OSWAAL EXAM TOOLS

ON TIPS NOTES

MATHEMATICS (B-1)

Note Making is a skill that we use in many walks of life : at school, university and in the world of work. However, accurate note making requires a thorough understanding of concepts. We, at Oswaal, have tried to encapsulate all the chapters from the given syllabus into the following ON TIPS NOTES. These notes will not only facilitate better understanding of concepts, but will also ensure that each and every concept is taken up and every chapter is covered in total. So, go ahead and use it to your advantage.... go get the **OSWAAL ADVANTAGE**!!

CHAPTER 1 : Relations And Functions

➢ Fundamentals

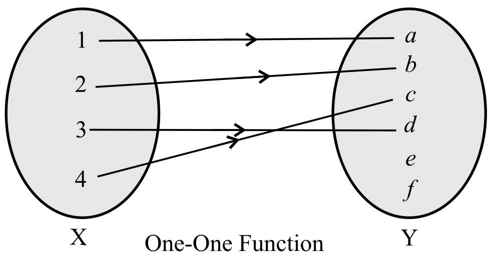
- A relation R from a non-empty set A to a non-empty set B is a subset of the Cartesian product $A \times B$.
- The set of all the first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R.
- The set of all second elements in a relation R from a set A to a set B is called the range of the relation R.
- The whole set B is called the co-domain of the relation R.

➢ Types of Relation

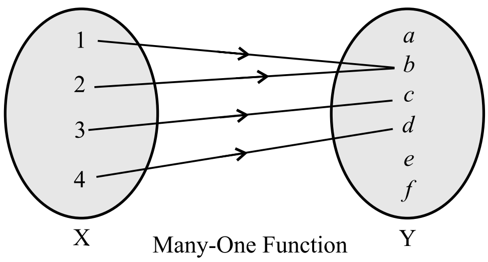
- **Empty Relation :** A relation R in a set A is called empty relation, if no element of A is related to any element of A, *i.e.*, $R = \phi \subset A \times A$.
- **Universal Relation :** A relation R in a set A is called universal relation, if each element of A is related to every element of A, *i.e.*, $R = A \times A$.
- A relation R in a set A is called
  (i) reflexive., if $(a_1, a_2) \in R$, for every $a \in A$.
  (ii) symmetric, if $(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.
  (iii) transitive, if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies that $(a_1, a_3) \in R$, for all $a_1, a_2, a_3 \in A$.
- **Equivalence Relation :** A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.

➢ Types of Function

- **One-One(Injective) and Many-One :** A function $f : X \to Y$ is defined to be one-one(or injective), if the images of distinct elements of X have distinct images in Y under f., *i.e.* for every $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Otherwise, f is called many-one.

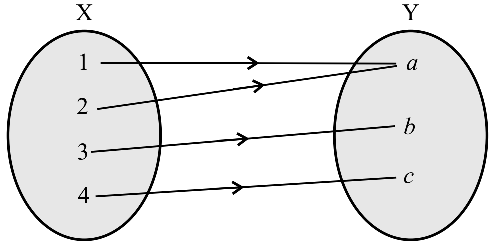


One-One Function



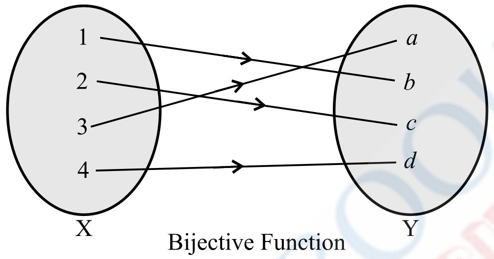
Many-One Function

- **Onto (Surjective) :** A function $f : X \to Y$ is said to be onto (or surjective), if every element of Y is the image of some element of X under f, *i.e.*, for every $y \in Y$, there exists an element x in X such that $f(x) = y$.



Onto Function

- **One-One and Onto(Bijective) :** *A* function $f : X \to Y$ is said to be one-one and onto (or bijective), if f is both one-one and onto.



Bijective Function

Composition of Functions

(i) Let $f : A \to B$ and $g : B \to C$ be two functions. Then, the composition of f and g, denoted by gof, is defined as the function $gof : A \to C$ given by

$$gof\,(x) = g(f\,(x)),\ \forall\ x \in A.$$

(ii) If $f : A \to B$ and $g : B \to C$ are one-one, then $gof : A \to C$ is also one-one.

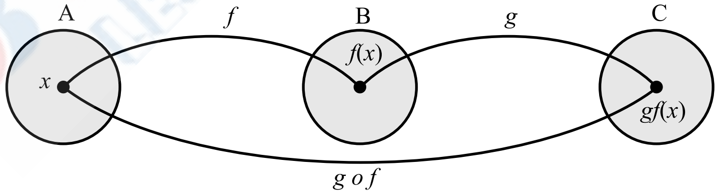
(iii) If $f : A \to B$ and $g : B \to C$ are onto, then $gof : A \to C$ is also onto.

However, converse of above stated results (ii) and (iii) need not be true. Moreover, we have the following results in this direction.

(iv) Let $f : A \to B$ and $g : B \to C$ be the given functions such that gof is one-one. Then, f is one-one.

(v) Let $f : A \to B$ and $g : B \to C$ be the given functions such that gof is onto. Then, g is onto.

$$gof\,(x) = g\,(f\,(x)),\ \forall\ x \in \text{A}.$$



Invertible Function

(i) A function $f : X \to Y$ is defined to be invertible, if there exists a function $g : Y \to X$ such that $gof = I_x$ and $fog = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}.

(ii) A function $f : X \to Y$ is invertible if and only if f is a bijective function.

(iii) If $f : X \to Y$, $g : Y \to Z$ and $h : Z \to S$ are functions, then $ho(gof) = (hog)of$.

(iv) Let $f : X \to Y$ and $g : Y \to Z$ be two invertible functions. Then gof is also invertible with $(gof)^{-1} = f^{-1}og^{-1}$.

Binary Operations

(i) A binary operation * on a set A is a function $* : A \times A \to A$. We denote $*(a, b)$ by $a * b$.

(ii) A binary operation * on the set X is called commutative, if $a * b = b * a$ for every $a, b \in X$.

(iii) A binary operation $* : A \times A \to A$ is said to be associative if $(a * b) * c = a * (b * c)$, for every $a, b, c \in A$.

(iv) Given a binary operation $* : A \times A \rightarrow A$, an element $e \in A$, if it exists, is called identity for the operation $*$, if $a * e = a = e * a, \forall a \in A$.

(v) Given a binary operation $* : A \times A \rightarrow A$, with the identity element e in A, an element $a \in A$, is said to be invertible with respect to the operation $*$, if there exists an element b in A such that $a * b = e = b * a$ and b is called the inverse of a and is denoted by a^{-1}.

CHAPTER 2 : Inverse Trigonometric Functions

➢ Inverse Trigonometric Functions

Functions	Domain	Range
$\sin^{-1}x$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, where $y = \sin^{-1}x$.
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$ $0 \le y \le \pi$, where $y = \cos^{-1} x$
$\text{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, where $y = \text{cosec}^{-1} x$
$\sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$ $0 \le y \le \pi$, where $y = \sec^{-1} x$
$\tan^{-1}x$	R	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, where $y = \tan^{-1} x$
$\cot^{-1}x$	R	$[0, \pi]$ $0 < y < \pi$, where $y = \cot^{-1} x$

Note : If no branch of an inverse trigonometric function is mentioned, then it means the principle value of branch of that function.

➢ Properties of Inverse Trigonometric Functions

1. (i) $\sin^{-1}\frac{1}{x} = \text{cosec}^{-1} x, x \ge 1 \text{ or } x \le -1$

(ii) $\cos^{-1}\frac{1}{x} = \sec^{-1} x, x \ge 1 \text{ or } x \le -1$

(iii) $\tan^{-1}\frac{1}{x} = \cot^{-1} x, x > 0$

2. (i) $\sin^{-1}(-x) = -\sin^{-1} x, x \in [-1, 1]$

(ii) $\tan^{-1}(-x) = -\tan^{-1} x, x \in R$

(iii) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x, |x| \ge 1$

3. (i) $\cos^{-1}(-x) = \pi - \cos^{-1} x, x \in [-1, 1]$

(ii) $\sec^{-1}(-x) = \pi - \sec^{-1} x, |x| \ge 1$

(iii) $\cot^{-1}(-x) = \pi - \cot^{-1} x, x \in R$

4. (i) $\sin^{-1} x + \cos^{-1}x = \frac{\pi}{2}, x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, x \in R$

(iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}, |x| \geq 1$

5. (i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}, xy < 1$

$$= \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ if } x > 0, y > 0, xy > 1.$$

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}, xy > -1$

(iii) $2\tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}, |x| < 1$

6. (i) $2 \tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2}, |x| \leq 1$

(ii) $2\tan^{-1} x = \cos^{-1} \frac{1-x^2}{1+x^2}, x \geq 0$

7. (i) $\sin^{-1}x = \tan^{-1} \frac{x}{\sqrt{1-x^2}}$

(ii) $\cos^{-1}x = \tan^{-1} \frac{\sqrt{1-x^2}}{x}$

CHAPTER 3 : Matrices

➢ Fundamentals

- A matrix is an ordered rectangular array of numbers (or functions).
- A matrix having m rows and n columns is called a matrix of order $m \times n$.
- A matrix is said to be a row matrix if it has only one row.

$$B = \begin{bmatrix} -\frac{1}{2} & \sqrt{5} & 2 & 3 \end{bmatrix}$$

- A matrix is said to be a column matrix if it has only one column.

$$A = \begin{bmatrix} 0 \\ \sqrt{3} \\ -1 \\ 1/2 \end{bmatrix}$$

- A matrix in which the number of rows are equal to the number of columns, is said to be a square matrix.

$$A = \begin{bmatrix} 3 & -1 & 0 \\ \frac{3}{2} & 3\sqrt{2} & 1 \\ 4 & 3 & -1 \end{bmatrix}$$

- A square matrix is said to be a diagonal matrix if it's all non-diagonal elements are zero.

$$A = [4], B = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- A diagonal matrix is said to be a scalar matrix if its diagonal elements are equal.

$$A = [3], B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, C = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

- A square matrix in which elements in the diagonal are all 1 and rest are all zeroes is called an identity matrix. Identity matrix is denoted by I.

$$[1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- A matrix is said to be a zero matrix or null matrix if all its elements are zeroes.

$$[0], \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [0 \quad 0]$$

- Two matrices can be added if they are of the same order.
- If A, B are respectively $m \times n$, $k \times l$ matrices, then both AB and BA are defined if and only if $n = k$ and $l = m$.

Transpose of a Matrix

If $A = [a_{ij}]$ be an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A.

$$\text{If } A = \begin{bmatrix} 3 & 5 \\ \sqrt{3} & 1 \\ 0 & \frac{-1}{5} \end{bmatrix}_{3\times 2}, \text{ then } A' = \begin{bmatrix} 3 & \sqrt{3} & 0 \\ 5 & 1 & \frac{-1}{5} \end{bmatrix}_{2\times 3}$$

Properties :

(i) $(A^T)^T = A$.

(ii) $(kA)^T = kA^T$ (where k is any constant)

(iii) $(A + B)^T = A^T + B^T$

(iv) $(AB)^T = B^T A^T$

Symmetric and Skew Symmetric Matrix

(i) A square matrix $A = [a_{ij}]$ is said to be symmetric if $A^T = A$, *i.e.*, $a_{ij} = a_{ji}$ for all possible values of i and j.

(ii) A square matrix $A = [a_{ij}]$ is said to be a skew symmetric matrix if $A^T = -A$, *i.e.* $a_{ij} = -a_{ji}$ for all possible values of i and j.

Theorem 1 : For any square matrix A with real number entries, $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew symmetric matrix.

Theorem 2 : Any square matrix A can be expressed as the sum of a symmetric matrix and a skew symmetric matrix, *i.e.*,

$$A = \frac{(A + A^T)}{2} + \frac{(A - A^T)}{2}$$

➢ Elementary Operations

(i) *The interchange of any two rows or two columns.* Symbolically, the interchange of i^{th} and j^{th} rows is denoted by $R_i \leftrightarrow R_j$ and interchange of i^{th} and j^{th} column is denoted by $C_i \leftrightarrow C_j$.

For example, applying $R_1 \leftrightarrow R_2$ to $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & \sqrt{3} & 1 \\ 5 & 6 & 7 \end{bmatrix}$, we get $\begin{bmatrix} -1 & \sqrt{3} & 1 \\ 1 & 2 & 1 \\ 5 & 6 & 7 \end{bmatrix}$

(ii) *The multiplication of the elements of any row or column by a non zero number*. Symbolically, the multiplication of each element of the i^{th} row by k, where $k \neq 0$ is denoted by $R_i \to kR_j$

The corresponding column operation is denoted by $C_i \to kC_i$

For example, applying $C_3 \to \frac{1}{7}C_3$ to $B = \begin{bmatrix} 1 & 2 & 1 \\ -1 & \sqrt{3} & 1 \end{bmatrix}$, we get $\begin{bmatrix} 1 & 2 & \frac{1}{7} \\ -1 & \sqrt{3} & \frac{1}{7} \end{bmatrix}$

(iii) *The addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non zero number.*

Symbolically, the addition to the elements of i^{th} row, the corresponding elements of j^{th} row multiplied by k is denoted by $R_i \to R_i + kR_j$.

CHAPTER 4 : Determinants

➢ Fundamentals

- Only square matrices have determinants.
- For a matrix A, $|A|$ is read as determinant of A and not as modulus of A.

➢ Determinant of a matrix of order one

Let $A = [a]$ be the matrix of order 1, then determinant of A is defined to be equal to a.

➢ Determinant of a matrix of order two

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

➢ Determinant of a matrix of order three

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22})$$

➢ Properties of Determinant

Property 1 : If the rows and columns of a determinant are interchanged, then value of determinant remains to be the same.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Property 2 : If any two rows or columns of a determinant are interchanged, then sign of determinant is changed.

$$\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property 3 : If any two rows (columns) of a determinant are identical, the value of the determinant is zero.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$$

Property 4 : If each element of a row or column of a determinant is multiplied by a constant k, then its value is k times the given determinant.

$$\begin{vmatrix} ka_1 & ka_2 & ka_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Property 5 : If the elements of a row and column of determinant are expressed as sum of two (or more terms), then the determinant can be expressed as sum of two (or more) determinants.

$$\begin{vmatrix} a_1+k_1 & a_2+k_2 & a_3+k_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} k_1 & k_2 & k_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

Property 6 : If the each element of any row or column of a determinant, the equimultiples of corresponding elements of any other row or column are added then the value of the determinant remains unchanged.

$$\begin{vmatrix} a_1+\lambda b_1 & a_2+\lambda b_2 & a_3+\lambda b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

➢ Area of Triangle

Area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

➢ Inverse of a Matrix

Step 1: Find out Minor for every element.

Minor of an element a_{ij} of the determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column, and it is denoted by M_{ij}.

e.g., Minor of a_{21} of $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$= M_{21} = \begin{vmatrix} - & a_{12} & a_{13} \\ - & - & - \\ - & a_{32} & a_{33} \end{vmatrix}$$

$$= -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Step 2 : Find out Co-factors of every Minor.

Co-factor of an element a_{ij} is given by $A_{ij} = (-1)^{i+j} M_{ij}$.

Step 3 : Find out Adjoint.

The adjoint of a square matrix is the transpose of matrix of co-factors.

Step 4 : Find out determinant of A *i.e.* $|A|$.

Step 5 : Find out inverse.

$$A^{-1} = \frac{1}{|A|} adj\ A$$

➢ How to Solve Linear System of Equations ?

Consider the system of equations

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

The given system of equations can be written as :

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \text{ or } AX = B$$

$\therefore \quad X = A^{-1}B$

CHAPTER 5 : Continuity and Differentiability

➢ Continuity (Definition) :

If f be a real valued function on a subset of real numbers and let c be a point in its domain, then f is a continuous function at c *i.e.*,

$$\lim_{x\to c^+} f(x) = f(c) = \lim_{x\to c^-} f(x)$$

Obviously, if left hand limit and right hand limit and value of the function at $x = c$ exists and are equal to each other then f is continuous at $x = c$.

➢ Algebra of continuous functions :

Let f and g be two real functions continuous at $x = c$, then

(i) Sum of two functions is continuous at $x = c$ *i.e.*, $(f + g)(x)$ is defined as $f(x) + g(x)$ is continuous at $x = c$.

(ii) Difference of two functions is continuous at $x = c$ *i.e.*, $(f - g)(x)$ is defined as $f(x) - g(x)$ is continuous at $x = c$.

(iii) Product of two functions is continuous at $x = c$ *i.e.*, $(fg)(x)$ is defined as $f(x).g(x)$ is continuous at $x = c$.

(iv) Quotient of two functions is continuous at $x = c$. (Provided it is defined at $x = c$)

i.e., $\left(\frac{f}{g}\right)(x)$ is defined as $\frac{f(x)}{g(x)}[g(x) \neq 0]$ is continuous at $x = c$.

However, if $f(x) = \lambda$, then

(a) $\lambda.g$ is defined by $\lambda.g(x)$, then $\lambda.g$ is also continuous at $x = c$.

(b) Similarly, if $\frac{\lambda}{g}$ is defined as $\frac{\lambda}{g}(x) = \frac{\lambda}{g(x)}$, then $\frac{\lambda}{g}$ is also continuous at $x = c$.

➢ Differentiability (Definition) :

Let f be a real function and c is a point in its domain. The derivative of f at c is defined as

$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$, provided limit exists

$$= f'(c)=\left[\frac{d}{dx}f(x)\right]_{x=c}$$

$f'(x)$ is defined as $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

wherever the limit exists is defined to be the derivative of f. It is denoted as :

$$f'(x) \text{ or } \frac{d}{dx}[f(x)]$$

or if $f(x) = y$, then $f'(x) = \frac{dy}{dx}$.

Every differentiable function is continuous.

➢ Algebra of derivatives :

Let u, v be the functions of x.

(i) $(u \pm v)' = u' \pm v'$

(ii) $(uv)' = u'v + uv'$ (Leibnitz or Product Rule)

(iii) $\left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}$, where $v \neq 0$. (Quotient Rule)

➢ Derivative of composite functions :

Let t be a real valued function which is a composite of two functions u and v i.e., $f = vou$, put $u(x) = t$, $f = v(t)$

$$\therefore \quad \frac{df}{dx}=\frac{df}{dt}\cdot\frac{dt}{dx}$$

➢ Chain rule :

Let f is a real valued function which is a composite function of u, v and w i.e., $f = (wov)ou$. Put $u(x) = t$, $v(t) = s$, $f = w(s)$, then

$$\frac{df}{dx}=\frac{df}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}$$

➢ Derivatives of inverse trigonometric functions :

Function	Domain	Derivative
$\sin^{-1}x$	[–1, 1]	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}x$	[–1, 1]	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1}x$	R	$\frac{1}{1+x^2}$

$\cot^{-1}x$	R	$-\dfrac{1}{1+x^2}$
$\sec^{-1}x$	$(-\infty, -1] \cup [1, \infty]$	$\dfrac{1}{x\sqrt{x^2-1}}$
$\operatorname{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty]$	$-\dfrac{1}{x\sqrt{x^2-1}}$

➢ Implicit functions :

An equation in the form $f(x, y) = 0$ in which y is not expressible in terms of x is called as an implicit function of x and y. Both sides of equation are differentiated term wise then from this equation, $\frac{dy}{dx}$ is obtained. It may be noted that when a function of y occurs, then differentiate it w.r.t. y and multiply it by $\frac{dy}{dx}$. *e.g.*, to find $\frac{dy}{dx}$ from $\cos^2 y + \sin xy = 1$, we differentiate it as :

$$2\cos y(-\sin y)\frac{dy}{dx} + \cos(xy).\frac{d}{dx}(xy) = 0$$

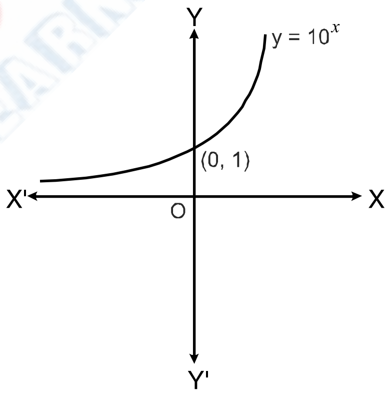
or $$-2\sin y\cos y\frac{dy}{dx} + \cos xy\left(x\frac{dy}{dx} + y\right) = 0$$

or $$\frac{dy}{dx} = \frac{y\cos xy}{2\sin y\cos y - x\cos(xy)}$$

➢ Exponential functions :

The exponential function with positive base $b > 1$, is the function $y = b^x$.

