UNIT - I : NUMBERS, QUANTIFICATION AND NUMERICAL APPLICATIONS

CHAPTER-1

NUMBERS AND QUANTIFICATION

Revision Notes

Modulo Arithmetic

Modulo Arithmetic is a special type of arithmetic that involves only integers and the operations used are addition, subtraction, multiplication and division. The only difference between modulo arithmetic and simple arithmetic that we have learned in our primary classes is that is modulo arithmetic all operations are performed regarding a positive integer *i.e.*, the modulus.

Let's revise the division theorem that tells us that for any two integers a and b where $b \neq 0$, there always exists unique integer a and r such that $a = qb + r$ and $0 \leq r < |b|$.

For example: $a = 39$, $b = 5$, we can find $q = 7$ and $r = 4$.

so, that $39 = 7 \times 5 + 4$.

Here, a is called dividend
b is called divisor
q is called quotient
r is called remainder

If $r = 0$, the we say b divides a or a is divisible by b. This established a natural **congruence relation** on the integers.

Key Words

Congruence Relation: In abstract algebra, a congruence relation (or simply congruence) is an equivalence relation on an algebraic structure (such as a group, ring, or vector space) that is compatible with the structure in the sense that algebraic operations done with equivalent elements will yield equivalent elements.

Congruence Modulo

Let m be a positive integer and I be the set of all integers. The relation "congruence modulo m" is defined on all a, $b \in I$ by $a \equiv b \pmod m$ if and only if m divides $(a - b)$.

The symbol "$\equiv \pmod m$" is read as "congruence modulo m".

"m divides $(a - b)$" or "m is a factor of $(a - b)$" is usually denoted by "$m|(a - b)$".

Clearly, $a \equiv b \pmod m$ if $(a - b)$ is a multiple of m,

i.e., $a = b + k\,m$ for some integer k.

Thus,

- $69 \equiv 25 \pmod 4$, since $4|(69 - 25)$ or 4 divides $(69 - 25)$ or $(69 - 25) = 44$ is a multiple of 4.
- $8 \not\equiv -8 \pmod 3$, since $8 - (-8) = 16$, which is not a multiple of 3.

Properties of Congruence Modulo

Property I: If $a \equiv b \pmod m$, then $b \equiv a \pmod m$

Proof: If $a \equiv b \pmod m$

then, $m|(a - b)$

or, $m|-(b - a)$

or, $m|(b - a)$

or, $b \equiv a \pmod m$ **Hence Proved**

Property II: If $a \equiv b \pmod m$, then $a \pmod m = b \pmod m$.

Proof: We know that, if $a \equiv b \pmod m$, then

$a - b = m\,k$ $\quad k \rightarrow$ integer constant

Let $k = k_1 - k_2$

so, $a - b = m\,(k_1 - k_2)$

$$a - b = mk_1 - mk_2$$
$$a - mk_1 = b - mk_2 = r$$

or, $r = a - m\,k_1$	$r = b - m\,k_2$
or, $a - r = m\,k_1$	$b - r = m\,k_2$
or, $a \equiv r \pmod m$	$b \equiv r \pmod m$
or, $r \equiv a \pmod m$ (from Property I)	$r \equiv b \pmod m$ (from Property I)

Thus, $a \pmod m = b \pmod m$ **Hence Proved.**

Property III: If $a \equiv b \pmod m$, then $(a \pm c) = (b \pm c) \pmod m$.

Proof: If $a \equiv b \pmod m$

then $m \mid (a - b)$

or, $m \mid [(a \pm c) - (b \pm c)]$

or, $a \pm c \equiv b \pm c \pmod m$ **Hence Proved.**

Property IV: If $a \equiv b \pmod m$, then $ac \equiv bc \pmod m$.

Proof: If $a \equiv b \pmod m$

then $a - b = m\,k$

$k \to$ integer constant

or, $ac - bc = m\,(ck)$

or, $ac - bc = mk_1$

$k_1 \to$ integer constant

or, $ac \equiv bc \pmod m$ **Hence Proved.**

Property V: If $a \equiv b \pmod m$ and $c \equiv d \pmod m$,

then, $(a + c) \equiv (b + d) \pmod m$.

Proof: $a \equiv b \pmod m$, $c \equiv d \pmod m$

$\Rightarrow m \mid (a - b)$, $m \mid (c - d)$

$\Rightarrow m \mid [(a - b) + (c - d)]$

$\Rightarrow m \mid [(a + c) - (b + d)]$

$\Rightarrow (a + c) \equiv (b + d) \pmod m$ **Hence Proved.**

Property VI: If $a \equiv b \pmod m$, and $c \equiv d \pmod m$,

then, $(a - c) \equiv (b - d) \pmod m$

Proof: $a \equiv b \pmod m$, $c \equiv d \pmod m$

$\Rightarrow m \mid (a - b)$, $m \mid (c - d)$

$\Rightarrow m \mid [(a - b) - (c - d)]$

$\Rightarrow m \mid (a - c) - (b - d)$

$\Rightarrow (a - c) \equiv (b - d) \pmod m$ **Hence Proved.**

Property VII: If $a \equiv b \pmod m$, and $c \equiv d \pmod m$,

then, $ac \equiv bd \pmod m$

Proof: $a \equiv b \pmod m$, $c \equiv d \pmod m$

$\Rightarrow m \mid (a - b)$, $m \mid (c - d)$

$\Rightarrow m \mid (ac - bc)$, $m \mid (bc - bd)$

$\Rightarrow m \mid [(ac - bc) + (bc - bd)]$

$\Rightarrow m \mid (ac - bd)$

$\Rightarrow ac \equiv bd \pmod m$ **Hence Proved.**

Property VIII: If x_1 is a solution of $ax \equiv b \pmod m$, then any other integer $x_2 \equiv x_1 \pmod m$ is also a solution.

Proof: Since, x_1 is a solution of $ax \equiv b \pmod m$, we have

$$ax_1 \equiv b \pmod m$$

But $x_2 \equiv x_1 \pmod m$, therefore $ax_2 \equiv ax_1 \pmod m$, *i.e.*, if m divides $(x_2 - x_1)$, then m also divides a $(x_2 - x_1)$, a being an integer.

Hence, on adding, we get by theorem I,

$$ax_2 + ax_1 \equiv ax_1 + b \pmod m$$

or, $$ax_2 \equiv b \pmod m$$

Hence, x_2 is also a solution of $ax \equiv b \pmod m$.

Property IX: The congruence modulo $ax \equiv b \pmod m$ has a solution iff the **greatest common divisor (GCD)** of a and m divides b. If the greatest common divisor d, of a and m, divides b, then congruence modulo has exactly d incongruent solutions.

Key Words

Greatest Common Divisor (GCD): The greatest common divisor (GCD) of two or more numbers is the greatest common factor number that divides them, exactly. It is also called the highest common factor (HCF). e.g.: The greatest common factor of 15 and 10 is 5, since both the numbers can be divided by 5.

Example 1. Prove that $35\,x \equiv 14 \pmod{21}$ has a solution. Also, find the number of incongruent solutions.

Solution: Here, the greatest common divisor (GCD) of 35 and 21 is 7; also 7 divides 14. Hence, the given congruence modulo has a solution and number of incongruent solutions (mod 21) is 7.

⊗	0	1	2	3	4	6	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	1	3	5	7	9
3	0	3	6	9	1	4	7	10	2	5	8
4	0	4	8	1	5	9	2	6	10	3	7
5	0	5	10	4	9	3	8	2	7	1	6
6	0	6	1	7	2	8	3	9	4	10	5
7	0	7	3	10	6	2	9	5	1	8	4
8	0	8	5	2	10	7	4	1	9	6	3
9	0	9	7	5	3	1	10	8	6	4	2
10	0	10	9	8	7	6	5	4	3	2	1

Amazing Facts

- In modulo arithmetic, an "even number" is one where it's "0 mod 2" — that is, it has a remainder of 0 when divided by 2. An odd number is "1 mod 2" (has remainder 1).
- Modular arithmetic is also used as a public key in cryptography systems, which are vital for modern commerce.
- In music theory, modulo 12 arithmetic is used to analyze the 12- tone equal temperament system, when notes separated by an octave of 12 semi-tones are treated as equivalent.

CHAPTER-2

NUMERICAL APPLICATIONS

Revision Notes

- **Alligation and Mixture:**

It is the rule that enables us to find the ratio in which two or more ingredients at the given price must be mixed to produce a **mixture** of a desired price. There are two types of questions asked in **alligation**.

Type 1: If two ingredients are mixed, then

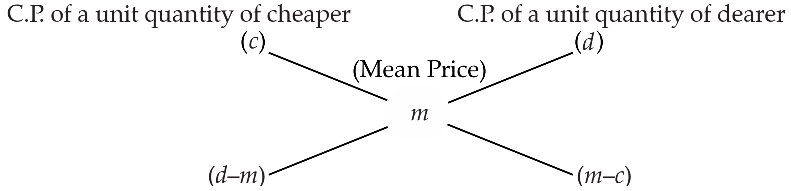
$$\left(\frac{\text{Quantity of cheaper}}{\text{Quantity of dearer}}\right) = \frac{(\text{C.P. of dearer}) - (\text{Mean Price})}{\text{Mean Price} - (\text{C.P. of cheaper})}$$

Key Words

Mixture: A mixture is a physical combination of two or more substances. The substances are known as ingredients of the mixtures.

Alligation: Alligation is an old and practical method of solving arithmetic problems related to mixtures of two or more ingredients.

We present as under:



$\therefore$ (Cheaper quantity) : (Dearer quantity) = $(d - m) : (m - c)$

Here, C.P. = Cost Price

Mean Price (m)= The cost price of a unit quantity of the mixture

Example 1: In what ratio must rice at ₹ 9.30 per kg be mixed with rice at ₹ 10.80 per kg so that the mixture be worth ₹ 10 per kg ?

Sol. By the rule of alligation, we have

C.P. of 1 kg rice of 1st kind (in paise) — 930 (c)

C.P. of 1 kg rice of 2nd kind (in paise) — 1080 (d)

Mean Price (in paise) 1000

$(d - m)$=80 $\quad$ $(m - c)$=70

$\therefore$ Required ratio = 80 : 70 = 8 : 7

Type 2: Suppose a container contains x units of liquids from which y units are taken out and replaced by water.

After n operations, the quantity of pure liquid = $\left[x\left(1-\dfrac{y}{x}\right)^n\right]$ units

Example 2: A container contains 50 litres of milk. From this container, 10 litres of milk was taken out and replaced by water. This process is repeated one more time. How much milk is now left in the container ?

Sol. Applying the method of replacement.

Amount of milk after 2 operations

$$= \left[50\left(1-\frac{10}{50}\right)^2\right]$$

$$= 50\times\frac{4}{5}\times\frac{4}{5} = 32 \text{ litres.}$$

Boats and Streams

Stream is the moving water in the river.

Upstream is the direction against the stream.

Downstream is the direction along the stream.

Still water is the state where water is considered to be stationary and the speed of the water in this case is zero.

- If the speed of a boat in still water is u km/h and the speed of the stream is v km/h, then:

  Speed of boat in downstream = $(u + v)$ km/h

  Speed of boat in upstream = $(u - v)$ km/h

- If the speed of boat in downstream is a km/h and the speed of boat in upstream is b km/h, then:

  Speed of boat in still water = $\dfrac{1}{2}(a + b)$ km/h

  Speed of stream = $\dfrac{1}{2}(a - b)$ km/h

Example 3: In a stream running at 2 km/h, a motorboat goes 6 km upstream and back again to the starting point in 33 minutes. Find the speed of the motorboat in still water.

Sol. Let the speed of the motorboat in still water be x km/h.

Then, Speed downstream = $(x + 2)$ km/h

Speed upstream = $(x - 2)$ km/h

$$\therefore \quad \frac{6}{x+2}+\frac{6}{x-2} = \frac{33}{60}=\frac{11}{20}$$

$$\Rightarrow \quad 11x^2 - 240x - 44 = 0$$

$$\Rightarrow \quad 11x^2 - 242x + 2x - 44 = 0$$

$$\Rightarrow \quad (x - 22)(11x + 2) = 0$$

$$\Rightarrow \quad x = 22$$

Hence, speed of motorboat in still water is 22 km/h.

Pipes and Cistern

Inlet: A pipe connected with a tank or a **cistern** or a **reservoir**, that fills it, is known as inlet.

Outlet: A pipe connected with a tank or cistern or a reservoir, emptying it, is called an outlet.

Key Words

Cistern: A tank for storing water, especially one supplying taps.
Reservoir: A large natural or artificial lake used as a source of supply water.

- If a pipe can fill a tank in x hours, then part of tank filled in 1 hour $= \frac{1}{x}$
- If a pipe can empty a full tank in y hours, then part of tank emptied in 1 hour $= \frac{1}{y}$
- If a pipe can fill a tank in x hours and another pipe can empty the full tank in y hours (where $y > x$), then on opening both pipes, the net part filled in 1 hour $= \left(\frac{1}{x} - \frac{1}{y}\right)$
- If a pipe can fill a tank in x hours and another pipe can empty the full tank in y hours (where $x > y$), then on opening both pipes, the net part emptied in 1 hour $= \left(\frac{1}{y} - \frac{1}{x}\right)$

Example 4: A cistern can be filled by a tap in 4 hours while it can be emptied by another tap in 9 hours. If both the taps are opened simultaneously then after how much time will the cistern get filled ?

Sol. Part filled by first cistern in 1 hour $= \frac{1}{4}$

Part emptied by second cistern 1 hour $= \frac{1}{9}$

So, net part filled in 1 hour $= \left(\frac{1}{4} - \frac{1}{9}\right) = \frac{5}{36}$

$\therefore$ The cistern will be filled in $\frac{36}{5}$ hours i.e., 7.2 hours

Races and Games

- **Races:** A contest of speed in running, riding, driving, sailing or rowing is called a race.
- **Race Course:** The ground or path on which contests are made is called race course.
- **Starting Point:** The point from which a race begins is known as a starting point.
- **Winning Point or Goal:** The point set to bound a race is called a winning point or a goal.
- **Winner:** The person who first reaches the winning point is called winner.
- **Dead Heat Race:** If all the persons contesting a race reach the goal exactly at the same time, then the race is said to be a dead heat race.
- **Start:** Suppose A and B are two contestants in a race. If before the start of the race, A is at the starting point and B is ahead of A by 15 metres, then we say that 'A gives B, a start of 15 metres'.

  So, in case to cover a race of 100 metres, A will have to cover 100 metres while B will have to cover only (100 – 15) metres = 85 metres.

  In other words, we can say that to cover a race of 100 metres 'A can give B 15 metres' or 'A can give B a start of 15 metres' or 'A beats B by 15 metres'.
- **Games:** A game of 100, means that the person among the contestants who scores 100 points first is the winner.

  If A scores 100 points while B scores only 80 points, then we say that 'A can give B 20 points'.

  Example 5: Aman can run 1 km in 3 min 10 s and Binay can cover the same distance in 3 min 20 s. By what distance can A beat B ?

  Sol. Clearly, Aman beats Binay by 10 s.

  Distance covered by Binay in 10 s $= \left(\frac{1000}{200} \times 10\right)$ m $= 50$ m [$\because$ 1 km = 1000 m and 3 min 20 s = 200 s]

  Thus, A beats B by 50 metres.

  Example 6: In a game of 80 points, Ajay can give Beena 5 points and Seema 15 points. Then how many points Beena can give Seema in a game of 60 ?

  Sol. Ajay : Beena = 80 : 75, Ajay : Seema = 80 : 65

  $$\frac{\text{Beena}}{\text{Seema}} = \left(\frac{\text{Beena}}{\text{Ajay}} \times \frac{\text{Ajay}}{\text{Seema}}\right) = \left(\frac{75}{80} \times \frac{80}{65}\right)$$

  $$= \frac{15}{13} = \frac{60}{52} = 60 : 52 = 15 : 13$$

$\therefore$ In a game of 15, Beena can give Seema 13 points.

Inequalities: Two real numbers or two algebraic expression related by the symbol '<', '>', '≤', or '≥', form an inequality.

Key Facts

- Inequalities are arguably used more than often in 'real life' than equalities. Businesses use inequalities to control inventory, plan production lines, produce pricing models and for shipping warehousing goods and materials.
- Roads have speed limits, certain movies have age restrictions and the time it takes you to walk to the pack are all examples of inequalities. Inequalities do not represent an exact amount, but instead represent a limit of what is allowed or what is possible.
- Alligation is one of the simple and illustrative methods in pharmacy to calculate the proportion of any two solutions to be mixed to prepare final solution of required concentration.

Types of inequalities

- **Numerical inequality:** An inequality which does involve any variable is called a numerical inequality.
  e.g.: $4 > 2, 8 < 21$
- **Literal inequality:** An inequality which have variable is called a literal inequality.
  e.g.: $x < 7, y \geq 11, x \leq 4$
- **Strick inequality:** An inequality which have only $<$ or $>$ is called strick inequality.
  e.g.: $3x + y < 0, x > 7$
- **Slack inequality:** An inequality which have only $\geq$ or $\leq$ is called slack inequality.
  e.g.: $3x + 2y \leq 0, y \leq 4$
- **Linear inequality:** An inequality is said to be linear, if each variable occurs in first degree only and there is no term involving the product of the variables.
  e.g.: $ax + b \leq 0, ax + by + c > 0, ax \leq 4$
  An inequality in one variable in which degree of variable is 2, is called quadratic inequality in one variable.
  e.g.: $ax^2 + bx + c \geq 0, 3x^2 + 2x + 4 \leq 0$

❖ **Linear inequality in one variable**

A linear inequality which has only one variable, is called linear inequality in one variable.

e.g.: $ax + b < 0$, where $a \neq 0$

$4c + 7 \geq 0$

Replacement Set: The set from which values of the variables (involved in the inequality) are chosen is called the 'replacement set'.

Solution Set: A solution to an inequality is a number (chosen from replacement set) which, when substituted for the variable, make the inequality true. The set of all solutions of an inequality is called the 'solution set' of the inequality.

e.g.: Consider the inequality $x < 5$

	Replacement Set	Solution Set
(i)	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$	$\{1, 2, 3, 4\}$
(ii)	$\{-1, 0, 1, 2, 3, 5, 8\}$	$\{-1, 0, 1, 2, 3\}$
(iii)	$\{-5, 1, 9, 10\}$	$\{-5, 1\}$
(iv)	$\{6, 7, 8, 9, 10\}$	ϕ

Fundamental Facts

- Solution set always depends upon replacement set.
- If the replacement set is not given, then we shall take it as R(set of real numbers).

Rules for solving inequalities in one variable: The rules for solving inequalities are similar to those for solving equations except for multiplying or dividing by a negative number.

(i) If $a \geq b$, then $a \pm k \geq b \pm k$, where k is any real number.
(ii) If $a \geq b$, then ka is not always $\geq kb$
(iii) If $k > 0$ (i.e., positive), then $a \geq b \Rightarrow ka \geq kb$
(iv) If $k < 0$ (i.e., negative), then $a \geq b \Rightarrow ka \leq kb$

Thus, always reverse the sign of inequality while multiplying or dividing both sides of an inequality by a negative number.

Procedure to solve a linear inequality in one variable:

(i) Simplify both sides by collecting like terms.
(ii) Remove fraction (or decimals) by multiplying both sides by appropriate factor (L.C.M. of denominator or a power of 10 in case of decimals)
(iii) Isolate the variables on one side and all constant on the other side.
(iv) Make the coefficient of the variable equal to 1.
(v) Choose the solution set from the replacement set.

Example 7: Solve $\frac{3x-4}{2} \geq \frac{x+1}{4} - 1$. Show the graph of the solutions on number line.

Sol. We have, $\frac{3x-4}{2} \geq \frac{x+1}{4} - 1$

or, $\frac{3x-4}{2} \geq \frac{x-3}{4}$

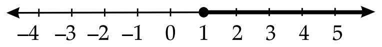
or, $2(3x-4) \geq (x-3)$

or, $6x - 8 \geq x - 3$

or, $5x \geq 5$

or, $x \geq 1$

The graphical representation of solution is given as:



❖ **Linear inequality in two variable**

The inequality of form $ax + by + c > 0$, $ax + by + c = 0$, or $ax + by + c < 0$ etc. where $a \neq 0$, $b \neq 0$ is called a linear equality in two variables x and y.

Graphical solution of Linear Inequalities in Two Variables

(i) The graph of the inequality $ax + by > c$ is one of the half planes and is called the solution region.

(ii) When the inequality involves the sign $\leq$ or $\geq$ then the points on the line are included in the solution region but if it has the sign $<$ or $>$ then the points on the line are not included in the solution region and it has to be drawn as a dotted line.

