors (A)



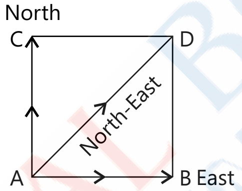
Interpretation :

Types of Vectors-

1. **Zero Vector -** Initial and terminal points coincide
2. **Unit Vector -** Magnitude is unity
3. **Coinitial Vectors -** Same initial points
4. **Collinear vectors -** Parallel to the same Line
5. **Equal Vectors -** Same magnitude and direction
6. **Negative of a vector -** Same magnitude, opp. direction

Properties Of Vectors(B)

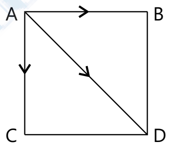
"Neither choose East nor choose north, always choose North-East and save your time".



Mnemonics

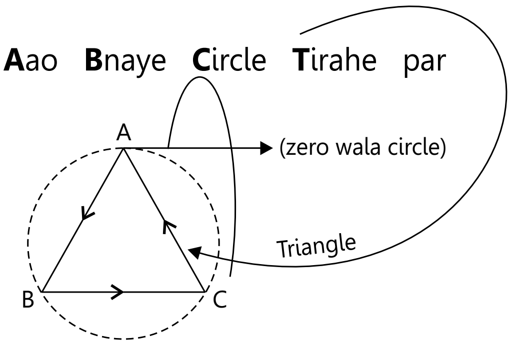
Interpretation :

The vector sum of two coinitial vectors is given by the diagonal of the parallelogram whose adjacent sides are given vectors.



$\vec{AB} + \vec{AC} = \vec{AD}$

Properties Of Vectors(C)



Interpretation:

The vector sum of the three sides of a triangle taken in order is $\vec{O}$ i.e

$\vec{AB} + \vec{BC} + \vec{CA} = \vec{O}$

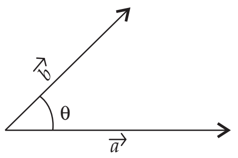
Topic-2 Dot Product of Vectors

Concepts Covered • *Properties of dot product,* • *Projection of a vector*

Revision Notes

1. Products of Two Vectors and Projection of Vectors

(a) **Scalar Product or Dot Product :** The dot product of two vectors $\vec{a}$ and $\vec{b}$ is defined by, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$.



Consider $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

- **Projection of a vector :** $\vec{a}$ on the other vector say $\vec{b}$ is given as $\left(\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}\right)$.
- **Projection of a vector :** $\vec{b}$ on the other vector say $\vec{a}$ is given as $\left(\dfrac{\vec{a}.\vec{b}}{|\vec{a}|}\right)$.

Key Words

Projection: The image of a geometrical figure reproduced on a line, plane or surface.
Scalar: A physical quantity that is completely described by its magnitude.

Know the Properties (Dot Product)

- **Properties/Observations of Dot product**
  - $\hat{i}.\hat{i} = |\hat{i}||\hat{i}|\cos 0 = 1$ or $\hat{i}.\hat{i} = 1 = \hat{j}.\hat{j} = \hat{k}\cdot\hat{k}$
  - $\hat{i}.\hat{j} = |\hat{i}||\hat{j}|\cos\dfrac{\pi}{2} = 0$ or $\hat{i}.\hat{j} = 0 = \hat{j}.\hat{k} = \hat{k}\cdot\hat{i}$
  - $\vec{a}\cdot\vec{b} \in R$, where R is real number *i.e.*, any scalar.
  - $\vec{a}\cdot\vec{b} = \vec{b}.\vec{a}$ (Commutative property of dot product).
  - $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ or $|\vec{a}| = 0$ or $|\vec{b}| = 0$.
  - If θ = 0, then $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$. Also $\vec{a}.\vec{a} = |\vec{a}|^2 = a^2$; as θ in this case is 0.

    Moreover if θ = π, then $\vec{a}\cdot$

    $\vec{b} = -|\vec{a}||\vec{b}|\cdot$
  - $\vec{a}.\left(\vec{b}+\vec{c}\right) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ (Distributive property of dot product).
  - $\vec{a}.\left(-\vec{b}\right) = -\left(\vec{a}.\vec{b}\right) = \left(-\vec{a}\right).\vec{b}$.