(i) The graph of $y = 10^x$ is shown in figure.



(ii) Domain = R.

(iii) Range = R^+. (all the real numbers)

(iv) The point (0, 1) always lies on the graph.

(v) It is an increasing function *i.e.*, as we move from left to right, the graph rises above.

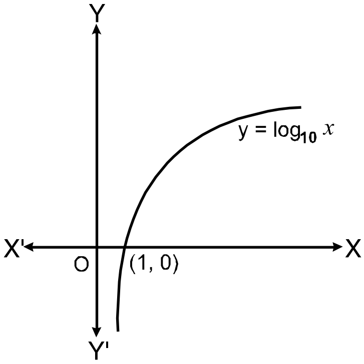
(vi) As $x \to -\infty$, $y \to 0$.

(vii) $\frac{d}{dx}(a^x) = a^x \log_e a$, $\frac{d}{dx}e^x = e^x$.

➢ Logarithmic functions :

Let $b > 1$ be real number, then $b^x = a$ may be written as $\log_b a = x$.

(i) The graph of $y = \log_{10} x$ is shown in the figure.



(ii) Domain $= R^+$, Range $= R$.

(iii) It is an increasing function.

(iv) As $x \to 0$, $y \to -\infty$.

(v) The function $y = e^x$ and $y = \log_e x$ are the mirror images of each other in the line $y = x$.

(vi) $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e, \frac{d}{dx}\log_e x = \frac{1}{x}$.

➢ Other properties of logarithm :

(i) $\log_b pq = \log_b p + \log_b q$

(ii) $\log_b \frac{p}{q} = \log_b p - \log_b q$

(iii) $\log_b p^x = x \log_b p$

(iv) $\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}$

➢ Logarithmic differentiation :

Whenever the functions are given in the form $y = [u(x)]^{v(x)}$

Using chain rule

$y = v(x) \log u(x)$ $\qquad$ $\frac{dy}{dx} = y\left[\frac{v(x)}{u(x)}.u'(x) + v'(x)\cdot\log u(x)\right]$

e.g.,

Let, $\qquad y = (\cos x)^{\sin x}$

Take log on both sides, simplify and differentiate

$$\log y = \log (\cos x)^{\sin x}$$

$$\log y = \sin x \log \cos x$$

$$\frac{1}{y}\frac{dy}{dx} = \cos x \log \cos x + \sin x \frac{-\sin x}{\cos x}$$

$$\therefore \quad \frac{dy}{dx} = (\cos x)^{\sin y}[\cos x \log \cos x - \sin x \tan x]$$

➢ Derivatives of functions of parametric form :

Let the given equations be

$$x = f(t), y = g(t),$$

where t is called the parameter.

$$\frac{dx}{dt} = f'(t), \frac{dy}{dt} = g'(t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

➢ Second order derivative :

Let, $y = f(x)$, then

$$\frac{dy}{dx} = f'(x)$$

If $f'(x)$ is differentiable, then it is again differentiated.

$$\text{L.H.S.} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D^2y = y'' = y_2$$

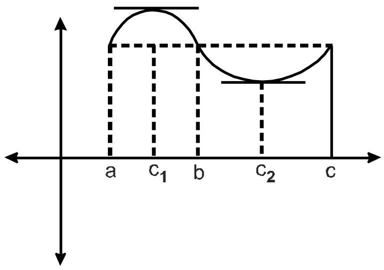
and $$\text{R.H.S.} = \frac{d}{dx}[f'(x)] = f''(x).$$

➢ Rolle's Theorem :

Let $f : [a, b] \to R$ be continuous function on closed interval $[a, b]$ and differentiable on open interval (a, b) such that $f(a) = f(b)$, where a, b are real numbers, then there exists some $c \in (a, b)$ such that $f'(c) = 0$.

From the figure, we observe that $f(a) = f(b)$. There exists a point $c_1 \in (a, b)$ such that $f'(c_1) = 0$ *i.e.,* tangent at c_1 is parallel to x-axis.

Similarly, $f(b) = f(c) \to f'(c_2) = 0$.



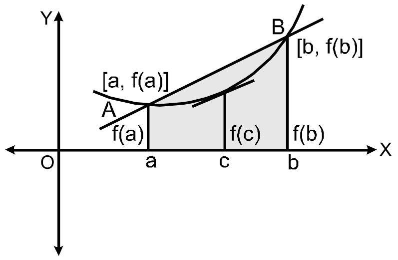
➢ Mean Value Theorem :

Let $f: [a, b] \to R$ be a continuous function on the closed interval $[a, b]$ and differentiable in the open interval (a, b). Then there exists some $c \in (a, b)$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Now we know $\frac{f(b)-f(a)}{b-a}$ is the slope of secant drawn between $A[a, f(a)]$ and $B[b, f(b)]$. We know the slope of the line joining (x_1, y_1) and (x_2, y_2) is $\frac{y_2-y_1}{x_2-x_1}$. The theorem states that there is a point $c \in (a, b)$, where $f'(c)$ is equal to the slope of AB.

In other words, there exists a point $c \in (a, b)$ such that the tangent at $x = c$ is parallel to AB.



CHAPTER 6 : Applications of Derivatives

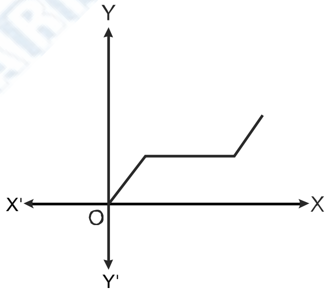
➢ Rate of change of quantities :

Let $y = f(x)$ be a function. If the change in one quantity y varies with another quantity x, then $\frac{dy}{dx} = f'(x)$ denotes the rate of change of y with respect to x. At $x = x_0$, $\left.\frac{dy}{dx}\right|_{x=x_0}$ of $f'(x)$ represents the rate of change of y w.r.t. x at $x = x_0$.

Let I be the open interval contained in the domain of real value function f.

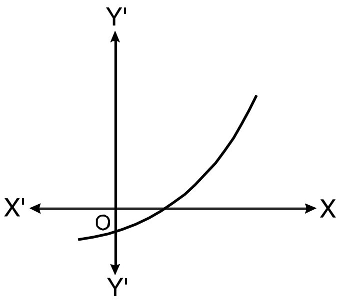
➢ Increasing function :

f is said to be increasing on I if $x_1 < x_2$ on I, then $f(x_1) \le f(x_2)$ for all $x_1, x_2 \in I$.



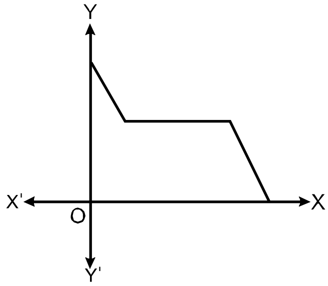
➢ Strictly increasing function :

f is said to be strictly increasing on I if $x_1 < x_2$ in $I \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$.



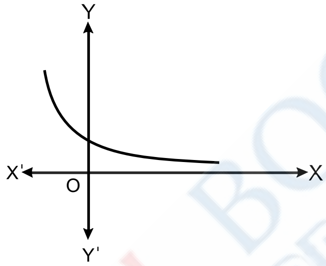
Decreasing function :

f is said to be decreasing function on I if $x_1 < x_2$ in I, then $f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in I$.



Strictly decreasing function :

f is said to be strictly decreasing function on I if $x_1 < x_2$ in $I \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$.



Increasing and decreasing function at x_0 :

Let x_0 be a point in the domain of definition of a real valued function f and there exists an open interval $I = (x_0 - h, x_0 + h)$ containing x_0 such that

(i) f is increasing at x_0, if $x_1 < x_2$ in $I \Rightarrow f(x_1) \leq f(x_2)$.

(ii) f is strictly increasing at x_0, if $x_1 < x_2$ in $I \Rightarrow f(x_1) < f(x_2)$.

(iii) f is decreasing at x_0, if $x_1 < x_2$ in $I \Rightarrow f(x_1) \geq f(x_2)$.

(iv) f is strictly decreasing at x_0, if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Test : Increasing/Decreasing/Constant Function :

Let f be a continuous function on $[a, b]$ and differentiable in an open interval (a, b), then

(i) f is increasing on $[a, b]$ if $f'(x) > 0$ for each $x \in (a, b)$.

(ii) f is decreasing on $[a, b]$ if $f'(x) < 0$ for each $x \in (a, b)$.

(iii) f is constant on $[a, b]$ if $f'(x) = 0$ for each $x \in (a, b)$.

Tangent to a curve :

Let $y = f(x)$ be the equation of curve. The equation of the tangent at (x_0, y_0) is

$$y - y_0 = m(x - x_0)$$

where, m = slope of the tangent

$$= \left[\frac{dy}{dx}\right]_{(x_0, y_0)} = f'(x_0)$$

Normal to the curve :

Let $y = f(x)$ be the equation of the curve. Equation of the normal is

$$y - y_0 = -\frac{1}{m}(x - x_0)$$

or $\quad x - x_0 + m(y - y_0) = 0$

where, $\quad m$ = slope of the tangent

$$= \left[\frac{dy}{dx}\right]_{(x_0, y_0)} = f'(x_0)$$

It may be noted that, slope of normal

$$= -\frac{1}{m} = \frac{-1}{f'(x_0)}$$

Particular case of tangent :

Let $m = \tan\theta$. If $\theta = 0$, $m = 0$, equation of tangent is $y - y_0 = 0$ *i.e.*, $y = y_0$.

If $\theta = \frac{\pi}{2}$, n is not defined.

$$\therefore \quad (x - x_0) = \frac{1}{m}(y - y_0)$$

When $\quad \theta = \frac{\pi}{2}, \cot\frac{\pi}{2} = 0$

$\therefore$ Equation of tangent is

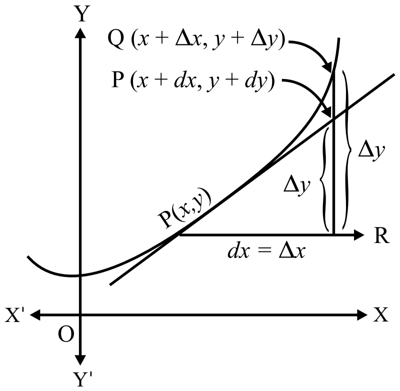
$$x - x_0 = 0 \text{ or } x = x_0.$$

Approximation :

Let $f : D \to R$, $D \subset R$ such that $y = f(x)$ and Δy is the increment in y corresponding to increment Δx in x, where

$$\Delta y = f(x + \Delta x) - f(x).$$

Now, (i) the differential of x denoted by dx is defined by $dx = \Delta x$.



(ii) The differential of y, denoted by dy defined by

$$dy = f'(x)\,dx$$

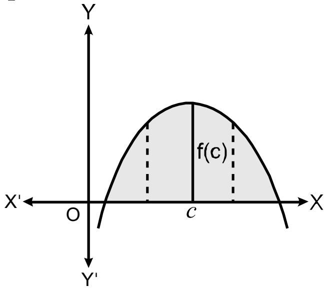
or

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$

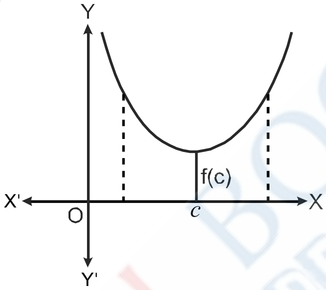
Maximum value, minimum value, extreme value :

Let f be a function defined in the interval I, then

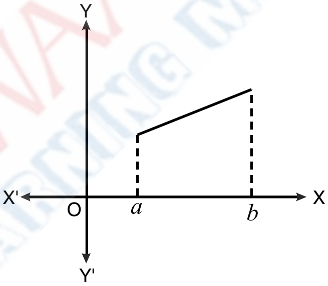
(i) Maximum value : If there exists a point c in I such that $f(c) \geq f(x)$, for all $x \in I$ then f is maximum in I. The point c is known as a point of maximum value in I.



(ii) Minimum value : If there exists a point c in I such that $f(c) \leq f(x)$, for all $x \in I$, then f is minimum in I. The point c is called as a point of minimum value in I.



(iii) Extreme value : If there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value in I, then f is having an extreme value in I. The point c is said to be an extreme point.



Local maxima and minima :

Let f be a real valued function and c be an interior point in the domain of f, then

(a) Local maxima : c is a point of local maxima if there is an $h > 0$, such that $f(c) \geq f(x)$ for all $x \in [c-h, c+h)$.

The value $f(c)$ is called local maximum value of f.

(b) Local Minima : c is a point of local minima if there is an $h > 0$, such that $f(c) \leq f(x)$ for all

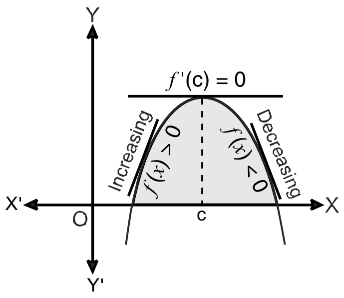
$$x \in (c-h, c+h).$$

The value of $f(c)$ is known as the local minimum value of f.

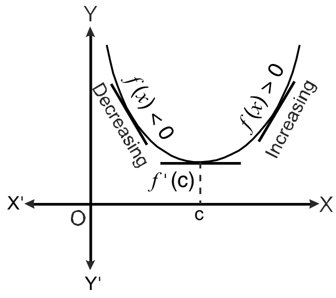
Geometrically :

(1) If $x = c$ is a point of local maxima of f, then

f is increasing $(f'(x) > 0)$ in the interval $(c-h, c)$ and decreasing $(f'(x) < 0)$ in the interval $(c, c+h)$. This implies $f'(c) = 0$.



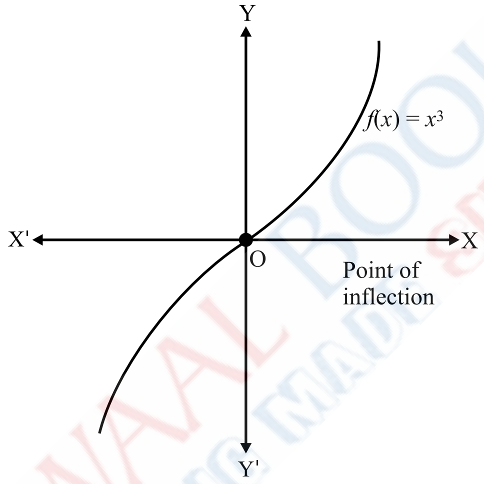
Maxima



Minima

Test of maxima and minima :

(1) Let f be a function on an open interval I and $c \in I$ be any point. If f has a local maxima or a local minima at $x = c$, then either $f'(c) = 0$ or f is not differentiable at c.



Let f be a continuous at a critical point c in it.

(2) If $f'(x)$ changes sign from positive to negative as x increases through c *i.e.,*

(i) $f'(x) > 0$ at every point in $(c - h, c)$

(ii) $f'(x) < 0$ at every point in $(c, c + h)$

where h is sufficiently small, there is a *point of local maxima.*

(3) If $f'(x)$ changes sign from negative to positive as x increases through c *i.e.,*

(i) $f'(x) < 0$ at every point in $(c - h, c)$

(ii) $f'(x) > 0$ at every point in $(c, c + h)$

where h is sufficiently small. Then, c is a *point of local minima.*

(4) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Such a point as called *point of inflection.*

Second derivative test of maxima and minima :

Let f be a function defined on an interval I and $c \in I$ and f be differentiable at c. Then,

(i) **Maxima :** $x = c$ is a local maxima if $f'(c) = 0$ and $f''(c) < 0$.

(ii) **Minima :** $x = 0$ is a local minima if $f'(c) = 0$ and $f''(c) > 0$.

Here, $f(c)$ is the local minimum value of f.

(iii) **Point of inflection :** If $f'(c) = 0$ and $f''(c) = 0$, test fails, then we apply first derivative test as x increases through c.

➢ Maximum and minimum values in a closed interval :

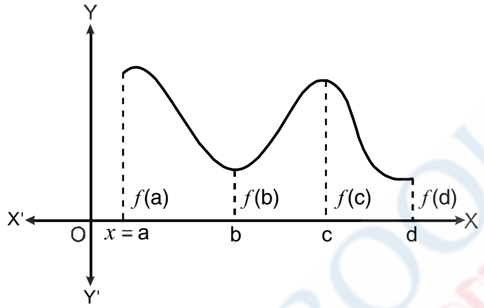
Consider the function $f(x) = x + 3, x \in [0, 1]$. Here, $f'(c) \neq 0$. It has neither maxima or minima. But $f(0) = 3$. This is the absolute minimum or global minimum or least value.

Further $f(1) = 4$. This is the absolute maximum or global maximum or greatest value.

Further consider f be any other continuous function having local maxima and minima.

➢ Absolute maxima and minima :

Let f be a continuous function on an interval $I = [a, b]$. Then, f has the absolute maximum value and attains atleast once in I. Similarly, f has the absolute minimum value and attains atleast once in I.



At $x = b$, there is a minima.

At $x = c$, there is a maxima.

At $x = a$, $f(a)$ is the greatest value or absolute maximum value.

At $x = d$, $f(d)$ is the least value or absolute minimum value.

➢ To find absolute maximum value or absolute minimum value :

(1) Find all the critical points *viz.* where $f'(x) = 0$ or f is not differentiable.

(2) Consider the end points also.

(3) Calculate the functional values at all the points found in steps (1) and (2).

(4) Identify the maximum and minimum values out of the values calculated in step (3). These are absolute maximum and absolute minimum values.

CHAPTER 7 : Integrals

➢ Some Basic Integrals Formulas

Derivatives	**Integrals (Anti derivatives)**
(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$;	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
Particularly, we note that $\frac{d}{dx}(x) = 1$;	$\int dx = x + C$
(ii) $\frac{d}{dx}(\sin x) = \cos x$;	$\int \cos x\, dx = \sin x + C$
(iii) $\frac{d}{dx}(\cos x) = -\sin x$;	$\int \sin x\, dx = -\cos x + C$

(iv) $\dfrac{d}{dx}(\tan x) = \sec^2 x;$ $\qquad \int \sec^2 x\, dx = \tan x + C$

(v) $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x;$ $\qquad \int \operatorname{cosec}^2 x\, dx = -\cot x + C$

(vi) $\dfrac{d}{dx}(\sec x) = \sec x \tan x;$ $\qquad \int \sec x \tan x\, dx = \sec x + C$

(vii) $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x;$ $\qquad \int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$

(viii) $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}};$ $\qquad \int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

(ix) $\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}};$ $\qquad \int \dfrac{dx}{1+x^2} = \tan^{-1} x + C$

(x) $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2};$ $\qquad \int \dfrac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$

(xi) $\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1+x^2};$ $\qquad \int e^x\, dx = e^x + C$

(xii) $\dfrac{d}{dx}(\operatorname{cosec}^{-1} x) = -\dfrac{1}{1+x^2};$ $\qquad \int \dfrac{1}{x}\, dx = \log|x| + C$

(xiii) $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}};$ $\qquad \int a^x dx = \dfrac{a^x}{\log a} + C$

(xiv) $\dfrac{d}{dx}(\operatorname{cosec}^{-1} x) = -\dfrac{1}{x\sqrt{x^2-1}};$

(xv) $\dfrac{d}{dx}(e^x) = e^x;$

(xvi) $\dfrac{d}{dx}(\log|x|) = \dfrac{1}{x};$

(xvii) $\dfrac{d}{dx}\left(\dfrac{a^x}{\log a}\right) = a^x;$

➢ Some Special Integrals Formulas

1. $\int \tan x\, dx = \log|\sec x| + C$
   or $-\log|\cos x| + C$

2. $\int \cot x\, dx = \log|\sin x| + C$

3. $\int \sec x\, dx = \log|\sec x + \tan x| + C$

4. $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C$
   or $\log\left|\tan \dfrac{x}{2}\right| + C$

5. $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\log\left|\dfrac{x-a}{x+a}\right| + C$

6. $\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a}\log\left|\dfrac{a+x}{a-x}\right| + C$

7. $\int \dfrac{dx}{x^2+a^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C$

8. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right|+C$

9. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+C$

10. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right|+C$

11. $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$

12. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

13. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

14. $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

➢ Some Important Rules while doing Integration

1. In case of rational function, if degree of numerator is equal or greater than the degree of denominator, then we first divide the numerator by denominator and write it as,

   $\frac{\text{Numerator}}{\text{Denominator}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Denominator}}$ and then integrate.