(iii) The common values of the variable form the required solution of the given system of linear inequalities in one variable.

(iv) The common part of coordinate plane is the required solution of the system of linear inequations in two variables when solved by graphical method.

Example 8: Solve the following system of inequalities graphically

Sol. $3x + 4y \leq 60$, $x + 3y \leq 30$, $x \geq 0$, $y \geq 0$.

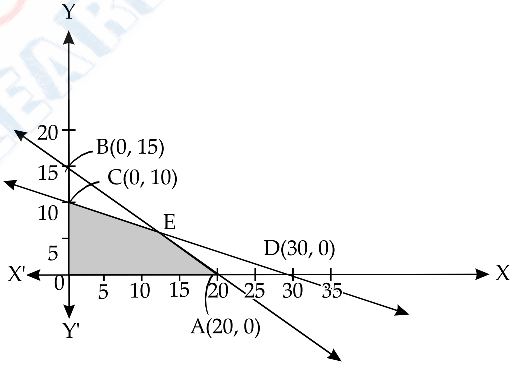
The inequalities are:

$3x + 4y \leq 60$ ...(1)

$x + 3y \leq 30$ ...(2)

$x \geq 0$ ...(3)

$y \geq 0$ ...(4)



We first draw the graph of line $3x + 4y = 60$, $x + 3y = 30$, $x = 0$ and $y = 0$

(a) The line $3x + 4y = 60$ passes through the points (20, 0) and (0, 15) and represented by the line AB. Putting $x = 0$, $y = 0$ in (1), we have $0 < 60$, which is true.

So, origin lies in this region.

$\therefore$ $3x + 4y \leq 60$ represents the region below the line AB and all the points on it.

(b) Further, $x + 3y = 30$ passes through (0, 10) and (30, 0). CD represents this line.

Put $x = 0$, $y = 0$ in (2), we have $0 < 30$, which is true. So, origin lies in this region $x + 3x \leq 30$. The inequality represent the region below the line CD and the line itself.

The solution region is OAEC as shown in figure.

UNIT - II : ALGEBRA

CHAPTER-3

MATRICES

Revision Notes

- **Matrix:**

A matrix is an ordered rectangular **array** of numbers (real or complex) or functions which are known as elements or the entries of the matrix. It is denoted by the uppercase letters i.e., A, B, C etc.

Key Words

Array: An array is a rectangular arrangement of objects in equal rows (horizontal) and equal columns (vertical). Everyday example of arrays include a muffin tray and an egg carton.

Consider a matrix A given as,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Here in matrix A depicted above, the horizontal lines of elements are said to constitute rows of the matrix A and vertical lines of elements are said to constitute columns of the matrix. Thus, matrix A has m **rows** and n **columns**. The array is enclosed by square brackets [ ], the parentheses ( ) or the double vertical bars || ||.

- A matrix having m rows and n columns is called a matrix of order $m \times n$ (read as 'm by n' matrix). A matrix 'A' of order $m \times n$ is depicted as $A = [a_{ij}]_{m\times n}; i, j \in \mathbb{N}$.
- Also, in general, a_{ij} means an element lying in the i^{th} row and j^{th} column.
- No. of elements in the matrix $A = [a_{ij}]_{m\times n}$ is given as $(m)(n)$.

- **Types of Matrices:**

(i) Column matrix: A matrix having only one column is called a **column matrix** or **column vector**.

e.g., $A = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}_{3\times 1}$, $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}_{2\times 1}$

General notation: $A = [a_{ij}]_{m\times 1}$

(ii) Row matrix: A matrix having only one row is called a **row matrix** or **row vector**.

e.g., $A = [2 \quad 5 \quad -4]_{1\times 3}$, $B = [\sqrt{2} \quad 4]_{1\times 2}$

General notation: $A = [a_{ij}]_{1\times n}$

(iii) Square matrix: It is a matrix in which the number of rows is equal to the number of columns *i.e.*, an $n\times n$ matrix is said to constitute a square matrix of order $n \times n$ and is known as a **square matrix of order** 'n'.

e.g., $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & -4 \\ 0 & -1 & -2 \end{bmatrix}_{3\times 3}$ is a square matrix of order 3.

General notation: $A = [a_{ij}]_{n\times n}$

(iv) Diagonal matrix: A square matrix $A = [a_{ij}]_{m\times n}$ is said to be diagonal matrix if all the elements, except those in the leading diagonal are zero i.e., $a_{ij} = 0$ for all $i \neq j$.

e.g., $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3}$ is a diagonal matrix of order 3.

- Also, there are more notation specifically used for the diagonal matrices. For instance, consider the matrix depicted above, it can also be written as diag (2 5 4) or diag [2, 5, 4]
- Note that the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{mm}$ of a square matrix $A=[a_{ij}]_{m\times n}$ of order m × m are said to constitute the principal diagonal or simply **the diagonal of the square matrix** A. These elements are known as **diagonal elements of matrix** A.

(v) Scalar matrix: A diagonal matrix $A=[a_{ij}]_{m\times n}$ is said to be a scalar matrix if its diagonal elements are equal.

i.e., $a_{ij}=\begin{cases}0, & \text{when } i\neq j\\ k, & \text{when } i=j \text{ for some constant } k\end{cases}$

e.g., $A=\begin{bmatrix}17 & 0 & 0\\ 0 & 17 & 0\\ 0 & 0 & 17\end{bmatrix}_{3\times3}$ is a scalar matrix of order 3.

(vi) Unit or Identity matrix: A square matrix $A=[a_{ij}]_{m\times n}$ is said to be an identity matrix if $a_{ij}=\begin{cases}1, \text{ if } i=j\\ 0, \text{ if } i\neq j\end{cases}$.

A unit matrix can also be defined as the scalar matrix each of whose diagonal elements is unity. We denote the identity matrix of order m by I_m or I.

e.g., $I_3=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}_{3\times3}$, $I_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}_{2\times2}$

(vii) Zero matrix or Null matrix: A matrix is said to be a **zero matrix** or **null matrix** if each of its elements is '0'.

e.g., $A=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}_{3\times3}$, $B=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}_{2\times2}$, $C=[0\ \ 0]_{1\times2}$

(viii) Triangular matrix:

(a) Lower triangular matrix: A square matrix is called a lower triangular matrix if all the entries above the main diagonal are zero.

e.g., $A=\begin{bmatrix}1 & 0 & 0\\ 1 & 2 & 0\\ 5 & 1 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 0 & 0\\ 1 & 4 & 0\\ 2 & 3 & 5\end{bmatrix}$

(b) Upper triangular matrix: A square matrix is called a **upper triangular matrix** if all the entries below the main diagonal are zero.

e.g., $A=\begin{bmatrix}1 & -8 & -1\\ 0 & 2 & 1\\ 0 & 0 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 2 & 1\\ 0 & 4 & 3\\ 0 & 0 & 5\end{bmatrix}$

► **Equality of Matrices:**

Two matrices A and B are said to be equal and written as $A = B$, if they are of the same order and their corresponding elements are identical i.e.,

$a_{ij} = b_{ij}$ i.e., $a_{11} = b_{11}, a_{22} = b_{22}, a_{32} = b_{32}$ etc.

► **Algebra of Matrices:**

- **Addition of Matrix:** If A and B are two $m \times n$ matrices, then another $m \times n$ matrix obtained by adding the corresponding elements of the matrices A and B is called the sum of the matrices A and B and is denoted by '$A + B$'.

  Thus, if $A=[a_{ij}], B=[b_{ij}] \Rightarrow A+B=[a_{ij}+b_{ij}]$.

Properties of matrix addition:

- Commutative property: $A + B = B + A$
- Associativity property:

  $A + (B + C) = (A+B) + C$
- Cancellation law:

  (i) Left cancellation: $A + B = A + C \Rightarrow B=C$

  (ii) Right cancellation: $B + A = C + A \Rightarrow B = C$
- **Subtraction of Matrices:**

  If A and B are two $m \times n$ matrices, then another $m \times n$ matrix obtained by subtracting the corresponding elements of the matrices A and B is called the subtraction of the matrices A and B and is denoted by '$A - B$'.

Thus if $A = [a_{ij}]$, $B = [b_{ij}]$, or $A - B = [a_{ij} - b_{ij}]$.

- **Multiplication of Matrices:**

We can only multiply two matrices if their dimensions are compatible , which means the number of columns in the first matrix is the same as the number of rows in the second matrix.

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, the product AB is an $m \times p$ matrix.

$AB = [c_{ij}]$, where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$.

(The entry in the i^{th} row and j^{th} column is denoted by the double subscript notation a_{ij}, b_{ij}, and c_{ij}. For instance, the entry a_{23} is the entry in the second row and third column.)

The definition of matrix multiplication indicates a row-by-column multiplication, where the entries in the i^{th} row of A are multiplied by the corresponding entries in the j^{th} column of B and then adding the results.

Let's take the following example, multiplying a 2×3 matrix with a 3×2 matrix, to get a 2×2 matrix as the product. The entries of the product matrix are called e_{ij} when they're in the i^{th} row and j^{th} column.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$$

Key Facts

- The term matrix was introduced by the 19th century English Mathematician James Sylvester, but it was his friend the Mathematics Arthur Cayley who developed the algebraic aspect of matrices in two papers in the 1850s.
- The English Mathematician Cuthbert Edmund Cullis was the first to use modern bracket notation for matrices in 1913.

To get e_{11}, multiply Row 1 of the first matrix by Column 1 of the second matrix.

$$e_{11} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 1(3) + 0(-1) + 1(2) = 5$$

To get e_{12}, multiply Row 1 of the first matrix by Column 2 of the second matrix.

$$e_{12} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix} = 1(5) + 0(0) + 1(-1) = 4$$

To get e_{21}, multiply Row 2 of the first matrix by Column 1 of the second matrix.

$$e_{21} = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 0(3) + 1(-1) + 2(2) = 3$$

To get e_{22}, multiply Row 2 of the first matrix by Column 2 of the second matrix.

$$e_{21} = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix} = 0(5) + 1(0) + 2(-1) = -2$$

Writing the product matrix, we get

$$\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & -2 \end{bmatrix}$$

Therefore, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & -2 \end{bmatrix}$

Properties of matrix multiplication :

- In the product AB, A is called the **pre-factor** and B is called the **post-factor**.
- If two matrices A and B are such that AB is possible then it is not necessary that the product BA is also possible.
- If A is an $m \times n$ matrix and both AB as well as BA are defined, then B will be an $n \times m$ matrix.
- If A is an $n \times n$ matrix and I_n be the unit matrix of order n, then $AI_n = I_n A = A$.
- Matrix multiplication is **associative** i.e., $A(BC) = (AB)C$.
- Matrix multiplication is **distributive** over the **addition** i.e., $A.(B + C) = AB + AC$.
- Matrix multiplication is not commutative.

Multiplication of a Matrix by a Scalar:

If a $m \times n$ matrix A is multiplied by a scalar k (say), then the new kA matrix is obtained by multiplying each element of matrix A by scalar k. Thus, if $A = [a_{ij}]$, and it is multiplied by a scalar k, then $kA = [ka_{ij}]$, i.e., $A = [a_{ij}]$.

$\Rightarrow kA = [ka_{ij}]$

e.g., $A = \begin{bmatrix} 2 & -4 \\ 5 & 6 \end{bmatrix} \Rightarrow 3A = \begin{bmatrix} 6 & -12 \\ 15 & 18 \end{bmatrix}$

Properties of scalar multiplication:

If A, B are two matrices of the same order and k, l are any scalars (numbers), then

(i) $(k + l)A = kA + lA$

(ii) $k(A + B) = kA + kB$

(iii) $(kl)A = k(lA) + l(kA)$

(iv) $IA = A$

(v) $(-1)A = -A$

Existence of non-zero matrices whose product is zero.

The product of two matrices can be zero without either factor being a zero matrix.

e.g., Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$

Here, $A \neq 0$ and $B \neq 0$.

Also, $AB = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Transpose of a Matrix:

If $A = [a_{ij}]_{m \times n}$ be a $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of matrix A is said to be a transpose of matrix A. The transpose of A is denoted by A' or A^T i.e., if $A^T = [a_{ji}]n \times m$.

For example, if $A = \begin{bmatrix} 5 & -4 & 1 \\ 0 & \sqrt{5} & 3 \end{bmatrix}$ then $A^T = \begin{bmatrix} 5 & 0 \\ -4 & \sqrt{5} \\ 1 & 3 \end{bmatrix}$

Properties of Transpose of Matrices:

- $(A+B)^T = A^T + B^T$
- $(A^T)^T = A$
- $(kA)^T = kA^T$, where k is any constant
- $(AB)^T = B^T A^T$
- $(ABC)^T = C^T B^T A^T$

Symmetric matrix: A square matrix $A = [a_{ij}]$ is said to be a symmetric matrix if $A^T = A$ *i.e.*, if $A = [a_{ij}]$, then $A^T = [a_{ji}] = [a_{ij}]$.

For example,

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

$$B = \begin{bmatrix} 2+i & 1 & 3 \\ 1 & 2 & 3+2i \\ 3 & 3+2i & 4 \end{bmatrix}$$

Note

- For any matrices AA^T and A^TA are symmetric matrices
- If A and B are two symmetric matrices of same order, then
  (i) AB is symmetric if and only if $AB = BA$.
  (ii) $A \pm B$, $AB + BA$ are also symmetric matrices.

► **Skew Symmetric Matrix:**

A square matrix $A = [a_{ij}]$ is said to be **a skew symmetric matrix** if $A^T = -A$ *i.e.*, if $A = [a_{ij}]$, then $A^T = [a_{ji}] = -[a_{ij}]$.

For example, $A = \begin{bmatrix} 0 & 1 & -5 \\ -1 & 0 & 5 \\ 5 & -5 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

Note

- All the diagonal elements in a skew-symmetric matrix are zero.
- If A and B are two symmetric matrices, then $AB - BA$ is a skew symmetric matrix.

Mnemonics

Concept: Types of Matrices

Mnemonics: Remember **C**hrist **S**ubah **D**oophar **S**hyam

Interpretations:

Remember	Christ	Subah	Doophar	Shyam
R	**C**	**S**	**D**	**S**
↓	↓	↓	↓	↓
Row Matrix	Column Matrix	Square Matrix	Diagonal Matrix	Scalar Matrix

Mnemonics: NITE

Interpretations:

N	**I**	**T**	**E**
↓	↓	↓	↓
Null Matrix	Identity Matrix	Triangular Matrix	Equal Matrix

Key Formulae

► For any square matrix A, the matrix $A + A^T$ is a symmetric and $A - A^T$ is always a skew symmetric matrix.

► A square matrix can be expressed as the sum of a symmetric and skew symmetric matrix *i.e.*, $A = \frac{1}{2}(P) + \frac{1}{2}(Q)$, where $P = A + A^T$ is a symmetric matrix and $Q = A - A^T$ is a skew symmetric matrix.

CHAPTER-4
DETERMINANTS

Revision Notes

- **Determinant:** A unique number (real or complex) can be associated to every square matrix $A = [a_{ij}]$ of order m. This number is called the determinant of the square matrix A, where $a_{ij} = (i, j)^{\text{th}}$ element of A.

  The determinant of matrix A is denoted by det A or $|A|$.

  - **Determinant of a square matrix of order 2**

    If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2, then $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

    It follows that the value of a determinant of order 2 is equal to the product of the elements along the principal diagonal *minus* the product of the off diagonal elements.

  Example 1: Let $A = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$, then $|A| = 5 \times 3 - (-2) \times 4 = 15 + 8 = 23$.

  - **Determinant of a square matrix of order 3**

    If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3, then $|A| = (-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

    or $|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{33}a_{21} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

    It follows that the value of a determinant of order 3 is the sum of the product of elements a_{ij} in first row with $(-1)^{i+j}$ times the determinant of a 2×2 sub matrix obtained by leaving the first row and column passing through elements.

- **Elementary properties of Determinants**

  Property I: The value of a determinant is not altered by inter changing its rows into columns and columns into rows, i.e.,

  $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & b_3 \end{vmatrix}$$

  Property II: If any two adjacent rows or columns of a determinant are interchanged, the sign of the determinant get changed but its numerical value remains unaltered, *i.e.*,

  $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

  Property III: If any two rows or columns of a determinant are same, the value of determinant is zero, i.e.,

  Let, $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ or, $\begin{vmatrix} a_1 & a_2 & a_1 \\ b_1 & b_2 & b_1 \\ c_1 & c_2 & c_1 \end{vmatrix} = 0$

  Property IV : If every element in a row or a column of a determinant is multiplied by the same non-zero constant k, then the value of the determinant gets multiplied by k, i.e.,

  $$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

  Property V : A determinant can be expressed as the sum of several determinants of the same order, i.e.,

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Property VI : The value of the determinant is not affected if the elements of a row or column are increased or diminished by the same multiple of the corresponding elements of any other row or column, i.e.,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+pb_1+qc_1 & b_1 & c_1 \\ a_2+pb_2+qc_2 & b_2 & c_2 \\ a_3+pb_3+qc_3 & b_3 & c_3 \end{vmatrix}$$

Property VII : If each element of a row or column of a determinant is zero, its value is zero, i.e.,

$$\begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

- **Singular Matrix & Non-Singular Matrix:**

(a) Singular matrix: A square matrix A is said to be singular if $|A| = 0$ i.e., its determinant is zero.

e.g., $$A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 12 \\ 1 & 1 & 3 \end{vmatrix}$$

$$= 1(15-12) - 2(12-12) + 3(4-5)$$
$$= 3 - 0 - 3 = 0$$

$\therefore$ A is singular matrix.

(b) Non-singular matrix : A square matrix A is said to be non-singular if $|A| \neq 0$.

e.g., $$A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 0\,(0-1) - 1(0-1) + 1(1-0)$$
$$= 0 + 1 + 1 = 2 \neq 0$$

$\therefore$ A is non-singular matrix.

- A square matrix A is **invertible** if and only if A is **non-singular**.

- **Minors:** Minors of an element a_{ij} of a determinant *(or a determinant corresponding to matrix* A) is the determinant obtained by deleting its i^{th} row and j^{th} column in which a_{ij} lies. Minor of a_{ij} is denoted by M_{ij}. Hence, we can get 9 minors corresponding to the 9 elements of a third order (i.e., 3×3) determinant.

- **Co-factors:** Cofactor of an element a_{ij}, denoted by A_{ij}, is defined by $A_{ij} = (-1)^{(i+j)} M_{ij}$, where M_{ij} is minor of a_{ij}. Sometimes C_{ij} is used in place of A_{ij} to denote the cofactor of element a_{ij}.

- **Adjoint of a Square Matrix:**

Let $A = [a_{ij}]$ be a square matrix. Also, assume $B = [A_{ij}]$, where A_{ij} is the cofactor of the elements a_{ij} in matrix A. Then the transpose B^T of matrix B is called the **adjoint of matrix** A and it is denoted by **"*adj* (A)".**

To find adjoint of a 2 × 2 matrix: If the adjoint of a square matrix of order 2 can be obtained by interchanging the diagonal elements and changing the signs of off-diagonal elements i.e. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $adj\ A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Example 2: consider a square matrix of order 3 as $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$, then in order to find the adjoint matrix A, we find a matrix B (formed by the co-factors of elements of matrix A as mentioned above in the definition)

i.e., $B = \begin{bmatrix} 15 & -2 & -6 \\ -10 & -1 & 4 \\ -1 & 2 & -1 \end{bmatrix}$. Hence, $adj\ A = B^T = \begin{bmatrix} 15 & -10 & -1 \\ -2 & -1 & 2 \\ -6 & 4 & -1 \end{bmatrix}$

- **Algorithm to find A^{-1} by Determinant Method:**

Step 1: Find $|A|$.

Step 2: If $|A| = 0$, then, write "A is a singular matrix and hence not invertible". Else write "A is a non-singular matrix and hence invertible".

Step 3: Calculate the co-factors of elements of matrix A.

Step 4: Write the matrix of co-factors of elements of A and then obtain its transpose to get *adjA* (i.e., adjoint A).

Step 5: Find the inverse of A by using the relation: $A^{-1} = \dfrac{1}{|A|}(adj\,A)$.

Note

For a 2 × 2 matrix, the inverse is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \text{ where } ad-bc \neq 0$$

Just swap the 'a' and 'd', negate the 'b' and 'c', then divide all by the determinant $ad - bc$.