Key Formulae

- Angle between two vectors $\vec{a}$ and $\vec{b}$ can be found by the expression given below :

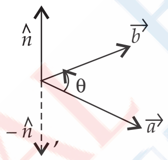
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \quad \text{or,} \quad \theta = \cos^{-1}\left(\frac{\vec{a}\,.\,\vec{b}}{|\vec{a}||\vec{b}|}\right).$$

Topic-3 Cross Product

Concepts Covered • *Properties of cross product,* • *Relationship between Vector product and scalar product*

Revision Notes

1. The cross product or vector product of two vectors $\vec{a}$ and $\vec{b}$ is defined by, $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. For better illustration, see figure.



Consider $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}.$$

- **Properties/Observations of Cross Product**
  - $\hat{i}\times\hat{i} = |\hat{i}||\hat{i}|\sin 0 = \vec{0}$ or $\hat{i}\times\hat{i} = \vec{0} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k}$.
  - $\hat{i}\times\hat{j} = |\hat{i}||\hat{j}|\sin\frac{\pi}{2}.\hat{k} = \hat{k}$ or $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$.
  - $\vec{a}\times\vec{b}$ is a vector $\vec{c}$ (say) then this vector $\vec{c}$ is perpendicular to both the vectors $\vec{a}$ and $\vec{b}$.
  - $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow \vec{a}\,||\,\vec{b}$ or, $\vec{a} = \vec{0}$, $\vec{b} = \vec{0}$.
  - $\vec{a}\times\vec{a} = \vec{0}$.
  - $\vec{a}\times\vec{b} \ne \vec{b}\times\vec{a}$ (Commutative property does not hold for cross product).
  - $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$ (Left distributive).
  - $(\vec{b}+\vec{c})\times\vec{a} = \vec{b}\times\vec{a} + \vec{c}\times\vec{a}$ (Right distributive).

  (Distributive property of the vector product or cross product)

2. Relationship between Vector product and Scalar product [Lagrange's Identity]

or $|\vec{a}\times\vec{b}|^2 + \left(\vec{a}\cdot\vec{b}\right)^2 = |\vec{a}|^2\cdot|\vec{b}|^2$

3. Cauchy-Schwarz inequality :

For any two vectors $\vec{a}$ and $\vec{b}$, always have $|\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|$.

Note :

- If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle, then the area of triangle can be obtained by evaluating $\frac{1}{2}|\vec{a}\times\vec{b}|$.
- If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then the area of parallelogram can be obtained by evaluating $|\vec{a}\times\vec{b}|$.
- The area of the parallelogram with diagonals $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|\vec{a}\times\vec{b}|$.

Key Formulae

- Angle between two vectors $\vec{a}$ and $\vec{b}$ in terms of cross-product can be found by the expression given here :

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} \text{ or } \theta = \sin^{-1}\left(\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\right).$$

CHAPTER-11

THREE DIMENSIONAL GEOMETRY

Topic-1 Direction Ratios and Direction Cosines

Concepts Covered • *Direction Ratios,* • *Direction Cosines*

• *Relationship between DC's of a line.*

Revision Notes

1. Direction Cosines of a Line :

- If A and B are two points on a given line L, then direction cosines of vectors $\vec{AB}$ and $\vec{BA}$ are the direction cosines (d.c.'s) of line L. Thus if α, β, γ are the direction-angles which the line L makes with the positive direction of X, Y, Z-axis respectively, then its d.c.'s are $\cos\alpha$, $\cos\beta$, $\cos\gamma$.
- If direction of line L is reversed, the direction angles are replaced by their **supplements angles** *i.e.*, $\pi-\alpha$, $\pi-\beta$, $\pi-\gamma$ and so are the d.c.'s *i.e.*, the direction cosines become $-\cos\alpha$, $-\cos\beta$, $-\cos\gamma$.

Key Words

Supplement angles: Two angles or arcs whose sum is 180° degrees.