2. To evaluate $\int \frac{ax+b}{px^2+qx+r}dx$, we write $ax + b = A\frac{d}{dx}(px^2 + qx + r) + B$.

3. To evaluate $\int \frac{ax+b}{\sqrt{px^2+qx+r}}dx$, we write $ax + b = A\frac{d}{dx}(px^2 + qx + r) + B$.

4. To evaluate $\int \frac{a\sin x+b\cos x}{p\sin x+q\cos x}dx$, we write

   $a \sin x + b \cos x = A(p \sin x + q \cos x) + B\frac{d}{dx}(p \sin x + q \cos x)$, find A, B and proceed.

5. To evaluate $\int \frac{x^2+1}{x^4+x^2+1}dx$, divide numerator and denominator by x^2 and make a perfect square as $\left[x\pm\frac{1}{x}\right]^2$ in denominator and substitute $x\pm\frac{1}{x}=t$ and then proceed.

6. To evaluate :

   (i) $\int \frac{1}{(ax+b)\sqrt{cx+d}}dx$, put $\sqrt{cx+d}=t$

   (ii) $\int \frac{1}{(ax+b)\sqrt{px^2+qx+r}}dx$, put $ax + b = \frac{1}{t}$

   (iii) $\int \frac{1}{(px^2+qx+r)\sqrt{ax+b}}dx$, put $\sqrt{ax+b}=t$

   (iv) $\int \frac{x}{(ax^2+b)\sqrt{cx^2+d}}dx$, put $\sqrt{cx^2+d}=t$

(v) $\int \frac{x}{(ax^2+b)\sqrt{cx^2+d}}dx$, put $x=\frac{1}{t}$ and $\sqrt{c+dt^2}=Z$

7. To evaluate integral $\int (px+q)\sqrt{ax^2+bx+c}\,dx$, we write $px+q = A\left[\frac{d}{dx}(ax^2+bx+c)\right]+B$

Integration by Partial Fraction

Choose the type of partial fraction from the table, find A, B and C, then proceed further by integrating each term separately.

S. No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$
	where, $x^2 + bx + c$ cannot be factorised further.	

Integration by Parts

$$\int f(x)\, g(x)\, dx = f(x) \int g(x)\, dx - \int \left[f'(x) \int g(x)\, dx\right] dx.$$

Fundamental Theorem of Definite Integral

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b)-F(a).$$

Definite Integral as the limit of the Sum

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+\ldots+f(a+(n-1)h)],$$

where, $h = \frac{b-a}{n} \to 0$ as $n \to \infty$.

Some Properties of Definite Integrals

$P_0 : \int_a^b f(x)\, dx = \int_a^b f(t)\, dt$

$P_1 : \int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$, in particular, $\int_a^a f(x)\, dx = 0$

$P_2 : \int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$

$P_3 : \int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$ (by property)

$P_4 : \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$ (by property)

$P_5 : \int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(2a - x)\, dx$ (by property)

$$P_6 : \int_0^{2a} f(x)\, dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

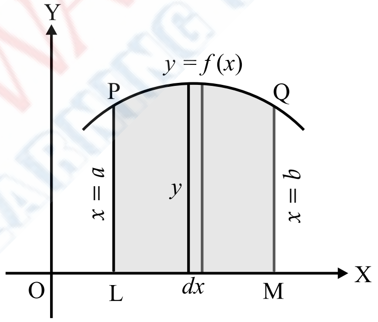
$P_7 : \int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$, if f is an even function *i.e.*, $f(-x) = f(x)$

$P_8 : \int_{-a}^{a} f(x) = 0$, if f is an odd function *i.e.*, $f(-x) = -f(x)$

CHAPTER 8 : Applications of the Integrals

- Area under simple curve $y = f(x)$ is given by

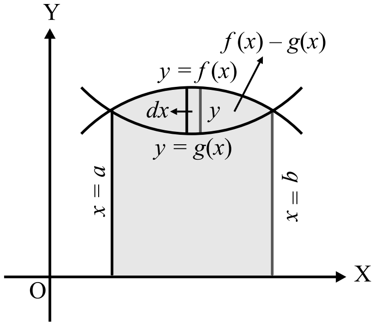
  $\int_a^b y\, dx = \int_a^b f(x)\, dx$



- Area between two curves $y = f(x)$ and $y = g(x)$ is given by

$$A = \int_a^b (f(x) - g(x))\, dx$$

$$= \int_a^b f(x)dx - \int_a^b g(x)\, dx$$



CHAPTER 9 : Differential Equations

Fundamentals

- An equation involving derivative (derivatives) of the dependent variable with respect to independent variable (variables) is called a differential equation, *e.g.*,

  (i) $x\dfrac{dx}{dy} + y = 0,$ (ii) $2\dfrac{d^2y}{dx^2} + y^3 = 0$

- The order of the highest order derivative of dependent variable with respect to independent variable involved in the differential equation is called the order of the differential equation, *e.g.*,

  $\dfrac{dx}{dy} + y = c,\ \dfrac{d^2y}{dx^2} + \dfrac{dx}{dy} + y = k$ involve derivatives

  whose highest orders are 1 and 2 respectively.
- When a differential equation is a polynomial equation in derivatives, the highest power (positive integral index), of the highest order derivative is known as degree of the differential equation, *e.g.*,

  (i) $\left(\dfrac{dy}{dx}\right)^2 + \dfrac{dx}{dy} + y = c$, the highest order derivative is $\dfrac{dx}{dy}$, its positive integral power is 2.

  $\therefore$ Its degree is two.

  (ii) $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} + \dfrac{dx}{dy} + 4 = 0$. Here, highest order derivative is $\dfrac{d^3y}{dx^3}$. Its positive integral power is 1.

  $\therefore$ Its degree is one.

Formation of Differential Equation

(i) Write the given equation.

(ii) Differentiate w.r.t. x successively as many times as the number of arbitrary constants involved in the given equation.

(iii) Eliminate the arbitrary constants.

The resulting equation is the required differential equation *e.g.*,

(a) The equation $y = mx$ has one arbitrary constant m.

$\therefore$ Differentiating, we get

$$\frac{dx}{dy} = m \qquad ...(1)$$

Eliminating m, $y = \left(\dfrac{dy}{dx}\right)x$ is the required differential equation.

(b) Consider the family of curves,

$$y = a\cos(x + b) \qquad ...(2)$$

Here, a and b are the arbitrary constants.

$$\frac{dx}{dy} = -a\sin(x + b) \qquad ...(3)$$

$$\frac{d^2y}{dx^2} = -a\cos(x + b) \qquad ...(4)$$

Eliminating a, b from (1) and (3)

$$\frac{d^2y}{dx^2} = -y$$

or,
$$\frac{d^2y}{dx^2} + y = 0$$

is the required differential equation.

Methods of Solving first order, first degree differential equations

(i) Variables are Separable : "When the equation may be expressed as $\frac{dx}{dy} = h(y)\, g(x)$, then we can write it as $\frac{dy}{h(y)} = g\,(x)\, dx$.

Integrating, we get the solution as

$$\int \frac{dy}{h(y)} = \int g\,(x)\, dx + C.$$

(ii) Homogeneous Differential Equation : Let us write the differential equation as $\frac{dx}{dy} = f\,(x, y)$.

Replacing x by λx and y by λy and we get $f\,(\lambda x, \lambda y)$

$$= \lambda^n f\,(x, y)$$

Then, the differential equation is homogeneous of degree n.

To solve such an equation

$$\frac{dx}{dy} = f\,(x, y) \text{ put } y = vx \text{ or } \frac{dx}{dy} = v + x\frac{dv}{dx} \qquad ...(5)$$

we get,

$$v + x\frac{dv}{dx} = f\,(v) \text{ or } x\,\frac{dv}{dx} = f\,(v) - v$$

$\therefore$ Solution is

$$\int \frac{dv}{f(v)-v} = \int \frac{dx}{x} + C$$

(iii) Linear Differential Equation :

(a) The linear differential equation is of the form, $\frac{dx}{dy} + Py = Q$, where P and Q are the functions of x.

This is a first order, first degree differential equation.

To solve the equation, we find the integrating factor

$$\text{I.F.} = e^{\int P dx}$$

Then, the solution is

$$ye^{\int P dx} = \int Q.e^{\int P.dx}\, dx + C.$$

(b) When the equation is of the form $\frac{dx}{dy} + Px = Q$, where P and Q are the functions of y.

$$\text{I.F.} = e^{\int P dy}$$

$\therefore$ Solution is

$$xe^{\int P dy} = \int Q.e^{\int P dy}\, dy + C.$$

CHAPTER 10 : Vectors

Fundamentals

- A quantity that has magnitude as well as direction is called a vector.
- The unit vector in the direction of $\vec{a}$ is given by $\frac{\vec{a}}{|\vec{a}|}$ and is represented by $\hat{a}$.

- Position vector of a point P(x, y, z) is given as $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$ and its magnitude as $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$, where O is the origin.
- The scalar components of a vector are its direction ratios, and represent its projections along the respective axes.
- The magnitude r, direction ratios (a, b, c) and direction cosines (l, m, n) of any vector are related as:

$$l = \frac{a}{r}, m = \frac{b}{r}, n = \frac{c}{r} \text{ and } l^2 + m^2 + n^2 = 1$$

- The sum of the vectors representing the three sides of a triangle taken in order is 0.

Addition of Vectors

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two vectors then $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$.

Scalar Multiplication

If $\vec{a}$ is a given vector and λ is a scalar, then $\lambda\vec{a}$ is a vector whose magnitude is $|\lambda\vec{a}| = |\lambda||\vec{a}|$. The direction of $\lambda\vec{a}$ is same as that of $\vec{a}$ if λ is positive and, opposite to that of $\vec{a}$ if λ is negative.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two vectors, then $\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$ and $\lambda\vec{b} = (\lambda b_1)\hat{i} + (\lambda b_2)\hat{j} + (\lambda a_3)\hat{k}$.

Vector Joining Two Points

If P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) are any two points, then

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\hat{i} + (y_2 + y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Section Formula

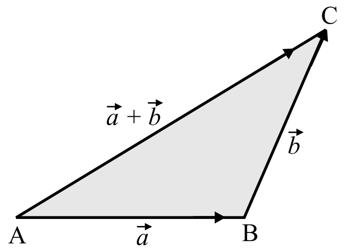
The position vector of a point R dividing the line segment joining the points P and Q whose position vectors are $\vec{a}$ and $\vec{b}$

(i) in the ratio $m : n$ internally, is given by $\dfrac{n\vec{a} + m\vec{b}}{m + n}$

(ii) in the ratio $m : n$ externally, is given by $\dfrac{m\vec{b} - n\vec{a}}{m - n}$

Triangle Law of Vector Addition

Let the vectors $\vec{a}$ and $\vec{b}$ be so positioned that initial point of one coincides with terminal point of the other, $\vec{a} = \overrightarrow{AB}$, $\vec{b} = \overrightarrow{BC}$. Then, the vector $\vec{a} + \vec{b}$ is represented by the third side of ΔABC *i.e.*,



$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

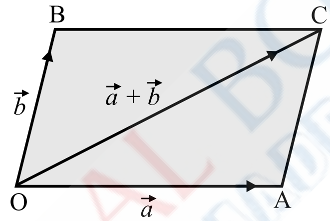
or $$\overrightarrow{AC}-\overrightarrow{AB}=\overrightarrow{BC}$$

or $$\overrightarrow{AC}-\overrightarrow{BC}=\overrightarrow{AB}$$

This is known as triangle law of addition.

➢ Parallelogram Law of Vector Addition

Parallelogram law of Addition : If the two vectors $\vec{a}$ and $\vec{b}$ are represented by the two adjacent sides OA and OB of a parallelogram OACB, then their sum $\vec{a}+\vec{b}$ is represented by the diagonal AC of parallelogram both in magnitude and direction through their common point O.



i.e., $$\overrightarrow{OA}+\overrightarrow{OB}=\overrightarrow{OC}$$

➢ Projection

Projection of vector $\vec{a}$ along vector $\vec{b}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

➢ Scalar or Dot Product

The scalar or dot product of two given vectors $\vec{a}$ and $\vec{b}$ having an angle θ between them is defined as :

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ are two vectors, then $\vec{a}\cdot\vec{b} = a_1b_1+a_2b_2+a_3b_3$.

➢ Properties of Scalar Product:

(a) $\vec{a}\cdot\vec{b}$ is a scalar quantity.

(b) When $\theta = 0$, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$

(c) When $\theta = \dfrac{\pi}{2}$, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\dfrac{\pi}{2} = 0$

$\Rightarrow$ When $\vec{a}\perp\vec{b}$, $\vec{a}\cdot\vec{b} = 0$

(d) When either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, $\vec{a}\cdot\vec{b} = 0$

Vector or Cross Product

The cross product of two vectors $\vec{a}$ and $\vec{b}$ having angle θ between them is given as $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{n}$,

where $\hat{n}$ is a unit vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$ If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two vectors, then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (b_1c_2 - b_2c_1)\hat{i} + (a_2c_1 - a_1c_2)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

Properties of Vector Product :

(a) $\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta . \hat{n}$

(i) If $\vec{a} = 0$ or $\vec{b} = 0$, $\vec{a}\times\vec{b} = 0$

(ii) $\vec{a} \,||\, \vec{b}$, $\vec{a}\times\vec{b} = 0$ (or θ = 0)

(b) $\vec{a}\times\vec{b}$ is a vector *i.e.*, $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

$\therefore \quad \vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$

$\Rightarrow \quad \vec{a}\times\vec{b}$ is not commutative.

(c) When $\theta = \frac{\pi}{2}, \vec{a}\times\vec{b} = |\vec{a}|\times|\vec{b}|\times\hat{n}$

or $\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|$

(d) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$

and $\hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$

$\Rightarrow \quad \hat{j}\times\hat{i} = -\hat{k}, \hat{k}\times\hat{j} = -\hat{i}, \hat{i}\times\hat{k} = -\hat{j}$

(e) $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|}$

(f) If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area = $\vec{a}\times\vec{b}$.

(g) If $\vec{a} . \vec{b}$ represents the adjacent sides of a triangle, then its area = $\frac{1}{2}\,|\vec{a}\times\vec{b}|$.

Angle between Two Vectors

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two vectors, then

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}.$$

Scalar Triple Product

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors, then the scalar $(\vec{a}\times\vec{b}).\vec{c}$ is called the scalar triple product of $\vec{a}$, $\vec{b}$ and $\vec{c}$ and is denoted by $[\vec{a}\ \vec{b}\ \vec{c}\,]$

$\therefore \qquad (\vec{a}\times\vec{b}).\vec{c} = [\vec{a}\ \vec{b}\ \vec{c}\,]$

Let, $\qquad \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

and $$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}\text{, then}$$

$$\vec{a}.(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Properties of Scalar Triple Product

1. $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$
2. $\vec{a}.(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b}).\vec{c}$
3. $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{c}\ \vec{b}]$
4. $[\vec{a}\ \vec{a}\ \vec{c}] = 0 = [\vec{a}\ \vec{b}\ \vec{b}]$
5. $\lambda[\vec{a}\ \vec{b}\ \vec{c}] = [\lambda\vec{a}\ \vec{b}\ \vec{c}] = [\vec{a}\ \lambda\vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \lambda\vec{c}]$

CHAPTER 11 : Three Dimensional Geometry

Fundamentals

- Direction cosines of a line are the cosines of the angles made by the line with positive directions of the co-ordinate axes.
- If l, m, n are the direction cosines of a line, then $l^2 + m^2 + n^2 = 1$.
- Direction cosines of a line joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are

  $\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$ where, PQ = $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$
- If l, m, n are the direction cosines and a, b, c are the direction ratios of a line, then

  $$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \frac{b}{\sqrt{a^2+b^2+c^2}}, n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$
- Skew lines are the lines in the space which are neither parallel nor intersecting. They lie in the different planes.
- Angle between skew lines is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.
- Angle between two lines

  $$\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$$

  OR

  $$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+a_2^2+a_3^2}.\sqrt{b_1^2+b_2^2+b_3^2}}\right|$$

Line

- Vector equation of a line that passes through the given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$
- Equation of a line through a point (x_1, y_1, z_1) and having directions cosines l, m, n (or, direction ratios a, b and c) is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \text{ or } \left(\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}\right)$$

- The vector equation of a line that passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$.
- Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

- If θ is the acute angle between the lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$, then θ is given by

$$\cos\theta = \frac{|\vec{b}_1.\vec{b}_2|}{|\vec{b}_1||\vec{b}_2|} \text{ or } \theta=\cos^{-1}\frac{|\vec{b}_1.\vec{b}_2|}{|\vec{b}_1||\vec{b}_2|}.$$

- If $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_1}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ are equations of two lines, then the acute angle θ between the two lines is given by $\cos\theta = |l_1l_2+m_1m_2+n_1n_2|$.
- The shortest distance between two skew lines is the length of the line segment perpendicular to both the lines.
- The shortest distance between the lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$ is

$$\left|\frac{(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)}{|\vec{b}_1\times\vec{b}_2|}\right|$$

- Shortest distance between the lines : $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is

$$\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

- Distance between parallel lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$ is (where b is in simplest form)

$$\left|\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$$

Plane

- The vector equation of a plane which is at a distance p from the origin, where $\hat{n}$ is the unit vector normal to the plane, is $\vec{r}.\hat{n}=p$.
- Equation of a plane which is at a distance p from the origin with direction cosines of the normal to the plane as l, m, n is $lx+my+nz=p$.
- The equation of a plane through a point whose position vector is $\vec{a}$ and perpendicular to the vector $\vec{n}$ is $(\vec{r}-\vec{a}).\vec{n}=0$ or $\vec{r}.\vec{n}=d$, where $d=\vec{a}.\vec{n}$.

- Equation of a plane perpendicular to a given line with direction ratios a, b, c and passing through a given point (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$
- Equation of a plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

- Vector equation of a plane that contains three non-collinear points having position vectors $\vec{a}, \vec{b}, \vec{c}$ is $(\vec{r} - \vec{a}).[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$.
- Equation of a plane that cuts the co-ordinates axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
- Vector equation of any plane that passes through the intersection of planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is $(\vec{r}.\vec{n}_1 - d_1) + \lambda(\vec{r}.\vec{n}_2 - d_2) = 0$, where λ is any non-zero constant.
- Cartesian equation of any plane that passes through the intersection of two given planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is $(A_1x + B_1y + C_1z + D_1) + \lambda(A_2x + B_2y + C_2z + D_2) = 0$.
- Two lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ are coplanar if $(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2) = 0$.
- Two lines $\frac{x - x_1}{a_1} \; \frac{y - y_1}{b_1} \; \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} \; \frac{y - y_2}{b_2} \; \frac{z - z_2}{c_2}$ are coplanar if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

- In vector form, if θ is the acute angle between the two planes, $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$

  then $\theta = \cos^{-1} \frac{|\vec{n}_1.\vec{n}_2|}{|\vec{n}_1|.|\vec{n}_2|}$.
- The acute angle θ between the line $\vec{r} = \vec{a} + \lambda\vec{b}$ and plane $\vec{r}.\vec{n}_1 = d_1$ is given by

$$\sin\theta = \frac{|\vec{b}.\vec{n}|}{|\vec{b}|.|\vec{n}|}$$

CHAPTER 12 : Linear Programming

- A linear programming problem deals with the optimization (maximization/ minimization) of a linear function of two variables {say x and y) known as objective function subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear constraints). A linear programming problem is a special type of optimization problem.
- **Objective Function :** Linear function, $Z = ax + by$, where a and b are constants, which has to be maximised or minimized is called a linear objective function.
- **Decision Variables :** In the objective function, $Z = ax + by$, x and y are called decision variables.
- **Constraints :** The linear inequalities or restrictions on the variables of an LPP are called constraints.