- **Properties associated with various Operation of Matrices & Determinants:**
  - **(a)** $AB = I = BA$
  - **(b)** $AA^{-1} = I$ or $A^{-1}I = A^{-1}$
  - **(c)** $(AB)^{-1} = B^{-1}A^{-1}$
  - **(d)** $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
  - **(e)** $(A^{-1})^{-1} = A$
  - **(f)** $(A^T)^{-1} = (A^{-1})^T$
  - **(g)** $A(adjA) = (adj\ A)A = |A|\ I$
  - **(h)** $adj(AB) = adj(B)\ adj\ (A)$
  - **(i)** $adj(A^T) = (adj\ A)^T$
  - **(j)** $(adj\ A)^{-1} = (adj\ A^{-1})$
  - **(k)** $|adj\ A| = |A|^{n-1}$, if $|A| \neq 0$, where n is of the order of A.
  - **(l)** $|AB| = |A|\,|B|$
  - **(m)** $|A\ adj\ A| = |A|^n$, where n is of the order of A.
  - **(n)** $|A^{-1}| = \dfrac{1}{|A|}$, if matrix A is invertible.
  - **(o)** $|A| = |A^T|$
  - $|kA| = k^n\ |A|$, where n is of the order of square matrix A and k is any scalar.
  - If A is a non-singular matrix of order n, then $adj\ (adj\ A) = |A|^{n-2} A$.
- **Uniqueness Theorem:** Let A be an invertible square matrix of order n. Suppose B and C are the two inverse of A.

  Then $AB = BA = I_n$ (by definition of inverse matrix)

  $AC = CA = I_n$

  Now, $B = BI_n = B(AC)$ [$\because$ Matrix multiplication is associative]

  $= (BA)C$

  $= I_nC$

  $= C$

  $\therefore B = C$, i.e., any two inverse of A are equal matrices.

  Hence, the inverse of A is unique.
- **Cramer's Rule:**

  Let the three nonhomogeneous linear equations be

  $$a_1x + b_1y + c_1z = d_1$$
  $$a_2x + b_2y + c_2z = d_2$$
  $$a_3x + b_3y + c_3z = d_3$$

  The solution of the system of linear equations is given by

  $$x = \frac{D_x}{D}, y = \frac{D_y}{D} \text{ and } z = \frac{D_z}{D}$$

  Where,

  $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

  $$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix},$$

  Provided that $D \neq 0$
- **Condition for consistency:**
  - **(a)** If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by

    $$x = \frac{D_x}{D},\ y = \frac{D_y}{D} \text{ and } z = \frac{D_z}{D}$$
  - **(b)** If $D = 0$ and $D_x = D_y = D_z = 0$, the given system of equations is consistent has infinitely many solutions.
  - **(c)** If $D = 0$ and atleast one of the determinants D_x, D_y, D_z is non-zero, then the given system of equations is inconsistent.

Key Facts

- In mathematics, the determinant is a scalar value that is a function of the entries of a square matrix.
- There are 10 main properties of determinants which include reflection property, all-zero property, proportionality or repetition property, switching property, scalar multiple property, sum property, invariance property, factor property, triangle property and determinant of factor property.

Mnemonics

Concept: Cramer's Rule

Mnemonics:

For X	**For Y**
Interpretations:	
Every **B**oy **F**inds **D**ogs **(E*D — B*F)** divided by (A*D — B*C) for X.	**A**ll **E**lephants **C**an't **F**orget **(A*F — E*C)** divided by (A*D — B*C) for Y.

$$2x - 3y = 9$$
$$x + 5y = -2$$

Comparing the above set of equations by

$$Ax + By = E$$
$$Cx + Dy = F$$

We, get $A = 2, B = -3, C = 1, D = 5, E = 9$ and $F = -2$

So, $$x = \frac{9\times5-(-3)\times(-2)}{2\times5-(-3)\times1}$$
$$= \frac{45-6}{10+3} = \frac{39}{13} = 3$$

and $$y = \frac{2\times(-2)-9\times1}{2\times5-(-3)\times1}$$
$$= \frac{-4-9}{10+3} = \frac{-13}{13} = -1$$

So, the solution set is (3, – 1).

UNIT - III : CALCULUS

CHAPTER-5

DIFFERENTIATION AND ITS APPLICATIONS

Revision Notes

- **Higher Order Derivatives of an Explicit Function**

Let the function $y = f(x)$ have a finite derivative $f'(x)$ in a certain interval (a, b), i.e., the derivative $f'(x)$ is also a function in this interval. If this function is differentiable, we can find the second derivative of the original function $y = f(x)$, which is denoted by various notations as

$$f'' = (f')' = \left(\frac{dy}{dx}\right)'$$
$$= \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$= \frac{d^2y}{dx^2}$$

Similarly, if f'' exists and is differentiable, we can calculate the third derivative of the function $f(x)$:

$$f''' = \frac{d^3y}{dx^3} = y'''$$

The result of taking the derivative n times is called the n^{th} derivative of $f(x)$ with respect to x and is denoted as

$$\frac{d^ny}{dx^n} = \frac{d^ny}{dx^n} \text{ (in Leibniz's notation)}$$

$$f^{(n)}(x) = y^{(n)}(x) \text{ (in Lagrange's notation)}$$

Thus, the notation of the nth order derivative in introduced inductively by sequential calculation of n derivatives starting from the first order derivative. Transition to the next higher-order derivative is performed using the recurrence formula

$$y^{(n)} = (y^{(n-1)})'$$

Higher Order Derivatives of an Implicit Function

The nth order derivative of an implicit function can be found by sequential (n times) differentiation of the equation $F(x, y) = 0$. At each step, after appropriate substitutions and transformations, we can obtain an explicit expression for the derivative, which depends only on the variables x and y, *i.e.*, the derivatives have the form

$$\begin{aligned} y' &= f_1(x, y) \\ y_n &= f_2(x, y), \ldots \\ y(n) &= f_n(x, y) \end{aligned}$$

Higher Order Derivatives of Parametric Function

Consider a function $y = f(x)$ given parametrically by the equations

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

The first derivative of this function is given by

$$y' = y'_x = \frac{y'_t}{x'_t}$$

Differentiating once more with respect to x, we find the second derivative:

$$y'' = y''_{xx} = \frac{(y'_x)'_t}{x'_t}$$

Example 1: Given the function $y = (2x + 1)^3 (x - 1)$. Find all derivatives of the n^{th} order from $n = 1$ to $n = 5$.

Sol. First we convert the given function into a polynomial:

$$\begin{aligned} y &= (2x + 1)^3 \\ &= (8x^3 + 12x^2 + 6x + 1).(x - 1) \\ &= 8x^4 + 12x^3 + 6x^2 + x - 8x^3 - 12x^2 - 6x - 1 \\ &= 8x^4 + 4x^3 - 6x^2 - 5x - 1. \end{aligned}$$

Now we successively calculate the derivatives from 1st to 5th order:

$$\begin{aligned} y' &= (8x^4 + 4x^3 - 6x^2 - 5x - 1)' \\ &= 32x^3 + 12x^2 - 12x - 5 \\ y'' &= (y')' = (32x^3 + 12x^2 - 12x - 5)' \\ &= 96x^2 + 24x - 12 \\ y''' &= (y'')' = (96x^2 + 24x - 12)' \\ &= 192x + 24 \\ y^{(4)} &= (y''')' = (192x + 24)' = 192 \\ y^{(5)} &= (y''')' = (192)' = 0. \end{aligned}$$

Increasing/Decreasing Functions

- A function $f(x)$ is said to be an increasing function in $[a, b]$, if as x increases, $f(x)$ also increases, i.e., if $\alpha, \beta \Leftrightarrow [a, b]$ and $\alpha > \beta$, $f(\alpha) > f(\beta)$.
  If $f'(x) \geq 0$ lies in (a, b), then $f(x)$ is an increasing function in $[a, b]$, provided $f(x)$ is continuous at $x = a$ and $x = b$.
- A function $f(x)$ is said to be a decreasing function in $[a, b]$, if as x increases, $f(x)$ decreases, i.e., if $\alpha, \beta \Leftrightarrow [a, b]$ and $\alpha > \beta \Rightarrow f(\alpha) < f(\beta)$.
  If $f'(x) = 0$ lies in (a, b), then $f(x)$ is a decreasing function in $[a, b]$, provided $f(x)$ is continuous at $x = a$ and $x = b$.
  - A function $f(x)$ is a constant function in $[a, b]$, if $f'(x) = 0$ for each $x \Leftrightarrow (a, b)$.
  - By monotonic function $f(x)$ in interval I, we mean that f is either only increasing in I or only decreasing in I.
- Finding the intervals of increasing and/or decreasing of a function:

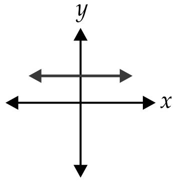
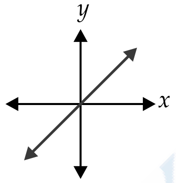
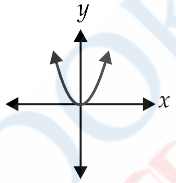
  Algorithm

  Step 1: Consider the function $y = f(x)$.

  Step 2: Find $f'(x)$.

  Step 3: Put $f'(x) = 0$ and solve to get the critical point(s).

  Step 4: The value(s) of x for which $f'(x) > 0$, $f(x)$ is increasing; and the value(s) of x for which $f'(x) < 0$, $f(x)$ is decreasing.

Function	Increasing/ Decreasing	Graph
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	

Example 2: Find the intervals in which the function $f(x)$ is strictly increasing where, $f(x) = 10 - 6x - 2x^2$.

Sol. Given, $f(x) = 10 - 6x - 2x^2$

$\therefore \quad f'(x) = -6 - 4x$

$f'(x) = -6 - 4x = 0$

$$\Rightarrow \qquad x = \frac{-6}{4} = \frac{-3}{2}$$

$\therefore$ The intervals are $\left(-\infty, \frac{-3}{2}\right)$ and $\left(\frac{-3}{2}, \infty\right)$

$f'(x) > 0$ in $\left(-\infty, \frac{-3}{2}\right)$

$\therefore$ $f(x)$ is strictly increasing in $\left(-\infty, \frac{-3}{2}\right)$

► **Maxima and Minima**

- **Understanding maxima and minima:**

  Consider $y = f(x)$ be a well defined function on an **interval** I, then

Key Words

Interval: In mathematics, an interval is a set of real numbers between two given numbers called end points of the interval.

(a) f is said to have a maximum value in I, if there exists a point c in I such that

$$f(c) > f(x), \text{ for all } x \Leftrightarrow I$$

The value corresponding to $f(c)$ is called the maximum value of f in I and the point c is called the point of maximum value of f in I.

(b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) < f(x)$, for all $x \Leftrightarrow I$.

The value corresponding to $f(c)$ is called the minimum value of f in I and the point c is called the **point of minimum value of f in I**.

(c) f is said to have an extreme value in I, if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I.

The value $f(c)$ in this case, is called an extreme value of f in I and the point c called an **extreme point**.

- Let f be a real valued function and also take a point c from its **domain**. Then

Key Words

Domain: The domain refers to the set of possible input values, the domain of a graph consists of all input values shown on the x-axis.

(i) c is called a point of **local maxima** if there exists a number $h > 0$ such that $f(c) > f(x)$, for all x in $(c - h, c + h)$. The value $f(c)$ is called the **local maximum value of** f.

(ii) c is called a point of **local minima** if there exists a number $h > 0$ such that $f(c) < f(x)$, for all x in $(c - h, c + h)$. The value $f(c)$ is called the **local minimum value** of f.

- **Critical points:**
  It is a point c (say) in the domain of a function $f(x)$ at which either $f'(x)$ vanishes, i.e., $f'(c) = 0$ or f is not differentiable.
- **First Derivative Test:**
  Consider $y = f(x)$ be a well defined function on an open interval I. Now proceed as have been mentioned in the following algorithm:

  Step 1: Find $\dfrac{dy}{dx}$.

  Step 2: Find the critical point(s) by putting $\dfrac{dy}{dx} = 0$. Suppose $c \Leftrightarrow I$ (where I is the interval) be any critical point and f be continuous at this point c. Then we may have following situations:

  ♦ $\dfrac{dy}{dx}$ changes sign from **positive to negative** as x increases through c, then the function attains a **local maximum** at $x = c$.

  ♦ $\dfrac{dy}{dx}$ changes sign from **negative to positive** as x increases through c, then the function attains a **local minimum** at $x = c$.

  ♦ $\dfrac{dy}{dx}$ **does not change sign** as x increases through c, then $x = c$ is **neither** a point of **local maximum nor** a point of **local minimum**. Rather in this case, the point $x = c$ is called the **point of inflection**.
- **Second Derivative Test:**
  Consider $y = f(x)$ be a well defined function on an open interval I and twice differentiable at a point c in the interval. Then we observe that:

  ♦ $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$
  The value $f(c)$ is called the local maximum value of f.

  ♦ $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$
  The value $f(c)$ is called the local minimum value of f.
  This test fails if $f'(c) = 0$ and $f''(c) = 0$. In such a case, we use **first derivative test** as discussed in the above.
- **Absolute maxima and absolute minima:**
  If f is a continuous function on a **closed interval** I, then f has the absolute maximum value and f attains it atleast once in I. Also f has the absolute minimum value and the function attains it atleast once in I.

  Algorithm

  Step 1: Find all the critical points of f in the given interval, i.e., find all the points x where either $f'(x) = 0$ or f is not differentiable.

  Step 2: Take the end points of the given interval.

  Step 3: At all these points (i.e., the points found in Step 1 and Step 2) calculate the values of f.

  Step 4: Identify the maximum and minimum value of f out of the values calculated in Step 3. This maximum value will be the **absolute maximum value** of f and the minimum value will be the **absolute minimum value** of the function f.

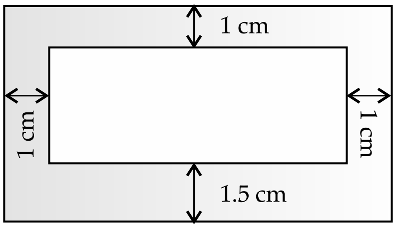
  Absolute maximum value is also called as **global maximum value** or **greatest value**. Similarly, absolute minimum value is called as **global minimum value** or the **least value**.

  Second Derivative Test:
  Consider $y = f(x)$ be a well defined function on an open interval I and twice differentiable at a point c in the interval. Then we observe that:

  Example 3: A printed page is to have total area of 80 sq. cm with a margin of 1 cm at the top and on each side and a margin of 1.5 cm at the bottom. What should be the dimensions of the page so that the printed area will be maximum?

Sol. Let x cm and y cm be the dimensions of the printed page.

$$xy = 80 \quad \text{...(i)}$$



Key Facts

- Maxima and minima is used to solve optimization problems such as maximizing profit, minimizing the amount of material used in manufacturing or finding the maximum height a rocket can reach.
- A function is called increasing if it increases as the input x moves from left to right, and is called decreasing if it decreases as x moves from left to right.
- In Computer Science, Calculus is used for machine learning, data mining, scientific computing, image processing, and creating the graphics and physics engines for video games, including the 3D visuals for simulations.

Let A (x sq. cm) be the printed area then,

$$A = (x-2)(y-5/2)$$

$$\Rightarrow \quad A = xy - \frac{5x}{2} - 2y + 5$$

$$\Rightarrow \quad A = 80 - \frac{5x}{2} - \frac{2\times 80}{x} + 5 \quad \text{[using (i)]}$$

$$\Rightarrow \quad A = 85 - \frac{5x}{2} - \frac{160}{x}$$

Differentiating w.r.t. x, we get

$$\frac{dA}{dx} = -\frac{5}{2} + \frac{160}{x^2}$$

Differentiating once again w.r.t. x

$$\frac{d^2A}{dx^2} = -\frac{320}{x^3}$$

For maximum and minimum Area,

$$\Rightarrow \quad \frac{dA}{dx} = 0$$

$$-\frac{5}{2} + \frac{160}{x^2} = 0$$

$$\Rightarrow \quad \frac{160}{x^2} = \frac{5}{2}$$

$$\Rightarrow \quad x^2 = 64$$

$$\Rightarrow \quad x = -8$$

Also, $\left(\frac{d^2A}{dx^2}\right)_{x=8} = -\frac{320}{512} < 0$

$\Rightarrow A$ is maximum at $x = 8$

From (i), $xy = 80$

$\Rightarrow \quad 8 \times y = 80$

$\Rightarrow \quad y = 10$

Therefore, the dimensions of the printed page are 10 cm and 8 cm.

Marginal Cost and Marginal Revenue

Derivatives are useful in analysing changes in cost, revenue, and profit. The concept of a marginal function is common in the fields of business and economics and implies the use of derivatives. The marginal cost is the derivative of the cost function. The marginal revenue is the derivative of the revenue function. The marginal profit is the derivative of the profit function, which is based on the cost function and the revenue function.

- The Fixed Cost (FC) is the amount of money we have to spend regardless of how many items we produce. FC can include things like rent, purchase costs of machinery, and salaries for office staff. We have to pay the fixed costs even if we don't produce anything.
- The Total Variable Cost (TVC) for q items is the amount of money we spend to actually produce them. TVC includes things like the materials we use, the electricity to run the machinery, gasoline for our delivery vans, maybe the wages of our production workers. These costs will vary according to how many items we produce.
- The Total Cost (TC) for x items is the total cost of producing them. It's the sum of the fixed cost and the total variable cost for producing q items.
- The Average Cost (AC) for x items is the total cost divided by x i.e., $\frac{\text{TC}}{x}$.

  Also, Average Fixed Cost = $\frac{\text{FC}}{x}$ and Average Variable Cost = $\frac{\text{TVC}}{x}$.
- If C(x) is the cost of producing x items, then the marginal cost $MC(x)$ is $MC(x) = C'(x)$.
- If R(x) is the revenue obtained from selling x items, then the marginal revenue $MR(x)$ is $MR(x) = R'(x)$.
- If $P(x) = R(x) - C(x)$ is the profit obtained from selling x items, then the marginal profit MP(x) is defined to be $MP(x) = P'(x) = MR(x) - MC(x) = R'(x) - C'(x)$.
- Break even point (BEP) is the point where the profit from the transaction is zero and the total sales is equal to total costs. Break even point is the inflection point where the revenue sales are same as the costs. At the break even point, there is zero profit or zero loss for the company.

  Break even point (BEP)= Total Sales – Total Costs
- Break even point is important for companies to understand the minimum business required to sustain any product or service. The units break even i.e. number of units to be sold at the break even pricing helps in evaluating the break even point. This method of evaluation is known as break even analysis.

Example 4: The total cost associated with provision of free mid-day meals to x students of a school in primary classes is given by $C(x)= 0.005x^3 - 0.02x^2 + 30x + 50$. If the marginal cost is given by rate of change $\frac{dC}{dx}$ of total cost, write the marginal cost of food for 300 students.

Sol. We have, $C(x) = 0.005x^3 - 0.02x^2 + 30x + 50$

$$\Rightarrow \quad \frac{dC}{dx} = C'(x) = 0.015x^2 - 0.04x + 30 + 0$$

At $x = 300$,

$C'(300) = 0.015(300)^2 - 0.04(300)+30 = 1368$

So, the marginal cost of food for 300 students is ₹ 1,368.

Mnemonics

Concept: Increasing and Decreasing Function

Mnemonics: Dead **Z**ombies **C**onsider **G**reen **L**emons

Interpretations:

Dead	Zombies	Consider	Green	Lemons
↓	↓	↓	↓	↓
Find **Derivative** of function $f(x)$ *i.e.*, $f'(x)$	Put Derivative equal to **zero** *i.e.*, $f'(x) = 0$	Get **Critical** points *i.e.*, values of x	If $f'(x)$ is **Greater** than zero *i.e.*, $f'(x) > 0$, then $f(x)$ is increasing	If $f'(x)$ is **Less** than zero i.e., $f'(x) < 0$, then $f(x)$ is decreasing

CHAPTER-6

INTEGRALS AND IT'S APPLICATIONS

Topic-1

Indefinite Integrals

Concepts Covered • *Formulae for Indefinite Integrals, Integration by substitution, Integration by parts, Integration by partial fraction.*

Revision Notes

- **Meaning of Integral of Function**

If differentiation of a function $F(x)$ is $f(x)$ i.e., if $\frac{d}{dx}[F(x)] = f(x)$, then we say that one integral or primitive or **anti-derivative** of $f(x)$ is $F(x)$ and in symbols, we write, $\int f(x)dx = F(x) + C$.

Key Word

Anti-derirative: In calculus, an anti-derivative, inverse derivative, primitive function, primitive integral or indefinite integral of a function f is a differentiable function F whose derivative is equal to the original function f. This can be stated symbolically as $F' = f$.

Therefore, we can say that integration is the inverse process of differentiation.

- **Indefinite Integral:** Let $f(x)$ be a function. Then family of all its primitives or anti derivatives is called the indefinite integral of $f(x)$ and denoted by $\int f(x)dx$.

