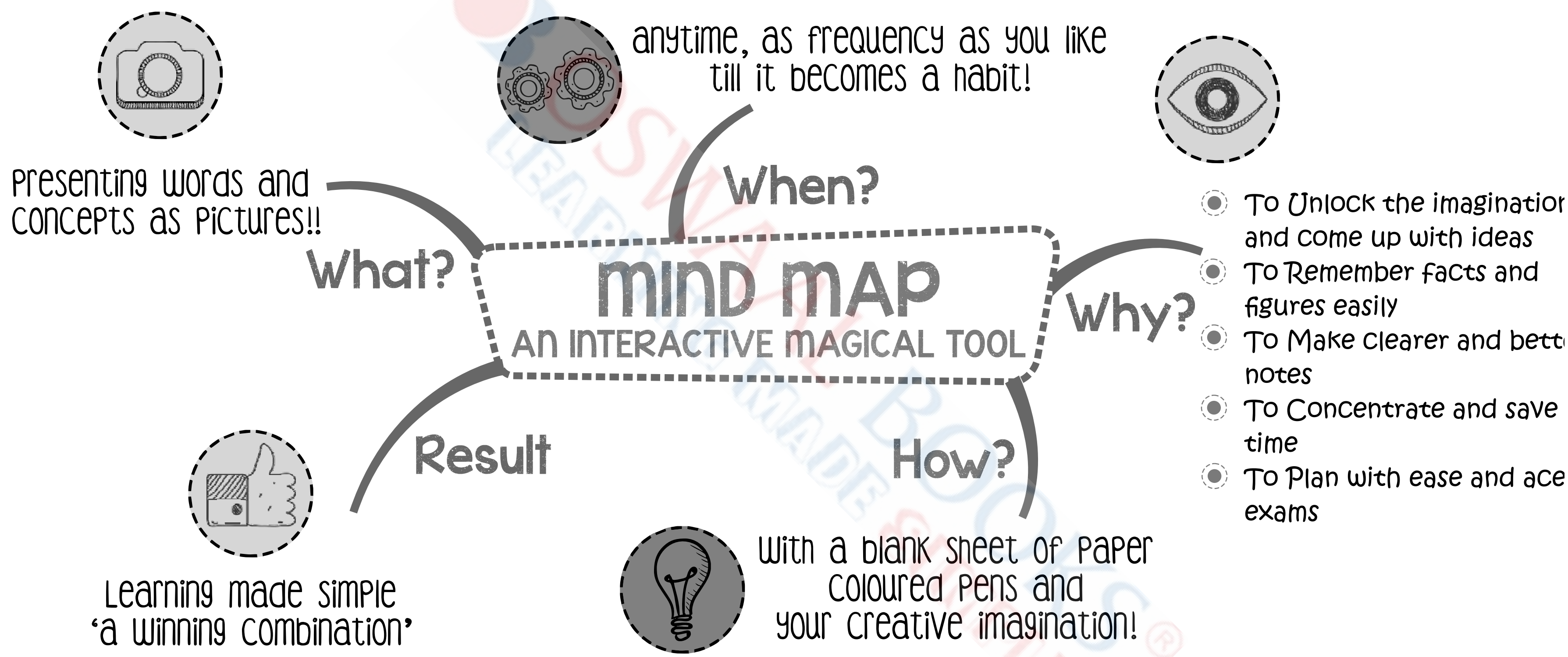
MIND MAPS
Learning Made Simple



What are Associations?

It's a technique connecting the core concept at the Centre to related concepts or ideas. Associations spreading out straight from the core concept are the First Level of Association. Then we have a Second Level of Association emitting from the first level and the chronology continues. The thickest line is the First Level of Association and the lines keep getting thinner as we move to the subsequent levels of association. This is exactly how the brain functions, therefore these Mind Maps. Associations are one powerful memory aid connecting seemingly unrelated concepts, hence strengthening memory.