The conditions $x \geq 0, y \geq 0$ are called non-negative constraints.

- **Feasible Region :** The common region determined by all the constraints including non-negative constraints $x \geq 0, y \geq 0$ of an LPP is called the feasible region for the problem.
- **Feasible Solutions :** Points within and on the boundary of the feasible region for an LPP represent feasible solutions.
- **Infeasible Solutions :** Any Point outside feasible region is called an infeasible solution.
- **Corner point method for solving a LPP :**

The method comprises of the following steps :

(1) Find the feasible region of the LPP and determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at that point.

(2) Evaluate the objective function $Z = ax + by$ at each corner point.

Let M and m, respectively denote the largest and the smallest values of Z.

(3) (i) When the feasible region is bounded, M and m are respectively, the maximum and minimum values of Z.

(ii) In case, the feasible region is unbounded.

(a) M is the maximum value of Z, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, Z has no maximum value.

(b) Similarly, m is the minimum of Z, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, Z has no minimum value.

CHAPTER 13 : Probability

Conditional Probability

If E and F are two events associated with the same sample space of a random experiment, then the conditional probability of the event E under the condition that the event F has occurred, written as P (E | F), is given by

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}, P(F) \neq 0$$

Properties of Conditional Probability

Let E and F be events associated with the sample space S of an experiment. Then :

(i) $P(S|F) = P(F|F) = 1$

(ii) $P[(A \cup B) \mid F] = P(A \mid F) + P(B \mid F) - P[(A \cap B \mid F)]$,

where A and B are any two events associated with S.

(iii) $P(E' \mid F) = 1 - P(E \mid F)$

Multiplication Theorem on Probability

Let E and F be two events associated with a sample space of an experiment. Then

$$P(E \cap F) = P(E)\, P(F \mid E),\ P(E) \neq 0$$
$$= P(F)\, P(E \mid F),\ P(F) \neq 0$$

If E, F and G are three events associated with a sample space, then

$$P(E \cap F \cap G) = P(E)\, P(F \mid E)\, P(G \mid E \cap F)$$

Independent Events

Let E and F be two events associated with a sample space S. If the probability of occurrence of one of them is not affected by the occurrence of the other, then we say that the two events are independent. Thus, two events E and F will be independent, if

(a) $P(F \mid E) = P(F)$, provided $P(E) \neq 0$

(b) $P(E \mid F) = P(E)$, provided $P(F) \neq 0$

Using the multiplication theorem on probability, we have

(c) $P(E \cap F) = P(E)\, P(F)$

Three events A, B and C are said to be mutually independent if all the following conditions hold :

$$P(A \cap B) = P(A)\, P(B)$$

$$P(A \cap C) = P(A)\, P(C)$$

$$P(B \cap C) = P(B)\, P(C)$$

and $\quad P(A \cap B \cap C) = P(A)\, P(B)\, P(C)$

Partition of a Sample Space

A set of events $E_1, E_2, \ldots, E_n$ is said to represent a partition of a sample space S if

(a) $E_t \cap E_j = \phi,\ i \neq 0;\ i, j = 1, 2, 3, \ldots\ldots, n$

(b) $E_t \cup E_j \cup \ldots \cup E_n = S$, and

(c) Each $E_i \neq \phi$, *i. e.*, $P(E_i) > 0$ for all $i = 1, 2, \ldots, n$

Theorem of Total Probability

Let $\{E_1, E_2, \ldots, E_n\}$ be a partition of the sample space S. Let A be any event associated with S, then

$$P(A) = \sum_{j=1}^{n} P(E_j)P(A \mid E_j)$$

Bayes' Theorem

If $E_1, E_2, \ldots, E_n$ are mutually exclusive and exhaustive events associated with a sample space, and A is any event of non-zero probability, then

$$P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)P(A \mid E_i)}$$

Random Variable and its Probability Distribution

A random variable is a real valued function whose domain is the sample space of a random experiment.

The probability distribution of a random variable X is the system of numbers

X :	x_1	x_2	....	x_n
P(X) :	p_1	p_2	....	p_n

where, $\quad p_1 > 0;\ i = 1, 2, \ldots, n,\ \sum_{i=1}^{n} p_i = 1$

Mean and Variance of a Random Variable

Let X be a random variable assuming values $x_1, x_2, ..., x_n$ with probabilities $p_1, p_2, ..., p_n$ respectively, such that $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$. Mean of X, denoted by μ [or expected value of X denoted by E (X)] is defined as

$$\mu = E(X) = \sum_{i=1}^{n} x_i p_i$$

and variance, denoted by σ^2, is defined as

$$\text{Var}(X) \text{ or } \sigma_x^2 = \sum_{i=1}^{n} (x_i - \mu) p_i$$

$$= E(X_i - \mu)^2$$

$$= E(X^2) - [E(X)]^2$$

Standard Deviation, $\sigma_x = \sqrt{\text{Var}(X)}$

Binomial Distribution

A random variable X taking values 0, 1, 2, ..., n is said to have a binomial distribution with parameters n and p, if its probability distribution is given by

$$P(X = r) = {}^nC_r p^r q^{n-r},$$

where, $q = 1 - p$ and $r = 0, 1, 2, ..., n$.

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ON TIPS NOTES

APPLIED MATHEMATICS (B-2)

Note-making is a skill that we use in many walks of life : at school, university and in the world of work. However, accurate Note-making requires a thorough understanding of concepts. We, at Oswaal, have tried to encapsulate all the chapters from the given syllabus into the following ON TIPS NOTES. These notes will not only facilitate better understanding of concepts, but will also ensure that each and every concept is taken up and every chapter is covered in totality. So go ahead and use these to your advantage.... go get the **OSWAAL ADVANTAGE!!**

UNIT – I
NUMBERS, QUANTIFICATION AND NUMERICAL APPLICATIONS

CHAPTER-1
NUMBERS AND QUANTIFICATION

- **Modulo Arithmetic**

  Modulo Arithmetic is a special type of arithmetic that involves only integers and the operations used are addition, subtraction, multiplication and division. The only difference between modulo arithmetic and simple arithmetic that we have learned in our primary classes is that is modulo arithmetic all operations are performed regarding a positive integer *i.e.*, the modulus.

  Let's revise the division theorem that tells us that for any two integers a and b where $b \neq 0$, there always exists unique integer q and r such that $a = qb + r$ and $0 \leq r < |b|$.

  For example: $a = 39$, $b = 5$, we can find $q = 7$ and $r = 4$.

  so, that $39 = 7 \times 5 + 4$.

  Here, a is called dividend
  b is called divisor
  q is called quotient
  r is called remainder

  If $r = 0$, the we say b divides a or a is divisible by b. This established a natural congruence relation on the integers.

- **Congruence Modulo**

  Let m be a positive integer and I be the set of all integers. The relation "congruence modulo m" is defined on all a, $b \in I$ by $a \equiv b \pmod{m}$ if and only if m divides $(a - b)$.

  The symbol "$\equiv \pmod{m}$" is read as "congruence modulo m".

  "m divides $(a - b)$" or "m is a factor of $(a - b)$" is usually denoted by "$m|(a - b)$".

  Clearly, $a \equiv b \pmod{m}$ if $(a - b)$ is a multiple of m,

  i.e., $a = b + k\,m$ *for some integer k.*

  Thus,

  ●$69 \equiv 25 \pmod{4}$, since $4|(69 - 25)$ or 4 divides $(69 - 25)$ or $(69 - 25) = 44$ is a multiple of 4.

  ●$8 \not\equiv -8 \pmod{3}$, since $8 - (-8) = 16$, which is not a multiple of 3.

- **Properties of Congruence Modulo**

  Property I: If $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$

  Property II: If $a \equiv b \pmod{m}$, then $a \pmod{m} = b \pmod{m}$.

  Property III: If $a \equiv b \pmod{m}$, then $(a \pm c) = (b \pm c) \pmod{m}$.

  Property IV: If $a \equiv b \pmod{m}$, then $ac \equiv bc \pmod{m}$.

  Property V: If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$,

  then, $(a + c) \equiv (b + d) \pmod{m}$.

  Property VI: If $a \equiv b \pmod{m}$, and $c \equiv d \pmod{m}$, *then,* $(a - c) \equiv (b - d) \pmod m$

Property VII: If $a \equiv b \pmod m$, and $c \equiv d \pmod m$,

then, $ac \equiv bd \pmod m$

Property VIII: If x_1 *is a solution of* $ax \equiv b \pmod m$, *then any other integer* $x_2 \equiv x_1 \pmod m$ *is also a solution.*

Property IX: The congruence modulo $ax \equiv b \pmod m$ has a solution iff the greatest common divisor of a and m divides b. If the greatest common divisor d, of a and m, divides b, then congruence modulo has exactly d incongruent solutions.

e.g.,: Prove that $35x \equiv 14 \pmod{21}$ has a solution. Also, find the number of incongruent solutions.

Solution: Here, the greatest common divisor (gcd) of 35 and 21 is 7; also 7 divides 14. Hence, the given congruence modulo has a solution and number of incongruent solutions (mod 21) is 7.

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CHAPTER-2
NUMERICAL APPLICATIONS

- **Alligation and Mixture:**

  It is the rule that enables us to find the ratio in which two or more ingredients at the given price must be mixed to produce a mixture of a desired price. There are two types of questions asked in alligation.

  Type 1: If two ingredients are mixed, then

$$\left(\frac{\text{Quantity of cheaper}}{\text{Quantity of dearer}}\right) = \frac{\text{(C.P. of dearer)} - \text{(Mean Price)}}{\text{Mean Price} - \text{(C.P. of cheaper)}}$$

  We present as under:

  C.P. of a unit quantity of cheaper — C.P. of a unit quantity of dearer

  (c) — (d)

  (Mean Price)

  m

  (d–m) — (m–c)

  ∴ (Cheaper quantity) : (Dearer quantity) = $(d - m) : (m - c)$

  Here, C.P. = Cost Price

  Mean Price (m)= The cost price of a unit quantity of the mixture

  Type 2: Suppose a container contains x units of liquids from which y units are taken out and replaced by water.

  After n operations, the quantity of pure liquid = $\left[x\left(1-\frac{y}{x}\right)^n\right]$ units

- **Boats and Streams**

  Stream is the moving water in the river.

  Upstream is the direction against the stream.

  Down stream is the direction along the stream.

  Still water is the state where water is considered to be stationary and the speed of the water in this case is zero.

  - If the speed of a boat in still water is u km/h and the speed of the stream is v km/h, then:

$$\text{Speed of boat in down stream} = (u + v) \text{ km/h}$$
$$\text{Speed of boat in up stream} = (u - v) \text{ km/h}$$

  - It the relative speed of boat in down stream is a km/h and the relative speed of boat in upstream is b km/h, then:

$$\text{Speed of boat in down water} = \frac{1}{2}(a + b) \text{ km/h}$$

$$\text{Speed of up stream} = \frac{1}{2}(a - b) \text{ km/h}$$

- **Partnership:** When two or more than two persons run a business jointly, they are called **partners** and the deal is known as **partnership**.
  - **Ratio of Division of Gains**

    (i) When investments of all the partners are for the same time, the gain or loss distributed among the partners in the ratio of their investments.

    Suppose A and B invest ₹ x and ₹ y respectively for a year in a business, then at the end of the year :

(A's share of profit) : (B's share of profit) = $x : y$

(ii) When investments are for different time periods, then equivalent capitals are calculated for a unit of time by taking (capital × number of units of time). Now, gain or loss is divided in the ratio of these capitals.
Suppose A invest ₹ x for p months and B invest ₹ y for q months, then
(A's share of profit) : (B's share of profit) = $xp : yq$

- **Working and Sleeping Partners**
  A partner who manages the business is known as a **working partner** and the one who simply invests the money is a **sleeping partner**.

➢ **Pipes and Cistern**

Inlet: A pipe connected with a tank or a cistern or a reservoir, that fills it, is known as inlet.

Outlet: A pipe connected with a tank or cistern or a reservoir, emptying it, is called an outlet.

- If a pipe can fill a tank in x hours, then part of tank filled in 1 hour = $\frac{1}{x}$
- If a pipe can empty a full tank in y hours, then part of tank emptied in 1 hour = $\frac{1}{y}$
- If a pipe can fill a tank in x hours and another pipe can empty the full tank in y hours (where $y > x$), then on opening both pipes, the net part filled in 1 hour = $\left(\frac{1}{x}-\frac{1}{y}\right)$
- If a pipe can fill a tank in x hours and another pipe can empty the full tank in y hours (where $x > y$), then on opening both pipes, the net part emptied in 1 hour = $\left(\frac{1}{y}-\frac{1}{x}\right)$

➢ **Races and Games**

- **Races:** A contest of speed in running, riding, driving, sailing or rowing is called a race.
- **Race Course:** The ground or path on which contests are made is called race course.
- **Starting Point:** The point from which a race begins is known as a starting point.
- **Winning Point or Goal:** The point set to bound a race is called a winning point or a goal.
- **Winner:** The person who first reaches the winning point is called winner.
- **Dead Heat Race:** If all the persons contesting a race reach the goal exactly at the same time, then the race is said to be a dead heat race.
- **Start:** Suppose A and B are two contestants in a race. If before the start of the race, A is at the starting point and B is ahead of A by 15 metres, then we say that 'A gives B, a start of 15 metres'.
  So, in case to cover a race of 100 metres, A will have to cover 100 metres while B will have to cover only (100 – 15) metres = 85 metres.
  In other words, we can say that to cover a race of 100 metres 'A can give B 15 metres' or 'A can give B a start of 15 metres' or 'A beats B by 15 metres'.
- **Games:** A game of 100, means that the person among the contestants who scores 100 points first is the winner.
  If A scores 100 points while B scores only 80 points, then we say that 'A can give B 20 points'.

Therefore, 1st January 1996 was a Tuesday.

➢ **Inequalities:** Two real numbers or two algebraic expression related by the symbol '<', '>', '≤', or '≥', form an inequality.

Types of inequalities

- **Numerical inequality:** An inequality which does involve any variable is called a numerical inequality.
  e.g.: $4 > 2, 8 < 21$
- **Literal inequality:** An inequality which have variable is called a literal inequality.
  e.g.: $x < 7, y \geq 11, x \leq 4$
- **Strick inequality:** An inequality which have only < or > is called strick inequality.
  e.g.: $3x + y < 0, x > 7$
- **Slack inequality:** An inequality which have only ≥ or ≤ is called slack inequality.
  e.g.: $3x + 2y \leq 0, y \leq 4$
- **Linear inequality:** An inequality is said to be linear, if each variable occurs in first degree only and there is no term involving the product of the variables.
  e.g.: $ax + b \leq 0, ax + by + c > 0, ax \leq 4$
  An inequality in one variable in which degree of variable is 2, is called quadratic inequality in one variable.
  e.g.: $ax^2 + bx + c \geq 0, 3x^2 + 2x + 4 \leq 0$

❖ **Linear inequality in one variable**

A linear inequality which has only one variable, is called linear inequality in one variable.

e.g.: $ax + b < 0$, where $a \neq 0$

Replacement Set: The set from which values of the variables (involved in the inequality) are chosen is called the 'replacement set'.

Solution Set: A solution to an inequality is a number (chosen from replacement set) which, when substituted for the variable, make the inequality true. The set of all solutions of an inequality is called the 'solution set' of the inequality.

Remarks:
- Solution set always depends upon replacement set.
- If the replacement set is not given, then we shall take it as R(set of real numbers).

Rules for solving inequalities in one variable: The rules for solving inequalities are similar to those for solving equations except for multiplying or dividing by a negative number.

(i) If $a \geq b$, then $a \pm k \geq b \pm k$, where k is any real number.

(ii) If $a \geq b$, then ka is not always $\geq kb$

(iii) If $k > 0$ (*i.e.*, positive), then $a \geq b \Rightarrow ka \geq kb$

(iv) If $k < 0$ (*i.e.*, negative), then $a \geq b \Rightarrow ka \leq kb$

Thus, always reverse the sign of inequality while multiplying or dividing both sides of an inequality by a negative number.

Procedure to solve a linear inequality in one variable:

(i) Simplify both sides by collecting like terms.

(ii) Remove fraction (or decimals) by multiplying both sides by appropriate factor (L.C.M. of denominator or a power of 10 in case of decimals)

(iii) Isolate the variables on one side and all constant on the other side.

(iv) Make the coefficient of the variable equal to 1.

(v) Choose the solution set from the replacement set.

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UNIT – II : ALGEBRA

CHAPTER-3
MATRICES

➢ **Matrix:**

A matrix is an ordered rectangular array of numbers (real or complex) or functions which are known as elements or the entries of the matrix. It is denoted by the uppercase letters *i.e.*, *A*, *B*, *C* etc.

Consider a matrix *A* given as,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Here in matrix *A* depicted above, the horizontal lines of elements are said to constitute rows of the matrix *A* and vertical lines of elements are said to constitute columns of the matrix. Thus, matrix *A* has *m* **rows** and *n* **columns**. The array is enclosed by square brackets [ ], the parentheses ( ) or the double vertical bars || ||.

- A matrix having *m* rows and *n* columns is called a matrix of order $m \times n$ (read as '*m* by *n*' matrix). A matrix 'A' of order $m \times n$ is depicted as $A = [a_{ij}]_{m\times n}; i, j \in N$.
- Also, in general, a_{ij} means an element lying in the i^{th} row and j^{th} column.

- No. of elements in the matrix $A=[a_{ij}]_{m\times m}$ is given as $(m)(n)$.

➢ **Types of Matrices:**

(i) Column matrix: A matrix having only one column is called a **column matrix** or **column vector**.

General notation: $A=[a_{ij}]_{m\times 1}$

(ii) Row matrix: A matrix having only one row is called a **row matrix** or **row vector**.

General notation: $A=[a_{ij}]_{1\times n}$

(iii) Square matrix: It is a matrix in which the number of rows is equal to the number of columns *i.e.*, an $n\times n$ matrix is said to constitute a square matrix of order $n \times n$ and is known as a **square matrix of order** 'n'.

General notation: $A=[a_{ij}]_{n\times n}$

(iv) Diagonal matrix: A square matrix $A=[a_{ij}]_{m\times m}$ is said to be diagonal matrix if all the elements, except those in the leading diagonal are zero *i.e.*, $a_{ij}=0$ for all $i\neq j$.

- Also, there are more notation specifically used for the diagonal matrices. For instance, consider the matrix depicted above, it can also be written as diag $(a_{11}, a_{22}, \ldots, a_{mm})$ or diag $[a_{11}, a_{22}, \ldots, a_{mm}]$
- Note that the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{mm}$ of a square matrix A = $[a_{ij}]_{m\times m}$ of order m × m are said to constitute the principal diagonal or simply **the diagonal of the square matrix *A***. These elements are known as **diagonal elements of matrix *A***.

(v) Scalar matrix: A diagonal matrix $A=[a_{ij}]_{m\times m}$ is said to be a scalar matrix if its diagonal elements are equal.

i.e., $a_{ij}=\begin{cases}0, & \text{when } i\neq j\\ k, & \text{when } i=j \text{ for some constant } k\end{cases}$

(vi) Unit or Identity matrix: A square matrix $A=[a_{ij}]_{m\times m}$ is said to be an identity matrix if $a_{ij}=\begin{cases}1, & \text{if } i=j\\ 0, & \text{if } i\neq j\end{cases}$.