Formulae for Indefinite Integrals

(a) $\int x^n dx = \frac{x^{n+1}}{n+1} + k,\ n \neq -1$

(b) $\int \frac{1}{x} dx = \log|x| + k$

(c) $\int a^x dx = \frac{1}{\log a} a^x + k$

(d) $\int e^{ax} dx = \frac{1}{a} e^{ax} + k$

(e) $\int \frac{1}{ax+b} dx = \frac{1}{a} \log|ax+b| + k$

(f) $\int \lambda dx = \lambda x + k$, where 'λ' is a none zero constant.

(g) $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + k$

(h) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + k$

(i) $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log\left|x + \sqrt{x^2 - a^2}\right| + k$

(j) $\int \frac{1}{\sqrt{x^2 + a^2}} dx = \log\left|x + \sqrt{x^2 + a^2}\right| + k$

(k) $\int \sqrt{x^2 - a^2} dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left|x + \sqrt{x^2 - a^2}\right| + k$

(l) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + k$

Example 1: Find: $\int \frac{dx}{4x^2-9}$

Sol. Let, $I = \int \frac{dx}{4x^2-9} = \frac{1}{4}\int \frac{dx}{x^2-\frac{9}{4}}$

$$= \frac{1}{4}\int \frac{dx}{x^2-\left(\frac{3}{2}\right)^2} = \frac{1}{4}\,\frac{1}{2\times\frac{3}{2}}\log\left|\frac{x-\frac{3}{2}}{x+\frac{3}{2}}\right| + C = \frac{1}{12}\log\left|\frac{2x-3}{2x+3}\right| + C$$

Key Facts

- In real life, integrations are used in various fields such as engineering, where engineers use integrals to find the shape of building.
- In Physics, integrals used in the centre of gravity etc.
- Integrals are used in the field of graphical representation, where three-dimensional models are demonstrated.

- **Methods of Integration**

(a) Integration by Substitution Method: In this method, we change the integral $\int f(x)dx$, where independent variable is x, to another integral in which independent variable is t (say different from x) such that x and t are related by $x = g(t)$.

Let $u = \int f(x)dx$ then, $\frac{du}{dx} = f(x)$

Again as $x = g(t)$ so we have $\frac{dx}{dt} = g'(t)$

Now $\frac{du}{dt} = \frac{du}{dx}\cdot\frac{dx}{dt} = f(x)\cdot g'(t)$

On integrating both sides w.r.t. t, we get

$$\int\left(\frac{du}{dt}\right)dt = \int f(x)g'(t)dt$$

$$\Rightarrow u = \int f[g(t)]g'(t)dt$$

i.e., $\int f(x)dx = \int f[g(t)]g'(t)dt$, where $x = g(t)$.

So, it is clear that substituting $x = g(t)$ in $\int f(x)dx$ will give us the same result as obtained by putting $g(t)$ in place of x and $g'(t)dt$ in place of dx.

Example 2: Find: $\int \frac{x}{e^{x^2}}dx$

Sol. Let, $I = \int \frac{x}{e^{x^2}}dx = \int e^{-x^2}x\,dx$

Put $-x^2 = t$

$\Rightarrow$ $-2x\,dx = dt$

$\Rightarrow$ $x\,dx = -\frac{1}{2}dt$

$\therefore$ $I = \int e^t\left(-\frac{1}{2}dt\right)$

$$= -\frac{1}{2}\int e^t dt$$

$$= -\frac{1}{2}e^t + C$$

$$= -\frac{1}{2}e^{-x^2} + C$$

(b) Integration by Parts:

If u and v be two functions of x, then

$$\int \underset{\text{I}}{u}, \underset{\text{II}}{v}\,dx = u\left(\int v dx\right) - \int\left\{\frac{du}{dx}\int v dx\right\}dx$$

Example 3: Find: $\int \log x\, dx$

Sol. Let us assume here log x is the first function and constant 1 is the second function. Then the integral of the second function is x.

Therefore,

$$\int(\log x.1)dx = \log x\int 1dx - \int\left[\frac{d}{dx}(\log x)\int 1dx\right]dx$$

$$= (\log x).x - \int\frac{1}{x}x\,dx$$

$$= x\log x - x + C$$

Note

We can choose first function as the one whose initials comes first in the word. 'ILATE', where
I - Inverse Trigonometric function
L - Logarithm function
A - Algebraic function
T - Trigonometric function
E - Exponential function

(C) Integration by partial Fractions: Consider $\frac{f(x)}{g(x)}$ defines a rational polynomial function.

If the degree of numerator i.e., $f(x)$ is greater than or equal to the degree of denominator i.e., $g(x)$ then, this type of rational function is called an improper rational function. And if degree of $f(x)$ is smaller than the degree of denominator i.e., $g(x)$ then this type of rational function is called a proper rational function.

In rational polynomial function if the degree (i.e., highest power of the variable) of numerator (Nr) is greater than or equal to the degree of denominator (Dr), then (without) any doubt always perform the division i.e,. divide the Nr by Dr before doing anything and thereafter use the following:

$$\frac{\text{Numerator}}{\text{Denominator}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Denominator}}$$

Table Demonstrating Partial Fractions of Various Forms

From of the Rational Functions	Form of the Rational Functions
$\frac{px+q}{(x-a)(x-b)}, x\neq a,b$	$\frac{A}{x-a}+\frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}\; x\neq a$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} x\neq a,b,c$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)} x\neq a,b$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}\; x\neq a$ where $x^2 + bx + c$ can't be factorized further.	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

Example 4: Evaluate: $\int\frac{3x+1}{(x+1)^2(x+3)}dx$

Sol. Let $$I = \int \frac{3x+1}{(x+1)^2(x+3)}dx$$

Let $$\frac{3x+1}{(x+1)^2(x+3)} = \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+3} \quad ...(i)$$

$$3x + 1 = A(x + 1)(x + 3) + B(x + 3) + C(x + 1)^2$$

$\Rightarrow$ $$3x + 1 = x^2(A + C) + x(4A + B + 2C) + (3A + 3B + C)$$

$\Rightarrow$ $$A + C = 0, 4A + B + 2C = 3, 3A + 3B + C = 1$$

On solving, we get $A = 2, B = –1, C = –2$

From (ii),

$$\frac{3x+1}{(x+1)^2(x+3)} = \frac{2}{x+1}-\frac{1}{(x+1)^2}-\frac{2}{x+3}$$

$$\therefore \quad I = \int\frac{2}{x+1}dx-\int\frac{1}{(x+1)^2}dx-\int\frac{2}{x+3}dx$$

$$= 2\log|x+1| + \frac{1}{x+1} - 2\log|x+3| + C$$

$$= 2\log\frac{(x+1)}{(x+3)}+\frac{1}{x+1}+C$$

- **Special form of Integrals:**

(a) $\int e^x\{f(x)+f'(x)\}dx = e^x f(x) + C$ **(b)** $\int\frac{f'(x)}{f(x)}dx = \log|f(x)| + C$

Topic-2 Definite Integrals

Concepts Covered • *Properties of definite integrals*

Revision Notes

- **Meaning of Definite Integral of Function**

If $\int f(x)dx = F(x)$, be an integral of $f(x)$, then $F(b) – F(a)$ is called the definite integral of $f(x)$ between the limits a and b and in symbols it is written as $\int_a^b f(x)dx = [F(x)]_a^b = F(b) – F(a)$. Moreover, the definite integral gives a unique and definite value (numeric value) of anti-derivative of the function between the given intervals. It acts as a substitute for evaluating the area analytically.

- **Properties of definite Integral**

(a) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ **(b)** $\int_a^b f(x)dx = \int_a^b f(t)dt$ **(c)** $\int_a^b f(x)dx = \int_a^c f(x)dx+\int_c^b f(x)dx, a<c<b$

(d) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ **(e)** $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(f) $$\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx \text{ if } f(x) \text{ is an even function i.e., } f(-x) = f(x) \\ 0, \text{ if } f(x) \text{ is an odd function i.e., } f(-x) = -f(x)\end{cases}$$

(g) $$\int_{-a}^a f(x)dx = \int_0^a \{f(x)+f(-x)\}dx$$

(h) $$\int_0^{2a} f(x)dx = \int_0^{2a}\{f(x)+f(2a-x)\}dx$$

(i) $$\int_o^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x) \\ 0 \quad ; \text{ if } f(2a-x) = -f(x)\end{cases}$$

Mnemonics

Concept:

SeCond FundAmental Theorem of Definite Integration

f c c   F a d   F b F a

Continuous   Closed   anti   derivative

Mnemonics:

fcc is small **f**ashionable **C**lothes

Counter **F**or **a**dorable **d**resses

at **Fb**b **Fa**shion

Closed   anti   Continuous   derivative

F(b) – F(a)

Interpretations: Let *f* be a continuous function defined on a closed interval [*a*,*b*] and F be an anti derivative of *f*.

Then $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$, where *a* and *b* are called limit of Integration.

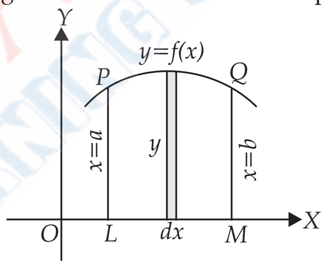
Topic-3 Application of Integrals

Concepts Covered • *Area under simple curves,* • *Consumer surplus,* • *Producer surplus*

Revision Notes

► **Area Under Simple Curves:**

(i) Let us find the area bounded by the **curve** $y = f(x)$, X-axis and the ordinates $x = a$ and $x = b$. Consider the area under the curve as composed by large number of thin vertical strips.



Key Words

Curve: A **curve** is a continuous and smooth flowing line without any sharp turns. One way to recognize a curve is that it bends and changes its direction at least once.

- A **open curve** does not enclose any area within itself and it has two endpoints. Some of the open curves are given in the figure below.
- A **closed curve** has no end points and encloses an area (or a region). It is formed by joining the end points of an open curve together. e.g.: Circles, ellipses are formed from closed curves.
- A **simple curve** changes direction but does not cross itself while changing direction. A simple curve can be open and closed both.
- A **non-simple curve** crosses its own path

Arbitrary: In mathematics, "arbitrary" just means "for all".

For example: "For all a, b, $a + b = b + a$".

Another way to say this would be "$a + b = b + a$ for arbitrary a, b."

Ordinate: The Cartesian coordinate obtained by measuring parallel to the y-axis.

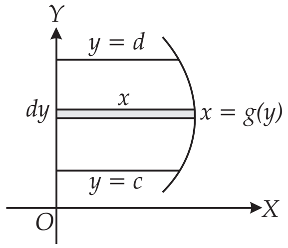
Let there be an **arbitrary** strip of height y and width dx.

Area of elementary strip $dA = y\,dx$, where $y = f(x)$. Total area A of the region between X-axis ordinates $x = a$, $x = b$ and the curve $y = f(x)$ = sum of areas of elementary thin strips across the region $PQML$.

$$A = \int_a^b y\,dx = \int_a^b f(x)\,dx$$

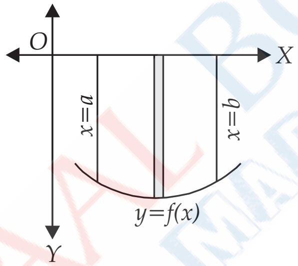
(ii) The area A of the region bounded by the curve $x = g(y)$, Y-axis and the lines $y = c$ and $y = d$ is given by

$$A = \int_c^d x\,dy$$

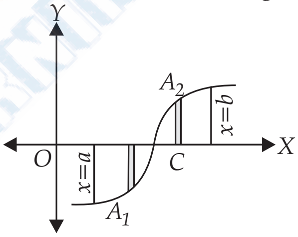


(iii) If the curve under consideration lies below X-axis, then $f(x) < 0$ from $x = a$ to $x = b$, the area bounded by the curve $y = f(x)$ and the **ordinates** $x = a$, $x = b$ and X-axis is negative. But, if the numerical value of the area is to be taken into consideration, then

$$\text{Area} = \left|\int_a^b f(x)dx\right|$$



(iv) It may also happen that some portion of the curve is above X-axis and some portion is below X-axis as shown in the figure. Let A_1 be the area below X-axis and A_2 be the area above the X-axis. Therefore, area bounded by the curve $y = f(x)$, X-axis and the ordinates $x = a$ and $x = b$ is given by $A = |A_1| + |A_2|$



Example 5 : Calculate the area under the curve of a function, $f(x) = 7 - x^2$, the limit is given as $x = -1$ to 2.

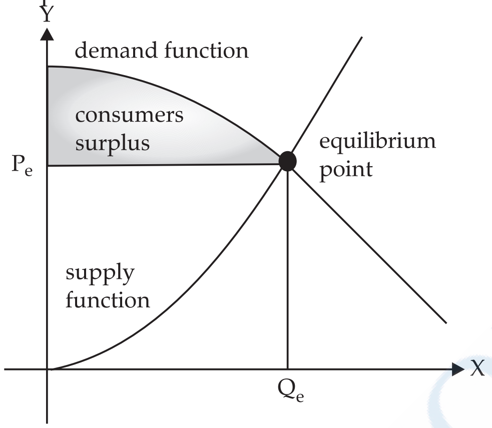
Sol.: Given function is, $f(x) = 7 - x^2$ and limit is $x = -1$ to 2

$$\begin{aligned}
\text{Area} &= \int_{-1}^{2}(7 - x^2)dx \\
&= \left(7x - \frac{1}{3}x^3\right)\Bigg|_{-1}^{2} \\
&= \left[7.2 - \frac{1}{3}(8)\right] - \left[7(-1) - \frac{1}{3}(-1)\right] \\
&= \left[\frac{(42-8)}{3}\right] - \left[\frac{(1-21)}{3}\right] \\
&= \frac{(34+20)}{3} \\
&= \frac{54}{3} \\
&= 18 \text{ sq. units}
\end{aligned}$$

Consumer Surplus

- It can be defined as the surplus that is retained with the consumer after he purchases a product for which he paid lesser than what he was able to.
- This is the difference between what the consumer pays and what he would have been willing to pay.

For example: If we would be willing to pay Rs. 50 for a ticket to see a drama, but we can buy a ticket for ₹ 40. In this case, the consumer surplus is ₹ 10.



In the graph, we can see that consumer's surplus is the area of the region bounded above by the demand function and below by the line that represents the unit market price.

The consumers' surplus is given by

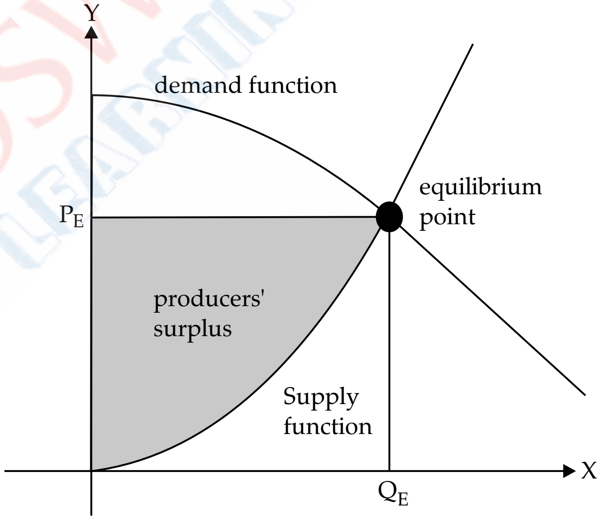
$$CS = \int_0^{Q_e} D(x)dx - Q_e.P_e$$

Where, $D(x)$ = Demand Function
Q_e = Quantity Sold
P_e = Unit Market Price

Producer Surplus

- It can be defined as the surplus that is retained with the producer after he sells a product for which he accepted more than what he was expected to receive.
- This is the difference between the price a firm receives and the price it would be willing to sell it at.

For example: If a firm would sell a good at ₹ 4, but the market price is ₹ 7, the producer surplus is ₹ 3.



In the graph, we can see that producer's surplus is the area of the region bounded above by the line that represents the price and below by the supply curve.

The producer's surplus is given by

$$PS = Q_e.P_e - \int_0^{Q_e} S(x)dx$$

Where, $S(x)$ = Supply Function
Q_e = Quantity Supplied
P_e = Unit Market Price

Example 6 : Find the producer's surplus defined by the supply curve $S(x) = 4x + 8$ for the supply of 5 units.

Sol.: Here,

$$S(x) = 4x + 8 \text{ and } Q_e = 5$$
$$P_e = S(5) = 4(5) + 8 = 28$$

Since,

$$PS = Q_e.P_e - \int_0^{Q_e} S(x)dx$$

Therefore, $$PS = (5\times28) - \int_0^5 (4x+8)dx$$

$$= 140 - \left[4\left(\frac{x^2}{2}\right) + 8x\right]_0^5$$

$$= 140 - (50 + 40)$$
$$= 50 \text{ units}$$

Hence the producer's surplus = 50 units.

CHAPTER-7

DIFFERENTIAL EQUATIONS AND MODELING

Topic-1 Basic Differential Equations

Concepts Covered • *Order of differential Equation,* • *Degree of differential Equation,* • *Solution of differential equation,* • *Variable separation method*

Revision Notes

- **Differential Equation:**
  An equation consisting of an independent variable, dependent variable and differential coefficients of dependent variable with respect to the independent variable is known as differential equation.

  e.g. : **(i)** $\frac{d^2y}{dx^2} = -a^2y,$

  (ii) $\frac{dy}{dx} = \frac{x+y}{x^2},$

  (iii) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = p\frac{dy}{dx}$

- **Order of Differential Equation:** The order of a differential equation is the order of the highest derivative appearing in the differential equation.

  e.g.: $\left(\frac{d^3y}{dx^3}\right)^2 - 3\left(\frac{dy}{dx}\right)^3 + 2 = 0$ is the differential equation of order 3 because highest order derivative of y w.r.t. x is $\frac{d^3y}{dx^3}$.

- **Degree of Differential Equation:** The degree of the differential equation is the degree (power) of the highest order derivative, when the **differential coefficient** has been made free from the radicals and fractions.

Key Word

Differential coefficient: Differential coefficient is the result of mathematical differentiation.

e.g.: $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{2}{3}} = 3\frac{d^2y}{dx^2}$ is the differential equation of degree 3, because the power of highest order derivative $\frac{d^2y}{dx^2}$ is 3 (after cubing).

- **Formation of the Differential Equation:**

  If the equation of the family of curves is given then its differential equation is obtained by **eliminating arbitrary constants** occurring in equation with the help of equation of the curve and the equations obtained by differentiating the equation of the curve.

- **Algorithm for the Formation of the Differential Equation:**

  Step 1: Write down the given equation of the curve.

  Step 2: Differentiate the given equation with respect to the independent variable as many times as the number of arbitrary constants.

  Step 3: Eliminate the arbitrary constants by using given equation and the equations obtained by the differentiation in step 2.

Example 1 : Find the differential equation formed from the equation $y = mx + c$, where m and c are arbitrary constants.

Sol. Given, $y = mx + c$

or $\frac{dy}{dx} = m$

or $\frac{d^2y}{dx^2} = 0$

So, the required differential equation is $\frac{d^2y}{dx^2} = 0$.

- **Solution of Differential Equations:**

  (a) General solution: The solution which contains as many as arbitrary constants as the order of the differential equations, e.g., $y=\alpha \cos x+\beta \sin x$ is the general solution of $\frac{d^2y}{dx^2}+y=0$.

  Here, the differential equation is of second order and there are two arbitrary constants *i.e.*, α and β in the general solution.

  (b) Particular solution: Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution.

  e.g., $y = 3 \cos x + 2 \sin x$ is a particular solution of the differential equation $\frac{d^2y}{dx^2}+y=0$.

  (c) Solution of Differential Equation by Variable Separable Method: A variable separable form of the differential equation is the one which can be expressed in the form of $f(x)\,dx = g(y)\,dy$. The solution is given by $\int f(x)dx = \int g(y)dy+k$, or $\int g(y)\,dy = \int f(n)\,dn + k$, where k is the constant of integration.

  Example 2 : Find the solution of the differential equation $x\frac{dy}{dx} = \frac{y}{1+\log x}$.

