UNIT – I : RELATIONS AND FUNCTIONS

CHAPTER-1
RELATIONS AND FUNCTIONS

Revision Notes

Relations

1. Definition

A relation R, from a non-empty set A to another non-empty set B is mathematically as an subset of $A \times B$. Equivalently, any subset of $A \times B$ is a relation from A to B.

Thus, R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$

$\Leftrightarrow R \subseteq \{(a, b) : a \in A, b \in B\}$

Illustrations :

(a) Let $A = \{1, 2, 4\}$, $B = \{4, 6\}$. Let $R = \{(1, 4), (1, 6), (2, 4), (2, 6), (4, 4) (4, 6)\}$. Here $R \subseteq A \times B$ and therefore R is a relation from A to B.

(b) Let $A = \{1, 2, 3\}$, $B = \{2, 3, 5, 7\}$, Let $R = \{(2, 3), (3, 5), (5, 7)\}$. Here $R \not\subset A \times B$ and therefore R is not a relation from A to B. Since $(5, 7) \in R$ but $(5, 7) \notin A \times B$.

(c) Let $A = \{-1, 1, 2\}$, $B = \{1, 4, 9, 10\}$ let $a \in A$ and $b \in B$ and $a\,R\,b$ means $a^2 = b$ then, $R = \{(-1, 1), (1, 1), (2, 4)\}$.

Note :

- A relation from A to B is also called a relation from A into B.
- $(a, b) \in R$ is also written as aRb (read as a is related to b).

- Let A and B be two non-empty finite sets having p and q elements respectively. Then $n(A \times B) = n(A) \cdot n(B) = pq$. Then total number of subsets of $A \times B = 2^{pq}$. Since each subset of $A \times B$ is a relation from A to B, therefore total number of relations from A to B will be 2^{pq}.

2. Domain & range of a relation

(a) Domain of a relation : Let R be a relation from A to B. The domain of relation R is the set of all those elements $a \in A$ such that $(a, b) \in R \;\forall\, b \in B$.

Thus, Dom.$(R) = \{a \in A : (a, b) \in R \;\forall\, b \in B\}$.

That is, the domain of R is the set of first components of all the ordered pairs which belong to R.

(b) Range of a relation : Let R be a relation from A to B. The range of relation R is the set of all those elements $b \in B$ such that $(a, b) \in R \;\forall\, a \in A$.

Thus, Range of $R = \{b \in B : (a, b) \in R \;\forall\, a \in A\}$.

That is, the range of R is the set of second components of all the ordered pairs which belong to R.

(c) Co-domain of a relation : Let R be a relation from A to B. Then B is called the co-domain of the relation R. So we can observe that co-domain of a relation R from A into B is the set B as a whole.

Illustrations : Let $a \in A$ and $b \in B$ and

(i) Let $A = \{1, 2, 3, 7\}$, $B = \{3, 6\}$. If aRb means $a < b$.

Then we have $R = \{(1, 3), (1, 6), (2, 3), (2, 6), (3, 6)\}$.

Here, Dom.$(R) = \{1, 2, 3\}$, Range of $R = \{3, 6\}$, Co-domain of $R = B = \{3, 6\}$

(ii) Let $A = \{1, 2, 3\}$, $B = \{2, 4, 6, 8\}$.

If $R_1 = \{(1, 2), (2, 4), (3, 6)\}$, and $R_2 = \{(2, 4\}, (2, 6), (3, 8), (1, 6)\}$

Then both R_1 and R_2 are related from A to B because

$R_1 \subseteq A \times B$, $R_2 \subseteq A \times B$

Here, Dom(R_1) = {1, 2, 3}, Range of R_1 = {2, 4, 6};

Dom(R_2) = {2, 3, 1}, Range of R_2 = {4, 6, 8}

3. Types of relations from one set to another set

(a) Empty relation : A relation R from A to B is called an empty relation or a void relation from A to B if $R = \phi$.

For example, Let $A = \{2, 4, 6\}, B = \{7, 11\}$

Let $R = \{(a, b) : a \in A, b \in B \text{ and } a - b \text{ is even}\}$.

Here R is an empty relation.

(b) Universal relation : A relation R from A to B is said to be the universal relation if $R = A \times B$.

For example, Let $A = \{1, 2\}, B = \{1, 3\}$

Let $R = \{(1, 1), (1, 3), (2, 1), (2, 3)\}$.

Here, $R = A \times B$, so relation R is a universal relation.

Note :
- The void relation *i.e.*, ϕ and universal relation *i.e.*, $A \times A$ on A are respectively the smallest and largest relations defined on the set A. Also these are also called **Trivial Relations** and other relation is called a **Non-Trivial Relation**.
- The relations $R = \phi$ and $R = A \times A$ are two **extreme relations**.

(c) Identity relation : A relation R defined on a set A is said to be the identity relation on A if

$$R = \{(a, b) : a \in A, b \in A \text{ and } a = b\}$$

Thus identity relation $R = \{(a, a) : \forall\, a \in A\}$

The identity relation on set A is also denoted by I_A.

For example, Let $A = \{1, 2, 3, 4\}$,

Then $I_A = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$.

But the relation given by $R = \{(1, 1), (2, 2), (1, 3), (4, 4)\}$

is not an identity relation because element of I_A is not related to elements 1 and 3.

Note :
- In an identity relation on A every element of A should be related to itself only.

(d) Reflexive relation : A relation R defined on a set A is said to be reflexive if $a\,R\,a\ \forall\, a \in A$ *i.e.*, $(a, a) \in R\ \ \forall\, a \in A$.

For example, Let $A = \{1, 2, 3\}$ and R_1, R_2, R_3 be the relations given as

$R_1 = \{(1, 1), (2, 2), (3, 3)\}$,

$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3)\}$ and

$R_3 = \{(2, 2), (2, 3), (3, 2), (1, 1)\}$

Here R_1 and R_2 are reflexive relations on A but R_3 is not reflexive as $3 \in A$ but $(3, 3) \notin R_3$.

Note :
- The universal relation on a non-void set A is reflexive.
- The identity relation is always a reflexive relation but the converse may or may not be true. As shown in the example above, R_1 is both identity as well as reflexive relation on A but R_2 is only reflexive relation on A.

(e) Symmetric relation : A relation R defined on a set A is symmetric if

$(a, b) \in R \Rightarrow (b, a) \in R\ \forall\, a, b \in A$ *i.e.*, $aRb \Rightarrow bRa$ (*i.e.*, whenever aRb then bRa).

For example, Let $A = \{1, 2, 3\}$,

$R_1 = \{(1, 2), (2, 1)\}, R_2 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$.

$R_3 = \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$

$R_4 = \{(1, 3), (3, 1), (2, 3)\}$

Here R_1, R_2 and R_3 are symmetric relations on A. But R_4 is not symmetric because $(2, 3) \in R_4$ but $(3, 2) \notin R_4$.

(f) Transitive relation : A relation R on a set A is transitive if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

i.e., aRb and $bRc \Rightarrow aRc$.

For example, Let $A = \{1, 2, 3\}$,

$R_1 = \{(1, 2), (2, 3), (1, 3), (3, 2)\}$

and $R_2 = \{(1, 3), (3, 2), (1, 2)\}$

Here R_2 is transitive relation whereas R_1 is not transitive because $(2, 3) \in R_1$ and $(3, 2) \in R_1$ but $(2, 2) \notin R_1$.

(g) Equivalence relation : Let A be a non-empty set, then a relation R on A is said to be an equivalence relation if

(i) R is reflexive *i.e.*, $(a, a) \in R \ \forall \ a \in A$ *i.e.*, aRa.

(ii) R is symmetric *i.e.*, $(a, b) \in R \Rightarrow (b, a) \in R \ \forall \ a, b \in A$ *i.e.*, $aRb \Rightarrow bRa$.

(iii) R is transitive *i.e.*, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R \ \forall \ a, b, c \in A$ *i.e.*, aRb and $bRc \Rightarrow aRc$.

For example, Let $A = \{1, 2, 3\}$

$R = \{(1, 2), (1, 1), (2, 1), (2, 2), (3, 3)\ (1, 3), (3, 1), (3, 2), (2, 3)\}$

Here R is reflexive, symmetric and transitive. So R is an equivalence relation on A.

Equivalence classes : Let A be an equivalence relation in a set A and let $a \in A$. Then, the set of all those elements of A which are related to a, is called equivalence class determined by a and it is denoted by $[a]$. Thus, $[a] = \{b \in A : (a, b) \in A\}$

Note :

- Two equivalence classes are either disjoint or identical.
- An equivalence relation R on a set A partitions the set into mutually disjoint equivalence classes.
- An important property of an equivalence relation is that it divides the set into pair-wise disjoint subsets called **equivalence classes** whose collection is called **a partition of the set**.

  Note that the union of all equivalence classes give the whole set.

  e.g., Let R denotes the equivalence relation in the set Z of integers given by $R = \{(a, b) : 2 \text{ divides } a - b\}$. Then the equivalence class [0] is $[0] = [0, \pm 2, \pm 4, \pm 6, ...]$.