A unit matrix can also be defined as the scalar matrix each of whose diagonal elements is unity. We denote the identity matrix of order m by I_m or I.

(vii) Zero matrix or Null matrix: A matrix is said to be a **zero matrix** or **null matrix** if each of its elements is '0'.

(viii) Triangular matrix:

(a) Lower triangular matrix: A square matrix is called a lower triangular matrix if all the entries above the main diagonal are zero.

(b) Upper triangular matrix: A square matrix is called a **upper triangular matrix** if all the entries below the main diagonal are zero.

➢ **Equality of Matrices:**

Two matrices A and B are said to be equal and written as $A = B$, if they are of the same order and their corresponding elements are identical *i.e.*, $a_{ij} = b_{ij}$ *i.e.*, $a_{11} = b_{11}, a_{22} = b_{22}, a_{32} = b_{32}$ etc.

➢ **Algebra of Matrices:**

- **Addition of Matrix:** If A and B are two $m \times n$ matrices, then another $m \times n$ matrix obtained by adding the corresponding elements of the matrices A and B is called the sum of the matrices A and B and is denoted by '$A + B$'.

  Thus, if $A=[a_{ij}], B=[b_{ij}] \Rightarrow A+B=[a_{ij}+b_{ij}]$.

Properties of matrix addition:

- Commutative property: $A + B = B + A$
- Associativity property: $A + (B + C) = (A+B) + C$
- Cancellation law: **(i)** Left cancellation: $A + B = A + C \Rightarrow B=C$

  (ii) Right cancellation: $B + A = C + A \Rightarrow B = C$
- **Subtraction of Matrices:**

  If A and B are two $m \times n$ matrices, then another $m \times n$ matrix obtained by subtracting the corresponding elements of the matrices A and B is called the subtraction of the matrices A and B and is denoted by '$A - B$'. Thus if $A = [a_{ij}], B = [b_{ij}]$, or $A - B = [a_{ij} - b_{ij}]$.
- **Multiplication of Matrices:**

  We can only multiply two matrices if their dimensions are compatible , which means the number of columns in the first matrix is the same as the number of rows in the second matrix.

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, the product AB is an $m \times p$ matrix.
$AB = [c_{ij}]$, where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}$.
(The entry in the i^{th} row and j^{th} column is denoted by the double subscript notation a_{ij}, b_{ij}, and c_{ij}. For instance, the entry a_{23} is the entry in the second row and third column.)

Properties of matrix multiplication :

- In the product AB, A is called the **pre-factor** and B is called the **post-factor**.
- If two matrices A and B are such that AB is possible then it is not necessary that the product BA is also possible.
- If A is an $m \times n$ matrix and both AB as well as BA are defined, then B will be an $n \times m$ matrix.
- If A is an $n \times n$ matrix and I_n be the unit matrix of order n, then $AI_n = I_n A = A$.
- Matrix multiplication is **associative** *i.e.*, $A(BC) = (AB)C$.
- Matrix multiplication is **distributive** over the **addition** *i.e.*, $A.(B + C) = AB + AC$.
- Matrix multiplication is not commutative.

➢ **Multiplication of a Matrix by a Scalar :**

If a $m \times n$ matrix A is multiplied by a scalar k (say), then the new kA matrix is obtained by multiplying each element of matrix A by scalar k. Thus, if $A = [a_{ij}]$, and it is multiplied by a scalar k, then $kA = [ka_{ij}]$, *i.e.*, $A = [a_{ij}]$.

$\Rightarrow kA = [ka_{ij}]$

Properties of scalar multiplication:

If A, B are two matrices of the same order and k, l are any scalars (numbers), then

(i) $(k + l)A = kA + lA$ **(ii)** $k(A + B) = kA + kB$
(iii) $(kl)A = k(lA) + l(kA)$ **(iv)** $IA = A$ **(v)** $(-1)A = -A$

➢ **Transpose of a Matrix :**

If $A = [a_{ij}]_{m \times m}$ be a $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of matrix A is said to be a transpose of matrix A. The transpose of A is denoted by A' or A^T *i.e.*, if $A^T = [a_{ji}]_{n \times m}$.

Properties of Transpose of Matrices :

- $(A+B)^T = A^T+B^T$
- $(A^T)^T = A$
- $(kA)^T = kA^T$, where k is any constant
- $(AB)^T = B^TA^T$
- $(ABC)^T = C^TB^TA^T$

➢ **Symmetric matrix :** A square matrix $A = [a_{ij}]$ is said to be a symmetric matrix if $A^T = A$ *i.e.*, if $A = [a_{ij}]$, then $A^T = [a_{ij}] = [a_{ji}]$.

Note
• For any matrices AA^T and A^TA are symmetric matrices • If A and B are two symmetric matrices of same order, then (i) AB is symmetric if and only if $AB = BA$. (ii) $A \pm B$, $AB + BA$ are also symmetric matrices.

➢ **Skew Symmetric Matrix :**

A square matrix $A = [a_{ij}]$ is said to be **a skew symmetric matrix** if $A^T = -A$ *i.e.*, if $A = [a_{ij}]$, then $A^T = [a_{ji}] = -[a_{ij}]$.

Note
• All the diagonal elements in a skew-symmetric matrix are zero. • If A and B are two symmetric matrices, then $AB - BA$ is a skew symmetric matrix.

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CHAPTER-4

DETERMINANTS

➢ **Determinant :** A unique number (real or complex) can be associated to every square matrix $A = [a_{ij}]$ of order m. This number is called the determinant of the square matrix A, where $a_{ij} = (i, j)^{th}$ element of A.

The determinant of matrix A is denoted by det A or $|A|$.

- **Determinant of a square matrix of order 2**

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2, then $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

It follows that the value of a determinant of order 2 is equal to the product of the elements along the principal diagonal *minus* the product of the off diagonal elements.

- **Determinant of a square matrix of order 3**

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3,

then $|A| = (-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

or $|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{33}a_{21} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

It follows that the value of a determinant of order 3 is the sum of the product of elements a_{ij} in first row with $(-1)^{i+j}$ times the determinant of a 2×2 sub-matrix obtained by leaving the first row and column passing through elements.

➢ **Elementary properties of Determinants**

Property I: The value of a determinant is not altered by inter changing its rows into columns and columns into rows, i.*e.*,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & b_3 \end{vmatrix}$$

Property II: If any two adjacent rows or columns of a determinant are interchanged, the sign of the determinant get changed but its numerical value remains unaltered, *i.e.,*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property III: If any two rows or columns of a determinant are identical, the value of determinant is zero, *i.e.,*

Let, $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ or, $\begin{vmatrix} a_1 & a_2 & a_1 \\ b_1 & b_2 & b_1 \\ c_1 & c_2 & c_1 \end{vmatrix} = 0$

Property IV : If every element in a row or a column of a determinant is multiplied by the same non-zero constant k, then the value of the determinant gets multiplied by k, *i.e.,*

$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property V : A determinant can be expressed as the sum of several determinants of the same order, *i.e.,*

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Property VI : The value of the determinant is not affected if the elements of a row or column are increased or diminished by the same multiple of the corresponding elements of any other row or column, *i.e.,*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+pb_1+qc_1 & b_1 & c_1 \\ a_2+pb_2+qc_2 & b_2 & c_2 \\ a_3+pb_3+qc_3 & b_3 & c_3 \end{vmatrix}$$

Property VII : If each element of a row or column of a determinant is zero, its value is zero, *i.e.,*

$$\begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

➢ **Singular Matrix & Non-Singular Matrix:**

(a) Singular matrix : A square matrix A is said to be singular if $|A| = 0$ *i.e.,* its determinant is zero.

(b) Non-singular matrix : A square matrix A is said to be non-singular if $|A| \neq 0$.

- A square matrix A is **invertible** if and only if A is **non-singular**.

➢ **Minors :** Minors of an element a_{ij} of a determinant *(or a determinant corresponding to matrix* A*)* is the determinant obtained by deleting its i^{th} row and j^{th} column in which a_{ij} lies. Minor of a_{ij} is denoted by M_{ij}. Hence, we can get 9 minors corresponding to the 9 elements of a third order (*i.e.*, 3 × 3) determinant.

➢ **Co-factors :** Cofactor of an element a_{ij}, denoted by A_{ij}, is defined by $A_{ij} = (-1)^{(i+j)} M_{ij}$, where M_{ij} is minor of a_{ij}. Sometimes C_{ij} is used in place of A_{ij} to denote the cofactor of element a_{ij}.

➢ **Adjoint of a Square Matrix:**

Let $A = [a_{ij}]$ be a square matrix. Also, assume $B = [A_{ij}]$, where A_{ij} is the cofactor of the elements a_{ij} in matrix A. Then the transpose B^T of matrix B is called the **adjoint of matrix** A and it is denoted by **"*adj* (A)".**

To find adjoint of a 2 × 2 matrix: If the adjoint of a square matrix of order 2 can be obtained by interchanging the diagonal elements and changing the signs of off-diagonal elements *i.e.* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $adj\,A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

➢ **Algorithm to find A^{-1} by Determinant Method:**

Step 1 : Find $|A|$.

Step 2 : If $|A| = 0$, then, write "A is a singular matrix and hence not invertible". Else write "A is a non-singular matrix and hence invertible".

Step 3 : Calculate the co-factors of elements of matrix A.

Step 4 : Write the matrix of co-factors of elements of A and then obtain its transpose to get *adjA* (*i.e.*, adjoint A).

Step 5 : Find the inverse of A by using the relation: $A^{-1} = \frac{1}{|A|}(adj\,A)$.

Note
For a 2 × 2 matrix, the inverse is: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$ Just swap the 'a' and 'd', negate the 'b' and 'c', then divide all by the determinant $ad - bc$.

➢ **Properties associated with various Operation of Matrices & Determinants:**

(a) $AB = I = BA$

(b) $AA^{-1} = I$ or $A^{-1} I = A^{-1}$

(c) $(AB)^{-1} = B^{-1}A^{-1}$

(d) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(e) $(A^{-1})^{-1} = A$

(f) $(A^T)^{-1} = (A^{-1})^T$

(g) $A(adjA) = (adj\,A)A = |A|\,I$

(h) $adj(AB) = adj(B)\,adj\,(A)$

(i) $adj(A^T) = (adj\,A)^T$

(j) $(adj\,A)^{-1} = (adj\,A^{-1})$

(k) $|adj\,A| = |A|^{n-1}$, if $|A| \neq 0$, where n is of the order of A.

(l) $|AB| = |A|\,|B|$

(m) $|A\,adj\,A| = |A|^n$, where n is of the order of A.

(n) $|A^{-1}| = \frac{1}{|A|}$, if matrix A is invertible.

(o) $|A| = |A^T|$

- $|kA| = k^n\,|A|$, where n is of the order of square matrix A and k is any scalar.
- If A is a non-singular matrix of order n, then $adj\,(adj\,A) = |A|^{n-2} A$.

➢ **Cramer's Rule:**

Let the three nonhomogeneous linear equations be

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

The solution of the system of linear equations is given by

$$x = \frac{D_x}{D},\ y = \frac{D_y}{D} \text{ and } z = \frac{D_z}{D}$$

Where,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}, \text{ Provided that } D \neq 0$$

- **Condition for consistency:**

(a) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by

$$x = \frac{D_x}{D}, \; y = \frac{D_y}{D} \text{ and } z = \frac{D_z}{D}$$

(b) If $D = 0$ and $D_x = D_y = D_z = 0$, the given system of equations is consistent has infinitely many solutions.

(c) If $D = 0$ and atleast one of the determinants D_x, D_y, D_z is non-zero, then the given system of equations is inconsistent.

- **Leontief Input-Output Model**

Based on the assumption that each industry in the economy has two types of demands: external demand (from outside the system) and internal demand (demand placed on one industry by another in the same system), the Leontief model represents the economy as a system of linear equations.

The main goal of the Leontief input output model is to balance the total amount of goods produced to the total demand that for production.

Total Output = Intermediate demand + Final demand

Consider a simple economic model consisting of two industries A_1 and A_2 where each produces only one type of product. Assume that each industry consumes part of its own output and rest from the other industry for its operation. The industries are thus interdependent. Further assume that whatever is produced that is consumed. That is the total output of each industry must be such as to meet its own demand, the demand of the other industry and the external demand (final demand).

Our aim is to determine the output levels of each of the two industries in order to meet a change in final demand, based on knowledge of the current outputs of the two industries, of course under the assumption that the structure of the economy does not change.

Let a_{ij} be the rupee value of the output of A_i consumed by A_j, $i, j = 1, 2$

Let x_1 and x_2 be the rupee value of the current outputs of A_1 and A_2 respectively.

Let d_1 and d_2 be the rupee value of the final demands for the outputs of A_1 and A_2 respectively.

The assumptions lead us to frame the two equations

$$a_{11} + a_{12} + d_1 = x_1; \qquad a_{21} + a_{22} + d_2 = x_2 \qquad \text{...(i)}$$

Let, $b_{ij} = \dfrac{a_{ij}}{x_1} i, j = 1, 2$

$$b_{11} = \frac{a_{11}}{x_1};\; b_{12} = \frac{a_{12}}{x_2};\; b_{21} = \frac{a_{21}}{x_1};\; b_{22} = \frac{a_{22}}{x_2}$$

The equations (i) take the form

$$b_{11}x_1 + b_{12}x_2 + d_1 = x_1 \qquad b_{21}x_1 + b_{22}x_2 + d_2 = x_2$$

The above equations can be rearranged as

$$(1 - b_{11})x_1 - b_{12}x_2 = d_1$$
$$-b_{21}x_1 + (1 - b_{22})x_2 = d_2$$

The matrix form of the above equations is

$$\begin{bmatrix} 1 - b_{11} & -b_{12} \\ -b_{21} & 1 - b_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$(I - B)X = D$$

where, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } D = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

By solving we get $\boxed{X = (I - B)^{-1}D}$

The matrix B is known as Technology matrix or Input Output matrix or Leontief matrix.

Hawkins-Simons Conditions

(i) $|I - B| > 0$

(ii) The diagonal elements of the leontief matrix I-B should all be positive i.e., $1-b_{11}$, $1-b_{22}$ should all be positive or b_{11}, b_{33} should all be less than 1.

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UNIT – III : CALCULUS

CHAPTER-5

DIFFERENTIATION AND ITS APPLICATIONS

➢ **Higher Order Derivatives of an Explicit Function**

Let the function $y = f(x)$ have a finite derivative $f'(x)$ in a certain interval (a, b), *i.e.*, the derivative $f'(x)$ is also a function in this interval. If this function is differentiable, we can find the second derivative of the original function $y = f(x)$, which is denoted by various notations as

$$f'' = (f')' = \left(\frac{dy}{dx}\right)'$$
$$= \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$= \frac{d^2y}{dx^2}$$

Similarly, if f'' exists and is differentiable, we can calculate the third derivative of the function $f(x)$:

$$f''' = \frac{d^3y}{dx^3} = y'''$$

The result of taking the derivative n times is called the n^{th} derivative of $f(x)$ with respect to x and is denoted as

$$\frac{d^ny}{dx^n} = \frac{d^ny}{dx^n} \text{ (in Leibniz's notation)}$$

$$f^{(n)}(x) = y^{(n)}(x) \text{ (in Lagrange's notation)}$$

Thus, the notation of the nth order derivative in introduced inductively by sequential calculation of n derivatives starting from the first order derivative. Transition to the next higher-order derivative is performed using the recurrence formula

$$y^{(n)} = (y^{(n-1)})'$$

➢ **Higher Order Derivatives of an Implicit Function**

The nth order derivative of an implicit function can be found by sequential (n times) differentiation of the equation $F(x, y) = 0$. At each step, after appropriate substitutions and transformations, we can obtain an explicit expression for the derivative, which depends only on the variables x and y, *i.e.*, the derivatives have the form

$$y' = f_1(x, y)$$
$$y'' = f_2(x, y), \ldots$$
$$y^n = f_n(x, y)$$

➢ **Higher Order Derivatives of Parametric Function**

Consider a function $y = f(x)$ given parametrically by the equations

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

The first derivative of this function is given by

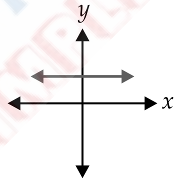
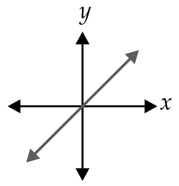
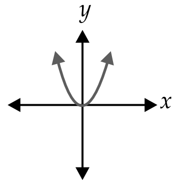
$$y' = y'_x = \frac{y'_t}{x'_t}$$

Differentiating once more with respect to x, we find the second derivative:

$$y'' = y''_{xx} = \frac{(y'_x)'_t}{x'_t}$$

➢ **Increasing/Decreasing Functions**

- A function $f(x)$ is said to be an increasing function in $[a, b]$, if as x increases, $f(x)$ also increases, *i.e.*, if α, β c $[a, b]$ and $\alpha > \beta$, $f(\alpha) > f(\beta)$.
  If $f'(x) \geq 0$ lies in (a, b), then $f(x)$ is an increasing function in $[a, b]$, provided $f(x)$ is continuous at $x = a$ and $x = b$.
- A function $f(x)$ is said to be a decreasing function in $[a, b]$, if as x increases, $f(x)$ decreases, *i.e.*, if α, β c $[a, b]$ and $\alpha > \beta \Rightarrow f(\alpha) < f(\beta)$.
  If $f'(x)$ [0 lies in (a, b), then $f(x)$ is a decreasing function in $[a, b]$, provided $f(x)$ is continuous at $x = a$ and $x = b$.
  - ♦ A function $f(x)$ is a constant function in $[a, b]$, if $f'(x) = 0$ for each x c (a, b).
  - ♦ By monotonic function $f(x)$ in interval I, we mean that f is either only increasing in I or only decreasing in I.
- Finding the intervals of increasing and/or decreasing of a function:
  Algorithm
  Step 1: Consider the function $y = f(x)$.
  Step 2: Find $f'(x)$.
  Step 3: Put $f'(x) = 0$ and solve to get the critical point(s).
  Step 4: The value(s) of x for which $f'(x) > 0$, $f(x)$ is increasing; and the value(s) of x for which $f'(x) < 0$, $f(x)$ is decreasing.

Function	Increasing/Decreasing	Graph
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	

➢ **Maxima and Minima**

- **Understanding maxima and minima:**
  Consider $y = f(x)$ be a well defined function on an interval I, then
  (a) f is said to have a maximum value in I, if there exists a point c in I such that $f(c) > f(x)$, for all x c I
  The value corresponding to $f(c)$ is called the maximum value of f in I and the point c is called the **point of maximum value of f in I**.
  (b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) < f(x)$, for all x c I.
  The value corresponding to $f(c)$ is called the minimum value of f in I and the point c is called the **point of minimum value of f in I**.
  (c) f is said to have an extreme value in I, if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I.
  The value $f(c)$ in this case, is called an extreme value of f in I and the point c called an **extreme point**.
- **Let f be a real valued function and also take a point c from its domain. Then**
  (i) c is called a point of **local maxima** if there exists a number $h > 0$ such that $f(c) > f(x)$, for all x in $(c - h, c + h)$. The value $f(c)$ is called the **local maximum value of f**.
  (ii) c is called a point of **local minima** if there exists a number $h > 0$ such that $f(c) < f(x)$, for all x in $(c - h, c + h)$. The value $f(c)$ is called the **local minimum value** of f.

- **Critical points:**
  It is a point c (say) in the domain of a function $f(x)$ at which either $f'(x)$ vanishes, *i.e.*, $f'(c) = 0$ or f is not differentiable.
- **First Derivative Test:**
  Consider $y = f(x)$ be a well defined function on an open interval I. Now proceed as have been mentioned in the following algorithm:

  Step 1: Find $\frac{dy}{dx}$.

  Step 2: Find the critical point(s) by putting $\frac{dy}{dx} = 0$. Suppose c c I (where I is the interval) be any critical point and f be continuous at this point c. Then we may have following situations:

  ♦ $\frac{dy}{dx}$ changes sign from **positive to negative** as x increases through c, then the function attains a **local maximum** at $x = c$.