Sol. $x\frac{dy}{dx} = \frac{y}{1+\log x}$

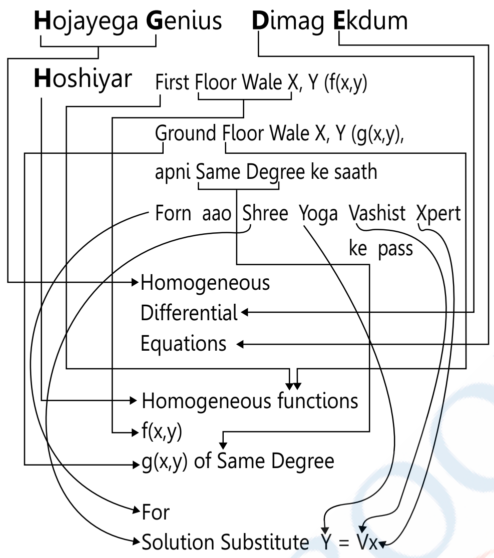
$\equiv \frac{dy}{y} = \int \frac{1}{x(1+\log x)}dx$

$\log|y| = \log|1+\log x| + \log C$

$y = C(1+\log x)$

Mnemonic

Homogeneous Differential Equation



Interpretation : Differential equation can be expressed in the form $\frac{dy}{dx} = f(x, y)$ or $\frac{dx}{dy} = g(x, y)$ where $f(x,y)$ and $g(x,y)$ are homogeneous functions of sum is called a homogeneous Differential equation. These equations can be solved by substituting $y = vx$ so that dependent variable y is changed to another variable v, where v is some unknown function.

Topic-2 Application of Differential Equations

Concepts Covered • *Growth and Decay model*

Revision Notes

- **Law of Natural Growth:** Let $x(t)$ be the population at any time t. Assume that population grows at a rate directly proportional to the amount of present population. Then the differential equation governing this phenomenon is the linear equation.

  $$\frac{dx}{dt} = kx, \text{ where } k \text{ is the proportionality constant}$$

  Here, $k > 0$, since this is phenomenon.

  The solution of the D.E. is

  $$x(t) = Ce^{kt},$$

  Where C is constant of integration. Here, C, k are determined from the two given (initial).

  Law of Natural Decay:

  The differential equation

  $$\frac{dm}{dt} = -km, k > 0$$

  describes the decay phenomenon, where it is assumed that the material $m(t)$ at any time t decays at a rate which is proportional to the amount present. The solution is

  $$m(t) = ce^{-kt}$$

It initially at, $t = 0$, m_0 is the amount present then

$$m(t) = m_0 e^{-kt}$$

Example 3: A population grows at the rate of 8% per year. How long does it take for the population to double ? Use differential equation for it.

Sol. Let P_0 be the initial population and let the population after t years be P. Then,

$$\frac{dP}{dt} = \frac{8P}{100}$$

$$\frac{dP}{dt} = \frac{2P}{25}$$

$$\frac{dP}{P} = \frac{2dt}{25}$$

Amazing Facts

1. Differential equations began with Leibniz, the Bernoulli brothers and others from the 1680s, not long after Newton's `fluxional equations' in the 1670s. Applications were made largely to geometry and mechanics; is operimetrical problems were exercises in optimization.
2. A differential equation will typically have an infinite number of solutions, for essentially the same reason that a function will typically have an infinite number of anti-derivatives.
3. For many important differential equations, it is not possible to find a formula for a solution, for essentially the same reason that, for some functions, we are not able to find a formula for an anti-derivative.

   e.g.: $\frac{dy}{dt} = e^{-t^2}$.

$$\int \frac{dP}{P} = \frac{2}{25}\int dt$$

$$\log P = \frac{2}{25}t + C \quad \text{...(i)}$$

At $t = 0$, $P = P_0$

Therefore, $\log P_0 = \frac{2}{25}(0) + C$

$\Rightarrow \qquad C = \log P_0$

Substituting the value of C in eq. (i), we get

$$\log P = \frac{2}{25}t + \log P_0$$

$$\log \frac{P}{P_0} = \frac{2}{25}t$$

$$t = \frac{25}{2}\log \frac{P}{P_0}$$

When $P = 2P_0$, $t = \frac{25}{2}\log \frac{2P_0}{P_0} = \frac{25}{2}\log 2$

Thus, the population is doubled in $\frac{25}{2}\log\ 2$ years.

UNIT - IV : PROBABILITY DISTRIBUTIONS

CHAPTER-8

PROBABILITY

Topic-1 Expectation and Variance of Probability Distribution

Concepts Covered • *Discrete and continuous random variables,* • *Probability mass function for discrete and continuous random variables,* • *Expected Value,* • *Variance*

Revision Notes

- **Random Variable:** A random variable (r.v.) is a real valued function defined on a **sample space** S and taking value in (– ∞, ∞) or whose possible values are numerical outcomes of a **random experiment**.

Key Word

Sample space: A sample space is a collection or a set of possible outcomes of a random experiment. It is a range set of values of random variable.

Random experiment: A random experiment is any well-defined procedure that produces an observable outcome that could not be perfectly predicted in advance.

- **Types of Random Variable:** Random variables are classified into two types namely discrete and continuous random variables. These are important for practical application in the field of Mathematics and statistics. The above types of random variable are defined with examples as follows.
  - **Discrete random variable:** A variable which can assume finite number of possible values or an infinite sequence of countable real numbers is called a discrete random variable.

Examples of discrete random variable:

- Marks obtained in a test.
- Number of red marbles in a jar.
- Number of telephone calls at a particular time.
- Number of cars sold by a car dealer in one month, etc.,

Probability Mass Function of Discrete Random Variable:

If X is a discrete random variable with distinct values $x_1, x_2, ..., x_n, ...$, then the function, denoted by $P_X(x)$ and defined by $P_X(x) = p(x) = \begin{cases} P(X = x_i) = p_i = p(x_i) & \text{if} \quad x = x_i, i = 1, 2, ..., n, ... \\ 0 & \text{if} \quad x \neq x_i \end{cases}$

This is defined to be the probability mass function or discrete probability function of X. The probability mass function $p(x)$ must satisfy the following conditions

(i) $p(x_i) \geq 0 \; \forall \; i,$ **(ii)** $\sum_{i=1}^{\infty} p(x_i)$

Example 1: The number of cars in a household is given below.

No. of cars	0	1	2	3	4
No. of Household	30	320	380	190	80

Estimate the probability mass function. Verify $p(x_i)$ is a probability mass function.

Sol. Let X be the number of cars

$X = x_i$	Number of Household	$p(x_i)$
0	30	0.03
1	320	0.32
2	380	0.38
3	190	0.19
4	80	0.08
Total	1000	1.00

Key Facts

- A probability distribution depicts the expected outcomes of possible values for a given data generating process.
- Probability distributions help to model our world, enabling us to obtain estimates of the probability that a certain event may occur, or estimate the variability of occurrence.
- Probability distributions help to forecast power failures and network outages.

(i) $p(x_i) \geq 0 \; \forall \; i$ and

(ii) $\sum_{i=1}^{\infty} p(x_i) = p(0) + p(1) + p(2) + p(3) + p(4)$

$= 0.03 + 0.32 + 0.38 + 0.19 + 0.08 = 1$

Hence $p(x_i)$ is a probability mass function.

Note:

For $X = 0$, the probability 0.03, comes from $\frac{30}{1000}$, the other probabilities are estimated similarly.

Discrete Distribution Function

The discrete cumulative distribution function or distribution function of a real valued discrete random variable X takes the countable number of points $x_1, x_2, \ldots$ with corresponding probabilities $p(x_1), p(x_2), \ldots.$ and then the cumulative distribution function is defined by

$$F_X(x) = P(X \leq x), \text{ for all } x \in R$$

i.e.,

$$F_X(x) = \sum_{x_i \leq x} p(x_i)$$

For instance, suppose we have a family of two children. The sample space

$S = \{bb, bg, gb, gg\}$, where b = boy and g = girl

Let X be the random variable which counts the number of boys. Then, the values (X) corresponding to the sample space are 2, 1, 1, and 0.

Hence, the probability mass function of X is

$X = x$	0	1	2
$p(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Then, we can from a cumulative distribution function of X is

$X = x$	0	1	2
$p(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$F_X(x) = P(X \leq x)$	$\frac{1}{4}$	$\frac{1}{4}+\frac{1}{2}=\frac{3}{4}$	$\frac{3}{4}+\frac{1}{4}=1$

- **Continuous random variable**

  A random variable X which can take on any value (integral as well as fraction) in the interval is called continuous random variable.

Examples of continuous random variable

- The amount of water in a 10 ounce bottle.
- The speed of a car.
- Electricity consumption in kilowatt hours.
- Height of people in a population.
- Weight of students in a class.
- The length of time taken by a truck driver to go from Chennai to Madurai, etc.

Probability Mass Function of Continuous Random Variable

The probability that a random variable X takes a value in the interval $[t_1, t_2]$ (open or closed) is given by the integral of a function called the probability density function $f_X(x)$:

$$P(t_1 \le X \le t_2) = \int_{t_1}^{t_2} f_X(x)dx.$$

Other names that are used instead of probability density function include density function, continuous probability function, integrating density function.

The probability density function $f_X(x)$ or simply by $f(x)$ must satisfy the following conditions.

(i) $f(x) \ge \forall\ x$ and (ii) $\int_{-\infty}^{\infty} f(x)dx = 1.$

Example 2: A continuous random variable X has the following p.d.f $f(x) = ax, 0 \le x \le 1.$

Determine the constant a and also find $P\left[X \le \frac{1}{2}\right]$

Sol. Since, $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_0^1 ax\ dx \Rightarrow a\int_0^1 xdx = 1$$

$$\Rightarrow \quad a\left(\frac{x^2}{2}\right)_0^1 = 1$$

$$\Rightarrow \quad \frac{a}{2}(1-0) = 1$$

$$\Rightarrow \quad a = 2$$

$$P\left[x \le \frac{1}{2}\right] = \int_{-\infty}^{\frac{1}{2}} f(x)dx$$

$$= \int_{-\infty}^{\frac{1}{2}} 2xdx = \frac{1}{4}$$

Continuous Distribution Function

If X is a continuous random variable with the probability density function $f_X(x)$, then the function $F_X(x)$ is defined by $F_X(x) = P[X \le x] = \int_{-\infty}^{x} f(t)dt,\ -\infty < x < \infty$ is called the distribution function (d.f.) or sometimes the cumulative distribution function (c.d.f) of the continuous random variable X.

Properties of cumulative distribution function

The function $F_X(x)$ or simply $F(x)$ has the following properties

(i) $0 \le F(x) \le 1, -\infty < x < \infty$

(ii) $F(-\infty) = \lim_{x\to-\infty} F(x) = 0$ and

$F(+\infty) = \lim_{x\to-\infty} F(x) = 1.$

(iii) $F(x)$ is a monotone, non-decreasing function; that is, $F(a) \le F(b)$ for $a < b$.

(iv) $F(x)$ is continuous from the right; that is, $\lim_{h\to0} F(x+h) = F(x).$

(v) $F'(x) = \frac{d}{dx}F(x) = f(x) \geq 0$

(vi) $F'(x) = \frac{d}{dx}F(x) = f(x) \Rightarrow dF(x) = f(x)dx$, $dF(x)$ is know as probability differential of X.

(vii) $P(a \leq x \leq b) = \int_a^b f(x)dx$

$$= \int_{-\infty}^{b} f(x)dx - \int_{-\infty}^{a} f(x)dx$$

$$= P(X \leq b) - P(X \leq a)$$

$$= F(b) - F(a)$$

Example 3: Suppose, the life in hours of a radio tube has the following p.d.f

$$f(x) = \begin{cases} \frac{100}{x^2}, & \text{when} \quad x \geq 100 \\ 0, & \text{when} \quad x < 100 \end{cases}$$

Find the distribution function.

Sol. $F(x) = \int_{-\infty}^{x} f(t)dt$

$$= \int_{100}^{x} \frac{100}{t^2}dt, \; x \geq 100$$

$$= \left[\frac{100}{-t}\right]_{100}^{x}, \; x \geq 100$$

$$F(x) = \left[1 - \frac{100}{x}\right], \; x \geq 100$$

- **Expected value:** The expected value is a weighted average of the values of a random variable may assume. The weights are the probabilities.

  Let X be a discrete random variable with probability mass function (p.m.f.) $p(x)$. Then, its expected value is defined by

  $$E(X) = \sum_x xp(x) \quad ...(1)$$

  If X is a continuous random variable and $f(x)$ is the value of its probability density function at x, the expected value of X is

  $$E(X) = \int_{-\infty}^{\infty} xf(x)dx \quad ...(2)$$

- **Variance:** The variance is a weighted average of the squared deviations of a random variable from its mean. The weights are the probabilities. The mean of a random variable X, defined in (1) and (2), was a measure of central location of the density of X. The variance of a random variable X will be a measure of the spread or dispersion of the density of X or simply the variability in the values of a random variable.

  The variance of X is defined by

  $$Var(X) = \sum [x - E(X)]^2 p(x) \quad ...(3)$$

  If X is discrete random variable with probability mass function $p(x)$.

  $$Var(X) = \int_{-\infty}^{\infty} [x - E(X)]^2 f_X(dx)dx \quad ...(4)$$

  if X is continuos random variable with probability density function $f_X(x)$.

  Expected value of $[X - E(X)]^2$ is called the variance of the random variable.

  i.e., $Var(X) = E[X - E(X)]^2 = E(X^2) - [E(X)]^2 \quad ...(5)$

where $$E(X^2) = \begin{cases} \sum_x x^2 p(x), & \text{if } X \text{ is Discrete Random Variable} \\ \int_{-\infty}^{\infty} x^2 f(x)dx, & \text{if } X \text{ is Continuous Random Variable} \end{cases}$$

Example 4: Determine the mean and variance of the random variable X having the following probability distribution.

$X = x$	1	2	3	4	5	6	7	8	9	10
$P(x)$	0.15	0.10	0.10	0.01	0.08	0.01	0.05	0.02	0.28	0.20

Sol. Mean of the random variable

$$X = E(X) = \sum_x x P_X(x)$$

$$= (1 \times 0.15) + (2 \times 0.10) + (3 \times 0.10) + (4 \times 0.01) + (5 \times 0.08) + (6 \times 0.01) + (7 \times 0.05) + (8 \times 0.02) + (9 \times 0.28) + (10 \times 0.20)$$

$$E(X) = 6.18$$

$$E(X^2) = \sum_x x^2 P_X(x)$$

$$= (1^2 \times 0.15) + (2^2 \times 0.10) + (3^2 \times 0.10) + (4^2 \times 0.01) + (5^2 \times 0.08) + (6^2 \times 0.01) + (7^2 \times 0.05) + (8^2 \times 0.02) + (9^2 \times 0.28) + (10^2 \times 0.20)$$

$$= 50.38$$

Variance of the Random Variable $X = V(X) = E(X^2) - [E(X)]^2$

$$= 50.38 - (6.18)^2$$

$$= 12.19$$

Therefore, the mean and variance of the given discrete distribution are 6.18 and 12.19 respectively.

Example 5: Consider a random variable X with probability density function

$$f(x) = \begin{cases} 4x^3, \text{ if } 0 < x < 1 \\ 0, \text{ otherwise} \end{cases}$$

Find E(X) and V(X).

Sol. We know that, $E(X) = \int_{-\infty}^{\infty} x f(x)dx$

$$= \int_0^1 x\, 4x^3\, dx$$

$$= 4\left[\frac{x^5}{5}\right]_0^1$$

$$E(X) = \frac{4}{5}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)dx$$

$$= \int_0^1 x^2\, 4x^3 dx$$

$$= 4\left[\frac{x^6}{6}\right]_0^1$$

$$= \frac{4}{6} = \frac{2}{3}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \frac{2}{3} - \left[\frac{4}{5}\right]^2$$

$$= \frac{2}{75}$$

Mnemonics

Concept: Independent and Mutually exclusive events.

I Is not **ME**

ME Is not **I**

Here, **I** : Independent Events

ME: Mutually Exclusive events

Topic-2 Binomial and Poisson Distributions

Concepts Covered • *Bernoulli Trials,* • *Binomial Distribution,* • *Poisson Distribution*

Revision Notes

- **Bernoulli Trials**

Trials of a random experiment are called Bernoulli trials, if they satisfy the following four conditions:

(a) The trials should be finite in numbers.

(b) The trials should be independent of each other.

(c) Each of the trial yields exactly two outcomes *i.e.*, success or failure.

(d) The probability of success or the failure remains the same in each of the trial.

> If an experiment is repeated n times under the similar conditions, we say that n trials of the experiment have been made.

- **Binomial Distribution**

Let E be an event. Let p = probability of success in one trial (*i.e.*, occurrence of event E in one trial) and, $q = 1 - p$ = probability of failure in one trial (*i.e.*, non-occurrence of event E in one trial).

Let X = number of successes (*i.e.*, number of times event E occurs in n trials)

Then, Probability of X successes in n trials is given by the relation:

$$P(X = r) = P(r) = {}^nC_r p^r q^{n-r},$$

where $r = 0, 1, 2, 3, ..., n$; p = probability of success in one trial and $q = 1 - p$ = probability of failure in one trial.

> **Note :**
> The result $P(X = r) = P(r) = {}^nC_r p^r q^{n-r}$ can be used only when :
> (i) the probability of success in each trial is the same.
> (ii) each trial must surely result in either a success or a failure.

- **Special Cases**

- $P(X = r)$ or $P(r)$ is also called the probability of occurrence of event E exactly r times in n trials.
- Here ${}^nC_r = \dfrac{n!}{r!(n-r)!}$
- Note that ${}^nC_r p^r q^{n-r}$ is the$(r + 1)^{\text{th}}$ term in the binomial expansion of $(q + p)^n$.
- Mean $= \sum_{r=0}^{n} r.P(r) = np$
- Variance $= \sum_{r=0}^{n} r^2.P(r) - (\text{Mean})^2 = npq$
- Recurrence formula, $P(x = r + 1)$

$$= \left(\frac{n-r}{r+1}\right)\left(\frac{p}{q}\right)P(r)$$

- A Binomial distribution with n Bernoulli trials and probability of success in each trial as p is denoted by $B(n, p)$. Here n and p are known as the parameters of binomial distribution.
- The expression $P(x = r)$ or $P(r)$ is called the probability function of the binomial distribution.

Example 6 : In a box containing 100 bulbs, 10 are defective. Find the probability that out of a sample of 5 bulbs, none is defective **is.**

Sol. The repeated selections of defective bulbs from a box are Bernoulli trials. Let X denotes the number of defective bulbs out of a sample of 5 bulbs.

Probability of getting a defective bulb,

$$P = \frac{10}{100} = \frac{1}{10}$$

$$\therefore \quad q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$$

Clearly, X has a binomial distribution with $n = 5$ and $p = \frac{1}{10}$

$$\therefore \quad P(X = x) = {}^nC_x q^{n-x} p^x = {}^5C_x \left(\frac{9}{10}\right)^{5-x} \left(\frac{1}{10}\right)^x$$

$$P(\text{none of the bulbs is defective}) = P(X=0)$$

$$= {}^5C_0 \cdot \left(\frac{9}{10}\right)^5 = 1.\left(\frac{9}{10}\right)^5 = \left(\frac{9}{10}\right)^5$$

Poisson Distribution

It is a discrete probability distribution. The Poisson distribution is a particular limiting form of Binomial distribution when p (or q) is very small and n is large enough so that np (or nq) is a finite constant say m.

A probability distribution of a random variable x is called Poisson distribution if x can assume non-negative integral values only and the distribution is given by

$$P(r) = P(X = r) = \begin{cases} \dfrac{e^{-m} m^r}{r!}, & r = 0, 1, 2 \ldots \\ 0, & r \neq 0, 1, 2 \ldots \end{cases}$$

Here, the value of $e = 2.7183$, it is the base of the natural system of logarithms.

Examples of Poisson Variates

(i) The number of cars passing through a certain street in time t.

(ii) The number of defective screws per box of 100 screws.

(iii) The number of deaths in a district in one year by a rare disease.

(iv) The number of printing mistakes at each page of the book.

- **Requirements for a Poisson Distribution**
  (1) Random variable (x) is the number of occurrences of an events **over some interval**.
  (2) Occurrences must be **random**
  (3) Occurrences must be **independent** of each other
  (4) Occurrences must be **uniformly distributed** over the interval being used.
- **Constants of the Poisson Distribution**
  (i) Mean, $E(X) = m = np$
  (ii) Variance, $V(X) = m = np$

Example 7 : If $P(X = 2) = 9P(X = 4) + 90\ P(X = 6)$ in Poisson distribution, then find $E(X)$.

Sol. Let $P(X = r) = \dfrac{e^{-m} m^r}{r!}$ $r = 0, 1, 2 \ldots, \infty$

Given, $P(X = 2) = 9P(X = 4) + 90P(X = 6)$

$$\Rightarrow \quad \frac{e^{-m} m^2}{2!} = 9\frac{e^{-m} m^9}{9!} + 90\frac{e^{-m} m^6}{6!}$$

$$\Rightarrow \quad m^4 + 3m^2 - 4 = 0$$

$$\Rightarrow \quad (m^4 + 4)(m^2 - 1) = 0$$

$$\Rightarrow \quad m = 1 \qquad (\because \text{ other values are not admissible})$$

Thus, $E(X) = m = 1$

Difference between Binomial and Poisson Distributions

- The binomial distribution is affected by the sample size n and the probability p, whereas the Poisson distribution is affected only by the mean μ.