4. Tabular representation of a relation

In this form of representation of a relation R from set A to set B, elements of A and B are written in the first column and first row respectively. If $(a, b) \in R$ then we write '1' in the row containing a and column containing b and if $(a, b) \notin R$ then we write '0' in the same manner.

For example, Let $A = \{1, 2, 3\}, B = \{2, 5\}$ and $R = \{(1, 2), (2, 5), (3, 2)\}$, then

R	**2**	**5**
1	1	0
2	0	1
3	1	0

5. Inverse relation

Let $R \subseteq A \times B$ be a relation from A to B. Then, the inverse relation of R, to be denoted by R^{-1}, is a relation from B to A defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$

Thus $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1} \ \forall \ a \in A, b \in B$.

Clearly, Dom.(R^{-1}) = Range of R, Range of R^{-1} = Dom.(R).

Also, $(R^{-1})^{-1} = R$.

For example, Let $A = \{1, 2, 4\}, B = \{3, 0\}$ and let $R = \{(1, 3), (4, 0), (2, 3)\}$ be a relation from A to B, then $R^{-1} = \{(3, 1), (0, 4), (3, 2)\}$.

Functions

1. Domain : If a function is expressed in the form $y = f(x)$, then domain of f means set of all those real values of x for which y is real (*i.e.*, y is well-defined).

Remember the following points :

(a) Negative number should not occur under the square root (even root) *i.e.*, expression under the square root sign must be always ≥ 0.

(b) Denominator should never be zero.

(c) For $\log_b a$ to be defined, $a > 0, b > 0$ and $b \neq 1$. Also note that $\log_b 1$ is equal to zero *i.e.*, 0.

2. Range : If a function is expressed in the form $y = f(x)$, then range of f means set of all possible real values of y corresponding to every value of x in its domain.

Remember the following points :

(a) At first find the domain of the given function.

(b) If the domain does not contain an interval, then find the values of y putting these values of x from the domain. The set of all these values of y obtained will be the range.

(c) If domain is the set of all real numbers R or set of all real numbers except a few points, then express x in terms of y and from this find the real values of y for which x is real and belongs to the domain.

3. Function as a special type of relation : A relation f from a set A to another set B is said be a function (or mapping) from A to B if with every element (say x) of A, the relation f relates a unique element (say y) of B. This y is called f - image of x. Also x is called pre-image of y under f.

4. Difference between relation and function : A relation from a set A to another set B is any subset of $A \times B$; while a function f from A to B is a subset of $A \times B$ satisfying following conditions :

(a) For every $x \in A$, there exists $y \in B$ such that $(x, y) \in f$.

(b) If $(x, y) \in f$ and $(x, z) \in f$ then, $y = z$.

S. No.	Function	Relation
(i)	Each element of A must be related to some element of B.	There may be some elements of A which are not related to any element of B.
(ii)	An element of A should not be related to more than one element of B.	An element of A may be related to more than one element of B.

5. Real valued function of a real variable : If the domain and range of a function f are subsets of R (the set of real numbers), then f is said to be **a real valued function of a real variable** or **a real function.**

6. Some important real functions and their domain & range

S. No. Function	Representation	Domain	Range
(i) Identity function	$I(x) = x \ \forall \ x \in R$	R	R
(ii) Modulus function or Absolute value function	$f(x) = \lvert x \rvert = \begin{cases} -x, \text{if } & x < 0 \\ x, \text{ if } & x \geq 0 \end{cases}$	R	$[0, \infty)$
(iii) Greatest integer function or Integral function or Step function	$f(x) = [x] \ \forall x \in R$	R	Z
(iv) Smallest integer function	$f(x) = [x] \ \forall \ x \in R$	R	Z
(v) Signum function	$f(x) = \begin{cases} \dfrac{\lvert x \rvert}{x}, \text{if } x \neq 0 \\ 0, \text{ if } \ x = 0 \end{cases}$ *i.e.*, $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$	R	$\{-1, 0, 1\}$
(vi) Exponential function	$f(x) = a^x, \ \forall \ a > 0, a \neq 1$	R	$(0, \infty)$
(vii) Logarithmic function	$f(x) = \log_a x, \ \forall \ a \neq 1, a > 0$ and $x > 0$	$(0, \infty)$	R

7. Types of Function

(a) One-one function (Injective function or Injection) : A function $f : A \to B$ is one-one function or injective function if distinct elements of A have distinct images in B.

Thus, $f : A \to B$ is one-one $\Leftrightarrow f(a) = f(b) \Rightarrow a = b, \forall a, b \in A$

$\Leftrightarrow a \neq b \Rightarrow f(a) \neq f(b) \ \forall \ a, b \in A.$

- If A and B are two sets having m and n elements respectively such that $m \leq n$, then total number of one-one functions from set A to set B is ${}^nC_m \times m!$ *i.e.*, ${}^nP_m$.
- If $n(A) = n$, then the number of injective functions defined from A onto itself is $n!$.

ALGORITHM TO CHECK THE INJECTIVITY OF A FUNCTION

STEP 1 : Take any two arbitrary elements a, b in the domain of f.

STEP 2 : Put $f(a) = f(b)$.

STEP 3 : Solve $f(a) = f(b)$. If it gives $a = b$ only, then f is a one-one function.

(b) Onto function (Surjective function or Surjection) : A function $f : A \rightarrow B$ is onto function or a surjective function if every element of B is the f - image of some element of A. That implies $f(A) = B$ or range of f is the co-domain of f.

Thus, $f : A \rightarrow B$ is onto $\Leftrightarrow f(A) = B$ *i.e.*, range of f = co-domain of f.

ALGORITHM TO CHECK THE SURJECTIVITY OF A FUNCTION

STEP 1 : Take an element $b \in B$, where B is the co-domain of the function.

STEP 2 : Put $f(x) = b$.

STEP 3 : Solve the equation $f(x) = b$ for x and obtain x in terms of b. Let $x = g(b)$.

STEP 4 : If for all values of $b \in B$, the values of x obtained from $x = g(b)$ are in A, then f is onto. If there are some $b \in B$ for which values of x, given by $x = g(b)$, is not in A. Then f is not onto.

(c) One-one onto function (Bijective function or Bijection) : A function $f : A \rightarrow B$ is said to be one-one onto or bijective if it is both one-one and onto *i.e.*, if the distinct elements of A have distinct images in B and each element of B is the image of some element of A.

Also note that a bijective function is also called a one-to-one function or one-to-one correspondence.

If $f : A \rightarrow B$ is a function such that,

(i) f is one-one $\Rightarrow n(A) \leq n(B)$.

(ii) f is onto $\Rightarrow n(B) \leq n(A)$.

(iii) f is one-one onto $\Rightarrow n(A) = n(B)$.

For an ordinary finite set A, a one-one function $f : A \rightarrow A$ is necessarily onto and an onto function $f : A \rightarrow A$ is necessarily one-one for every finite set A.

(d) Identity function : The function $I_A : A \rightarrow A$; $I_A(x) = x, \forall x \in A$ is called an identity function on A.

> **Note :**
> - Domain (I_A) = A and Range $(I_A) = A$.

(e) Equal function : Two functions f and g having the same domain D are said to be equal if $f(x) = g(x)$ for all $x \in D$.

8. Constant and Types of Variables

(a) Constant : A constant is a symbol which retains the same value throughout a set of operations. So, a symbol which denotes a particular number is a constant. Constants are usually denoted by the symbols a, b, c, k, l, m, ... etc.

(b) Variable : It is a symbol which takes a number of values *i.e.*, it can take any arbitrary values over the interval on which it has been defined. *For example,* if x is a variable over R (set of real numbers) then we mean that x can denote any arbitrary real number. Variables are usually denoted by the symbols $x, y, z, u, v, \ldots$ etc.

(i) Independent variable : The variable which can take an arbitrary value from a given set is termed as an independent variable.

(ii) Dependent variable : The variable whose value depends on the independent variable is called a dependent variable.

9. Defining a Function

Consider A and B be two non-empty sets, then a rule f which associates **each element of** A **with a unique element of** B is called a function or the mapping from A to B or f maps A to B. If f is a mapping from A to B, then we write $f : A \rightarrow B$ which is read as 'f is mapping from A to B' or 'f is a function from A to B'.

If f associates $a \in A$ to $b \in B$, then we say that 'b **is the image of the element** a **under the function** f' or 'b **is the** f **- image of** a' or '**the value of** f **at** a' and denotes it by $f(a)$ and we write $b = f(a)$. The element a is called the **pre-image** or **inverse-image of** b.

Thus for a bijective function from A to B,

(a) A and B should be non-empty.

(b) Each element of A should have image in B.

(c) No element of A should have more than one image in B.

(d) If A and B have respectively m and n number of elements then the **number of functions defined from** A **to** B **is** n^m.