- So, a line in space has two set of d.c.'s *viz* $\pm\cos\alpha$, $\pm\cos\beta$, $\pm\cos\gamma$.
- The d.c.'s are generally denoted by l, m, n. Also $l^2 + m^2 + n^2 = 1$ and so we can deduce that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Also $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.
- The d.c.'s of a line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are $\pm\frac{x_2-x_1}{AB}, \pm\frac{y_2-y_1}{AB}, \pm\frac{z_2-z_1}{AB}$;

where AB is the distance between the points A and B *i.e.,* $AB = \left|\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\right|$

2. Direction Ratios of a Line :

Any three numbers a, b, c (say) which are proportional to d.c.'s *i.e.,* l, m, n of a line are called the **direction ratios** (d.r.'s) of the line. Thus, $a = \lambda l$, $b = \lambda m$, $c = \lambda n$ for any $\lambda \in R - \{0\}$.

Consider, $\dfrac{l}{a} = \dfrac{m}{b} = \dfrac{n}{c} = \dfrac{1}{\lambda}$ (say)

or $\quad l = \dfrac{a}{\lambda}, m = \dfrac{b}{\lambda}, n = \dfrac{c}{\lambda}$

or $\quad \left(\dfrac{a}{\lambda}\right)^2 + \left(\dfrac{b}{\lambda}\right)^2 + \left(\dfrac{c}{\lambda}\right)^2 = 1$ [Using $l^2 + m^2 + n^2 = 1$]

or $\quad \lambda = \pm\sqrt{a^2+b^2+c^2}$

Therefore, $\quad l = \pm\dfrac{a}{\sqrt{a^2+b^2+c^2}}$,

$$m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

- The d.r.'s of a line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ or $x_1 - x_2$, $y_1 - y_2$, $z_1 - z_2$.
- Direction ratios are sometimes called as **Direction Numbers.**

3. Relation Between the Direction Cosines of a Line :

Consider a line L with d.c's l, m, n. Draw a line passing through the origin and $P(x, y, z)$ and parallel to the given line L. From P draw a perpendicular PA on the X-axis, suppose $OP = r$

Now in ΔOAP, $\angle PAO = 90°$

we have, $\quad \cos\alpha = \dfrac{OA}{OP} = \dfrac{x}{r}$ or $x = lr$.

Similarly we can obtain

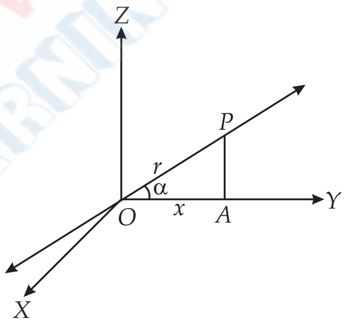
$$y = mr \text{ and } z = nr.$$

Therefore, $x^2 + y^2 + z^2 = r^2(l^2 + m^2 + n^2)$

But we know that

$$x^2 + y^2 + z^2 = r^2$$

Hence, $\quad l^2 + m^2 + n^2 = 1.$



Key Formulae

1. Distance Formula :

The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by the expression

$$AB = \left|\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\right| \text{ units.}$$

2. Section Formula :

The co-ordinates of a point Q which divides the line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the ratio $m : n$

(a) internally, are $\left(\dfrac{(mx_2+nx_1)}{m+n}, \dfrac{(my_2+ny_1)}{m+n}, \dfrac{(mz_2+nz_1)}{m+n}\right)$

(b) externally, are $\left(\dfrac{mx_2-nx_1}{m-n}, \dfrac{my_2-ny_1}{m-n}, \dfrac{mz_2-nz_1}{m-n}\right)$.

Amazing Facts

- The largest 3D shape in the world is a Rhombicosidecahedron. It is an Archimedian solid. It has 20 faces that are triangular, 30 faces that are squares, and 12 are that are pentagons. This shape has 120 edges and 60 vertices.
- The Louvre pyramid is a beautiful installation that is perfect example of a 3D shape *i.e.*, square pyramid. It is situated in the city of Paris in the prestigious museum of the Louvre.