- In a binomial distribution the possible values of the random variable x are 0, 1, ... n but a Poisson distribution has possible x values of 0, 1, ..., with no upper limit.

Key Terms

- Expansion of e^x

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

- Expansion of e^{-x}

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$$

Key Facts

- Binomial distribution describes the distribution of binary data from a finite sample. Thus, it gives the probability of getting r events out of n trials.
- Poisson distribution describes the distribution of binary data from a infinite sample. Thus, it gives the probability of getting r events in a population.

Topic-3 Normal Distribution

Concepts Covered • *Normal Distribution,* • *Standard Normal Distribution*

Revision Notes

- Normal distribution is a continuous probability distribution in which the relative frequencies of a continuous variable are distributed according to normal probability. In simple words, it is a symmetrical distribution in which the frequencies are distributed evenly about the mean of distribution.

  e.g.,: (i) Height and intelligence are approximately normally distributed.

  (ii) Measurement of errors also often have a normal distribution.

  The Normal (or Gaussian) Distribution is defined by the probability density function

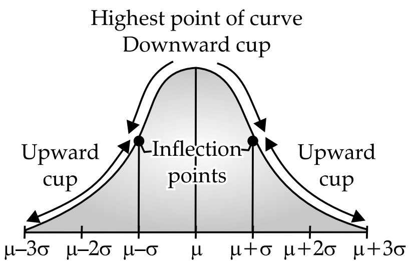
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \text{ for } -\infty < x < \infty$$

  where μ and $\sigma > 0$ are parameters of the distribution.

  Clearly, $f(x)$ is non-negative and $\int_{-\infty}^{\infty} f(x)dx = 1$.

  The notation $N(\mu, \sigma^2)$ means normally distributed with mean μ and variance σ^2. If we say $x \sim N(\mu, \sigma^2)$ we mean that x is distributed $N(\mu, \sigma^2)$.

- **Normal distribution is diagrammatically represented as follows:**



Normal curve

Normal distribution is limiting case of binomial distribution under the following conditions:

(i) n, the number of trials is infinitely large, *i.e.*, $n \to \infty$

(ii) neither p(nor q) is very small, The normal distribution of a variable when represented graphically, takes the shape of a symmetrical curve, known as the Normal Curve. The curve is asymptotic to X-axis on its either side.

Chief Characteristics or Properties of Normal Probability distribution and Normal probability Curve.

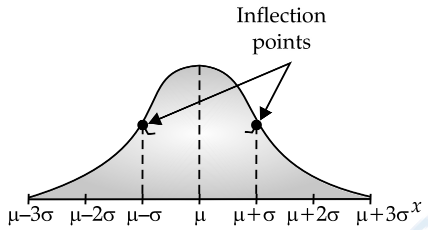
The normal probability curve with mean μ and standard deviation σ has the following properties:

(i) The curve is bell- shaped and symmetrical about the line $x = \mu$

(ii) Mean, median and mode of the distribution coincide.

(iii) x-axis is an asymptote to the curve, (tails of the curve never touches the horizontal (X) axis).

(iv) No portion of the curve lies below the X-axis as $f(x)$ being the probability function can never be negative.



(v) The points of inflection of the curve are $x = \mu \pm \sigma$

(vi) The curve of a normal distribution has a single peak *i.e.*, it is a unimodal.

(vii) As x increases numerically, $f(x)$ decreases rapidly, the maximum probability occurring at the point $x = \mu$ and is given by $[p(x)]\text{max} = \dfrac{1}{\sigma\sqrt{2\pi}}$

(viii) The total area under the normal curve is equal to unity and the percentage distribution of area under the normal curve is given below.

(a) About 68.27% of the area falls between $\mu - \sigma$ and $\mu + \sigma$

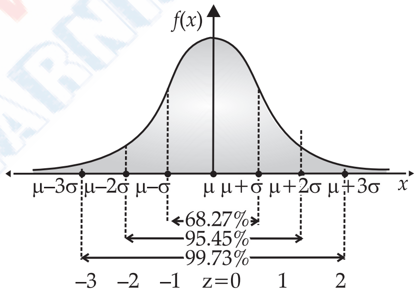
$P(\mu - \sigma < X < \mu + \sigma) = 0.6826$

(b) About 95.5% of the area falls between $\mu - 2\sigma$ and $\mu + 2\sigma$

$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$

(c) About 99.7% of the area falls between $\mu - 3\sigma$ and $\mu + 3\sigma$

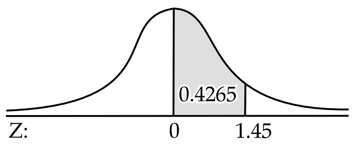
$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$



Standard Normal Distribution

A random variable $Z = (X - \mu)/\sigma$ follows the standard normal distribution. Z is called the standard normal variate with mean 0 and standard deviation 1 *i.e.*, $Z \sim N(0, 1)$. Its Probability density function is given by :

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty < z < \infty$$



1. The area under the standard normal curve in equal to 1.
2. 68.26% of the area under the standard normal curve lies between Z = –1 and Z = 1
3. 95.44% of the area lies between Z = – 2 and Z = 2
4. 99.74% of the area lies between Z = – 3 and Z = 3

Standard Normal Distribution Table

This table provides the area between the mean and some Z score.

For example, when Z score = 1.45

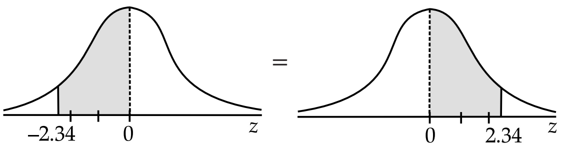
the area = 0.4265.

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3980	0.3997	0.4015
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.5	04938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	04985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.489	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998
3.5	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998
3.6	0.4998	0.4998	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.7	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.8	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.9	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000

Example: Find the area under the standard normal curve from $z = -2.34$ to 0.

Sol. The area from $z = -2.34$ to 0 is the same as the area from $z = 0$ to 2.34. (See Figure) By Table 1, the area from 0 to 2.34 is 0.4904. Therefore, the area from $z = -2.34$ to 0 is also 0.4904.

Area from $z = -2.34$ to 0 Equals Area from $z = 0$ to 2.34



To find areas other than those between a given z value and $z = 0$, we use Table A together with addition or subtraction of areas we find in Table 1.

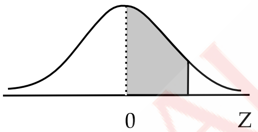
1. For areas extending from one side of the mean $z = 0$ to the other side, we add areas found in Table 1.
2. For areas completely on one side of the mean $z = 0$ (but not bordering $z = 0$), we subtract areas found in Table 1.
3. The area extending from $z = 0$ and including the entire right half of the graph is 0.5000. Likewise, the area extending from $z = 0$ and including the entire left half of the graph is 0.5000.

Key Facts

- Normal distribution is also called as Gaussian distribution, named after mathematician **Karl Friedrich Gauss**'.

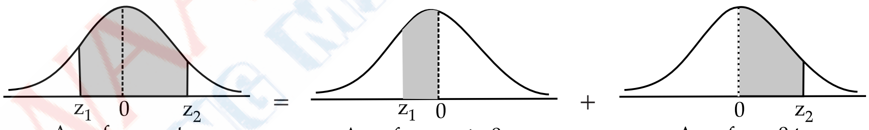
- **Patterns for Finding Areas Under the Standard Normal Curve:**

(a) Area between a given z value and 0



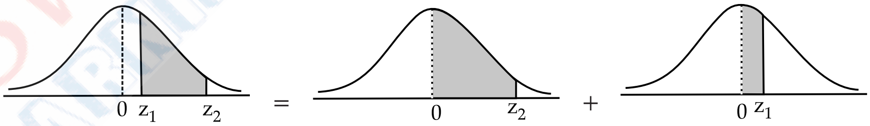
use Table 1

(b) Area between z values on either side of 0



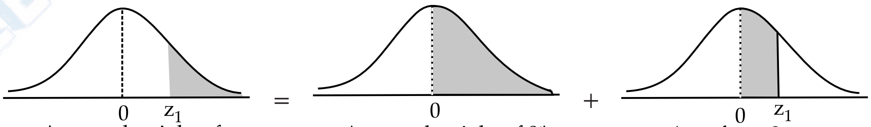
Area from z_1 to z_2 | Area from z_1 to 0 | Area from 0 to z_2

(c) Area between z value on same side of 0.



Area between z_1 and z_2 | Area from 0 to z_2 | Area from 0 to z_1

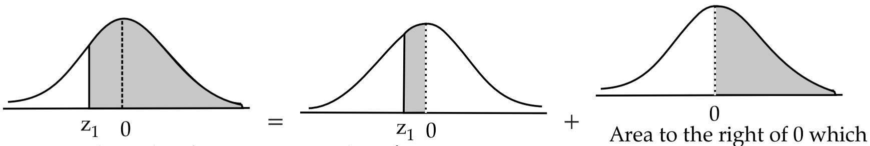
(d) Area to the right of a positive z value or to the left of a negative z value



Area to the right of z_1 | Area to the right of 0* | Area from 0 to z_1

This area = 0.5000 since the area under the entire curve is 1 and the area to the right of 0 is half the area under the entire curve

(e) Area of the right of a negative z value or to the left of a positive z value.



Area to the right of z_1 | Area from z_1 to 0 | Area to the right of 0 which is 0.5000

UNIT - V : INFERENTIAL STATISTICS

CHAPTER-9

INFERENTIAL STATISTICS

Revision Notes

Inferential statistics is mainly used to derive estimates about a large group (or population) and draw conclusions on the data, based on hypotheses testing methods.

Key Words

Inferential: Characterized by or involving conclusions reached on the basis of evidence and reasoning.

- **Population:** The group of individuals considered under study is called as population. The word population here refers not only to people but to all items that have been chosen for the study. Thus in statistics, population can be number of bikes manufactured in a day or week or month, number of cars manufactured in a day or week or month, number of fans, TVs, chalk pieces, people, students, girls, boys, any manufacturing products, etc.

  Finite and infinite population: When the number of observations/individuals/products is countable in a group, then it is a finite population.

  Example: Weights of students of class XII in a school.

  When the number of observations/individuals/products is uncountable in a group, then it is an infinite population.

  Example: Number of grains in a sack, number of germs in the body of a sick patient.
- **Sample and Sample Size:** A selection of a group of individuals from a population in such a way that it represents the population is called as sample and the number of individuals included in a sample is called the sample size.
- **Sampling:** Sampling is the procedure or process of selecting a sample from a population. Sampling is quite often used in our day-to-day practical life.
- **Parameter:** The statistical constants of the population like mean (μ), variance (σ^2) are referred as population parameters.

  Statistic: Any statistical measure computed from sample is known as statistics.

$$\mu \text{ (mean)} = \frac{\sum_{i=1}^{n} x_i}{N}$$

$$\sigma^2 \text{(variance)} = \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{N}$$

$$\sigma \text{(standard deviation, SD)} = \sqrt{\sum_{i=1}^{n} \frac{(x_i - \mu)^2}{N}}$$

where N is the population size
- **Errors in a Sample:** A sample is a part of the whole population. A sample drawn from the population depends upon chance and as such all the characteristics of the population may not be present in the sample drawn from the same population. The errors involved in the collection, processing and analysis of the data may be broadly classified into two categories namely,

  (i) Sampling Errors

  (ii) Non-Sampling Errors

  (i) Sampling Errors: Errors, which arise in the normal course of investigation or enumeration on account of chance, are called sampling errors. Sampling errors are inherent in the method of sampling. They may arise accidentally without any bias or prejudice.

  (ii) Non-Sampling Errors: The errors that arise due to human factors which always vary from one investigator to another in selecting, estimating or using measuring instruments (tape, scale) are called Non-Sampling errors.
- **Types of Sampling:** There are various techniques of sampling, but they can be broadly grouped into two:
  - Random probability sampling.
  - Non Random or Non Probability sampling.

We will consider only random (Probability) sampling.

Random sampling or Probability sampling: Random sampling refers to selection of samples from the population in a random manner. A random sample is one where each and every item in the population has an equal chance of being selected.

The following are different types of probability sampling:

(i) Simple Random Sampling: Simple random sampling is the randomized selection of a small segment of individuals or members from a whole population. It provides each individual or member of a population with an equal and fair chance of being chosen. This is one of the most common method of sampling.

e.g.: There are 1000 students in a school, we want to select a simple random sample of 100 students. We can assign a number to every student in the school from 1 to 1000 and select randomly 100 numbers.

(ii) Systematic Sampling: Systematic sampling is the selection of specific individuals or members from entire population. The selection often follows a pre-determined interval (k). The systematic sampling method is comparable to the simple random sampling method; however it is less complicated to conduct.

e.g.: Out of 1000 students of school, we want to select a sample of 100 students. All students of the school are arranged in alphabetical order and assigned a number 1 to 1000. Now, we randomly select a number (say 4) from first 10 numbers and then every 10th student in the list is selected *i.e.*, 4, 14, 24, ... and end up with sample of 100 members.

(iii) Stratified Sampling: Stratified sampling includes the partitioning of a population into subclasses with notable distinctions and variances. This method is useful when population is dispersed.

e.g., Suppose a school has its branches in 15 cities. We want to select the sample of 100 students. It is difficult to select students from each branch. So first select any 5 branches and then select students from each branch.

- **Sampling Distribution:** Sampling distribution of a statistic is the frequency distribution which is formed with various values of a statistic computed from different samples of the same size drawn from the same population.

For instance if we draw a sample of size n from a given finite population of size N, then the total number of possible samples are ${}^{N}C_n = \dfrac{N!}{n!(N-n)!} = k$(say). For each of these k samples we can compute some statistic $t = t(x_1, x_2, x_3, \ldots x_n)$, in particular the mean $\bar{x}$, the variance S^2, etc., is given below:

Sample Number	Statistic		
	t	$\bar{x}$	S^2
1	t_1	$\bar{x}_1$	S_1^2
2	t_2	$\bar{x}_2$	S_2^2
3	t_3	$\bar{x}_3$	S_3^2
⋮	⋮	⋮	⋮
k	t_k	$\bar{x}_k$	S_k^2

Key Facts

- Inferential statistics helps to suggest explanations for a situation or phenomenon. It allows us to draw conclusions based on extrapolations, and is in that way fundamentally different from descriptive statistics that merely summarize the data that has actually been measured.
- Inferential statistics are powerful tools for making inference that rely on frequencies and probabilities.
- Inferential statistics can improve one's ability to make decisions, form predictions, and conduct research.

The set of the values of the statistic so obtained, one for each sample, constitutes the sampling distribution of the statistic.

Example: The sampling distribution of the mean for $n = 2$ is given below:

Sample Number	Outcomes	Sample mean $\bar{x}_i$
1	1, 3	2
2	1, 9	5
3	3, 9	6
4	6, 7	6.5
5	7, 10	8.5

- **Standard Error:** The standard deviation of the sampling distribution of statistic known as its Standard Error abbreviated as S.E. The Standard Errors (S.E.) of some of the well-known statistics, for large samples, are given below, where n is the sample size, σ^2 is the population variance.

S. No	Statistic	Standard Error
1.	Sample mean ($\bar{x}$)	$\frac{\sigma}{\sqrt{n}}$
2.	Observed sample proportion (p)	$\sqrt{\frac{PQ}{n}}$
3.	Sample standard deviation (s)	$\sqrt{\frac{\sigma^2}{2n}}$
4.	Sample variance (s^2)	$\sigma^2\sqrt{\frac{2}{n}}$

Example 1 : A die is thrown 9000 times and a throw of 3 or 4 is observed 3240 times. Find the standard error of the proportion for an unbiased die.

Sol.: If the occurrence of 3 or 4 on the die is called a success, then

Sample size = 9000; Number of Success = 3240

$$\text{Sample proportion} = p = \frac{3240}{9000} = 0.36$$

Population proportion (P) = Prob (getting 3 or 4 when a die is thrown)

$$= \frac{1}{6}+\frac{1}{6} = \frac{2}{6}$$

$$= \frac{1}{3} = 0.3333$$

Thus $P = 0.3333$ and $Q = 1 - P = 1 - 0.3333 = 0.6667$.

The S.E. for sample proportion is given by

$$\text{S.E.} = \sqrt{\frac{PQ}{n}}$$

$$= \sqrt{\frac{(0.3333)(0.6667)}{9000}}$$

$$= 0.00406$$

Hence the standard error for sample proportion is S.E. = 0.00406.

- **Statistical Inferences:** One of the main objectives of any statistical investigation is to draw inferences about a population from the analysis of samples drawn from that population. Statistical Inference provides us how to estimate a value from the sample and test that value for the population. This is done by the two important classifications in statistical inference,

  (i) Estimation;

  (ii) Testing of Hypothesis

  - **Estimation:** The method of obtaining the most likely value of the population parameter using statistic is called estimation.
  - **Estimator:** Any sample statistic which is used to estimate an unknown population parameter is called an estimator *i.e.*, an estimator is a sample statistic used to estimate a population parameter.
  - **Estimate:** To estimate an unknown parameter of the population, concept of theory of estimation is used. There are two types of estimation namely.
  - **Point Estimation:** When a single value is used as an estimate, it is called as point estimation.
  - **Interval Estimation:** An interval within which the parameter would be expected to lie is called interval estimation.

- **Statistical Hypothesis:** Statistical hypothesis is some assumption or statement, which may or may not be true, about a population.

There are two types of statistical hypothesis
(i) Null hypothesis
(ii) Alternative hypothesis

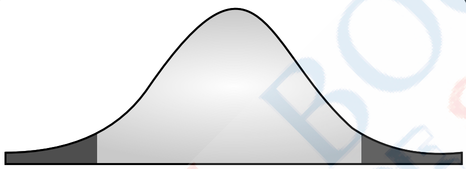
(i) Null Hypothesis: According to prof. R.A. Fisher, "Null hypothesis is the hypothesis which is tested for possible rejection under the assumption that it is true", and it is denoted by H_0.

For example: If we want to find the population mean has a specified value μ_0, then the null hypothesis H_0 is set as follows $H_0: \mu = \mu_0$

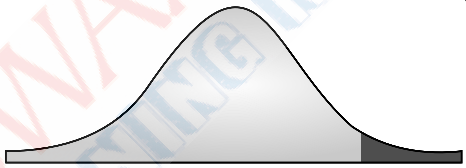
(ii) Alternative Hypothesis: Any hypothesis which is complementary to the null hypothesis is called as the alternative hypothesis and is usually denoted by H_1.

For example: If we want to test the null hypothesis that the population has specified mean μ *i.e.*, $H_0: \mu = \mu_0$ then the alternative hypothesis could be any one among the following:
(a) $H_1: \mu \neq \mu_0$
(b) $H_1: \mu > \mu_0$
(c) $H_1: \mu < \mu_0$

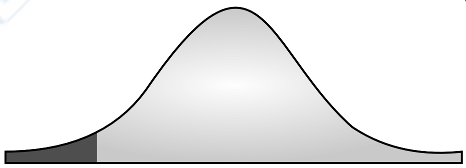
The alternative hypothesis in $H_1: \mu \neq \mu_0$ is known as two tailed alternative test. Two tailed test is one where the hypothesis about the population parameter is rejected for the value of sample statistic falling into either tails of the sampling distribution. When the hypothesis about the population parameter is rejected only for the value of sample statistic falling into one of the tails of the sampling distribution, then it is known as one-tailed test. Here $H_1: \mu > \mu_0$ and $H_1: \mu < \mu_0$ are known as one tailed alternative.


Two tailed diagram

Right tailed test: $H_1: \mu > \mu_0$ is said to be right tailed test where the rejection region or critical region lies entirely on the right tail of the normal curve.


Right tailed diagram

Left tailed test: $H_1: \mu < \mu_0$ is said to be left tailed test where the critical region lies entirely on the left tail of the normal curve. (diagram)


Left tailed diagram

- **Types of Errors in Hypothesis:** There is every chance that a decision regarding a null hypothesis may be correct or may not be correct. There are two types of errors. They are
  Type I error: The error of rejecting H_0 when it is true.
  Type II error: The error of accepting when H_0 it is false.
- **Critical Region or Rejection Region:** A region corresponding to a test statistic in the sample space which tends to rejection of H_0 is called critical region or region of rejection. The region complementary to the critical region is called the region of acceptance.
- **Level of Significance:** The probability of type I error is known as level of significance and it is denoted by α. The level of significance is usually employed in testing of hypothesis are 5% and 1%. The level of significance is always fixed in advance before collecting the sample information.
- **Critical Values or Significant Values:** The value of test statistic which separates the critical (or rejection) region and the acceptance region is called the critical value of significant value. It depend upon.