10. Domain, Co-domain and Range of A function

The **set** A **is called the domain** of the function f and the **set** B **is called the co- domain.** The set of the images of all the elements of A under the function f is called the **range of the function** f and is denoted as $f(A)$.

Thus range of the function f is $f(A)=\{f(x) : x \in A\}$.

Clearly $f(A) = B$ for a bijective function.

Note :

- It is necessary that every f-image is in B; but there may be some elements in B which are not the f-images of any element of A *i.e.*, whose pre-image under f is not in A.
- Two or more elements of A may have same image in B.
- $f : x \to y$ means that under the function f from A to B, an element x of A has image y in B.
- Usually we denote the function f by writing $y = f(x)$ and read it as 'y **is a function of** x'.

Know the Facts

1. **(i)** A relation R from A to B is an empty relation or void relation if $R = \phi$
   (ii) A relation R on a set A is an empty relation or void relation if $R = \phi$
2. **(i)** A relation R from A to B is a universal relation if $R = A \times B$.
   (ii) A relation R on a set A is an universal relation if $R = A \times A$.
3. A relation R on a set A is reflexive if $aRa, \forall a \in A$.
4. A relation R on a set A is symmetric if whenever aRb, then bRa for all $a, b \in A$.
5. A relation R on a set A is transitive if whenever aRb and bRc then aRc for all $a, b, c \in A$.
6. A relation R on A is identity relation if $R = \{(a, a) \forall a \in A\}$ *i.e.*, R contains only elements of the type $(a, a) \forall a \in A$ and it contains no other element.
7. A relation R on a non-empty set A is an equivalence relation if the following conditions are satisfied :
   (i) R **is reflexive** *i.e.*, for every $a \in A$, $(a, a) \in R$ *i.e.*, aRa.
   (ii) R **is symmetric** *i.e.*, for $a, b \in A$, $aRb \Rightarrow bRa$ *i.e.*, $(a, b) \in R \Rightarrow (b, a) \in R$.
   (iii) R **is transitive** *i.e.*, for all $a, b, c \in A$, we have, aRb and $bRc \Rightarrow aRc$ *i.e.*, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$.

TYPES OF INTERVALS

(i) Open Intervals : If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying strictly between a and b is called an open interval. It is denoted by $]a, b[$ or (a, b) *i.e.*, $\{x \in R : a < x < b\}$.

(ii) Closed Intervals : If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying between a and b such that it includes both a and b as well is known as a closed interval. It is denoted by $[a, b]$ *i.e.*, $\{x \in R : a \le x \le b\}$.

(iii) Open Closed Interval : If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying between a and b such that it excludes a and includes only b is known as an open closed interval. It is denoted by $]a, b]$ or $(a, b]$ *i.e.*, $\{x \in R : a < x \le b\}$.

(iv) Closed Open Interval : If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying between a and b such that it includes only a and excludes b is known as a closed open interval. It is denoted by $[a, b[$ or $[a, b)$ *i.e.*, $\{x \in R : a \le x < b\}$.

❑❑

CHAPTER-2

INVERSE TRIGONOMETRIC FUNCTIONS

Revision Notes

As we have learnt in class XI, the domain and range of trigonometric functions are given below :

S. No.	Function	Domain	Range
(i)	sine	R	$[-1, 1]$
(ii)	cosine	R	$[-1, 1]$
(iii)	tangent	$R-\left\{x : x=(2n+1)\frac{\pi}{2} ; n \in Z\right\}$	R
(iv)	cosecant	$R-\{x : x = n\pi, n \in Z\}$	$R-(-1, 1)$
(v)	secant	$R-\left\{x : x=(2n+1)\frac{\pi}{2} ; n \in Z\right\}$	$R-(-1, 1)$
(vi)	cotangent	$R-\{x : x = n\pi, n \in Z\}$	R

1. **Inverse function**

We know that if $f : X \to Y$ such that $y = f(x)$ is one-one and onto, then we define another function $g : Y \to X$ such that $x = g(y)$, where $x \in X$ and $y \in Y$, which is also one-one and onto.

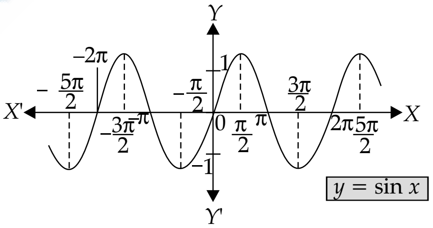
In such a case, Domain of g = Range of f

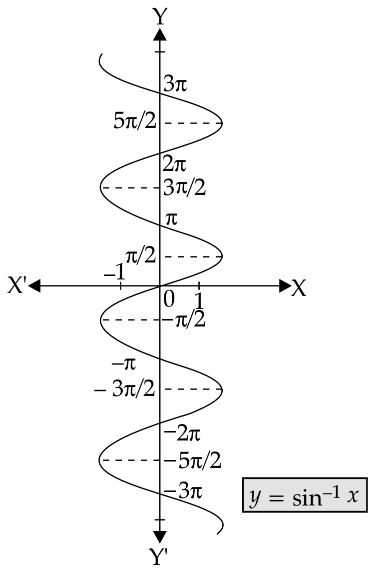
and Range of g = Domain of f

g is called the inverse of f $g = f^{-1}$

or Inverse of $g = g^{-1} = (f^{-1})^{-1} = f$

The graph of sine function is shown here :

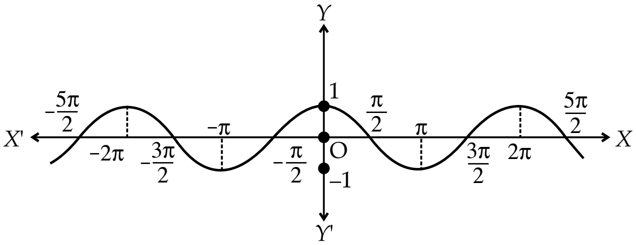




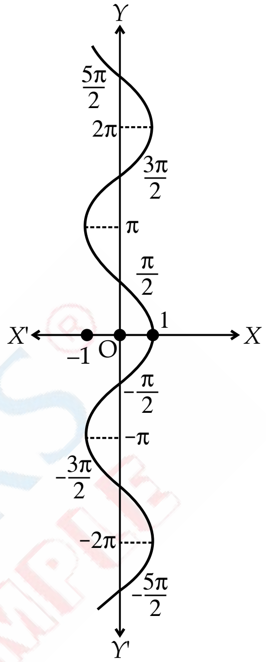
Principal value of branch function $\sin^{-1}$: It is a function with domain $[-1, 1]$ and range $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right], \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and so on corresponding to each interval, we get a branch of the function $\sin^{-1} x$. The branch with range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is called the principal value branch. Thus, $\sin^{-1} : [-1, 1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Principal value branch of function $\cos^{-1}$: The graph of the function $\cos^{-1}$ is as shown in figure. Domain of the function $\cos^{-1}$ is $[-1, 1]$. Its range in one of the intervals $(-\pi, 0)$, $(0, \pi)$, $(\pi, 2\pi)$, etc. is one-one and onto with the range $[-1, 1]$. The branch with range $(0, \pi)$ is called the principal value branch of the function $\cos^{-1}$.

Thus, $\cos^{-1} : [-1, 1] \to [0, \pi]$



$y = \cos x$

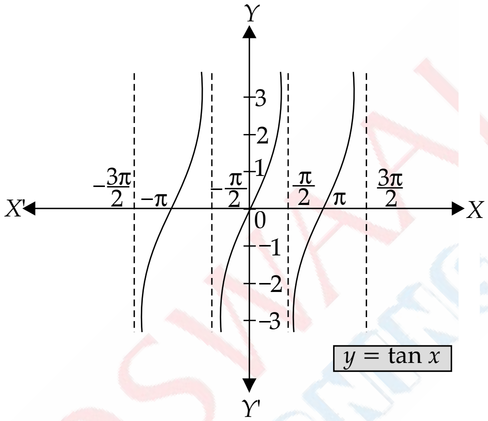


$y = \cos^{-1} x$

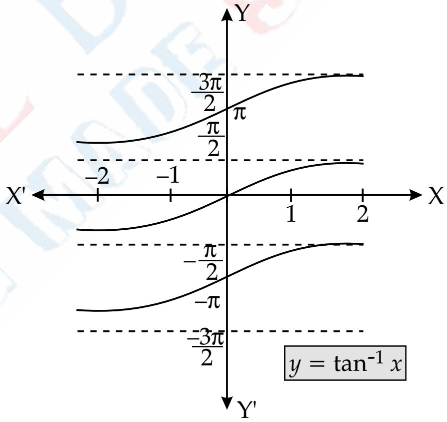
Principal value branch function $\tan^{-1}$: The function $\tan^{-1}$ is defined whose domain is set of real numbers and range is one of the intervals,

$$\left(\frac{-3\pi}{2}, \frac{\pi}{2}\right), \left(\frac{-\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots\ldots$$

Graph of the function is as shown in the figure :