Mnemonics

Direction Cosines

1 glass **L** e **M** o **N** juice

$l^2 + m^2 + n^2 = 1$

Dance **C**horeographer **P**refer **D**ieting

Direction Cosines Positive Direction

Direction Ratios

Director Remo a **P**rofessional **D**ancer

Direction Ratios Proportional Direction

Choreographer created

Cosines

3 **L**ifetime **M**ovies with **N**ew faces **a b c**

l m n

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}},\ m = \frac{b}{\sqrt{a^2+b^2+c^2}},\ n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Interpretation :

Direction cosines of a line are the cosines of the angles made by the line with the positive directions of the co. ordinate axes. If l, m, n are the D. cs of a line, then $l^2+m^2+n^2=1$

Topic-2

Lines & Its Equations in Different forms

Concepts Covered
- *Equation of line in cartesian and vector form,*
- *Shortest distance between lines*
- *Skew lines*
- *Condition of parallelism and perpendicularity of lines.*

Revision Notes

1. Equation of a Line passing through two given points :

Consider the two given points as $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ with position vectors $\vec{a}$ and $\vec{b}$ respectively. Also assume $\vec{r}$ as the position vector of any arbitrary point $P(x, y, z)$ on the line L passing through A and B.
Thus

$\vec{OA} = \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k},\ \vec{OB} = \vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k},\ \vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

(a) Vector equation of a line : Since the points A, B and P all lie on the same line which means that they are all collinear points.

Further it means, $\vec{AP} = \vec{r} - \vec{a}$ and $\vec{AB} = \vec{b} - \vec{a}$ are collinear vectors, *i.e.*,

$$\vec{AP} = \lambda\vec{AB}$$

or $$\vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a})$$

or $$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}),\ \text{where } \lambda \in R.$$

This is the vector equation of the line.

(b) Cartesian equation of a line : By using the vector equation of the line $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, we get

$$x\hat{i} + y\hat{j} + z\hat{k} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + \left[(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}\right]$$

On equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$, we get

$x = x_1 + \lambda(x_2 - x_1),\ y = y_1 + \lambda(y_2 - y_1),\ z = z_1 + \lambda(z_2 - z_1)$ ...(i)

On eliminating λ, we have

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

2. Angle between two lines :

(a) When d.r.'s or d.c.'s of the two lines are given :

Consider two lines L_1 and L_2 with d.r.'s in proportion to a_1, b_1, c_1 and a_2, b_2, c_2 respectively ; d.c.'s as l_1, m_1, n_1 and l_2, m_2, n_2. Consider

$\vec{b}_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b}_2 = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$.

These vectors $\vec{b}_1$ and $\vec{b}_2$ are parallel to the given lines L_1 and L_2. So in order to find the angle between the lines L_1 and L_2, we need to get the angle between the vectors $\vec{b}_1$ and $\vec{b}_2$.

So the acute angle θ between the vectors $\vec{b}_1$ and $\vec{b}_2$ (and hence lines L_1 and L_2) can be obtained as,

$$\vec{b}_1 . \vec{b}_2 = |\vec{b}_1||\vec{b}_2|\cos\theta$$

Thus, $$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

- Also, in terms of d.c.'s : $\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$.
- Sine of angle is given as :

$$\sin\theta = \left|\frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|.$$

(b) When Vector equations of two lines are given :

Consider vector equations of lines L_1 and L_2 as $\vec{r}_1 = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r}_2 = \vec{a}_2 + \mu\vec{b}_2$ respectively.

Then, the acute angle θ between the two lines is given by the relation

$$\cos\theta = \left|\frac{\vec{b}_1 . \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|.$$

(c) When Cartesian equation of two lines are given:

Consider the lines L_1 and L_2 in Cartesian form as,

$$L_1 : \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

$$L_2 : \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

Then the acute angle θ between the lines L_1 and L_2 can be obtained by, $\cos\theta = \left|\dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$

Note :

- For two perpendicular lines : $a_1a_2 + b_1b_2 + c_1c_2 = 0$, $l_1l_2 + m_1m_2 + n_1n_2 = 0$.
- For two parallel lines : $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$; $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$.

3. Shortest Distance between two Lines :

If two lines are in the same plane *i.e.*, they are coplanar, they will intersect each other if they are non-parallal. Hence, the shortest distance between them is zero. If the lines are parallel then the shortest distance between them will be the perpendicular distance between the lines *i.e.*, the length of the perpendicular drawn from a point on one line onto the other line. Adding to this discussion, in space, there are lines which are neither intersecting nor parallel. In fact, such pair of lines are non-coplanar and are called the skew lines.