(i) The level of significance.

(ii) The alternative hypothesis whether it is two-tailed or single tailed.

- ***t*-Test:** The *t*-test is a test in statistics that is used for testing hypotheses regarding the mean of a small sample taken population when the standard deviation of the population is not known. *t*-Test was first invented by William Sealy Gosset, in 1908. Since he used the pseudo name as 'Student' when publishing his method in the paper titled 'Biometrika', the test came to be known as Student's T Test. There are many types of *t*-test. Two of them are:
  - The one-sample *t*-test, which is used to compare the mean of a population with a theoretical value.
  - The two-sample *t*-test, which is used to compare the mean of two independent given samples.

One Sample *t*-Test Procedure:

Step 1: Define the Null Hypothesis (H_0) and Alternate Hypothesis (H_1)

Example 2 : H_0: Sample mean ($\bar{X}$) = Hypothesized Population mean (μ)

H_1: Sample mean ($\bar{X}$) $\neq$ Hypothesized Population mean (μ)

The alternate hypothesis can also state that the sample mean is greater than or less than the comparison mean.

Step 2: Compute the test statistic (t)

$$t = \frac{Z}{\text{S.E.}} = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$$

Where S.E. is the standard error

Step 3: Find the *t*-critical from the *t*-Table

Use the degree of freedom and the alpha level (0.05) to find the t -critical.

Step 4: Determine if the computed test statistic falls in the rejection region.

Alternately, simply compute the P-value. If it is less than the significance level (0.05 or 0.01), reject the null hypothesis.

Example 2: We have the potato yield from 12 different farms. We know that the standard potato yield for the given variety is $\mu = 20$.

x = [21.5, 24.5, 18.5, 17.2, 14.5, 23.2, 22.1, 20.5, 19.4, 18.1, 24.1, 18.5]

Test if the potato yield from these farms is significantly better than the standard yield.

Sol.: Step 1: Define the Null and Alternate Hypothesis

$$H_0 : \bar{X} = 20$$
$$H_1 : \bar{X} > 20$$
$$n = 12$$

Since this is one sample t test, the degree of freedom = $n - 1 = 12 - 1 = 11$.

Let's set alpha = 0.05, to meet 95% confidence level.

Step 2: Calculate the Test Statistic (t)

(i) Calculate sample mean

$$\bar{X} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

$$= \frac{\begin{matrix}21.5 + 24.5 + 18.5 + 17.2 + 14.5 + 23.2 \\ +22.1 + 20.5 + 19.4 + 18.1 + 24.1 + 18.5\end{matrix}}{12}$$

$$= 20.175$$

(ii) Calculate sample standard deviation

$$\sigma = \sqrt{\frac{\begin{matrix}(x_1 - \bar{X})^2 + (x_2 - \bar{X})^2 \\ +(x_3 - \bar{X})^2 + \ldots + (x_n - \bar{X})^2\end{matrix}}{n-1}} = 3.0211$$

(iii) Substitute in the t Statistic formula

$$t = \frac{Z}{\text{S.E.}} = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$$

$$= \frac{20.175 - 20}{\frac{3.0211}{\sqrt{12}}} = 0.2006$$

Step 3: Find the t-Critical

Confidence level = 0.95, alpha = 0.05. For one tailed test, look under 0.05 column. For d.f. (degree of freedom) = 12 – 1 = 11, t-Critical = 1.796.

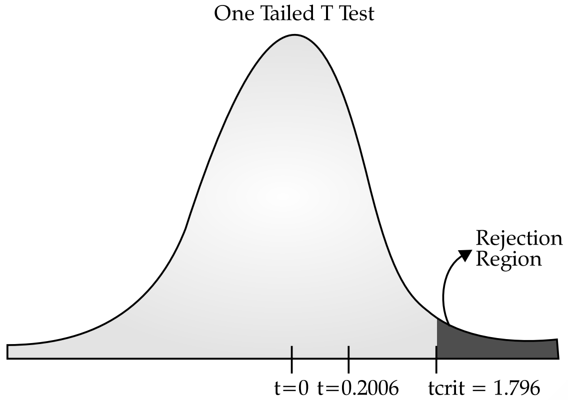
Note: Subtracting one from the sample size(n) gives the degrees of freedom (d.f.).

***t* Table**

cum.prob	$t_{.50}$	$t_{.75}$	$t_{.80}$	$t_{.85}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$	$t_{.999}$	$t_{.9995}$
one-tall	**0.50**	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**	**0.0005**
two-talls	**1.00**	**0.50**	**0.40**	**0.30**	**0.20**	**0.10**	**0.05**	**0.02**	**0.01**	**0.002**	**0.001**
df											
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.885	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.335	4.501	5.041
9	0.000	0.703	0.883	1.100	1.389	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.685	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.000	0.684	0.56	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.686	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
Z	0.000	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.090	3.297
	0%	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
	Confidence Level										

Step 4: Does it fall in rejection region ?

Since the computed t-Statistic is less than the t-critical, it does not fall in the rejection region.



Clearly, the calculated t statistic does not fall in the rejection region. So, we do not reject the null hypothesis.

Two Sample t-Test

The two sample t-test is a test that is used to compare the mean of two groups of samples. It is meant for evaluating whether the means of the two sets of data are statistically significantly different from each other.

Let the two independent samples be $x_1, x_2, x_3, \ldots.., x_{n1}$ and $y_1, y_2, y_3, \ldots.., y_{n2}$ with means $\bar{x} = \dfrac{\sum x}{n_1}$ and $\bar{y} = \dfrac{\sum y}{n_2}$ from two normal population with means μ_1 and μ_2 and common variance σ^2(unknown).

Let $s_1^2 = \dfrac{1}{n_1 - 1}\sum(x-\bar{x})^2$ and $s_2^2 = \dfrac{1}{n_2 - 1}\sum(y-\bar{y})^2$

Thus, standard error can be given by $s = \sqrt{\dfrac{\sum(x-\bar{x})^2 + \sum(y-\bar{y})^2}{n_1+n_2-2}}$

The t-test formula is given as: $t = \dfrac{\bar{x}-\bar{y}}{s}\sqrt{\dfrac{n_1 n_2}{n_1+n_2}}$

The significance of t is tested in the same way as in one sample t-test with degree of freedom $n_1 + n_2 - 2$ to test the hypothesis $\mu_1 = \mu_2$.

Example: Find the t-test value for the following given two sets of values: 7, 2, 9, 8 and 1, 2, 3, 4.

Sol. For first data set:

Number of terms in first set *i.e.* $n_1 = 4$

Calculate mean value for first data set using formula:

$$\bar{x} = \frac{\sum x}{n_1} = \frac{7+2+9+8}{4} = \frac{26}{4} = 6.5$$

For second data set:

Number of terms in second set *i.e.* $n_2 = 4$

Calculate mean value for second data set using formula:

$$\bar{y} = \frac{\sum y}{n_2} = \frac{1+2+3+4}{4} = \frac{10}{4} = 2.5$$

Construct the following table for standard error:

x	$x-\bar{x}$	$(x-\bar{x})^2$	y	$y-\bar{y}$	$(y-\bar{y})^2$
7	0.5	0.25	1	– 1. 5	2.25
2	– 4. 5	20.25	2	– 0. 5	0.25
9	2.5	6.25	3	0.5	0.25
8	1.5	2.25	4	1.5	2.25
		$\sum(x-\bar{x})^2 = 29$			$\sum(y-\bar{y})^2 = 5$

Now, compute the standard error, s using formula,

$$s = \sqrt{\frac{\sum(x-\bar{x})^2 + \sum(y-\bar{y})^2}{n_1+n_2-2}}$$

$$s = \sqrt{\frac{29+5}{4+4-2}} = \sqrt{\frac{34}{6}}$$

$$= \sqrt{5.666} = 2.380$$

Now using t-test formula:

$$t = \frac{\bar{x}-\bar{y}}{s}\sqrt{\frac{n_1 n_2}{n_1+n_2}}$$

$$t = \frac{6.5-2.5}{2.380}\sqrt{\frac{4\times 4}{4+4}}$$

$$= \frac{4}{2.380}\sqrt{2}$$

$$= 1.6806 \times 1.414 = 2.376$$

Hence, t-test value for the two data sets is 2.38.

Mnemonics

Concept: Errors in Null Hypothesis

Mnemonics: a = you "observed" a difference that did not exist.

b = you were Blinded by the truth.

Interpretations: In statistical hypothesis testing:

A **type I error** (α) is the incorrect rejection of a true null hypothesis (a "false positive"), that is, a null hypothesis incorrectly rejected.

While, a **type II error** (β) is the failure to reject a false null hypothesis (a "false negative"), that is, a null hypothesis is not rejected when it is in fact false.

UNIT - VI: INDEX NUMBERS AND TIME BASED DATA

CHAPTER-10

TIME SERIES ANALYSIS

Revision Notes

- A time series consists of a set of observations arranged in chronological order (either ascending or descending). Time Series has an important objective to identify the variations and try to eliminate the variations and also helps us to estimate or predict the **future values**.

Key Words

Future value: Future value is a value of an investment or asset on a specific date in the future.

- It helps in the analysis of past behaviour.
- It helps in forecasting and for future plans.
- It helps in evaluation of current achievements.
- It helps in making comparative studies between one time period and others.

The following series is the example of time series:

Year	1971	1981	1991	2001	2011	2021
Population (Crore)	43.9	54.2	68.3	84.9	102.6	121.5

- **Components of Time Series:** There are four types of components in a time series. They are as follows:

(i) Secular Trend, **(ii)** Seasonal Variations, **(iii)** Cyclic Variations and **(iv)** Irregular Variations.

(i) Secular Trend or Simple Trend or Long term movement: Secular trend refers to the general tendency of data to increase or decrease or stagnate over a long period of time. Time series relating to Economic, Business, and Commerce may show an upward or increasing tendency. Whereas, the time series relating to death rates, birth rates, share prices, etc. may show a downward or decreasing tendency.

The symbol of 'T' is used for denoting long term trend in the formulae relating to analysis of Time Series.

(ii) Seasonal variations: As the name suggests, tendency movements are due to nature which repeat themselves periodically in every seasons. These variations repeat themselves in less than one year time. It is measured in an interval of time. Seasonal variations may be influenced by natural force, social customs and traditions. These variations are the results of such factors which uniformly and regularly rise and fall in the magnitude. For example, in summers the sale of ice-cream increases and at the time of Diwali the sale of diyas, crackers, etc. go up.

The symbol of 'S' is used for denoting Seasonal variations in the formulae relating to analysis of Time Series.

(iii) Cyclical variations: Cyclical variations are due to the ups and downs recurring after a period from time to time. These are due to the business cycle and every organization has to phase all the four phases of a business cycle some time or the other. Prosperity or boom, recession, depression, and recovery are the four phases of a business cycle.

The symbol of 'C' is used for denoting Cyclic variations in the formulae relating to analysis of Time Series.

(iv) Irregular or Random variations: These variations do not have particular pattern and there is no regular period of time of their occurrences. These are accidently changes which are purely random or unpredictable. Normally they are short-term variations, but its occurrence sometimes has its effect so intense that they may give rise to new cyclic or other movements of variations. For example floods, wars, earthquakes, Tsunami, strikes, lockouts etc...

The symbol of 'I' is used for denoting Irregular variations in the formulae relating to analysis of Time Series.

- **Mathematical Model for a Time Series**

There are two common models used for decomposition of a time series into its components, namely additive and multiplicative model.

(i) Additive Model: The model assumes that the observed value is the sum of all the four components of time series. *i.e.*,

$$Y = T + S + C + I$$

where, Y = Original value, T = Trend Value, S = Seasonal component, C = Cyclic component, I = Irregular component

The additive model assumes that all the four components operate independently. It also assumes that the behaviour of components is of an additive character.

(ii) Multiplicative Model: This model assumes that the observed value is obtained by multiplying the trend (T) by the rates of other three components.

i.e., $Y = T \times S \times C \times I$

where, Y = Original value, T = Trend Value, S = Seasonal component, C = Cyclic component, I = Irregular component

This model assumes that the components due to different causes are not necessarily independent and they can affect one another. It also assumes that the behaviour of components is of a multiplicative character.

- **Measurement of Trends**

Following are the methods by which we can measure the trend.

(I) Method of Moving Averages.

(II) Method of Least Squares.

(I) Method of Moving Averages: Moving Averages Method gives a trend with a fair degree of accuracy. In this method, we take arithmetic mean of the values for a certain time span. The time span can be three-years, four-years, five-years and so on depending on the data set and our interest. We will see the working procedure of this method.

Procedure:

(i) Decide the period of moving averages (three-years, four-years).

(ii) In case of odd years, averages can be obtained by calculating,

$$\frac{a+b+c}{3}, \frac{b+c+d}{3}, \frac{c+d+e}{3}, \frac{d+e+f}{3}, ...$$

(iii) If the moving average is an odd number, there is no problem of centring it, the average value will be centered besides the second year for every three years.

(iv) In case of even years, averages can be obtained by calculating,

$$\frac{a+b+c+d}{4}, \frac{b+c+d+e}{4}, \frac{c+d+e+f}{4}, \frac{d+e+f+g}{4}, ...$$

(v) If the moving average is an even number, the average of first four values will be placed between 2nd and 3rd year, similarly the average of the second four values will be placed between 3rd and 4th year. These two averages will be again averaged and placed in the 3rd year. This continues for rest of the values in the problem. This process is called as centring of the averages.

Example 1 : Calculate three-yearly moving averages of number of students studying in a higher secondary school in a particular village from the following data.

Year	Number of Students
2011	332
2012	317
2013	357
2014	392
2015	402
2016	405
2017	410
2018	427
2019	435
2020	438

Key Facts

- Retail stores often use time series analysis to analyse how their total sales is trending over time.
- Heights of ocean tides and counts of sunspots, are the examples of time series analysis.

Sol. Computation of three-yearly moving averages.

Year	Number of Students	3- yearly moving Total	3- yearly moving Averages
2011	332	—	—
2012	317	1006	335.33
2013	357	1066	355.33
2014	392	1151	383.67
2015	402	1199	399.67
2016	405	1217	405.67
2017	410	1242	414.00
2018	427	1272	424.00
2019	435	1300	433.33
2020	438	—	—

(II) Method of Least Squares: The line of best fit is a line from which the sum of the deviations of various points is zero. This is the best method for obtaining the trend values. It gives a convenient basis for calculating the line of best fit for the time series. It is a mathematical method for measuring trend. Further the sum of the squares of these deviations would be least when compared with other fitting methods. So, this method is known as the Method of Least Squares and satisfies the following conditions:

(i) The sum of the deviations of the actual values of Y and $\bar{Y}$ (estimated value of Y) is Zero. *i.e.*, $(Y - \bar{Y}) = 0$.

(ii) The sum of squares of the deviations of the actual values of Y and (estimated value of Y) is least. *i.e.*, $(Y - \bar{Y})^2$ is least.

(iii) The straight line trend is represented by the equation

$$Y = a + bX \quad \text{...(1)}$$

where, Y is the actual value, X is time, *a*, *b* are constants

(iv) The constants '*a*' and '*b*' are estimated by solving the following two normal equations

$$\Sigma Y = n\,a + b\,\Sigma X \quad \text{....(2)}$$

$$\Sigma Y = a\,\Sigma X + b\,\Sigma X^2 \quad \text{....(3)}$$

where, '*n*' = number of years given in the data.

(v) By taking the mid-point of the time as the origin, we get

$$\Sigma X = 0$$

(vi) When $\Sigma X = 0$, the two normal equations reduces to

$$\Sigma Y = na + b\,(0);\ a = \frac{\Sigma Y}{n} = \bar{Y}$$

$$\Sigma XY = a(0) + b\,\Sigma X\,;\ b = \frac{\Sigma XY}{\Sigma X^2}$$

The constant '*a*' gives the mean of Y and '*b*' gives the rate of change (slope).

(vii) By substituting the values of '*a*' and '*b*' in the trend equation (1), we get the Line of Best Fit.

Example: Given below are the data relating to the sales of a product in a district.

Fit a straight line trend by the method of least squares and tabulate the trend values.

Year	2015	2016	2017	2018	2019	2020	2021	2022
Sales	6.7	5.3	4.3	6.1	5.6	7.9	5.8	6.1

Sol. Computation of trend values by the method of least squares.

In case of EVEN number of years, let us consider

$$X = \frac{(x - \text{Arithimetic mean of two middle years})}{0.5}$$

Year (x)	**Sales** (Y)	$X = \dfrac{(x-2018.5)}{0.5}$	**XY**	X^2	**Trend value** (Y_t)
2015	6.7	– 7	– 46.9	49	5.6167
2016	5.3	– 5	– 26.5	25	5.7190
2017	4.3	– 3	– 12.9	9	5.8214
2018	6.1	– 1	– 6.1	1	5.9238
2019	5.6	1	5.6	1	6.0262
2020	7.9	3	23.7	9	6.1286
2021	5.8	5	29.0	25	6.2310
2022	6.1	7	42.7	49	6.3333
$n = 8$	47.8	$\Sigma X = 0$	8.6	**168**	

$$a = \frac{\Sigma Y}{n} = \frac{47.8}{8} = 5.975$$

$$b = \frac{\Sigma XY}{\Sigma X^2} = \frac{8.6}{168} = 0.05119$$

Therefore, the required equation of the straight line trend is given by

$Y \quad = a + bX\,;\ Y = 5.975 + 0.05119\,X$.

When $X = 2015$, $\quad X = 1995,\ Y_t = 5.975 + 0.05119\left(\dfrac{2015 - 2018.5}{0.5}\right) = 5.6167$

When $X = 2016$, $\quad X = 1996,\ Y_t = 5.975 + 0.05119\left(\dfrac{2016 - 2018.5}{0.5}\right) = 5.7190$

Similarly, other values can be obtained.

Mnemonics

Concept: Components of Times Series

Mnemonics: **T**rendy **S**iya **C**ame **I**n

Interpretations:

Trendy	Siya	Came	In
↓	↓	↓	↓
Secular Trend	Seasonal Variations	Cyclic Variations	Irregular Variations

UNIT - VII: FINANCIAL MATHEMATICS

CHAPTER-11

FINANCIAL MATHEMATICS

Revision Notes

- **Perpetuity:** Perpetuity in the financial system is a situation where a stream of cash flow payments continues indefinitely or is an **annuity** that has no end. In valuation analysis, perpetuities are used to find the present value of a company's future projected cash flow stream and the company's terminal value. Essentially, a perpetuity is a series of cash flows that keep paying out forever.

Key Words

Annuity: A fixed sum of money paid to someone each year, typically for the rest of their life.

- **Perpetuity Formula:** There are two different annual perpetual valuations; perpetuity with flat or constant annuity and perpetuity with a growing annuity. Perpetuity gives a business the value of its cash flow, an essential slice of data as it aids in determining the firm's total cash flow in a single year.

  (1) Flat Perpetuity: Perpetuity is also known as regular flat perpetuity. In this cash flow is constant for each year. This perpetuity formula is the simplest, and it is straightforward as it doesn't include terminal value. It is the basic formula for the price of perpetuity. Calculate the present value of flat perpetuity we only need to divide the cash flows/payments by the discount rate.

  Present value of Perpetuity

  $$= \frac{\text{Cash flow}}{\text{Interest rate or yield}}$$

  (2) Growing Perpetuity: A growing perpetuity is the same as regular or flat perpetuity, but the difference is that the cash flow is growing each year. The present value of growing perpetuity formula factors in long term growth. This version is used to calculate the terminal value in a stream of cash flows for valuation purposes is always more complicated.

  Present value of Perpetuity

  $$= \frac{\text{Payment}}{\text{Interest rate} - \text{Growth rate}}$$

Example 1 : Company 'ABC' pays ₹ 2 in dividends annually and estimates that they will pay the dividends indefinitely. How much are investors willing to pay for the dividend with a required rate of return of 5% ?

Sol. Given, Payment amount = ₹ 2, Interest rate or yield = 5% or 0.05

Present value of Perpetuity

$$= \frac{\text{cash flow}}{\text{Interest rate or yield}}$$

$$= \frac{2}{0.05} = ₹\ 40$$

An investor will consider investing in the company if the stock price is ₹ 40 or less.

Taking the above example, imagine if the ₹ 2 dividend is expected to grow annually by 2%.

Given, Growth = 2% or 0.02.