$y = \tan x$

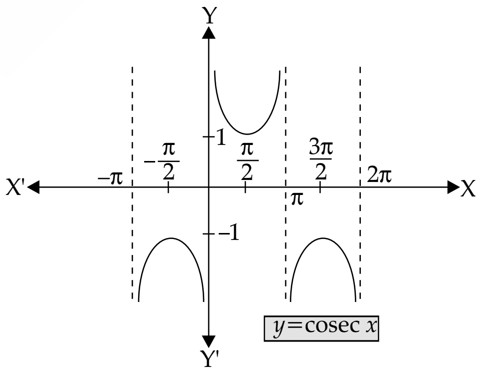


$y = \tan^{-1} x$

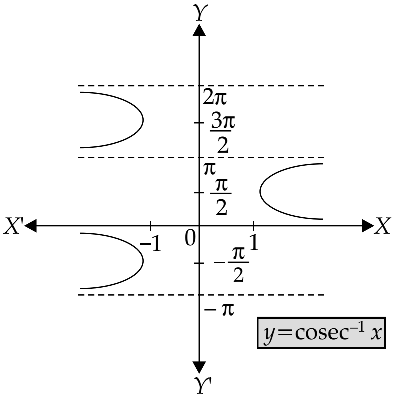
The branch with range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is called the principal value branch of function $\tan^{-1}$. Thus, $\tan^{-1} : R \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Principal value branch of function cosec^{-1} : The graph of function cosec^{-1} is shown in the figure. The cosec^{-1} is defined on a function whose domain is $R - (-1, 1)$ and the range is any one of the interval,

$$\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] - \{\pi\}, \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}, \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}, \ldots$$



$y = \text{cosec}\, x$



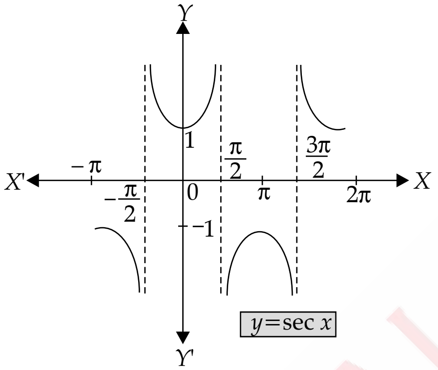
$y = \text{cosec}^{-1} x$

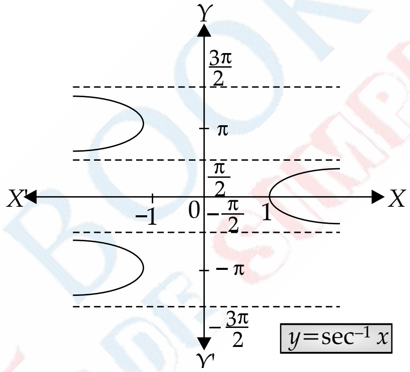
The function corresponding to the range $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\}$ is called the principal value branch of cosec^{-1}.

Thus, $\text{cosec}^{-1}: R-(-1,1)\rightarrow\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$.

Principal value branch of function $\sec^{-1}$: The graph of function $\sec^{-1}$ is shown in figure. The $\sec^{-1}$ is defined as a function whose domain R – (– 1, 1) and range is $[-\pi, 0]-\left[\frac{-\pi}{2}\right]$, $[0, \pi]-\left\{\frac{\pi}{2}\right\}$, $[\pi, 2\pi]-\left\{\frac{3\pi}{2}\right\}$, etc. Function corresponding to range $[0, \pi]-\left\{\frac{\pi}{2}\right\}$ is known as the principal value branch of $\sec^{-1}$.

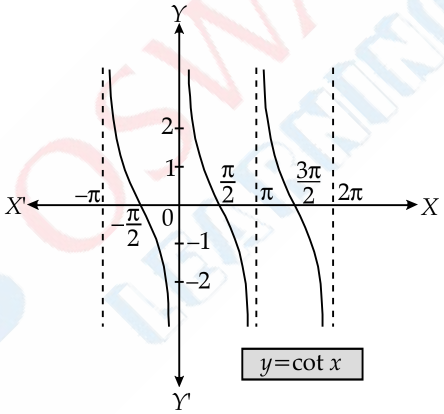
Thus, $\sec^{-1}: R-(-1,1)\rightarrow[0,\pi]-\left\{\frac{\pi}{2}\right\}$

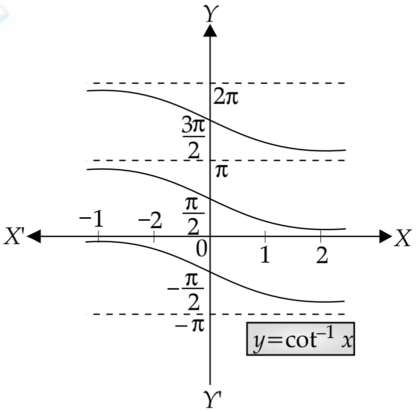




The principal value branch of function $\cot^{-1}$:

The graph of function $\cot^{-1}$ is shown below :





The $\cot^{-1}$ function is defined on function whose domain is R and the range is any of the intervals, $(-\pi, 0)$, $(0, \pi)$, $(\pi, 2\pi)$, ...

The function corresponding to $(0, \pi)$ is called the principal value branch of the function $\cot^{-1}$.

Then, $\cot^{-1}: R \rightarrow (0, \pi)$

The principal value branch of trigonometric inverse functions is as follows :

Inverse Function	Domain	Principal Value Branch
$\sin^{-1}$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$
cosec^{-1}	$R - (-1, 1)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1}$	$R - (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\tan^{-1}$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}$	R	$(0, \pi)$

(3) Principal Value :

Numerically smallest angle is known as the principal value.

Finding the principal value : For finding the principal value, following algorithm can followed :

STEP 1 : First draw a trigonometric circle and mark the quadrant in which the angle may lie.

STEP 2 : Select anti-clockwise direction for 1st and 2nd quadrants and clockwise direction for 3rd and 4th quadrants.

STEP 3 : Find the angles in the first rotation.

STEP 4 : Select the numerically least (magnitude wise) angle among these two values. The angle thus found will be the principal value.

STEP 5 : In case, two angles one with positive sign and the other with the negative sign qualify for the numerically least angle then, it is the convention to select the angle with positive sign as principal value.

The principal value is never numerically greater than π.

(4) To simplify inverse trigonometric expressions, following substitutions can be considered :

Expression	Substitution
$a^2 + x^2$ or $\sqrt{a^2+x^2}$	$x = a\tan\theta$ or $x = a\cot\theta$
$a^2 - x^2$ or $\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$x^2 - a^2$ or $\sqrt{x^2-a^2}$	$x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ or $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$	$x^2 = a^2\cos 2\theta$
$\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$	$x = a\sin^2\theta$ or $x = a\cos^2\theta$
$\sqrt{\frac{x}{a+x}}$ or $\sqrt{\frac{a+x}{x}}$	$x = a\tan^2\theta$ or $x = a\cot^2\theta$

Note the following and keep them in mind :

- The symbol $\sin^{-1} x$ is used to denote the **smallest angle** whether positive or negative, such that the sine of this angle will give us x. Similarly $\cos^{-1} x$, $\tan^{-1} x$, $\text{cosec}^{-1} x$, $\sec^{-1} x$ and $\cot^{-1} x$ are defined.
- You should note that $\sin^{-1} x$ can be written as **arcsin** x. Similarly, other Inverse Trigonometric Functions can also be written as arccos x, arctan x, arcsec x etc.
- Also note that $\sin^{-1} x$ (and similarly other Inverse Trigonometric Functions) **is entirely different from** $(\sin x)^{-1}$. In fact, $\sin^{-1} x$ is the measure of an angle in Radians whose sine is x whereas $(\sin x)^{-1}$ is $\frac{1}{\sin x}$ (which is obvious as per the laws of exponents).
- Keep in mind that these inverse trigonometric relations are **true only in their domains** *i.e.*, they are valid only for some values of 'x' for which inverse trigonometric functions are well defined.