Key Fact

1. Equation of a line in space passing through a given point and parallel to a given vector :

Consider the line L is passing through the given point $A(x_1, y_1, z_1)$ with the position vector $\vec{a}$, $\vec{d}$ is the given vector with d.r.'s a, b, c and $\vec{r}$ is the position vector of any arbitrary point $P(x, y, z)$ on the line.

Thus, $\overrightarrow{OA} = \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{d} = a\hat{i} + b\hat{j} + c\hat{k}$.

(a) Vector equation of a line : As the line L is parallel to given vector $\vec{d}$ and points A and P are lying on the line so, $\overrightarrow{AP}$ is parallel to the $\vec{d}$.

or $\quad \overrightarrow{AP} = \lambda\vec{d}$, where $\lambda \in R$ *i.e.*, set of real numbers

or $\quad \vec{r} - \vec{a} = \lambda\vec{d}$

or $\quad \vec{r} = \vec{a} + \lambda\vec{d}$.

This is the vector equation of line.

(b) Parametric equations : If d.r.'s of the line are a, b, c, then by using $\vec{r} = \vec{a} + \lambda\vec{d}$, we get

$$x\hat{i} + y\hat{j} + z\hat{k} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + \lambda\left(a\hat{i} + b\hat{j} + c\hat{k}\right)$$

Now, as we equate the coefficients of $\hat{i}, \hat{j}, \hat{k}$, we get the parametric equations of line given as,

$$x = x_1 + \lambda a, y = y_1 + \lambda b, z = z_1 + \lambda c.$$

- Co-ordinates of any point on the line considered here are $(x_1 + \lambda a, y_1 + \lambda b, z_1 + \lambda c)$.

Key Word

<u>**Parametric Equation:**</u> It is a type of equation that employs an independent variable called parameter (often denoted by t) and in which dependent variables are defined as continuous functions of the parameter and are not dependent on another existing variable.

(c) **Cartesian equation of a line :** If we eliminate the parameter λ from the parametric equations of a line, we get the Cartesian equation of line as

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

- If l, m, n are the d.c.'s of the line, then Cartesian equation of line becomes

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

- **Skew Lines :** Two straight lines in space which are neither parallel nor intersecting are known as the skew lines. They lie in different planes and are non-coplanar.
- **Line of Shortest distance :** There exists unique line perpendicular to each of the skew lines L_1 and L_2, and this line is known as the line of shortest distance (S.D.).

UNIT – V: LINEAR PROGRAMMING

CHAPTER-12
LINEAR PROGRAMMING

Revision Notes

Linear programming problems: Problems which minimize or maximize a linear function z subject to certain conditions determined by a set of linear inequalities with non-negative variables are known as linear programming problems.

Objective function: A linear function $z = ax + by$, where a and b are constants which has to be maximised or minimised according to a set of given conditions, is called as linear objective function.

Decision variables: In the objective function $z = ax + by$, the variables x, y are said to be decision variables.

Constraints: The restrictions in the form of inequalities on the variables of a linear programming problems are called constraints. The condition $x \geq 0$, $y \geq 0$ are known as non-negative restrictions.

Key Terms

Feasible region: The common region determined by all the constraints including non-negative constraints $x, y \geq 0$ of linear programming problem is known as the feasible region.

Feasible solution: Points with in and on the boundary of the feasible region represents feasible solutions of constraints.

In the feasible region, there are infinitely many points (solutions) which satisfy the given conditions.

Theorem 1: Let R be the feasible region for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where variables x and y are subject to constraints described by linear inequalities, the optimal value must occur at a corner point (vertex) of the feasible region.

Theorem 2: Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function Z has both maximum and minimum values of R and each of these occurs at a corner point (vertex) of R.

However, if the feasible region is unbounded, the optimal value obtained may not be maximum or minimum.

Mnemonics

LLP parameters

N	**O**	**C**
↓	↓	↓
Non-negative variables	Objective function	Constraints

Key Facts

- Linear programming is often used for problems where no exact solution is known, for example for planning traffic flows.
- The goal of linear programming is to maximize or minimize specified objectives, such as profit or cost. This process is known as optimization.
- Linear programming is heavily used in microeconomics and company management, such as planning, product, transportation, technology and other issues, either to maximize the income or minimize the costs of a production scheme.