  ♦ $\frac{dy}{dx}$ changes sign from **negative to positive** as x increases through c, then the function attains a **local minimum** at $x = c$.

  ♦ $\frac{dy}{dx}$ **does not change sign** as x increases through c, then $x = c$ is **neither** a point of **local maximum nor** a point of **local minimum**. Rather in this case, the point $x = c$ is called the **point of inflection**.
- **Second Derivative Test:**
  Consider $y = f(x)$ be a well defined function on an open interval I and twice differentiable at a point c in the interval. Then we observe that:

  ♦ $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$
  The value $f(c)$ is called the local maximum value of f.

  ♦ $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$
  The value $f(c)$ is called the local minimum value of f.
  This test fails if $f'(c) = 0$ and $f''(c) = 0$. In such a case, we use **first derivative test** as discussed in the **above.**
- **Absolute maxima and absolute minima:**
  If f is a continuous function on a **closed interval** I, then f has the absolute maximum value and f attains it atleast once in I. Also f has the absolute minimum value and the function attains it atleast once in I.

  Algorithm

  Step 1: Find all the critical points of f in the given interval, *i.e.*, find all the points x where either $f'(x) = 0$ or f is not differentiable.

  Step 2: Take the end points of the given interval.

  Step 3: At all these points (*i.e.*, the points found in Step 1 and Step 2) calculate the values of f.

  Step 4: Identify the maximum and minimum value of f out of the values calculated in Step 3. This maximum value will be the **absolute maximum value** of f and the minimum value will be the **absolute minimum value** of the function f.

  Absolute maximum value is also called as **global maximum value** or **greatest value**. Similarly, absolute minimum value is called as **global minimum value** or the **least value**.

➢ **Marginal Cost and Marginal Revenue**

Derivatives are useful in analyzing changes in cost, revenue, and profit. The marginal cost is the derivative of the cost function. The marginal revenue is the derivative of the revenue function. The marginal profit is the derivative of the profit function, which is based on the cost function and the revenue function.

- The Fixed Cost (FC) is the amount of money we have to spend regardless of how many items we produce. FC can include things like rent, purchase costs of machinery, and salaries for office staff. We have to pay the fixed costs even if we don't produce anything.
- The Total Variable Cost (TVC) for q items is the amount of money we spend to actually produce them. TVC includes things like the materials we use, the electricity to run the machinery, gasoline for our delivery vans, maybe the wages of our production workers. These costs will vary according to how many items we produce.
- The Total Cost (TC) for x items is the total cost of producing them. It's the sum of the fixed cost and the total variable cost for producing q items.

- The Average Cost (AC) for x items is the total cost divided by x *i.e.*, $\frac{\text{TC}}{x}$.

  Also, Average Fixed Cost = $\frac{\text{FC}}{x}$ and Average Variable Cost = $\frac{\text{TVC}}{x}$.
- If C(x) is the cost of producing x items, then the marginal cost $MC(x)$ is $MC(x) = C'(x)$.
- If R(x) is the revenue obtained from selling x items, then the marginal revenue $MR(x)$ is $MR(x) = R'(x)$.
- If $P(x) = R(x) - C(x)$ is the profit obtained from selling x items, then the marginal profit MP(x) is defined to be $MP(x) = P'(x) = MR(x) - MC(x) = R'(x) - C'(x)$.
- If R(x) is the revenue obtained from selling x items, then the marginal revenue $MR(x)$ is $MR(x) = R'(x)$.
- If $P(x) = R(x) - C(x)$ is the profit obtained from selling x items, then the marginal profit MP(x) is defined to be $MP(x) = P'(x) = MR(x) - MC(x) = R'(x) - C'(x)$.
- Break even point (*BEP*) is the point where the profit from the transaction is zero and the total sales is equal to total costs. Break even point is the inflection point where the revenue sales are same as the costs. At the break even point, there is zero profit or zero loss for the company.

  Break even point (*BEP*) = $\frac{\text{Fixed Costs}}{(\text{Selling Price} - \text{Variable Costs})}$

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UNIT – IV : PROBABILITY DISTRIBUTIONS

CHAPTER-6
PROBABILITY DISTRIBUTIONS

Expectation and Variance of Probability Distribution

➢ **Random Variable:** A random variable (r.v.) is a real valued function defined on a sample space S and taking value in $(-\infty, \infty)$ or whose possible values are numerical outcomes of a random experiment.

➢ **Types of Random Variable:** Random variables are classified into two types namely discrete and continuous random variables. These are important for practical application in the field of Mathematics and statistics. The above types of random variable are defined with examples as follows.

- **Discrete random variable:** A variable which can assume finite number of possible values or an infinite sequence of countable real numbers is called a discrete random variable.

Examples of discrete random variable:

- Marks obtained in a test.
- Number of telephone calls at a particular time.

Probability Mass Function of Discrete Random Variable:

If X is a discrete random variable with distinct values $x_1, x_2, ..., x_n, ...$, then the function, denoted by $P_X(x)$ and

defined by $P_X(x) = p(x) = \begin{cases} P(X = x_i) = p_i = p(x_i) & \text{if} \quad x = x_i,\ i = 1, 2, ..., n, ... \\ 0 & \text{if} \quad x \neq x_i \end{cases}$

This is defined to be the probability mass function or discrete probability function of X. The probability mass function $p(x)$ must satisfy the following conditions

(i) $p(x_i) \geq 0\ \forall\ i$, **(ii)** $\sum_{i=1}^{\infty} p(x_i)$

Discrete Distribution Function

The discrete cumulative distribution function or distribution function of a real valued discrete random variable X takes the countable number of points $x_1, x_2, ...$ with corresponding probabilities $p(x_1), p(x_2), ...$ and then the cumulative distribution function is defined by

$$F_X(x) = P(X \le x), \text{ for all } x \in R$$

i.e., $F_X(x) = \sum_{x_i \le x} p(x_i)$

- **Continuous random variable**

  A random variable X which can take on any value (integral as well as fraction) in the interval is called continuous random variable.

  Examples of continuous random variable
- The amount of water in a 10 ounce bottle.
- Height of people in a population.

Probability Mass Function of Continuous Random Variable

The probability that a random variable X takes a value in the interval $[t_1, t_2]$ (open or closed) is given by the integral of a function called the probability density function $f_X(x)$:

$$P(t_1 \le X \le t_2) = \int_{t_1}^{t_2} f_X(x)dx.$$

Other names that are used instead of probability density function include density function, continuous probability function, integrating density function.

The probability density function $f_X(x)$ or simply by $f(x)$ must satisfy the following conditions.

(i) $f(x) \ge \forall x$ and (ii) $\int_{-\infty}^{\infty} f(x)dx = 1.$

Continuous Distribution Function

If X is a continuous random variable with the probability density function $f_X(x)$, then the function $F_X(x)$ is defined by $F_X(x) = P[X \le x] = \int_{-\infty}^{x} f(t)dt, \; -\infty < x < \infty$ is called the distribution function (d.f.) or sometimes the cumulative distribution function (c.d.f) of the continuous random variable X.

- **Expected value:** The expected value is a weighted average of the values of a random variable may assume. The weights are the probabilities.

  Let X be a discrete random variable with probability mass function (p.m.f.) $p(x)$. Then, its expected value is defined by

$$E(X) = \sum_x xp(x) \quad \text{...(1)}$$

If X is a continuous random variable and $f(x)$ is the value of its probability density function at x, the expected value of X is

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx \quad \text{...(2)}$$

- **Variance:** The variance is a weighted average of the squared deviations of a random variable from its mean. The weights are the probabilities. The mean of a random variable X, defined in (1) and (2), was a measure of central location of the density of X. The variance of a random variable X will be a measure of the spread or dispersion of the density of X or simply the variability in the values of a random variable.

  The variance of X is defined by

$$Var(X) = \sum [x - E(X)]^2 p(x) \quad \text{...(3)}$$

If X is discrete random variable with probability mass function $p(x)$.

$$Var(X) = \int_{-\infty}^{\infty} [x - E(X)]^2 f_X(dx)dx \quad \text{...(4)}$$

if X is continuos random variable with probability density function $f_X(x)$.

Expected value of $[X - E(X)]^2$ is called the variance of the random variable.

i.e., $Var(X) = E[X - E(X)]^2 = E(X^2) - [E(X)]^2$ ...(5)

where $E(X^2) = \begin{cases} \sum_x x^2 p(x), & \text{if } X \text{ is Discrete Random Variable} \\ \int_{-\infty}^{\infty} x^2 f(x)dx, & \text{if } X \text{ is Continuous Random Variable} \end{cases}$

Binomial and Poisson Distributions

Revision Notes

➢ **Bernoulli Trials**

Trials of a random experiment are called Bernoulli trials, if they satisfy the following four conditions :

(a) The trials should be finite in numbers.
(b) The trials should be independent of each other.
(c) Each of the trial yields exactly two outcomes *i.e.*, success or failure.
(d) The probability of success or the failure remains the same in each of the trial.

If an experiment is repeated n times under the similar conditions, we say that n trials of the experiment have been made.

➢ **Binomial Distribution**

Let E be an event. Let p = probability of success in one trial (*i.e.*, occurrence of event E in one trial) and, $q = 1 - p$ = probability of failure in one trial (*i.e.*, non-occurrence of event E in one trial).

Let X = number of successes (*i.e.*, number of times event E occurs in n trials)

Then, Probability of X successes in n trials is given by the relation:

$$P(X = r) = P(r) = {}^nC_r p^r q^{n-r},$$

where $r = 0, 1, 2, 3, \ldots., n;$ p = probability of success in one trial and $q = 1 - p$ = probability of failure in one trial.

Note :
The result $P(X = r) = P(r) = {}^nC_r p^r q^{n-r}$ can be used only when :
(i) the probability of success in each trial is the same.
(ii) each trial must surely result in either a success or a failure.

➢ **Special Cases**

- $P(X = r)$ or $P(r)$ is also called the probability of occurrence of event E exactly r times in n trials.
- Here ${}^nC_r = \dfrac{n!}{r!(n-r)!}$
- Note that ${}^nC_r p^r q^{n-r}$ is the$(r + 1)^{\text{th}}$ term in the binomial expansion of $(q + p)^n$.
- Mean $= \sum_{r=0}^{n} r.P(r) = np$
- Variance $= \sum_{r=0}^{n} r^2.P(r) - (\text{Mean})^2 = npq$
- Recurrence formula, $P(x = r + 1) = \left(\dfrac{n-r}{r+1}\right)\left(\dfrac{p}{q}\right)P(r)$
- A Binomial distribution with n Bernoulli trials and probability of success in each trial as p is denoted by $B(n, p)$. Here n and p are known as the parameters of binomial distribution.
- The expression $P(x = r)$ or $P(r)$ is called the probability function of the binomial distribution.

➢ **Poisson Distribution**

It is a discrete probability distribution. The Poisson distribution is a particular limiting form of Binomial distribution when p (or q) is very small and n is large enough so that np (or nq) is a finite constant say m.

A probability distribution of a random variable x is called Poisson distribution if x can assume non-negative integral values only and the distribution is given by

$$P(r) = P(X = r) = \begin{cases} \dfrac{e^{-m}m^r}{r!}, & r = 0, 1, 2\ldots \\ 0, & r \neq 0, 1, 2\ldots \end{cases}$$

Here, the value of $e = 2.7183$, it is the base of the natural system of logarithms.

Examples of Poisson Variates

(i) The number of cars passing through a certain street in time t.
(ii) The number of defective screws per box of 100 screws.

- **Requirements for a Poisson Distribution**
  (1) Random variable (x) is the number of occurrences of an events **over some interval**.
  (2) Occurrences must be **random**
  (3) Occurrences must be **independent** of each other
  (4) Occurrences must be **uniformly distributed** over the interval being used.
- **Constants of the Poisson Distribution**
  (i) Mean, $E(X) = m = np$
  (ii) Variance, $V(X) = m = np$

➢ **Difference between Binomial and Poisson Distributions**

- The binomial distribution is affected by the sample size n and the probability p, whereas the Poisson distribution is affected only by the mean μ.
- In a binomial distribution the possible values of the random variable x are 0, 1, ... n but a Poisson distribution has possible x values of 0, 1, ..., with no upper limit.

Normal Distribution

➢ Normal distribution is a continuous probability distribution in which the relative frequencies of a continuous variable are distributed according to normal probability. In simple words, it is a symmetrical distribution in which the frequencies are distributed evenly about the mean of distribution.

e.g., : **(i)** Height and intelligence are approximately normally distributed.
(ii) Measurement of errors also often have a normal distribution.

The Normal (or Guassian) Distribution is defined by the probability density function

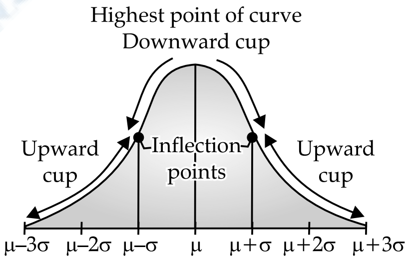
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \text{ for } -\infty < x < \infty$$

where μ and σ > 0 are parameters of the distribution.

Clearly, f(x) is non-negative and $\int_{-\infty}^{\infty} f(x)dx = 1$.

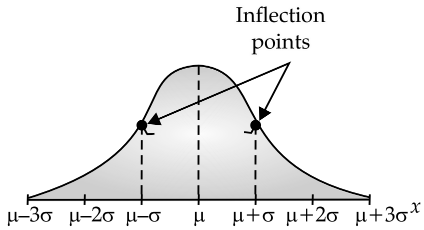
The notation $N(\mu, \sigma^2)$ means normally distributed with mean μ and variance σ^2. If we say $x \sim N(\mu, \sigma^2)$ we mean that x is distributed $N(\mu, \sigma^2)$.

➢ **Normal distribution is diagrammatically represented as follows:**



Normal curve

Normal distribution is limiting case of binomial distribution under the following conditions:
(i) n, the number of trials is infinitely large, *i.e.*, $n \to \infty$
(ii) neither p(nor q) is very small,
The normal distribution of a variable when represented graphically, takes the shape of a symmetrical curve, known as the Normal Curve. The curve is asymptotic to x-axis on its either side.
Chief Characteristics or Properties of Normal Probability distribution and Normal probability Curve.
The normal probability curve with mean μ and standard deviation σ has the following properties:
(i) The curve is bell- shaped and symmetrical about the line $x = \mu$
(ii) Mean, median and mode of the distribution coincide.
(iii) x-axis is an asymptote to the curve, (tails of the curve never touches the horizontal (x) axis).
(iv) No portion of the curve lies below the x-axis as $f(x)$ being the probability function can never be negative.



(v) The points of inflection of the curve are $x = \mu \pm \sigma$

(vi) The curve of a normal distribution has a single peak *i.e.*, it is a unimodal.

(vii) As x increases numerically, $f(x)$ decreases rapidly, the maximum probability occurring at the point $x = \mu$ and is given by $[p(x)]\text{max} = \dfrac{1}{\sigma\sqrt{2\pi}}$

(viii) The total area under the normal curve is equal to unity and the percentage distribution of area under the normal curve is given below.

(a) About 68.27% of the area falls between $\mu - \sigma$ and $\mu + \sigma$

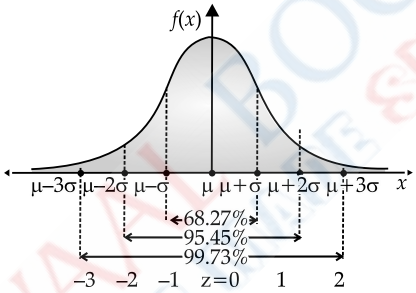
$$P(\mu - \sigma < X < \mu + \sigma) = 0.6826$$

(b) About 95.5% of the area falls between $\mu - 2\sigma$ and $\mu + 2\sigma$

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$$

(c) About 99.7% of the area falls between $\mu - 3\sigma$ and $\mu + 3\sigma$

$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$$



➢ **Standard Normal Distribution**

A random variable $Z = (X - \mu)/\sigma$ follows the standard normal distribution. Z is called the standard normal variate with mean 0 and standard deviation 1 *i.e.*, $Z \sim N(0, 1)$. Its Probability density function is given by :

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty < z < \infty$$

1. The area under the standard normal curve in equal to 1.

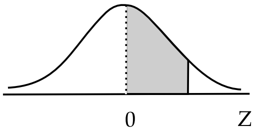
2. 68.26% of the area under the standard normal curve lies between Z = –1 and Z = 1

3. 95.44% of the area lies between Z = – 2 and Z = 2

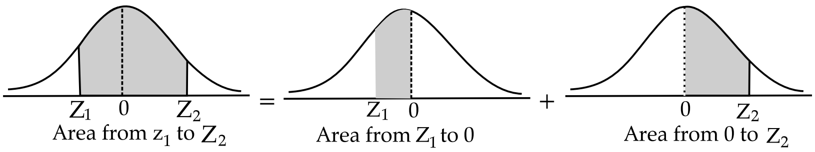
4. 99.74% of the area lies between Z = – 3 and Z = 3

➢ **Patterns for Finding Areas Under the Standard Normal Curve :**

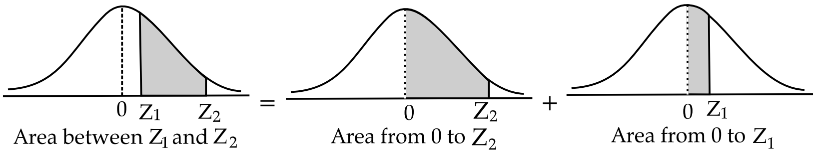
(a) Area between a given Z value and 0



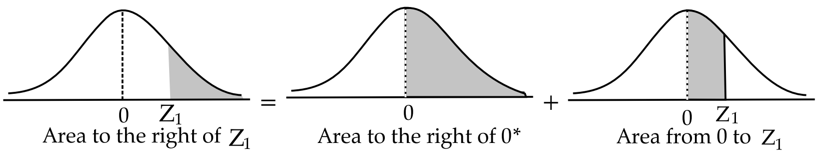
(b) Area between Z values on either side of 0



(c) Area between Z value on same side of 0.

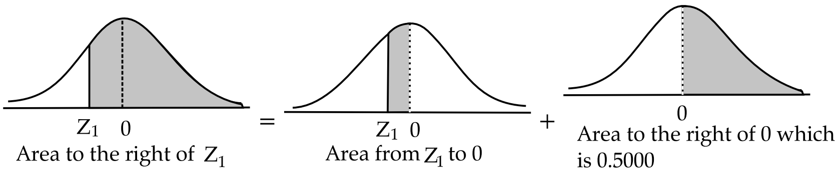


(d) Area to the right of a positive Z value or to the left of a negative Z value



This area = 0.5000 since the area under the entire curve is 1 and the area to the right of 0 is half the area under the entire curve

(e) Area of the right of a negative Z value or to the left of a positive Z value.