Know the Formula

TRIGONOMETRIC FORMULAE (ONLY FOR REFERENCE) :

- **Relation between trigonometric ratios**

(a) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ **(b)** $\tan\theta = \frac{1}{\cot\theta}$ **(c)** $\cot\theta = \frac{\cos\theta}{\sin\theta}$

(d) $\text{cosec}\,\theta = \frac{1}{\sin\theta}$ **(e)** $\sec\theta = \frac{1}{\cos\theta}$

- **Trigonometric Identities**

(a) $\sin^2\theta + \cos^2\theta = 1$ **(b)** $\sec^2\theta = 1 + \tan^2\theta$ **(c)** $\text{cosec}^2\theta = 1 + \cot^2\theta$

- **Addition/subtraction/ formulae & some related results**

(a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ **(b)** $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(c) $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(d) $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(e) $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ **(f)** $\cot(A \pm B) = \frac{\cot B \cot A \mp 1}{\cot B \pm \cot A}$

- **Multiple angle formulae involving A, $2A$ & $3A$**

(a) $\sin 2A = 2 \sin A \cos A$ **(b)** $\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$

(c) $\cos 2A = \cos^2 A - \sin^2 A$ **(d)** $\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$

(e) $\cos 2A = 2\cos^2 A - 1$ **(f)** $2\cos^2 A = 1 + \cos 2A$

(g) $\cos 2A = 1 - 2\sin^2 A$ **(h)** $2\sin^2 A = 1 - \cos 2A$

(i) $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$ **(j)** $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

(k) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$ **(l)** $\sin 3A = 3\sin A - 4\sin^3 A$

(m) $\cos 3A = 4\cos^3 A - 3\cos A$ **(n)** $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

- **Transformation of sums/differences into products & vice-versa :**

(a) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$ **(b)** $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(c) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$ **(d)** $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

(e) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$

(f) $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$

(g) $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$

(h) $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$

➢ **Relations in different measures of Angle**

(a) Angle in Radian Measure = (Angle in degree measure) $\times \frac{\pi}{180^\circ}$rad

(b) Angle in Degree Measure = (Angle in radian measure) $\times \frac{180^\circ}{\pi}$

(c) θ (in radian measure) $= \frac{l}{r} = \frac{\text{arc}}{\text{radius}}$

Also following are of importance as well :

(a) 1 right angle = 90°

(b) 1° = 60′, 1′ = 60″

(c) $1^\circ = \frac{\pi}{180^\circ} = 0{\cdot}01745$ radians (Approx.)

(d) 1 radian = 57°17′45″ or 206265 seconds.

➢ **General Solutions :**

(a) $\sin x = \sin y \Rightarrow x = n\pi + (-1)^n y$, where $n \in Z$.

(b) $\cos x = \cos y \Rightarrow x = 2n\pi \pm y$, where $n \in Z$.

(c) $\tan x = \tan y \Rightarrow x = n\pi + y$, where $n \in Z$.

➢ **Relation in Degree & Radian Measures :**

Angles in Degree	0°	30°	45°	60°	90°	180°	270°	360°
Angles in Radian	0°	$\left(\frac{\pi}{6}\right)$	$\left(\frac{\pi}{4}\right)$	$\left(\frac{\pi}{3}\right)$	$\left(\frac{\pi}{2}\right)$	(π)	$\left(\frac{3\pi}{2}\right)$	(2π)

➢ **Trigonometric Ratio of Standard Angles :**

Degree	0°	30°	45°	60°	90°
sin x	**0**	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	**1**
cos x	**1**	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	**0**
tan x	**0**	$\frac{1}{\sqrt{3}}$	**1**	$\sqrt{3}$	∞
cot x	∞	$\sqrt{3}$	**1**	$\frac{1}{\sqrt{3}}$	**0**
cosec x	∞	**2**	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	**1**
sec x	**1**	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	**2**	∞

➢ **Trigonometric Ratios of Allied Angles :**

Angles (→) / T – Ratios (↓)	$\frac{\pi}{2} - \theta$	$\frac{\pi}{2} + \theta$	$\pi - \theta$	$\pi + \theta$	$\frac{3\pi}{2} - \theta$	$\frac{3\pi}{2} + \theta$	$2\pi - \theta$ or $-\theta$	$2\pi + \theta$
sin	**cos θ**	**cos θ**	**sin θ**	**– sin θ**	**– cos θ**	**– cos θ**	**– sin θ**	**sin θ**
cos	**sin θ**	**– sin θ**	**– cos θ**	**– cos θ**	**– sin θ**	**sin θ**	**cos θ**	**cos θ**

tan	**cot θ**	**– cot θ**	**– tan θ**	**tan θ**	**cot θ**	**– cot θ**	**– tan θ**	**tan θ**
cot	**tan θ**	**– tan θ**	**– cot θ**	**cot θ**	**tan θ**	**– tan θ**	**– cot θ**	**cot θ**
sec	**cosec θ**	**– cosec θ**	**– sec θ**	**– sec θ**	**– cosec θ**	**cosec θ**	**sec θ**	**sec θ**
cosec	**sec θ**	**sec θ**	**cosec θ**	**– cosec θ**	**– sec θ**	**– sec θ**	**– cosec θ**	**cosec θ**

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UNIT – II : ALGEBRA

CHAPTER-3
MATRICES

Revision Notes

Matrices and operations

1. **MATRIX - BASIC INTRODUCTION :**

A matrix is an ordered rectangular array of numbers (real or complex) or functions which are known as **elements** or the **entries** of the matrix. It is denoted by the **uppercase letters** *i.e. A, B, C* etc.

Consider a matrix A given as,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Here in matrix A the horizontal lines of elements are said to constitute **rows** and vertical lines of elements are said to constitute **columns** of the matrix. Thus, matrix A has m **rows** and n **columns**. The array is enclosed by square brackets [ ], the parentheses ( ) or the double vertical bars || ||.

- A matrix having m rows and n columns is called a matrix of order $m\times n$ (read as 'm by n' matrix). A matrix A of order $m \times n$ is depicted as $A = [a_{ij}]_{m \times n}$; $i, j \in N$.
- Also in general, a_{ij} means an element lying in the i^{th} row and j^{th} column.
- Number of elements in the matrix $A = [a_{ij}]_{m \times n}$ is given as mn.

2. **TYPES OF MATRICES :**

(i) Column matrix : A matrix having only one column is called a **column matrix** or **column vector**.

$e.g: \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}_{3\times1}, \begin{bmatrix} 4 \\ 5 \end{bmatrix}_{2\times1}$

General notation : $A = [a_{ij}]_{m\times1}$

(ii) Row matrix : A matrix having only one row is called a **row matrix** or **row vector**.

$e.g: \begin{bmatrix} 2 & 5 & -4 \end{bmatrix}_{1\times3}, \begin{bmatrix} \sqrt{2} & 4 \end{bmatrix}_{1\times2}$

General notation : $A = [a_{ij}]_{1\times n}$

(iii) Square matrix : It is a matrix in which the number of rows is equal to the number of columns *i.e.,* an $n\times n$ matrix is said to constitute a square matrix of order $n\times n$ and is known as a **square matrix of order 'n'.**

$e.g: \begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & -4 \\ 0 & -1 & -2 \end{bmatrix}_{3\times3}$ is a square matrix of order 3.

General notation : $A = [a_{ij}]_{n\times n}$

(iv) Diagonal matrix : A square matrix $A = [a_{ij}]_{m\times m}$ is said to be a **diagonal matrix** if all the elements, except those in the leading diagonal are zero *i.e.*, $a_{ij} = 0$, for all $i \neq j$.

$e.g: \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times3}$ is a diagonal matrix of order 3.

- Also there are **more notations** specifically used for the diagonal matrices. For instance, consider the matrix given above, it can also be written as diag (2, 5, 4) or diag [2, 5, 4].
- Note that the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{mm}$ of a square matrix $A = [a_{ij}]_{m\times m}$ of **order** μ are said to constitute the **principal diagonal** or simply **the diagonal of the square matrix** A. These elements are known as **diagonal elements of matrix** A.

(v) Scalar matrix : A diagonal matrix $A = [a_{ij}]_{m\times m}$ is said to be a **scalar matrix** if its diagonal elements are equal.

$$i.e., a_{ij} = \begin{cases} 0, & \text{when } i \neq j \\ k, & \text{when } i = j \text{ for some constant } k \end{cases}$$

$e.g: \begin{bmatrix} 17 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 17 \end{bmatrix}_{3\times3}$ is a scalar matrix of order 3.

(vi) Unit or Identity matrix : A square matrix $A = [a_{ij}]_{m\times m}$ is said to be an **identity matrix** if $a_{ij} = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j \end{cases}$

A **unit matrix** can also be defined as the **scalar matrix** in which all diagonal elements are equal to **unity**. We denote the identity matrix of order m by I_m or I.

$$e.g: I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}.$$

(vii) Zero matrix or Null matrix : A matrix is said to be a **zero matrix** or **null matrix** if each of its elements is '0'.

$$e.g: \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}, [0 \ \ 0]_{1\times2}.$$

(viii) Horizontal matrix : A $m\times n$ matrix is said to be a **horizontal matrix** if $m < n$.

$$e.g: \begin{bmatrix} 1 & 2 & 5 \\ 4 & 8 & -9 \end{bmatrix}_{2\times3}$$

(ix) Vertical matrix : A $m\times n$ matrix is said to be a **vertical matrix** if $m > n$.

$$e.g: \begin{bmatrix} -5 & -1 \\ 8 & -9 \\ 4 & 0 \end{bmatrix}_{3\times2}$$

(x) Triangular matrix :

(a) Lower triangular matrix : A square matrix is called a **lower triangular matrix** if $a_{ij} = 0$ when $i < j$.

$$e.g: \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 5 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 2 & 3 & 5 \end{bmatrix}$$

(b) Upper triangular matrix : A square matrix is called an **upper triangular matrix** if $a_{ij} = 0$ when $i > j$.

$$e.g: \begin{bmatrix} 1 & -8 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

3. EQUALITY OF MATRICES :

Two matrices A and B are said to be equal and written as $A = B$, if they are of the **same order** and their **corresponding elements are identical** *i.e.* $a_{ij} = b_{ij}$ *i.e.*, $a_{11} = b_{11}$, $a_{22} = b_{22}$, $a_{32} = b_{32}$ etc.