UNIT – VI: PROBABILITY

CHAPTER-13

PROBABILITY

Topic-1 Conditional Probability and Multiplication Theorem on Probability

Concepts Covered • *Conditional Probability,* • *Multiplication Theorem of Probability*

Revision Notes

1. Basic Definition of Probability :

Let S and E be **the sample** space and **event** in an experiment respectively.

Key Words

Sample Space: A set in which all of the possible outcomes of a statistical experiment are represented as points.
Event: Event is a subset of a sample space. *e.g.*: Event of getting odd outcome in a throw of a die.

Then, Probability

$$= \frac{\text{Number of Favourable Events}}{\text{Total number of Elementary Events}} = \frac{n(E)}{n(S)}$$

$$0 \le n(E) \le n(S)$$
$$0 \le P(E) \le 1$$

Hence, if $P(E)$ denotes the probability of occurrence of an event E, then $0 \le P(E) \le 1$ and $P(\bar{E}) = 1 - P(E)$ such that $P(\bar{E})$ denotes the probability of non-occurrence of the event E.

➲ **Note that $P(\bar{E})$ can also be represented as $P(E')$.**

2. Mutually Exclusive Or Disjoint Events :

Two events A and B are said to be mutually exclusive if occurrence of one prevents the occurrence of the other *i.e.*, they can't occur simultaneously.
In this case, sets A and B are disjoint *i.e.*, $A \cap B = \phi$.
Consider an example of throwing a die. We have the sample space as, $S = \{1, 2, 3, 4, 5, 6\}$
Suppose A = the event of occurrence of a number greater than 4 = $\{5, 6\}$
B = the event of occurrence of an odd number = $\{1, 3, 5\}$ and C = the event of occurrence of an even number = $\{2, 4, 6\}$

In these events, the events B and C are mutually exclusive events but A and B are not mutually exclusive events because they can occur together (when the number 5 comes up). Similarly A and C are not mutually exclusive events as they can also occur together (when the number 6 comes up).

3. Independent Events :

Two events are independent if the occurrence of one does not affect the occurrence of the other. Consider an example of drawing two balls one by one with replacement from a bag containing 3 red and 2 black balls.

Suppose A = the event of occurrence of a red ball in first draw

B = the event of occurrence of a black ball in the second draw.

Then, $P(A) = \frac{3}{5}, P(B) = \frac{2}{5}$

Here probability of occurrence of event B is not affected by the occurrence or non - occurrence of the event A. Hence events A and B are independent events.

4. Exhaustive Events :

Two or more events say A, B and C of an experiment are said to be exhaustive events, if

(a) their union is the total sample space

i.e., $A \cup B \cup C = S$

(b) the event A, B and C are disjoint in pairs

i.e., $A \cap B = \phi$, $B \cap C = \phi$ and $C \cap A = \phi$.

(c) $P(A) + P(B) + P(C) = 1$.

Consider an example of throwing a die. We have

$S = \{1, 2, 3, 4, 5, 6\}$

Suppose A = the event of occurrence of an even number = $\{2, 4, 6\}$ B = the event of occurrence of an odd number = $\{1, 3, 5\}$ and C = the event of getting a number multiple of 3 = $\{3, 6\}$

In these events, the events A and B are exhaustive events as $A \cup B = S$ but the events A and C or the events B and C are not exhaustive events as $A \cup C \neq S$ and similarly $B \cup C \neq S$.

- If A and B are mutually exhaustive events, then we always have

  $P(A \cap B) = 0$ [As $n(A \cap B) = n(\phi) = 0$]

  $\therefore \quad P(A \cup B) = P(A) + P(B)$.

- If A, B and C are mutually exhaustive events, then we always have

  $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

Mnemonics

Concept: Independent and Mutually exclusive events.

I **I**s not ME

ME **I**s not I

Here, **I**: Independent Events

ME: Mutually Exclusive events

5. Conditional Probability :

By the conditional probability, we mean the probability of occurrence of event A when B has already occurred.

The 'conditional probability of occurrence of event A when B has already occurred' is sometimes also called as probability of occurrence of event A w.r.t. B.