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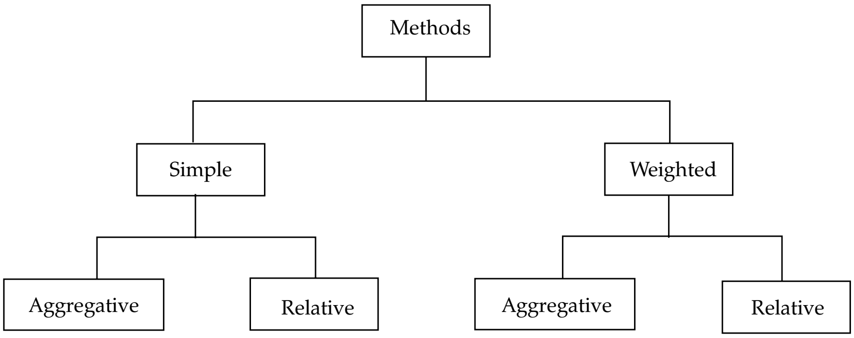
UNIT – V : INDEX NUMBERS AND TIME BASED DATA

CHAPTER-7
INDEX NUMBERS AND TIME BASED DATA

Index Numbers

- Index Numbers are indicators which reflect the changes over a specified period of time in price of different commodities, production, sales, cost of living etc. Index numbers are statistical methods used to measure the relative change in the level of a variable or group of variables with respect to time, geographical location or other characteristics such as income, profession etc. The variables may be
  (i) The price of a particular commodity. For example gold, silver, iron (or) a group of commodities. For example consumer goods, household food items etc...
  (ii) The volume of export and import, agricultural and industrial production.
  (iii) National income of a country, cost of living of persons belonging to a particular income group.
- **Classification of Index Numbers: Index numbers can be classified as follows:**
  (i) **Price Index Number:** It measures the general changes in the retail or wholesale price level of a particular commodity or group of commodities.
  (ii) **Quantity Index Number:** These are indices to measure the changes in the quantity of goods manufactured in a factory.
  (iii) **Cost of living Index Number:** These are intended to study the effect of change in the price level on the cost living of different classes of people.
- **Uses of the Index Numbers**
  1. Index numbers work as Economic Barometers.
  2. Index numbers help us in framing suitable policies.
  3. Index numbers are helpful in determining trends and tendencies.
  4. Index numbers are used to measure the purchasing power of money.
- **Construction of Index Numbers**
  There are two methods of construction of index numbers :



(i) Simple Aggregative Method: In this method of computing a price index, we express the total of commodity prices in a given year as a percentage of total commodity prices in the base year. In symbols, we have

$$\text{Simple aggregative price index } P_{01} = \frac{\Sigma P_1}{\Sigma P_0} \times 100$$

Where

ΣP_1 is the sum of all commodity prices in the current year

ΣP_0 is the sum of all commodity prices in the base year

(ii) Relative Aggregative Method: In this method, the index number is equal to the sum of price relatives divided by the number of items and is calculated by using the following formula:

$$P_{01} = \frac{\Sigma R}{N}$$

where, ΣR stands for the sum of price relatives,

i.e. $R = \dfrac{P_1}{P_0} \times 100$

N stands for the number of items

(iii) Weighted Aggregative Method: In general, all the commodities cannot be given equal importance, so we can assign weights to each commodity according to their importance and the index number computed from these weights are called as weighted index number. The weights can be production, consumption values. If 'w' is the weight attached to a commodity, then the price index is given by,

$$\text{Price Index, } p_{01} = \frac{\Sigma p_1 w}{\Sigma p_0 w} \times 100$$

Where,

p_1 = current year price

p_0 = base year price

q_1 = current year quantity

q_0 = base year quantity

here suffix '0' represents base year and '1' represents current year.

Laspeyre's price index number $P_{01}^{L} = \dfrac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$

Paasche's price index number $P_{01}^{P} = \dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$

Fisher's price index number $p_{01}^{F} = \sqrt{p_{01}^{L} \times p_{01}^{P}}$

$$p_{01}^{F} = \sqrt{\frac{\Sigma p_1 q_0 \, \Sigma p_1 q_1}{\Sigma p_0 q_0 \, \Sigma p_0 q_1}} \times 100$$

Marshall-Edgeworth's Index Number = $p_{01} = \dfrac{\Sigma p_1 q_0 + \Sigma p_1 q_1}{\Sigma p_0 q_0 + \Sigma p_0 q_1} \times 100$

Note : Fisher's price index number is the geometric mean between Laspeyre's price index number and Passche's price index number.

(iv) Weighted Average of Relative Method: In this method we make use of price relatives. When the base and current prices of a number of items along with weights or quantities are given, their weighted average of price relatives is given by

$$p_{01} = \frac{\sum \left(\frac{p_1}{p_0} \times 100 \right) \times w}{\sum w}$$

- **Test of Adequacy for an Index Number**

  Index numbers are studied to know the relative changes in price and quantity for any two years compared. There are two tests which are used to test the adequacy for an index number. The two tests are as follows,

  (i) Unit test

  (ii) Time Reversal Test

  - **Unit Test:** This is a common test which requires that an index number formula should be such that it does not affect the value of the index number, even if, the units of the price quotations are altered viz. price per kg, converted into price per quintal or vice versa. This test is satisfied by all the index formulae except the simple aggregative method under which the value of the index number changes radically, if the units of price quotations of any of the items included in the index number are changed.
  - **Time Reversal Test:** It is an important test for testing the consistency of a good index number. This test maintains time consistency by working both forward and backward with respect to time (here time refers to base year and current year). Symbolically the following relationship should be satisfied, $p_{01} \times p_{10} = 1$

    Fisher's index number formula satisfies the above relationship

    $$p_{01}^{F} = \sqrt{\frac{\sum p_1 q_0 \times \sum p_1 q_1}{\sum p_0 q_0 \times \sum p_0 q_1}}$$

    When the base year and current year are interchanged, we get

    $$p_{10}^{F} = \sqrt{\frac{\sum p_0 q_1 \times \sum p_0 q_0}{\sum p_1 q_1 \times \sum p_1 q_0}}$$

    $$p_{01}^{F} \times p_{10}^{F} = 1$$

- **Construction of Cost Living Index**

  Cost of Living Index Number is constructed to study the effect of changes in the price of goods and services of consumers for a current period as compared with base period. The change in the cost of living index number between any two periods means the change in income which will be necessary to maintain the same standard of living in both the periods. Therefore the cost of living index number measures the average increase in the cost to maintain the same standard of life.

  Methods of constructing Cost of Living Index Number

  The cost of living index number can be constructed by the following methods,

  (i) Aggregate Expenditure Method (or) Weighted Aggregative Method.

  (ii) Family Budget Method.

  (i) **Aggregate Expenditure Method:** This is the most common method used to calculate cost of living index number. In this method, weights are assigned to various commodities consumed by a group in the base year. In this method the quantity of the base year is used as weight.

  The formula is given by,

  $$\text{Cost of Living Index Number} = \frac{\sum p_1 q_0}{\sum p_0 q_0} \times 100$$

  Note : The formula for Aggregate Expenditure Method to calculate Cost of Living Index Number is same as formula of Laspeyre's Method.

  (ii) **Family Budget Method:** In this method, the weights are calculated by multiplying prices and quantity of the base year *i.e.*,

  $V = \Sigma p_0 q_0$. The formula is given by,

  $$\text{Cost of Living Index Number} = \frac{\sum PV}{\sum V}$$

  where, $P = \frac{p_1}{p_0} \times 100$ is the price relative.

  $V = \Sigma p_0 q_0$ is the value relative.

  Note : This method is same as the weighted average of price relative method.

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Unit-VI: Inferential Statistics

CHAPTER 8: Inferential Statistics

Inferential statistics is mainly used to derive estimates about a large group (or population) and draw conclusions on the data, based on hypothesis testing methods.

- **Population:** The group of individuals considered under study is called as population. The word population here refers not only to people but to all items that have been chosen for the study. Thus in statistics, population can be number of bikes manufactured in a day or week or month, number of cars manufactured in a day or week or month, number of fans, TVs, chalk pieces, people, students, girls, boys, any manufacturing products, etc.

  Finite and Infinite Population: When the number of observations/individuals/products is countable in a group, then it is a finite population.

  Example: Weights of students of class XII in a school.

  When the number of observations/individuals/products is uncountable in a group, then it is an infinite population.

  Example: Number of grains in a sack, number of germs in the body of a sick patient.

- **Sample and Sample Size:** A selection of a group of individuals from a population in such a way that it represents the population is called as sample and the number of individuals included in a sample is called the sample size.
- **Sampling:** Sampling is the procedure or process of selecting a sample from a population. Sampling is quite often used in our day-to-day practical life.
- **Parameter:** The statistical constants of the population like mean (μ), variance (σ^2) are referred as population parameters.

  Statistic: Any statistical measure computed from sample is known as statistics.

$$\mu \text{ (mean)} = \frac{\sum_{i=1}^{n} x_i}{N}$$

$$\sigma^2\text{(variance)} = \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{N}$$

$$\sigma\text{(standard deviation, SD)} = \sqrt{\sum_{i=1}^{n} \frac{(x_i - \mu)^2}{N}}$$

where N is the population size.

- **Errors in a Sample:** A sample is a part of the whole population. A sample drawn from the population depends upon chance and as such all the characteristics of the population may not be present in the sample drawn from the same population. The errors involved in the collection, processing and analysis of the data may be broadly classified into two categories namely,

  (i) Sampling Errors

  (ii) Non-Sampling Errors

  (i) Sampling Errors: Errors, which arise in the normal course of investigation or enumeration on account of chance, are called sampling errors. Sampling errors are inherent in the method of sampling. They may arise accidentally without any bias or prejudice.

  (ii) Non-Sampling Errors: The errors that arise due to human factors which always vary from one investigator to another in selecting, estimating or using measuring instruments (tape, scale) are called Non-Sampling errors.

➢ **Types of Sampling:** There are various techniques of sampling, but they can be broadly grouped into two:

- Random or probability sampling.
- Non-Random or Non-Probability sampling.

We will consider only random (Probability) sampling.

Random sampling or Probability sampling: Random sampling refers to selection of samples from the population in a random manner. A random sample is one where each and every item in the population has an equal chance of being selected.

The following are different types of probability sampling:

(i) Simple Random Sampling: Simple random sampling is the randomized selection of a small segment of individuals or members from a whole population. It provides each individual or member of a population with an equal and fair chance of being chosen. This is one of the most common method of sampling.

e.g.: There are 1000 students in a school, we want to select a simple random sample of 100 students. We can assign a number to every student in the school from 1 to 1000 and select randomly 100 numbers.

(ii) Systematic Sampling: Systematic sampling is the selection of specific individuals or members from entire population. The selection often follows a predetermined interval (k). The systematic sampling method is comparable to the simple random sampling method; however it is less complicated to conduct.

e.g.: Out of 1000 students of school, we want to select a sample of 100 students. All students of the school are arranged in alphabetical order and assigned a number 1 to 1000. Now, we randomly select a number (say 4) from first 10 numbers and then every 10^{th} student in the list is selected *i.e.*, 4, 14, 24, ... and end up with sample of 100 members.

(iii) Stratified Sampling: Stratified sampling includes the partitioning of a population into subclasses with notable distinctions and variances. This method is useful when population is dispersed.

e.g.: Suppose a school has its branches in 15 cities. We want to select the sample of 100 students. It is difficult to select students from each branch. So first select any 5 branches and then select students from each branch.

➢ **Sampling Distribution:** Sampling distribution of a statistic is the frequency distribution which is formed with various values of a statistic computed from different samples of the same size drawn from the same population.

For instance, if we draw a sample of size n from a given finite population of size N, then the total number of possible samples are ${}^{N}C_{n} = \dfrac{N!}{n!(N-n)!} = k$(say). For each of these k samples we can compute some statistic $t = t(x_1, x_2, x_3, \ldots x_n)$, in particular the mean x, the variance S^2, etc., is given below:

Sample Number	Statistic		
	t	$\bar{x}$	S^2
1	t_1	$\bar{x}_1$	S_1^2
2	t_2	$\bar{x}_2$	S_2^2
3	t_3	$\bar{x}_3$	S_3^2
⋮	⋮	⋮	⋮
k	t_k	$\bar{x}_k$	S_k^2

The set of the values of the statistic so obtained, one for each sample, constitutes the sampling distribution of the statistic.

➢ **Standard Error:** The standard deviation of the sampling distribution of statistic known as its Standard Error abbreviated as S.E. The Standard Errors (S.E.) of some of the well-known statistics, for large samples, are given below, where n is the sample size, σ^2 is the population variance.

S. No	Statistic	Standard Error
1.	Sample mean ($\bar{x}$)	$\frac{\sigma}{\sqrt{n}}$
2.	Observed sample proportion (p)	$\sqrt{\frac{PQ}{n}}$
3.	Sample standard deviation (s)	$\sqrt{\frac{\sigma^2}{2n}}$
4.	Sample variance (s^2)	$\sigma^2\sqrt{\frac{2}{n}}$

➢ **Statistical Inferences:** One of the main objectives of any statistical investigation is to draw inferences about a population from the analysis of samples drawn from that population. Statistical Inference provides us how to estimate a value from the sample and test that value for the population. This is done by the two important classifications in statistical inference,

(i) Estimation

(ii) Testing of Hypothesis

- **Estimation:** The method of obtaining the most likely value of the population parameter using statistic is called estimation.
- **Estimator:** Any sample statistic which is used to estimate an unknown population parameter is called an estimator *i.e.*, an estimator is a sample statistic used to estimate a population parameter.
- **Estimate:** To estimate an unknown parameter of the population, concept of theory of estimation is used. There are two types of estimation namely.
- **Point Estimation:** When a single value is used as an estimate, it is called as point estimation.
- **Interval Estimation:** An interval within which the parameter would be expected to lie is called interval estimation.

➢ **Statistical Hypothesis:** Statistical hypothesis is some assumption or statement, which may or may not be true, about a population.

There are two types of statistical hypothesis:

(i) Null hypothesis

(ii) Alternative hypothesis

(i) Null Hypothesis: According to Prof. R.A. Fisher, "Null hypothesis is the hypothesis which is tested for possible rejection under the assumption that it is true", and it is denoted by H_0.

For example: If we want to find the population mean having a specified value μ_0, then the null hypothesis H_0 is set as follows H_0: $\mu = \mu_0$

(ii) Alternative Hypothesis: Any hypothesis which is complementary to the null hypothesis is called as the alternative hypothesis and is usually denoted by H_1.

For example: If we want to test the null hypothesis that the population has specified mean μ *i.e.*,

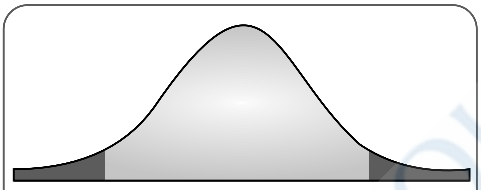
H_0: $\mu = \mu_0$ then the alternative hypothesis could be any one among the following:

(a) $H_1: \mu \neq \mu_0$

(b) $H_1: \mu > \mu_0$

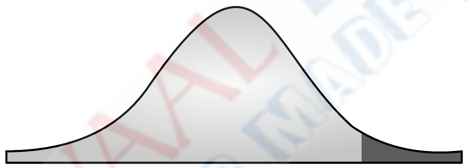
(c) $H_1: \mu < \mu_0$

The alternative hypothesis in $H_1: \mu \neq \mu_0$ is known as two tailed alternative test. Two tailed test is one where the hypothesis about the population parameter is rejected for the value of sample statistic falling into either tails of the sampling distribution. When the hypothesis about the population parameter is rejected only for the value of sample statistic falling into one of the tails of the sampling distribution, then it is known as one-tailed test. Here $H_1: \mu > \mu_0$ and $H_1: \mu < \mu_0$ are known as one-tailed alternative.



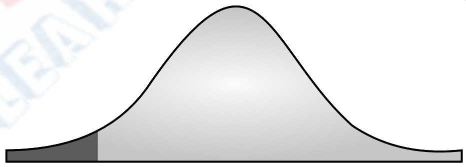
Two tailed diagram

Right tailed test: $H_1: \mu > \mu_0$ is said to be right tailed test where the rejection region or critical region lies entirely on the right tail of the normal curve.



Right tailed diagram

Left tailed test: $H_1: \mu < \mu_0$ is said to be left tailed test where the critical region lies entirely on the left tail of the normal curve.



Left tailed diagram

- **Types of Errors in Hypothesis:** There is every chance that a decision regarding a null hypothesis may be correct or may not be correct. There are two types of errors. They are

  Type I error: The error of rejecting H_0 when it is true.

  Type II error: The error of accepting H_0 when it is false.

- **Critical Region or Rejection Region:** A region corresponding to a test statistic in the sample space which tends to rejection of H_0 is called critical region or region of rejection. The region complementary to the critical region is called the region of acceptance.

- **Level of Significance:** The probability of type I error is known as level of significance and it is denoted by α. The levels of significance which are usually employed in testing of hypothesis are 5% and 1%. The level of significance is always fixed in advance before collecting the sample information.

- **Critical Values or Significant Values:** The value of test statistic which separates the critical (or rejection) region and the acceptance region is called the critical value or significant value. It depends upon:

(i) The level of significance.

(ii) The alternative hypothesis whether it is two-tailed or single tailed.

- **t-Test:** The t-test is a test in statistics that is used for testing hypothesis regarding the mean of a small sample of the population when the standard deviation of the population is not known. *t*-Test was first invented by William Sealy Gosset, in 1908. Since he used the pseudo name as 'Student' when publishing his method in the paper titled 'Biometrika', the test came to be known as Student's T Test. There are many types of t-test. Two of them are:
  - The one-sample *t*-test, which is used to compare the mean of a population with a theoretical value.
  - The two-sample *t*-test, which is used to compare the mean of two independent given samples.

One Sample *t*-Test Procedure:

Step 1: Define the Null Hypothesis (H_0) and Alternate Hypothesis (H_1)

Example: H_0: Sample mean ($\bar{X}$) = Hypothesized Population mean (μ)

H_1: Sample mean ($\bar{X}$) $\neq$ Hypothesized Population mean (μ)

The alternate hypothesis can also state that the sample mean is greater than or less than the comparison mean.

Step 2: Compute the test statistic (t)

$$t = \frac{Z}{\text{S.E.}} = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$$

where S.E. is the standard error

Step 3: Find the *t*-critical from the *t*-Table

Use the degree of freedom and the alpha level (0.05) to find the t -critical.

Step 4: Determine if the computed test statistic falls in the rejection region.

Alternately, simply compute the P-value. If it is less than the significance level (0.05 or 0.01), reject the null hypothesis.

***t* Table**

Cum. prob	$t_{.50}$	$t_{.75}$	$t_{.80}$	$t_{.85}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$	$t_{.999}$	$t_{.9995}$
One-tail	**0.50**	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**	**0.0005**
Two-tails	**1.00**	**0.50**	**0.40**	**0.30**	**0.20**	**0.10**	**0.05**	**0.02**	**0.01**	**0.002**	**0.001**
df											
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.885	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.335	4.501	5.041
9	0.000	0.703	0.883	1.100	1.389	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437

12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.685	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.000	0.684	0.56	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.686	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
Z	0.000	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.090	3.297
	0%	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
	Confidence Level										

Two Sample *t*-Test

The two sample *t*-test is a test that is used to compare the mean of two groups of samples. It is meant for evaluating whether the means of the two sets of data are statistically and significantly different from each other.

Let the two independent samples be $x_1, x_2, x_3, \ldots\ldots, x_{n_1}$ and $y_1, y_2, y_3, \ldots\ldots, y_{n_2}$ with means $\bar{x} = \dfrac{\sum x}{n_1}$ and $\bar{y} = \dfrac{\sum y}{n_2}$ from two normal population with means μ_1 and μ_2 and common variance σ^2(unknown).

Let $s_1^2 = \dfrac{1}{n_1 - 1}\sum(x-\bar{x})^2$ and $s_2^2 = \dfrac{1}{n_2 - 1}\sum(y-\bar{y})^2$

Thus, standard error can be given by $s = \sqrt{\dfrac{\sum(x-\bar{x})^2 + \sum(y-\bar{y})^2}{n_1 + n_2 - 2}}$

The *t*-test formula is given as: $t = \dfrac{\bar{x}-\bar{y}}{s}\sqrt{\dfrac{n_1 n_2}{n_1 + n_2}}$

The significance of *t* is tested in the same way as in one sample t-test with degree of freedom $n_1 + n_2 - 2$ to test the hypothesis $\mu_1 = \mu_2$.

Unit-VII: Index Number and Time Based Data

CHAPTER 9: Time Series

- A time series consists of a set of observations arranged in chronological order (either ascending or descending). Time Series has an important objective to identify the variations and try to eliminate the variations and also helps us to estimate or predict the future values.
  - It helps in the analysis of past behaviour.
  - It helps in forecasting future plans.
  - It helps in evaluation of current achievements.
  - It helps in making comparative studies between one time period and others.

  The following series is the example of time series:

Year	1971	1981	1991	2001	2011	2021
Population (Crores)	43.9	54.2	68.3	84.9	102.6	121.5

- **Components of Time Series:** There are four types of components in a time series. They are as follows:

  (i) Secular Trend, (ii) Seasonal Variations, (iii) Cyclic Variations and (iv) Irregular Variations.