4. ADDITION OF MATRICES :

If A and B are two $m \times n$ matrices, then another $m \times n$ matrix obtained by adding the corresponding elements of the matrices A and B is called the sum of the matrices A and B and is denoted by '$A + B$'.

Thus if $A = [a_{ij}]$, $B = [b_{ij}]$, ορ $A + B = [a_{ij} + b_{ij}]$.

Properties of matrix addition :

- Commutative property : $A + B = B + A$
- Associative property : $A + (B + C) = (A + B) + C$
- Cancellation laws : (i) Left cancellation : $A + B = A + C \Rightarrow B = C$
  (ii) Right cancellation : $B + A = C + A \Rightarrow B = C$.

5. MULTIPLICATION OF A MATRIX BY A SCALAR :

If a $m \times n$ matrix A is multiplied by a scalar k (say), then the new kA matrix is obtained by multiplying each element of matrix A by scalar k. Thus, if $A = [a_{ij}]$ and it is multiplied by a scalar k, then $kA = [ka_{ij}]$, *i.e.* $A = [a_{ij}]$ ορ $kA = [ka_{ij}]$.

e.g : $A = \begin{bmatrix} 2 & -4 \\ 5 & 6 \end{bmatrix}$ ορ $3A = \begin{bmatrix} 6 & -12 \\ 15 & 18 \end{bmatrix}$

6. MULTIPLICATION OF TWO MATRICES :

Let $A = [a_{ij}]$ be a $m \times n$ matrix and $B = [b_{jk}]$ be a $n \times p$ matrix such that the number of columns in A is equal to the number of rows in B, then the $m \times p$ matrix $C = [c_{ik}]$ such that $[c_{ik}] = \Sigma_{j=1}^{n} a_{ij} b_{jk}$ is said to be the product of the matrices A and B in that order and it is denoted by AB *i.e.* "$C = AB$".

Properties of matrix multiplication :

- Note that the product AB is defined only when the number of columns in matrix A is equal to the number of rows in matrix B.
- If A and B are $m \times n$ and $n \times p$ matrices, respectively, then the matrix AB will be an $m \times p$ matrix *i.e.*, order of matrix AB will be $m \times p$.
- In the product AB, A is called the **pre-factor** and B is called the **post-factor**.
- If two matrices A and B are such that AB is possible then it is not necessary that the product BA is also possible.
- If A is a $m \times n$ matrix and both AB as well as BA are defined, then B will be a $n \times m$ matrix.
- If A is a $n \times n$ matrix and I_n be the unit matrix of order n, then $A I_n = I_n A = A$.
- Matrix multiplication is **associative** *i.e.*, $A(BC) = (AB)C$.
- Matrix multiplication is **distributive** over the **addition** *i.e.*, $A.(B + C) = AB + AC$.
- Matrix multiplication is not commutative.

7. IDEMPOTENT MATRIX :

A square matrix A is said to be an idempotent matrix if $A^2 = A$.

For example, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ are idempotent matrices.

8. TRANSPOSE OF A MATRIX :

If $A = [a_{ij}]_{m \times n}$ be a $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of matrix A is said to be a **transpose of matrix** A. The transpose of A is denoted by A' or A^T *i.e.*, if $A^T = [a_{ji}]_{n \times m}$.

For example, $$\begin{bmatrix} 5 & -4 & 1 \\ 0 & \sqrt{5} & 3 \end{bmatrix}^T = \begin{bmatrix} 5 & 0 \\ -4 & \sqrt{5} \\ 1 & 3 \end{bmatrix}$$

PROPERTIES OF TRANSPOSE OF MATRICES :

(i) $(A + B)^T = A^T + B^T$

(ii) $(A^T)^T = A$

(iii) $(kA)^T = kA^T$, where k is any constant

(iv) $(AB)^T = B^T A^T$

(v) $(ABC)^T = C^T B^T A^T$

Symmetric and skew symmetric matrices

Symmetric matrix : A square matrix $A = [a_{ij}]$ is said to be a **symmetric matrix** if $A^T = A$. *i.e.*, if $A = [a_{ij}]$, then $A^T = [a_{ji}] = [a_{ij}]$ or $A^T = A$.

For example : $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \begin{bmatrix} 2+i & 1 & 3 \\ 1 & 2 & 3+2i \\ 3 & 3+2i & 4 \end{bmatrix}$

Skew symmetric matrix : A square matrix $A = [a_{ij}]$ is said to be a **skew symmetric matrix** if $A^T = -[A]$ *i.e.*, if $A = [a_{ij}]$, then $A^T = [a_{ji}] = -[a_{ij}]$ op $A^T = -A$.

For example : $\begin{bmatrix} 0 & 1 & -5 \\ -1 & 0 & 5 \\ 5 & -5 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

Orthogonal matrix : A matrix A is said to be **orthogonal** if $A.A^T = I$, where A^T is transpose of A.

Know the Facts

- Note that $[a_{ji}] = -[a_{ij}]$ op $[a_{ii}] = -[a_{ii}]$ op $2[a_{ii}] = 0$ (Replacing j by i). *i.e.*, all the diagonal elements in a skew symmetric matrix are zero.
- For any matrices, AA^T and A^TA are symmetric matrices.
- For a square matrix A, the matrix $A + A^T$ is a symmetric matrix and $A - A^T$ is always a skew-symmetric matrix.
- Also note that any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix

  i.e., $A = P + Q$ where $P = \dfrac{A+A^T}{2}$ is a symmetric matrix

  and $Q = \dfrac{A-A^T}{2}$ is a skew symmetric matrix.

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CHAPTER-4
DETERMINANTS

Revision Notes

Determinants, Minors & Co-factors.

Determinants, Minors & Co-factors

(a) Determinant : A unique number (real or complex) can be associated to every square matrix $A = [a_{ij}]$ of order m. This number is called the determinant of the square matrix A, where $a_{ij} = (i, j)^{th}$ element of A.

For instance, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then, determinant of matrix A is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = det (A) and its value is given by $ad - bc$.

(b) Minors : Minors of an element a_{ij} of a determinant *(or a determinant corresponding to matrix A)* is the determinant obtained by deleting its i^{th} row and j^{th} column in which a_{ij} lies. Minor of a_{ij} is denoted by M_{ij}. Hence, we can get 9 minors corresponding to the 9 elements of a third order (*i.e.*, 3×3) determinant.

(c) Co-factors : Cofactor of an element a_{ij}, denoted by A_{ij}, is defined by $A_{ij} = (-1)^{(i+j)} M_{ij}$, where M_{ij} is minor of a_{ij}. Sometimes C_{ij} is used in place of A_{ij} to denote the co-factor of element a_{ij}.

1. ADJOINT OF A SQUARE MATRIX :

Let $A = [a_{ij}]$ be a square matrix. Also, assume $B = [A_{ij}]$, where A_{ij} is the cofactor of the elements a_{ij} in matrix A. Then the transpose B^T of matrix B is called the **adjoint of matrix** A and it is denoted by **"*adj* (*A*)".**

To find adjoint of a 2 × 2 matrix : Follow this, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ορ $adj\,A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

For example, consider a square matrix of order 3 as $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$, then in order to find the adjoint matrix A, we find a matrix B (formed by the co-factors of elements of matrix A as mentioned above in the definition)

i.e., $B = \begin{bmatrix} 15 & -2 & -6 \\ -10 & -1 & 4 \\ -1 & 2 & -1 \end{bmatrix}$. Hence, $adj\,A = B^T = \begin{bmatrix} 15 & -10 & -1 \\ -2 & -1 & 2 \\ -6 & 4 & -1 \end{bmatrix}$

2. SINGULAR MATRIX AND NON-SINGULAR MATRIX :

(a) Singular matrix : A square matrix A is said to be singular if $|A| = 0$ *i.e.*, its determinant is zero.

e.g.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 12 \\ 1 & 1 & 3 \end{bmatrix}$$

$$= 1(15 - 12) - 2(12 - 12) + 3(4 - 5) = 3 - 0 - 3 = 0$$

$\therefore$ A is singular matrix.

$$B = \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix} = 12 - 12 = 0$$

$\therefore$ B is singular matrix.

(b) Non-singular matrix : A square matrix A is said to be non-singular if $|A| \neq 0$.

e.g.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= 0\,(0 - 1) - 1(0 - 1) + 1(1 - 0)$$

$$= 0 + 1 + 1 = 2 \neq 0$$

$\therefore$ A is non-singular matrix.