- $P(A|B) = \frac{P(A \cap B)}{P(B)}$, $B \neq \phi$ *i.e.,* $P(B) \neq 0$
- $P(B|A) = \frac{P(A \cap B)}{P(A)}$, $A \neq \phi$ *i.e.,* $P(A) \neq 0$
- $P(\bar{A}|B) = \frac{P(\bar{A} \cap B)}{P(B)}$, $P(B) \neq 0$
- $P(A|\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})}$, $P(\bar{B}) \neq 0$
- $P(\bar{A}|\bar{B}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})}$, $P(\bar{B}) \neq 0$
- $P(A|B) + P(\bar{A}|B) = 1, B \neq \phi$.

Key Facts

- Probability originated from a gambler's dispute in 1654 concerning the division of a stake between two players whose game was interrupted before it close.
- Quantum physics is an inherently probabilistic theory in that only probabilities for measurement outcomes can be determined.

Key Formulae

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ i.e., $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

(b) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

(c) $P(\bar{A} \cap B) = P(\text{only } B) = P(B - A) = P(B \text{ but not } A) = P(B) - P(A \cap B)$

(d) $P(A \cap \bar{B}) = P(\text{only } A) = P(A - B) = P(A \text{ but not } B) = P(A) - P(A \cap B)$

(e) $P(\bar{A} \cap \bar{B}) = P(\text{neither } A \text{ nor } B) = 1 - P(A \cup B)$

NOTE : EVENTS AND SYMBOLIC REPRESENTATIONS :

Verbal description of the event	Equivalent set notation
Event A	A
Not A	$\bar{A}$ or A'
A or B (occurrence of atleast one A or B)	$A \cup B$ or $A + B$
A and B (simultaneous occurrence of both A and B)	$A \cap B$ or AB
A but not B (A occurs but B does not)	$A \cap \bar{B}$ or $A - B$
Neither A nor B	$\bar{A} \cap \bar{B}$
Atleast one A, B or C	$A \cup B \cup C$
All the three A, B and C	$A \cap B \cap C$

Key Facts

- The probability of you being born was about 1 in 400 trillion.
- The probability of living 110 years or more is about 1 in 7 million.
- If you are in the group of 23 people, there is a 50% chance that 2 of them share a birthday. If you are in a group of 70 people, that probability jumps to over 99%.

Topic-2 Bayes' Theorem

Concept Covered • *Bayes' Theorem*

Revision Notes

BAYES' THEOREM :

If $E_1, E_2, E_3, \ldots\ldots, E_n$ are n non-empty events constituting a partition of sample space S *i.e.*, $E_1, E_2, E_3, \ldots, E_n$ are pair wise disjoint and $E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n = S$ and A is any event of non-zero probability, then

$$P(E_i|A) = \frac{P(E_i).P(A|E_i)}{\sum_{j=1}^{n} P(E_j)P(A|E_j)}, i = 1, 2, 3, ..., n$$

For example,

$$P(E_1|A) = \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2) + P(E_3).P(A|E_3)}$$

- Bayes' theorem is also known as the formula for the probability of causes.
- If $E_1, E_2, E_3, ..., E_n$ form a partition of S and A be any event, then
  $P(A) = P(E_1).P(A|E_1) + P(E_2).P(A|E_2) + ... + P(E_n).P(A|E_n)$ $[\because P(E_i \cap A) = P(E_i).P(A|E_i)]$
- The probabilities $P(E_1), P(E_2), ..., P(E_n)$ which are known before the experiment takes place are called **prior probabilities** and $P(A|E_n)$ are called **posterior probabilities**.

Topic-3 Random Variable and its Probability Distributions

Concepts Covered • *Random Variable*, • *Probability Distribution*

Revision Notes

1. RANDOM VARIABLE :

A random variable is a real valued function defined over the sample space of an experiment. In other words, a random variable is a real-valued function whose domain is the sample space of a random experiment. A random variable is usually denoted by uppercase letters X, Y, Z etc.

2. PROBABILITY DISTRIBUTION OF A RANDOM VARIABLE :

If the values of a random variable together with the corresponding probabilities are given, then this description is called a probability distribution of the random variable.

Key Terms

- **Discrete random variable :** It is a random variable which can take only finite or countable infinite number of values.
- **Continuous random variable :** A variable which can take any value between two given limits is called a continuous random variable.

Key Formulae

- Mean or Expectation of a random variable $X = \mu = \sum_{i=1}^{n} x_i P_i$

❑❑