Present value of Perpetuity

$$= \frac{\text{Payment}}{\text{Interest rate} - \text{Growth rate}}$$

$$= \frac{2}{0.05 - 0.02} = ₹\ 66.67$$

- **Sinking Fund:** A sinking fund is a type of fund that is created and set up purposely for repaying debt. The owner of the account sets aside a certain amount of money regularly and uses it only for a specific purpose. Often, it is used by corporations for bonds and deposit money to buy back issued bonds or parts of bonds before the maturity date arrives. It is also one way of enticing investors because the fund helps convince them that the issuer will not default on their payments.

  The formula to calculate the sinking fund is given below

$$\text{Sinking Fund, } A = \frac{\left[\left(1+\left(\frac{r}{m}\right)^{n\times m}\right)\right]-1}{\left(\frac{r}{m}\right)} \times P$$

where P = Periodic contribution to the sinking fund

r = Annualized rate of interest

n = No. of years

m = No. of payments per year

Example 2 : A sinking fund with a monthly periodic contribution of ₹ 1,500. The fund will be required to retire a newly taken debt (zero-coupon bonds) rose for the ongoing expansion project. Do the calculation of the amount of the sinking fund if the annualized rate of interest is 6%, and the debt will be repaid in 5 years. Given that $(1.005)^{60} = 1.3489$

Sol. Given, P = ₹ 1,500, r = 6% or 0.06, No. of years, n = 5 years and No. of payments per year, m = 12

$$\text{Sinking Fund, } A = \frac{\left[\left(1+\left(\frac{r}{m}\right)^{n\times m}\right)\right]-1}{\left(\frac{r}{m}\right)} \times P$$

$$A = \frac{\left[\left(1+\left(\frac{0.06}{12}\right)^{5\times 12}\right)\right]-1}{\left(\frac{0.06}{12}\right)} \times 1500$$

$$= \frac{1.3489 - 1}{0.005} \times 1500$$

$$= \frac{532.35}{0.005} = ₹\ 104,670$$

Key Facts

- Financial mathematics, also called analytical finance, mathematical finance and mathematical finance, is an inter-disciplinary subject of mathematics and finance.
- French mathematician Louis Bachelier is considered the author of the first scholarly work on mathematical finance, published in 1900. But mathematical finance emerged as a discipline in the 1970s, following the work of Fischer Black, Myron Scholes and Robert Merton on option pricing theory.

Advantages of sinking fund:

The following are the advantages of sinking fund:

(1) A sinking fund helps the company to pay its liability well in advance.

(2) A company is able to pay the debt in time because a company has already pulled a money well before.

(3) A sinking fund is also used to redeem the bond or any other liability in a mid-way also.

(4) A sinking fund also increases the goodwill of the company by paying the debt in time, and this will increase the faith of investors and attract more investment.

- **EMI (Equated Monthly Instalment):** An equated monthly instalment (EMI) is a fixed payment amount made by a borrower to a lender at a specified date each calendar month. Equated monthly instalments are used to pay off both interest and principal each month so that over a specified number of years, the loan is paid off in full. With most common types of loans—such as real estate mortgages, auto loans, and student loans—the borrower makes fixed periodic payments to the lender over the course of several years with the goal of retiring the loan.

The EMI system of loan repayment has following features:

(1) Each instalment contains both components of principal repayment and interest charges.

(2) Interest is calculated on reducing balance method.

(3) Interest component is higher in the beginning and progressively lower towards the end. That mean, the principal component of an EMI is lower during initial periods and higher during later periods.

(4) The amount of EMI depends on:

(i) The period of compounding *i.e.*, whether the compounding is yearly, half yearly, quarterly or monthly. If the compounding is more frequent, then the amount of EMI would be higher and vice-versa.

(ii) The rate of interest.

(iii) Period of repayment if the repayment period is more, then EMI would be lower and vice versa.

EMI can be calculated by two methods:

(i) Flat Rate method (FRM)

(ii) Reducing Balance Method (RBM)

Flat Rate Method: In flat rate method the principal amount remains same throughout the tenure and the interest is changed on it at a constant rate throughout the loan tenure: Suppose an amount of ₹ P is borrowed at flat rate r per rupee per month for a period of n months.

Then, Interest $= Pin$

$$\therefore \quad \text{EMI} = \frac{\text{Principal} + \text{Interest}}{n}$$

$$\text{or,} \quad \text{EMI} = \frac{P+Pni}{n} = P\left(\frac{l+in}{n}\right)$$

$$= P\left(i+\frac{1}{n}\right)$$

Example 3 : Navisha lakes a loan of ₹ 3,00,000 at an interest 10% compounded annually for a period of 3 years. Find EMI using flat rate method.

Sol. We have, P = Principal = ₹ 3,00,000

i = rate of interest per rupee per month

$$= \frac{10}{1200} = \frac{1}{120}$$

n = Number of instalments = 36

$$\therefore \quad \text{EMI} = ₹\, P\left(i + \frac{1}{n}\right)$$

$$= ₹\, 3,00,000\left(\frac{1}{120} + \frac{1}{36}\right)$$

$$= ₹\,(250 + 833.33)$$

$$= ₹\, 1083.33$$

Reducing Balance Method: In reducing balance method principal paid back gets deducted from the outstanding loan amount and the interest for the subsequent year is changed on the remaining deducted balance and not on the entire loan amount unlike the flat rate method.

The formula for EMI for Reduce Balance Method is:

$$\text{EMI} = \frac{(1+i)^n}{(1+i)^n - 1} \times (P \times i)$$

where $i = \dfrac{\text{monthly interest rate annual}}{12 \times 100}$

n = number of Instalments,

P = principal amount of the loan

Example 4 : Ram took a home loan of ₹ 50 lakhs at an 8.50 % interest for a 20 years loan tenure. What would be his EMI ? Given $(1.0070)^{240} = 5.3342$

Sol. Given,

$$i = \left[\frac{\left(\frac{\text{annual rate}}{12}\right)}{100}\right]$$

$$= \left[\frac{\left(\frac{8.50}{12}\right)}{100}\right]$$

$$= \frac{0.70833}{100}$$

$$= 0.0070$$

$$n = 20 \times 12 = 240$$

$$P = ₹\, 50,00,000$$

$$\text{Instalment Amount} = \frac{(1+0.0070)^{240}}{(1+0.0070)^{240} - 1} \times (50,00,000 \times 0.0070)$$

$$= \frac{5.3342}{5.3342 - 1} \times 35,000$$

$$= \frac{186,697}{4.3342}$$

$$= ₹\, 43,075.308 \sim ₹\, 43,075$$

➢ **Calculation of Returns:** All financial decisions involve same risk one may expect to get a return of 15% per annum in his investment but the risk of not able to achieve 15% return will always be there. Return is simply a reward for investing as all investing involves some risk.

A debt investment is a loan, and the return is just the loans interest rate. This is simply the ratio of the interest paid to the loan principal

$$\text{Return, } k = \frac{\text{interest paid}}{\text{loan amount}}$$

This formulation leads to the convenient idea that a return is what the investor receives divided by what he or she invests.

➢ **Rate of Return (ROR) or Nominal Rate of Return (NROR):** A Rate of Return (ROR) is the gain or loss of an investment over a certain period of time. In other words, the rate of return is the gain (or loss) compared to the cost of an initial investment, typically expressed in the form of a percentage. When the ROR is positive, it is considered a gain and when the ROR is negative, it reflects a loss on the investment.

The standard formula for calculating ROR is as follows:

Rate of Return

$$= \frac{\begin{array}{c}\text{Ending value of investment}\\ -\text{Initial value of investment}\end{array}}{\text{Beginning value of investment}} \times 100$$

➢ **Compounded Annual Growth Rate (CAGR):** Compounded annual growth rate (CAGR) depicts the cumulative performance of a particular variable over a significant period of time and is used to measure relative profitability of businesses. CAGR is often associated with specific parameters which indicate the performance of a company over a fixed period, such as sales, revenue, earnings, etc.

$$\text{CAGR} = \left(\frac{\text{Final value}}{\text{Initial value}}\right)^{1/n} - 1$$

Where n = investment period

Example 5 : Aritra invested ₹ 1,000 in a fund for three years. While the total Net Asset value (NAV) remained ₹ 1,000 for the first year, it increased to ₹ 1,100 in the second year. Upon maturity of this fund, final NAV stood at ₹ 1,300. Find compounded annual growth rate. Given that $(1.3)^{1/3} = 1.0913$.

Sol.

Year	NAV (in ₹)
I	1000
II	1100
III	1300

$$\text{CAGR} = \left(\frac{\text{Final value}}{\text{Initial value}}\right)^{1/n} - 1$$

$$= \left(\frac{13,00}{1,000}\right)^{\frac{1}{3}} - 1 = 1.0913 - 1$$

$$= 0.0913 = 9.13\%$$

➢ **Depreciation:** Depreciation allows a portion of the cost of a fixed asset to the revenue generated by the fixed asset. This is mandatory under the matching principle as revenues are recorded with their associated expenses in the accounting period when the asset is in use. This helps in getting a complete picture of the revenue generation transaction.

Example: If a delivery truck is purchased a company with a cost of ₹ 100,000 and the expected usage of the truck are 5 years, the business might depreciate the asset under depreciation expense as ₹ 20,000 every year for a period of 5 years.

There are three methods commonly used to calculate depreciation. They are:

(1) Linear or Straight line method

(2) Unit of production method

(3) Double-declining balance method

Three main inputs are required to calculate depreciation:

Useful life: This is the time period over which the organisation considers the fixed asset to be productive. Beyond its useful life, the fixed asset is no longer cost-effective to continue the operation of the asset.

Salvage value: Post the useful life of the fixed asset, the company may consider selling it at a reduced amount. This is known as the salvage value of the asset.

The cost of the asset: This includes taxes, shipping, and preparation/setup expenses.

Here, we have discuss only Linear or Straight line method for calculating depreciation.

- **Linear or Straight-line Depreciation Method:** This is the simplest method of all. It involves simple allocation of an even rate of depreciation every year over the useful life of the asset. The formula for straight line depreciation is:

  Annual Depreciation Expense

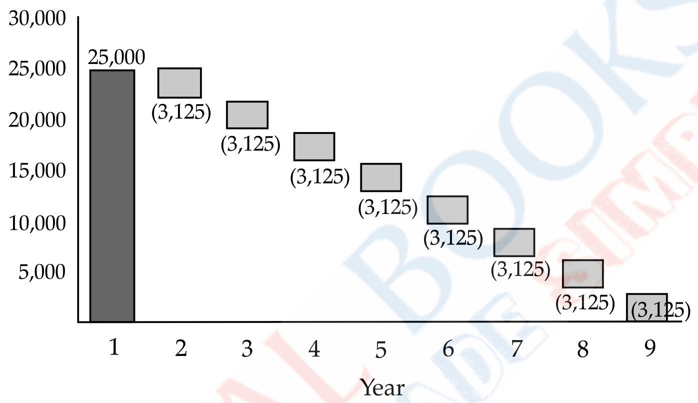
$$= \frac{\text{Asset cost} - \text{Residual value}}{\text{Useful life of the asset}}$$

Example 6 : Consider a piece of equipment that costs ₹ 25,000 with an estimated useful life of 8 years and a ₹ 0 salvage value. The depreciation expense per year for this equipment would be as follows:

Year#		1	2	3	4	5	6	7	8
Straight Line									
Opening Book Value		25,000	21,875	18,750	15,625	12,500	9,375	6,250	3,125
Depreciation	8	3,125	3,125	3,125	3,125	3,125	3,125	3,125	3,125
Ending Book Value	25,000	21,875	18,750	15,625	12,500	9,375	6,250	3,125	

$$\text{Depreciation Expense} = \frac{₹\,25{,}000 - ₹\,0}{8} = ₹\,3{,}125 \text{ per year}$$

Straight Line Depreciation



Mnemonics

Concept: Perpetuity

Mnemonics: "A **perpetuity** is a series of cash flows that keep paying out forever."

Interpretations: Per + Pet *i.e.*, Per Every Pet (animal etc.).

So that it can be linked to the statement "every pet will last in every stage of life" *i.e.*, Per Every Stage.

UNIT - VIII: LINEAR PROGRAMMING

CHAPTER-12

LINEAR PROGRAMMING

Revision Notes

- Problems which minimize or maximize a linear function Z subject to certain conditions determined by a set of linear inequalities with non-negative variables are known as linear programming problems.
- **Objective function:** A linear function $Z = ax + by$, where a and b are constants which has to be maximized or minimized according to a set of given conditions, is called as linear objective function.
- **Decision variables:** In the objective function $Z = ax + by$, the variables x, y are said to be decision variables.
- **Constraints:** The restrictions in the form of inequalities on the variables of a linear programming problem are called constraints. The condition $x \geq 0$, $y \geq 0$ are known as non-negative restrictions.
- **Different types of linear programming problems:** A few important linear programming problems are as follows:

(i) **Manufacturing problem:** In such problem, we determine: (a) Number of units of different products to be produced and sold. (b) Manpower required, machines hours needed, warehouse space available, etc. Objective function is to maximize profit.

(ii) **Diet problem:** Here, we determine the amount of different types of constituent or nutrients which should be included in the diet. Objective function is to minimize the cost of production.

(iii) **Transportation problem:** These problems deal with the cost of transportation which is to be minimized under given constraints.

(iv) **Assignment Problem:** The assignment problem is a special case of transportation problem where the number of sources and destinations are equal. Supply at each source and demand at each destination must be one.

(v) **Blending problem:** To determine the optimum amount of several constituents used in producing a set of products while determining the optimum quantity of each product to be produced.

(vi) **Investment problem:** To determine the amount of investment in fixed income securities to maximize the return on these investment.

- **Limitations of Linear Programming:**

(i) To specify an objective function in mathematical form is not an easy task.

(ii) Even if objective function is determined, it is difficult to determine social, institutional, financial and other constraints.

(iii) It is also possible that the objective function of constraints may not be directly specified by linear inequality equations.

- **Mathematical Formulation of LPP**

Let us take an example to understand how to formulate a LPP mathematically.

Example 1 : An electronic firm is undecided at the most profitable mix for its products. The products manufactured are transistors, resistors and carbon tubes with a profit of (per 100 units) ₹ 10, ₹ 6 and ₹ 4 respectively. To produce a shipment of transistors containing 100 units requires 1 hour of engineering, 10 hours of direct labour, and 2 hours of administrative service. To produce 100 units of resistors requires 1 hour, 4 hours and 2 hours of engineering, direct labour and administrative services respectively. For 100 units of carbon tubes it needs 1 hour, 6 hours and 5 hours of engineering direct labour and administrative services respectively.

There are 100 hours of engineering time, 600 hours of direct labour and 300 hours of administrative time available. Formulate the corresponding LPP.

Sol.: Let the firm produce X hundred units of transistors, Y hundred units of resistors and Z hundred units of carbon tubes. Then the total profit to be maximized from this output will be

$$P = 10X + 6Y + 4Z$$

This is our objective function.

Now production of X hundred units of transistors, Y hundred units of resistors and Z hundred units of carbon tubes will require X + Y + Z hours of engineering time, 10X + 4Y + 6Z hours of direct labour time and 2X + 2Y + 5Z hours of administrative service.

But the total time available for engineering, direct labour and administrative services is 100, 600 and 300 hours respectively.

Hence the constraints are:

$$X + Y + Z \le 100$$
$$10X + 4Y + 6Z \le 600$$
$$2X + 2Y + 5Z \le 300$$

with $X, Y, Z \ge 0$ as the non-negativity restriction.

Thus the formulation is

Maximize, $P = 10X + 6Y + 4Z$

subject to constraints: $X + Y + Z \le 100$

$$10X + 4Y + 6Z \le 600$$
$$2X + 2Y + 5Z \le 300$$
$$X, Y, Z \ge 0.$$

Key Facts

Linear programming is often used for problems where no exact solution is known, for example for planning traffic flows.

- The goal of linear programming is to maximize or minimize specified objectives, such as profit or cost. This process is know as optimization.
- Linear programming is heavily used in microeconomics and company management, such as planning, product, transportation, technology and other issues, either to maximize the income or minimize the costs of a production scheme.

- **Feasible region:** The common region determined by all the constraints including non-negative constraints $x, y \geq 0$ of linear programming problem is known as the feasible region.
- **Feasible solution:** Points within and on the boundary of the feasible region represents feasible solutions of constraints.
  In the feasible region, there are infinitely many points (solutions) which satisfy the given conditions.
- **Theorem 1:** Let R be the feasible region for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where variables x and y are subject to constraints described by linear inequalities, the optimal value must occur at a corner point (vertex) of the feasible region.
- **Theorem 2:** Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function R has both maximum and minimum values of R and each of these occurs at a corner point (vertex) of R.
  However, if the feasible region is unbounded, the optimal value obtained may not be maximum or minimum.
- **Corner or Extreme Point Method of Formulation of LPP**
  Step 1: Formulate the linear programming problem in x and y with given conditions.
  Step 2: Convert the inequality constraints into equality constraints and plot each line on the graph paper.
  Step 3: Find the feasible region and check if the feasible region is bounded or unbounded.
  Step 4: Evaluate the value of the objective function Z at each corner point. Let M be the greatest and m be the smallest value of the objective function Z.
  (i) **When the feasible region is bounded:** M and m are the maximum and minimum values of the objective function Z, respectively.
  (ii) **When the feasible region is unbounded:**
  (a) M is the maximum value of the objective function Z, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise the objective function has no maximum value.
  (b) m is the minimum value of the objective function Z, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, the objective function has no minimum value.

Example: A mill owner buys two types of machines A and B for his mill. Machine A occupies 1,000 sq.m of area and requires 12 men to operate it; while machine B occupies 1,200 sq.m of area and requires 8 men to operate it. The owner has 7,600 sq.m of area available and 72 men to operate the machines. If machine A produces 50 units and machine B produces 40 units daily, how many machines of each type should he buy to maximize the daily output? Use Linear Programming to find the solution.

Sol: The data given in the problem are as under:

	Machine A	Machine B	Maximum availability
Area needed	1,000 sq.m	1,200 sq.m	7,600 sq.m
Labour force	12	8	72
Daily output	50 units	40 units	—

Let x and y be the number of machines A and B respectively

$$1{,}000x + 1{,}200y \leq 7{,}600$$

or $$10x + 12y \leq 76$$

or $$5x + 6y \leq 38 \quad \text{...(i)}$$

$$12x + 8y \leq 72$$

or $$3x + 2y \leq 18 \quad \text{...(ii)}$$

$$x \geq 0, y \geq 0$$

Total output $$Z = 50x + 40y$$

Max. $Z = 50x + 40y$ subject to constraints

$$5x + 6y \leq 38$$
$$3x + 2y \leq 18$$
$$x \geq 0, y \geq 0$$

$3x + 2y = 18$

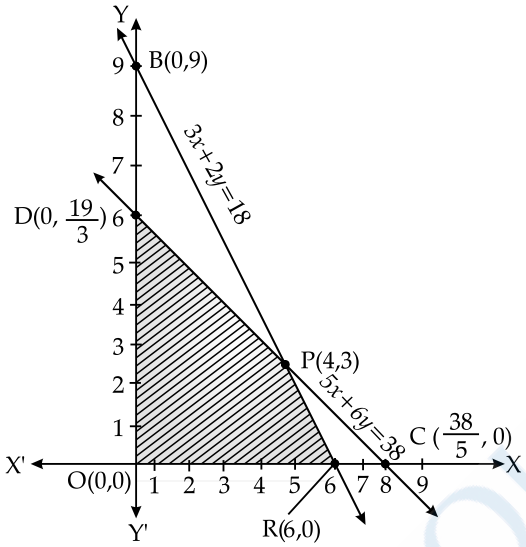
x	6	0	4
y	0	9	3

$5x + 6y = 38$

x	$\frac{38}{5}$	0	4
y	0	$\frac{19}{3}$	3

The vertices of the feasible region $ORPD$ are $O(0, 0)$, $R(6, 0)$, $P(4, 3)$ and $D\left(0, \frac{19}{3}\right)$.

The graph of these in equations is shown below:



Points	Value of Z = 50x + 40y
At O(0, 0)	Z = 0
At R(6, 0)	Z = 50 × 6 + 0 = 300
At P(4, 3)	Z = 50 × 4 + 40 × 3 = 320
At $D\left(0, \frac{19}{3}\right)$	$Z = 50 \times 0 + 40 \times \frac{19}{3}$ $= \frac{760}{3} = 253.33$

Thus, we see that Z is maximum at (4, 3)
i.e., Number of machine A = 4
Number of machine B = 3

Mnemonics

Concept: LPP Parameter
Mnemonics: NOC
Interpretations:

N	O	C
↓	↓	↓
Non-negative Variables	Objective function	Constraints