- A square matrix A is **invertible** if and only if A is **non-singular**.

3. ALGORITHM TO FIND A^{-1} BY DETERMINANT METHOD :

STEP 1 : Find $|A|$.

STEP 2 : If $|A| = 0$, then, write "A is a singular matrix and hence not invertible". Else write "A is a non-singular matrix and hence invertible".

STEP 3 : Calculate the cofactors of elements of matrix A.

STEP 4 : Write the matrix of cofactors of elements of A and then obtain its transpose to get *adj.A* (*i.e.*, adjoint A).

STEP 5 : Find the inverse of A by using the relation $A^{-1} = \frac{1}{|A|}(adj\,A)$.

4. PROPERTIES ASSOCIATED WITH VARIOUS OPERATIONS OF MATRICES AND THE DETERMINANTS :

(a) $AB = I = BA$

(b) $AA^{-1} = I$ or $A^{-1}\,I = A^{-1}$

(c) $(AB)^{-1} = B^{-1}A^{-1}$

(d) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(e) $(A^{-1})^{-1} = A$

(f) $(A^T)^{-1} = (A^{-1})^T$

(g) $A(adjA) = (adj\,A)A = |A|\,I$

(h) $adj(AB) = adj(B)\,adj\,(A)$

(i) $adj(A^T) = (adj\,A)^T$

(j) $(adj\,A)^{-1} = (adj\,A^{-1})$

(k) $|adj\,A| = |A|^{n-1}$, if $|A| \neq 0$, where n is of the order of A.

(l) $|AB| = |A|\,|B|$

(m) $|A\,adj\,A| = |A|^n$, where n is of the order of A.

(n) $|A^{-1}| = \frac{1}{|A|}$, if matrix A is invertible. **(o)** $|A| = |A^T|$

- $|kA| = k^n |A|$, where n is of the order of square matrix A and k is any scalar.
- If A is a non-singular matrix (*i.e.*, when $|A| \neq 0$) of order n, then $|adj\ A| = |A|^{n-1}$.
- If A is a non-singular matrix of order n, then $adj\ (adj\ A) = |A|^{n-2} A$.

5. AREA OF TRIANGLE :

Area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \text{ sq. units}$$

- Since area is a positive quantity, we take absolute value of the determinant.
- If the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear, then $\Delta = 0$.
- The equation of a line passing through the points (x_1, y_1) and (x_2, y_2) can be obtained by the expression given here :

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Solutions of System of Linear equations

SOLVING SYSTEM OF EQUATIONS BY MATRIX METHOD [INVERSE MATRIX METHOD]

(a) Homogeneous and Non-homogeneous system : A system of equations $AX = B$ is said to be a homogeneous system if $B = O$. Otherwise it is called a non-homogeneous system of equations.

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3$$

STEP 1 : Assume $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

STEP 2 : Find $|A|$. Now there may be following situations :

(i) $|A| \neq 0 \Rightarrow A^{-1}$ exists. It implies that the given system of equations is consistent and therefore, the system has **unique solution**. In that case, write

$$AX = B$$

$$\Rightarrow \quad X = A^{-1}B \quad \left[\text{where } A^{-1} = \frac{1}{|A|}(adj\ A)\right]$$

Then by using the definition of equality of matrices, we can get the values of x, y and z.

(ii) $|A| = 0 \Rightarrow A^{-1}$ does not exist. It implies that the given system of equations may be consistent or inconsistent. In order to check proceed as follow :

$\Rightarrow$ Find $(adj\ A)\ B$. Now, we may have either $(adj\ A)\ B \neq O$ or $(adj\ A)\ B = O$.

- If $(adj\ A)B = O$, then the given system may be consistent or inconsistent.

  To check, put $z = k$ in the given equations and proceed in the same manner in the new two variables system of equations assuming $d_i - c_ik$, $1 \leq i \leq 3$ as constant.
- And if $(adj\ A)\ B \neq O$, then the given system is inconsistent with no solutions.

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UNIT – III : CALCULUS

CHAPTER-5

CONTINUITY & DIFFERENTIABILITY

Revision Notes

Continuity

FORMULAE FOR LIMITS :

(a) $\lim_{x \to 0} \cos x = 1$

(b) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(c) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

(d) $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1$

(e) $\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1$

(f) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a, a > 0$

(g) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(h) $\lim_{x \to 0} \frac{\log_e(1+x)}{x} = 1$

(i) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

- For a function $f(x)$, $\lim_{x \to m} f(x)$ exists if $\lim_{x \to m^-} f(x) = \lim_{x \to m^+} f(x)$.
- A function $f(x)$ is continuous at a point $x = m$ if, $\lim_{x \to m^-} f(x) = \lim_{x \to m^+} f(x) = f(m)$, where $\lim_{x \to m^-} f(x)$ is **Left Hand Limit** of $f(x)$ at $x = m$ and $\lim_{x \to m^+} f(x)$ is **Right Hand Limit** of $f(x)$ at $x = m$. Also $f(m)$ is the value of function $f(x)$ at $x = m$.
- A function $f(x)$ is continuous at $x = m$ (say) if, $f(m) = \lim_{x \to m} f(x)$ *i.e.,* a function is continuous at a point in its domain if the **limit value of the function** at that point **equals** the value of the function at the same point.
- For a continuous function $f(x)$ at $x = m$, $\lim_{x \to m} f(x)$ can be directly obtained by evaluating $f(m)$.
- Indeterminate forms or meaningless forms :

  $\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 1^{\infty}, 0^0, \infty^0.$

Differentiability

Derivative of Some Standard Functions

(a) $\frac{d}{dx}(x^n) = nx^{n-1}$

(b) $\frac{d}{dx}(k) = 0$, where k is any constant

(c) $\frac{d}{dx}(a^x) = a^x \log_e a, a > 0$

(d) $\frac{d}{dx}(e^x) = e^x$

(e) $\frac{d}{dx}(\log_a x) = \frac{1}{x\log_e a} = \frac{1}{x}\log_a e$

(f) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(g) $\frac{d}{dx}(\sin x) = \cos x$

(h) $\frac{d}{dx}(\cos x) = -\sin x$

(i) $\frac{d}{dx}(\tan x) = \sec^2 x$

(j) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(k) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(l) $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

(m) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, x \in (-1, 1)$

(n) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, x \in (-1, 1)$

(o) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, x \in R$

(p) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, x \in R$

(q) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$, where $x \in (-\infty, -1) \cup (1, \infty)$

(r) $\frac{d}{dx}(\text{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$, where $x \in (-\infty, -1) \cup (1, \infty)$

Following derivatives should also be memorized by you for quick use :

(i) $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$

(ii) $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

➲ **Left Hand Derivative of *f*(*x*) at *x* = *m*,**

$$Lf'(m) = \lim_{x \to m^-} \frac{f(x) - f(m)}{x - m} \text{ and,}$$

Right Hand Derivative of *f*(*x*) at *x* = *m*,

$$Rf'(m) = \lim_{x \to m^+} \frac{f(x) - f(m)}{x - m}$$

For a **function to be differentiable** at a point, **LHD and RHD at that point should be equal**.

➲ **Derivative of *y* w.r.t. *x* :** $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$.

Also, for very-very small value h, $f'(x) = \frac{f(x+h) - f(x)}{h}$, (as $h \to 0$)

Relation between Continuity and Differentiability :

(i) If a function is differentiable at a point, it is continuous at that point as well.

(ii) If a function is not differentiable at a point, it may or may not be continuous at that point.

(iii) If a function is continuous at a point, it may or may not differentiable at that point.

(iv) If a function is discontinuous at a point, it is not be differentiable at that point.

Rules of Derivatives :

➲ Product or Leibniz's rule of derivatives : $\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$

➲ Quotient Rule of derivatives : $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2} = \frac{vu' - uv'}{v^2}$.

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CHAPTER-6

APPLICATIONS OF DERIVATIVES

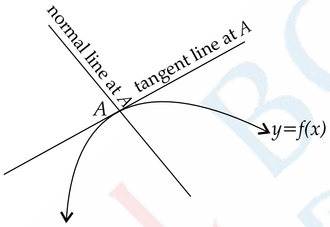
Revision Notes

Tangents and Normals

1. **Slope or gradient of a line :**

If a line makes an angle θ with the positive direction of X-axis in anti-clockwise direction, then tan θ is called the slope or gradient of the line. [Note that θ is taken as positive or negative accordingly as it is measured in anti-clockwise (*i.e.,* **from positive direction of X-axis to the positive direction of *Y*-axis)** or clockwise direction respectively.]

2. **Pictorial representation of tangent & normal :**



3. **Facts about the slope of a line :**

(a) If a line is parallel to x-axis (or perpendicular to y-axis), then its slope is 0 (zero).

(b) If a line is parallel to y-axis (or perpendicular to x-axis), then its slope is $\frac{1}{0}$ *i.e.,* not defined.