  (i) Secular Trend or Simple Trend or Long Term Movement: Secular trend refers to the general tendency of data to increase or decrease or stagnate over a long period of time. Time series relating to Economics, Business, and Commerce may show an upward or increasing tendency. Whereas, the time series relating to death rates, birth rates, share prices, etc. may show a downward or decreasing tendency.

  The symbol of 'T' is used for denoting long term trend in the formulae relating to analysis of Time Series.

  (ii) Seasonal variations: As the name suggests, tendency movements are due to nature which repeat themselves periodically in every seasons. These variations repeat themselves in less than one year time. It is measured in an interval of time. Seasonal variations may be influenced by natural force, social customs and traditions. These variations are the results of such factors which uniformly and regularly rise and fall in the magnitude. For example, in summers the sale of ice-cream increases and at the time of Diwali the sale of diyas, crackers, etc. go up.

  The symbol of 'S' is used for denoting Seasonal variations in the formulae relating to analysis of Time Series.

  (iii) Cyclic variations: Cyclic variations are due to the ups and downs recurring after a period from time to time. These are due to the business cycle and every organization has to face all the four phases of a business cycle some time or the other. Prosperity or boom, recession, depression, and recovery are the four phases of a business cycle.

  The symbol of 'C' is used for denoting Cyclic variations in the formulae relating to analysis of Time Series.

  (iv) Irregular or Random variations: These variations do not have particular pattern and there is no regular period of time of their occurrences. These are accidently changes which are purely random or unpredictable. Normally they are short-term variations, but its occurrence sometimes has its effect so intense that they may give rise to new cyclic or other movements of variations. For example floods, wars, earthquakes, Tsunami, strikes, lockdowns etc.

  The symbol of 'I' is used for denoting Irregular variations in the formulae relating to analysis of Time Series.

- **Mathematical Model for a Time Series**

  There are two common models used for decomposition of a time series into its components, namely additive and multiplicative model.

(i) Additive Model: The model assumes that the observed value is the sum of all the four components of time series. *i.e.,*

$$Y = T + S + C + I$$

where, Y = Original value, T = Trend Value, S = Seasonal component, C = Cyclic component, I = Irregular component

The additive model assumes that all the four components operate independently. It also assumes that the behaviour of components is of an additive character.

(ii) Multiplicative Model: This model assumes that the observed value is obtained by multiplying the trend (T) by the rates of other three components.

i.e.,
$$Y = T \times S \times C \times I$$

where, Y = Original value, T = Trend Value, S = Seasonal component, C = Cyclic component, I = Irregular component

This model assumes that the components due to different causes are not necessarily independent and they can affect one another. It also assumes that the behaviour of components is of a multiplicative character.

- **Measurement of Trends**

Following are the methods by which we can measure the trend.

(I) Method of Moving Averages.

(II) Method of Least Squares.

(I) Method of Moving Averages: Moving Averages Method gives a trend with a fair degree of accuracy. In this method, we take arithmetic mean of the values for a certain time span. The time span can be three-years, four-years, five-years and so on depending on the data set and our interest. We will see the working procedure of this method.

Procedure:

(i) Decide the period of moving averages (three-years, four-years).

(ii) In case of odd years, averages can be obtained by calculating,

$$\frac{a+b+c}{3}, \frac{b+c+d}{3}, \frac{c+d+e}{3}, \frac{d+e+f}{3}, ...$$

(iii) If the moving average is an odd number, there is no problem of centering it, the average value will be centered besides the second year for every three years.

(iv) In case of even years, averages can be obtained by calculating,

$$\frac{a+b+c+d}{4}, \frac{b+c+d+e}{4}, \frac{c+d+e+f}{4}, \frac{d+e+f+g}{4}, ...$$

(v) If the moving average is an even number, the average of first four values will be placed between 2^{nd} and 3^{rd} year, similarly the average of the second four values will be placed between 3^{rd} and 4^{th} year. These two averages will be again averaged and placed in the 3^{rd} year. This continues for rest of the values in the problem. This process is called as centering of the averages.

(II) Method of Least Squares: The line of best fit is a line from which the sum of the deviations of various points is zero. This is the best method for obtaining the trend values. It gives a convenient basis for calculating the line of best fit for the time series. It is a mathematical method for measuring trend. Further the sum of the squares of these deviations would be least when compared with other fitting methods. So, this method is known as the Method of Least Squares and satisfies the following conditions:

(i) The sum of the deviations of the actual values of Y and $\bar{Y}$ (estimated value of Y) is Zero. *i.e.,* $(Y - \bar{Y}) = 0$.

(ii) The sum of squares of the deviations of the actual values of Y and (estimated value of Y) is least. *i.e.,* $(Y - \bar{Y})^2$ is least.

(iii) The straight line trend is represented by the equation

$$Y = a + bX \qquad ...(1)$$

where, Y is the actual value, X is time, a, b are constants

(iv) The constants 'a' and 'b' are estimated by solving the following two normal equations

$$\Sigma Y = n\,a + b\,\Sigma X \quad \text{....(2)}$$
$$\Sigma XY = a\,\Sigma X + b\,\Sigma X^2 \quad \text{....(3)}$$

where, 'n' = number of years given in the data.

(v) By taking the mid-point of the time as the origin, we get

$$\Sigma X = 0$$

(vi) When $\Sigma X = 0$, the two normal equations reduces to

$$\Sigma Y = na + b\,(0);\ a = \frac{\Sigma Y}{n} = \overline{Y}$$

$$\Sigma XY = a(0) + b\,\Sigma X^2\,;\ b = \frac{\Sigma XY}{\Sigma X^2}$$

The constant 'a' gives the mean of Y and 'b' gives the rate of change (slope).

(vii) By substituting the values of 'a' and 'b' in the trend equation (1), we get the Line of Best Fit.

Unit-VIII: Financial Mathematics

CHAPTER 10: Financial Mathematics

- **Perpetuity:** Perpetuity in the financial system is a situation where a stream of cash flow payments continues indefinitely or is an annuity that has no end. In valuation analysis, perpetuities are used to find the present value of a company's future projected cash flow stream and the company's terminal value. Essentially, a perpetuity is a series of cash flows that keep paying out forever.
  - **Perpetuity Formula:** There are two different annual perpetual valuations; perpetuity with flat or constant annuity and perpetuity with a growing annuity. Perpetuity gives a business the value of its cash flow, an essential slice of data as it aids in determining the firm's total cash flow in a single year.

    (1) Flat Perpetuity: Perpetuity is also known as regular flat perpetuity. In this, cash flow is constant for each year. This perpetuity formula is the simplest, and it is straightforward as it doesn't include terminal value. It is the basic formula for the price of perpetuity. For calculating the present value of flat perpetuity we only need to divide the cash flows/payments by the discount rate.

    $$\text{Present value of Perpetuity} = \frac{\text{Cash flow}}{\text{Interest rate or yield}}$$

    (2) Growing Perpetuity: A growing perpetuity is the same as regular or flat perpetuity, but the difference is that the cash flow is growing each year. The present value of growing perpetuity formula factors in long term growth. This version is used to calculate the terminal value in a stream of cash flows for valuation purposes which is always more complicated.

    $$\text{Present value of Perpetuity} = \frac{\text{Payment}}{\text{Interest rate} - \text{Growth rate}}$$

- **Sinking Fund:** A sinking fund is a type of fund that is created and set up purposely for repaying debt. The owner of the account sets aside a certain amount of money regularly and uses it only for a specific purpose. Often, it is used by corporations for bonds and deposit money to buy back issued bonds or parts of bonds before the maturity date arrives. It is also one way of enticing investors because the fund helps convince them that the issuer will not default on their payments.

  Example: Let us consider a franchisee of 7-Eleven who issues ₹ 50,000 worth bonds with a sinking fund provision and establishes a sinking fund wherein the franchisee regularly deposits ₹ 500, with the intent of using it to buy back bonds slowly before they mature.

  The provision will then allow him to buy back the bonds at a lower price if the market price is lower or at face value if the market price goes higher. Eventually, the principal amount owed will be lower, depending on how much was bought back.

  However, it is important to remember that there is a certain limit to how many bonds can be bought back before the maturity date.

  The formula to calculate the sinking fund is given below

$$\text{Sinking Fund, } A = \frac{\left[\left(1+\left(\frac{r}{m}\right)\right)^{n\times m}\right]-1}{\left(\frac{r}{m}\right)}\times P$$

where P = Periodic contribution to the sinking fund

r = Annualized rate of interest

n = No. of years

m = No. of payments per year

- **Valuation of Bond:** Bonds are long-term debt securities issued by companies or government entities to raise debt finance. Investors who invest in bonds receive periodic interest payments, called coupon payments, and at maturity, they receive the face value of the bond along with the last coupon payment. Each payment received from the bonds, be it coupon payment or payment at maturity, is termed as cash flow for investors.

  Bond valuation is a process of calculating its fair price. Both investors and issuers use many different techniques, but most of them are based on one fundamental principle—that the fair price of a bond is equal to the present value of all future expected cash flows.

  Because of continued economic changes the market price of a bond is usually different from its par value. If its current market price is less than par value, a bond is traded at a discount. Conversely, if its current price is above par value, a bond is traded at a premium.

  - **Par Value or Face Value (P):** This is the actual money that is being borrowed by the lender or purchaser of bonds. Generally, it is 100 or 1000 per any bond. The principal amount borrowed by the lender is the number of bonds purchased multiplied by the par value.
  - **Tenure or Years of Maturity (N):** This describes the number of years that it takes for any bond to mature or when the issuer of bonds will return the par value to the purchaser of bonds.
  - **Yield to Maturity (YTM):** This can be described as the rate of return that the purchaser of a bond will get if the investor holds the bond till its maturity. Also, this could be the prevailing interest rate to calculate the current market price of the bond.
  - **Coupon Rate (C):** This is the periodic payment, usually half-yearly or yearly, given to the purchaser of the bonds as interest payments for purchasing the bonds from the issuer.

  Present Value formula for Bond Valuation

  $$\text{Bond Value} = \sum_{t=1}^{N}\frac{C}{(1+r)^t}+\frac{P}{(1+r)^N}$$

  where, C is a periodic coupon payment, r is the market interest rate or required rate of return, and P is the par value of a bond.

  If a bond has a fixed coupon rate, the formula above can be modified as follows:

  $$\text{Bond Value} = C\times\frac{1-(1+r)^{-N}}{r}+\frac{P}{(1+r)^N}$$

  Relative price approach: The relative price approach of bond valuation focuses on determining the required rate of return. Under such an approach, the bond is priced relative to yield to maturity (YTM) of a benchmark (usually a government bond of similar maturity). Then we should add to a benchmark a risk premium depending on the issuer's credit rating. The resulting interest rate is then used as the discount rate.

- **EMI (Equated Monthly Installment):** An equated monthly installment (EMI) is a fixed payment amount made by a borrower to a lender at a specified date each calendar month. Equated monthly installments are used to pay off both interest and principal each month so that over a specified number of years, the loan is paid off in full. With most common types of loans—such as real estate mortgages, auto loans, and student loans—the borrower makes fixed periodic payments to the lender over the course of several years with the goal of retiring the loan.

The EMI system of loan repayment has following features:

(1) Each installment contains both components of principal repayment and interest charges.

(2) Interest is calculated on reducing balance method.

(3) Interest component is higher in the beginning and progressively lower towards the end. That means, the principal component of an EMI is lower during initial periods and higher during later periods.

(4) The amount of EMI depends on:

- **(i)** The period of compounding *i.e.*, whether the compounding is yearly, half yearly, quarterly or monthly. If the compounding is more frequent, then the amount of EMI would be higher and vice-versa.
- **(ii)** The rate of interest. *i.e.*,
- **(iii)** Period of repayment if the repayment period is more, then EMI would be lower and vice versa.

The formula for EMI is:

$$\text{Installment Amount} = \frac{(1+i)^n}{(1+i)^n - 1} \times (P \times i)$$

where $i = \dfrac{\text{monthly interest rate annual}}{12 \times 100}$

n = number of installments,

P = principal amount of the loan

- **Calculation of Returns:** All financial decisions involve some risk. One may expect to get a return of 15% per annum in his investment but the risk of not able to achieve 15% return will always be there. Return is simply a reward for investing as all investing involves some risk.

  A debt investment is a loan, and the return is just the loan's interest rate. This is simply the ratio of the interest paid to the loan principal.

$$\text{Return, } k = \frac{\text{interest paid}}{\text{loan amount}}$$

  This formulation leads to the convenient idea that a return is what the investor receives divided by what he or she invests.

- **Rate of Return (ROR) or Nominal Rate of Return (NROR):** A Rate of Return (ROR) is the gain or loss of an investment over a certain period of time. In other words, the rate of return is the gain (or loss) compared to the cost of an initial investment, typically expressed in the form of a percentage. When the ROR is positive, it is considered a gain and when the ROR is negative, it reflects a loss on the investment.

  The standard formula for calculating ROR is as follows:

$$\text{Rate of Return} = \frac{\text{Ending value of investment} - \text{Initial value of investment}}{\text{Beginning value of investment}} \times 100$$

- **Compounded Annual Growth Rate (CAGR):** Compounded annual growth rate (CAGR) depicts the cumulative performance of a particular variable over a significant period of time and is used to measure relative profitability of businesses. CAGR is often associated with specific parameters which indicate the performance of a company over a fixed period, such as sales, revenue, earnings, etc.

$$\text{CAGR} = \left(\frac{\text{Final value}}{\text{Initial value}}\right)^{1/n} - 1$$

  where n = investment period

- **Depreciation:** Depreciation allows a portion of the cost of a fixed asset to the revenue generated by the fixed asset. This is mandatory under the matching principle as revenues are recorded with their associated expenses in the accounting period when the asset is in use. This helps in getting a complete picture of the revenue generation transaction.

  Example: If a delivery truck is purchased by a company with a cost of ₹ 1,00,000 and the expected usage of the truck is 5 years, the business might depreciate the asset under depreciation expense as ₹

20,000 every year for a period of 5 years.

There are three methods commonly used to calculate depreciation. They are:

(1) Linear or Straight line method

(2) Unit of production method

(3) Double-declining balance method

Three main inputs are required to calculate depreciation:

Useful life: This is the time period over which the organisation considers the fixed asset to be productive. Beyond its useful life, the fixed asset is no longer cost-effective to continue the operation of the asset.

Salvage value: Post the useful life of the fixed asset, the company may consider selling it at a reduced amount. This is known as the salvage value of the asset.

The cost of the asset: This includes taxes, shipping, and preparation/setup expenses.

Here, we have discuss only Linear or Straight line method for calculating depreciation.

- **Linear or Straight-line Depreciation Method:** This is the simplest method of all. It involves simple allocation of an even rate of depreciation every year over the useful life of the asset. The formula for straight line depreciation is:

$$\text{Annual Depreciation Expense} = \frac{\text{Asset cost} - \text{Residual value}}{\text{Useful life of the asset}}$$

Unit-IX: Linear Programming

CHAPTER 11: Linear Programming

- Problems which minimize or maximize a linear function Z subject to certain conditions determined by a set of linear inequalities with non-negative variables are known as linear programming problems.
- **Objective function:** A linear function $Z = ax + by$, where a and b are constants which has to be maximized or minimized according to a set of given conditions, is called as linear objective function.
- **Decision variables:** In the objective function $Z = ax + by$, the variables x, y are said to be decision variables.
- **Constraints:** The restrictions in the form of inequalities on the variables of a linear programming problem are called constraints. The condition $x \geq 0$, $y \geq 0$ are known as non-negative restrictions.
- **Different types of linear programming problems:** A few important linear programming problems are as follows:

  (i) Manufacturing problem: In such problem, we determine : (a) Number of units of different products to be produced and sold. (b) Manpower required, machines hours needed, warehouse space available, etc. Objective function is to maximize profit.

  (ii) Diet problem: Here, we determine the amount of different types of constituent or nutrients which should be included in the diet. Objective function is to minimize the cost.

  (iii) Transportation problem: These problems deal with the cost of transportation which is to be minimized under given constraints.

  (iv) Assignment Problem: The assignment problem is a special case of transportation problem where the number of sources and destinations are equal. Supply at each source and demand at each destination must be one.

  (v) Blending problem: To determine the optimum amount of several constituents used in producing a set of products while determining the optimum quantity of each product to be produced.

  (vi) Investment problem: To determine the amount of investment in fixed income securities to maximize the return on these investment.

- **Limitations of Linear Programming:**

  (i) To specify an objective function in mathematical form is not an easy task.

  (ii) Even if objective function is determined, it is difficult to determine social, institutional, financial and other constraints.

  (iii) It is also possible that the objective function of constraints may not be directly specified by linear inequality equations.

- **Mathematical Formulation of LPP**

  Let us take an example to understand how to formulate a LPP mathematically.

  Example: An electronic firm is undecided at the most profitable mix for its products. The products manufactured are transistors, resistors and carbon tubes with a profit of (per 100 units) ₹ 10, ₹ 6 and ₹ 4 respectively. To produce a shipment of transistors containing 100 units requires 1 hour of engineering, 10 hours of direct labour, and 2 hours of administrative service. To produce 100 units of resistors requires 1 hour, 4 hours and 2 hours of engineering, direct labour and administrative services respectively. For 100 units of carbon tubes it needs 1 hour, 6 hours and 5 hours of engineering direct labour and administrative services respectively.

  There are 100 hours of engineering time, 600 hours of direct labour and 300 hours of administrative time available. Formulate the corresponding LPP.

  Sol.: Let the firm produce X hundred units of transistors, Y hundred units of resistors and Z hundred units of carbon tubes. Then the total profit to be maximized from this output will be

  $$P = 10X + 6Y + 4Z$$

  This is our objective function.

  Now production of X hundred units of transistors, Y hundred units of resistors and Z hundred units of carbon tubes will require X + Y + Z hours of engineering time, 10X + 4Y + 6Z hours of direct labour time and 2X + 2Y + 5Z hours of administrative service.

  But the total time available for engineering, direct labour and administrative services is 100, 600 and 300 hours respectively.

  Hence the constraints are:

  $$X + Y + Z \leq 100$$
  $$10X + 4Y + 6Z \leq 600$$
  $$2X + 2Y + 5Z \leq 300$$

  with $X, Y, Z \geq 0$ as the non-negativity restriction.

  Thus the formulation is

  Maximize, $P = 10X + 6Y + 4Z$

  subject to constraints: $X + Y + Z \leq 100$

  $$10X + 4Y + 6Z \leq 600$$
  $$2X + 2Y + 5Z \leq 300$$
  $$X, Y, Z \geq 0$$

- **Feasible region:** The common region determined by all the constraints including non-negative constraints $x, y \geq 0$ of linear programming problem is known as the feasible region.

- **Feasible solution:** Points within and on the boundary of the feasible region represents feasible solutions of constraints.

  In the feasible region, there are infinitely many points (solutions) which satisfy the given conditions.

- **Theorem 1:** Let R be the feasible region for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where variables x and y are subject to constraints described by linear inequalities, the optimal value must occur at a corner points (vertex) of the feasible region.

- **Theorem 2:** Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function R has both maximum and minimum values of R and each of these occurs at a corner points (vertex) of R.

  However, if the feasible region is unbounded, the optimal value obtained may not be maximum or minimum.

- **Corner or Extreme Point Method of Formulation of LPP**

  Step 1: Formulate the linear programming problem in x and y with given conditions.

  Step 2: Convert the inequality constraints into equality constraints and plot each line on the graph paper.

  Step 3: Find the feasible region and check if the feasible region is bounded or unbounded.

  Step 4: Evaluate the value of the objective function Z at each corner point. Let M be the greatest and m be the smallest value of the objective function Z.

  (i) When the feasible region is bounded: M and m are the maximum and minimum values of the objective function Z, respectively.

  (ii) When the feasible region is unbounded:

  (a) M is the maximum value of the objective function Z, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, the objective function has no maximum value.

  (b) m is the minimum value of the objective function Z, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, the objective function has no minimum value.

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