(c) It two lines are perpendicular, then product of their slopes equals – 1 *i.e.,* $m_1 \times m_2 = -1$. Whereas, for two parallel lines, their slopes are equal *i.e.,* $m_1 = m_2$. (Here in both the cases, m_1 and m_2 represent the slopes of respective lines).

4. **Equation of Tangent at (x_1, y_1) :**

$(y - y_1) = m_T(x - x_1)$, where m_T is the slope of tangent such that $m_T = \left[\frac{dy}{dx}\right]_{\text{at}(x_1, y_1)}$

5. **Equation of Normal at (x_1, y_1) :**

$(y - y_1) = m_N(x - x_1)$, where m_N is the slope of normal such that $m_N = \dfrac{-1}{\left[\frac{dy}{dx}\right]_{\text{at}(x_1, y_1)}}$

Note that $m_T \times m_N = -1$ which is obvious because tangent and normal are perpendicular to each other. In other words, the tangent and normal lines are inclined at right angle to each other.

6. **Acute angle between the two curves whose slopes m_1 and m_2 are known :**

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1.m_2}\right| \text{ ορ } \theta = \tan^{-1}\left|\frac{m_2 - m_1}{1 + m_1.m_2}\right|.$$

It is absolutely sufficient to find one angle **(generally the acute angle)** between the two curves. Other angle between the curve is given by $\pi - \theta$.

Note that if the curves cut **orthogonally** (*i.e.,* they cut each other at right angles), then it means $m_1 \times m_2 = -1$, where m_1 and m_2 represent the slopes of the tangents of curves at the intersection point.

7. **Finding the slope of a line $ax + by + c = 0$:**

STEP 1 : Express the given line in the standard slope-intercept form $y = mx + c$ *i.e.,* $y = \left(-\frac{a}{b}\right)x - \frac{c}{b}$.

STEP 2 : By comparing to the standard form $y = mx + c$, we can conclude $-\frac{a}{b}$ is the slope of given line $ax + by + c = 0$.

Increasing/Decreasing Functions

1. A function $f(x)$ is said to be an increasing function in $[a, b]$, if as x increases, $f(x)$ also increases *i.e.*, if $\alpha, \beta \in [a, b]$ and $\alpha > \beta, f(\alpha) > f(\beta)$.

   If $f'(x) \geq 0$ lies in (a, b), then $f(x)$ is an increasing function in $[a, b]$, provided $f(x)$ is continuous at $x = a$ and $x = b$.

2. A function $f(x)$ is said to be a **decreasing function** in $[a, b]$, if, as x increases, $f(x)$ decreases *i.e.*, if $\alpha, \beta \in [a, b]$ and $\alpha > \beta \Rightarrow f(\alpha) < f(\beta)$.

   If $f'(x) \leq 0$ lies in (a, b), then $f(x)$ is a decreasing function in $[a, b]$ provided $f(x)$ is continuous at $x = a$ and $x = b$.

➲ A function $f(x)$ is a **constant function** in $[a, b]$ if $f'(x) = 0$ for each $x \in (a, b)$.

➲ By **monotonic function** $f(x)$ in interval I, we mean that f is either **only increasing** in I or **only decreasing** in I.

3. **Finding the intervals of increasing and/or decreasing of a function :**

ALGORITHM

STEP 1 : Consider the function $y = f(x)$.

STEP 2 : Find $f'(x)$.

STEP 3 : Put $f'(x) = 0$ and solve to get the critical point(s).

STEP 4 : The value(s) of x for which $f'(x) > 0$, $f(x)$ is increasing; and the value(s) of x for which $f'(x) < 0$, $f(x)$ is decreasing.

Maxima and Minima

1. **Understanding maxima and minima :**

   Consider $y = f(x)$ be a well defined function on an interval I, then

   (a) f is said to have a **maximum value** in I, if there exists a point c in I such that $f(c) > f(x)$, for all $x \in I$.
   The value corresponding to $f(c)$ is called the maximum value of f in I and the point c is called the **point of maximum value of** f **in** I.

   (b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) < f(x)$, for all $x \in I$.
   The value corresponding to $f(c)$ is called the minimum value of f in I and the point c is called the **point of minimum value of** f **in** I.

   (c) f is said to have an **extreme value** in I, if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I.
   The value $f(c)$ in this case, is called an extreme value of f in I and the point c called an **extreme point**.

Know the Terms

1. Let f be a real valued function and also take a point c from its domain, then

   (i) c is called a point of **local maxima** if there exists a number $h > 0$ such that $f(c) > f(x)$, for all x in $(c - h, c + h)$.
   The value $f(c)$ is called the **local maximum value of** f.

   (ii) c is called a point of **local minima** if there exists a number $h > 0$ such that $f(c) < f(x)$, for all x in $(c - h, c + h)$.
   The value $f(c)$ is called the **local minimum value of** f.

2. **Critical points**

   It is a point c (say) in the domain of a function $f(x)$ at which either $f'(x)$ vanishes *i.e.*, $f'(c) = 0$ or f is not differentiable.

3. **First Derivative Test :**

   Consider $y = f(x)$ be a well defined function on an open interval I. Now proceed as have been mentioned in the following algorithm :

STEP 1 : Find $\frac{dy}{dx}$.

STEP 2 : Find the critical point(s) by putting $\frac{dy}{dx} = 0.$ Suppose $c \in I$ (where I is the interval) be any critical point point and f be continuous at this point c. Then we may have following situations :

- $\frac{dy}{dx}$ changes sign from **positive to negative** as x increases through c, then the function attains a **local maximum** at $x = c$.
- $\frac{dy}{dx}$ changes sign from **negative to positive** as x increases through c, then the function attains a **local minimum** at $x = c$.
- $\frac{dy}{dx}$ **does not change sign** as x increases through c, then $x = c$ is **neither** a point of **local maximum nor** a point of **local minimum**. Rather in this case, the point $x = c$ is called the **point of inflection**.

4. **Second Derivative Test :**

Consider $y = f(x)$ be a well defined function on an open interval I and twice differentiable at a point c in the interval. Then we observe that :

- $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$.
  The value $f(c)$ is called the local maximum value of f.
- $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$
  The value $f(c)$ is called the local minimum value of f.

This test fails if $f'(c) = 0$ and $f''(c) = 0$. In such a case, we use **first derivative test** as discussed above.

5. **Absolute maxima and absolute minima :**

If f is a continuous function on a **closed interval** I, then f has the absolute maximum value and f attains it atleast once in I. Also f has the absolute minimum value and the function attains it atleast once in I.

ALGORITHM

STEP 1 : Find all the critical points of f in the given interval, *i.e.*, find all the points x where either $f'(x) = 0$ or f is not differentiable.

STEP 2 : Take the end points of the given interval.

STEP 3 : At all these points (*i.e.*, the points found in STEP 1 and STEP 2) calculate the values of f.

STEP 4 : Identify the maximum and minimum value of f out of the values calculated in STEP 3. This maximum value will be the **absolute maximum value** of f and the minimum value will be the **absolute minimum value** of the function f.

Absolute maximum value is also called as **global maximum value** or **greatest value**. Similarly absolute minimum value is called as **global minimum value** or the **least value.**

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UNIT – V : LINEAR PROGRAMMING

CHAPTER-7

LINEAR PROGRAMMING

Revision Notes

Linear programming problems : Problems which minimize or maximize a linear function z subject to certain conditions determined by a set of linear inequalities with non-negative variables are known as linear programming problems.

Objective function : A linear function $z = ax + by$, where a and b are constants which has to be maximised or minimised according to a set of given conditions, is called as linear objective function.

Decision variables : In the objective function $z = ax + by$, the variables x, y are said to be decision variables.

Constraints : The restrictions in the form of inequalities on the variables of a linear programming problems are called constraints. The condition $x \geq 0$, $y \geq 0$ are known as non-negative restrictions.

Know the Terms

Feasible region : The common region determined by all the constraints including non-negative constraints $x, y \geq 0$ of linear programming problem is known as the feasible region.

Feasible solution : Points with in and on the boundary of the feasible region represents feasible solutions of constraints.

In the feasible region, there are infinitely many points (solutions) which satisfy the given conditions.

Theorem 1 : Let R be the feasible region for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where variables x and y are subject to constraints described by linear inequalities, the optimal value must occur at a corner point (vertex) of the feasible region.

Theorem 2 : Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function Z has both maximum and minimum values of R and each of these occurs at a corner point (vertex) of R.

However, if the feasible region is unbounded, the optimal value obtained may not be maximum or minimum.

Different types of linear programming problems are : A few important linear programming problems are as follows :

1. **Manufacturing problem :** In such problem, we determine :
   (i) Number of units of different products to be produced and sold.
   (ii) Men power required, machines hours needed, warehouse space available, etc. Objective function is to maximise profit.
2. **Diet problem :** Here we determine the amount of different types of constituent or nutrients which should be included in the diet.
   Objective function is to minimise the cost of production.
3. **Transportation problem :** These problems deal with the cost of transportation which is to be minimised under given constraints.

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