MATHEMATICS (B-1)

Relations and Functions

Types of Relations

Trivial Relations

- **Empty relations:** A relation $R : A \to A$ is empty if $a \not R b \ \forall \ a, b \in A$. $R = \phi \subset A \times A$. For *eg* : $R = \{(a,b) : a = b^2\}$, $A = \{1,5,10\}$
- **Universal relation:** A relation $R : A \to A$ is universal if $a \ R \ b \ \forall, a,b \in A, R = A \times A$. if $R = \phi$, then R is universal.
- **Reflexive relation:** A relation $R : A \to A$ is reflexive if $aRa \ \forall \ a \in A$
- **Symmetric relation:** A relation $R : A \to A$ is symmetric if $aRb \Rightarrow bRa \ \forall \ a, b \in A$
- **Transitive relation:** A relation $R : A \times A$ is transitive if $aRb, bRc \Rightarrow aRc \ \forall \ a,b,c \in A$.

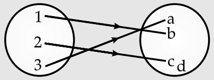
Reflexive + Symmetric + Transitive →

Equivalence relation (reflexive, symmetric, transitive *e.g.*, Let T = the set of all triangles in a plane and $R : T \to T$ defined by $R = \{(T_1, T_2)\} : T_1$ is congruent to $T_2\}$. Then, R is equivalence.

Types of Functions

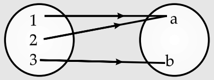
- **One-one (injective):** $f : x \to y$ is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ $\forall \ x_1, x_2 \in x$. Other wise, f is many- one, f_1 is one-one.

f_1



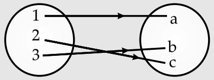
- **Onto (surjective):** $f : x \to y$ is onto it for energy $y \in Y, \exists \ x \in X$ S.t. $f(x) = y$, f_2 is onto.

f_2



- **Bijective:** $f : x \to y$ is both one-one and onto, then f is bijective. f_3 is bijective.

f_3



Composition of functions and Invertible functions

- **Composition of functions:** The composition of functions $f : A \to B$ and $g : B \to C$ is denoted by gof, and is defined as $gof : A \to C$ given by $gof(x) = g(f(x)) \ \forall x \in A$. *e.g.* let $A = N$ and $f, g : N \to N$ such that $f(x) = x^2$ and $g(x) = x^3$ $\forall \ x \in N$. Then $gof(2) = g(f(2)) = g(2^2) = 4^3 = 64$.
- **Invertible functions:** A function $f : x \to y$ is invertible, if $\exists$ a function $g : y \to x$ such that $gof = I_x$ and $fog = I_y$. Then, g is the inverse of f. If f is invertible, then it is both one-one and onto and vice-versa. For *eg.* If $f(x) = x$ and $f : N \to N$, then f is invertible.
  Theorem 1 : If $f : x \to y$, $g : y \to z$ and $h : z \to s$ are functions, then $ho(got) = (hog)of$.
  Theorem 2 : Let $f : x \to y$ and $g : y \to z$ be two invertible functions, then gof is invertible and $(gof)^{-1} = f^{-1} \text{o} \ g^{-1}$.

Binary operations

- **Definition and its types:** A binary operation '*' on a set A is a function $* : A \times A \to A$ denoted by $a * b$ *i.e.* $\forall$ $a,b, \in$ A, $a * b \in$ A. Commutative if $a * b = b * a \ \forall \ a,b \in A$. Associative if $(a * b) * c = a * (b * c) \ \forall a,b, \in$ A. $e \in A$ is identity if $a * e = a = e * a \ \forall a \in A$. and $b \in A$ is the inverse of $a \in A$, if. $a * b = e = b * a$. Addition is a binary operation on the set of integers.

MATHEMATICS (B-1)

Inverse Trigonometric Functions

Domain and range of inverse trigonometric functions

(i) $y = \sin^{-1}x$. Domain = [–1,1], Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $y = \cos^{-1}x$. Domain = [–1,1] Range = $[0, \pi]$

(iii) $y = \text{cosec}^{-1}x$. Domain = R–(–1,1), Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

(iv) $y = \sec^{-1}x$. Domain = R–(–1,1), Range = $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

(v) $y = \tan^{-1}x$. Domain = R, Range = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(vi) $y = \cot^{-1}x$. Domain = R, Range = $(0, \pi)$.

Trigonometric functions

(I) $\sin : R \to [-1,1]$

(ii) $\cos : R \to [-1,1]$

(iii) $\tan : R - \left\{x : x = (2n+1)\frac{\pi}{2}, n \in Z\right\} \to R$

(iv) $\cot : R - \{x : x = n\pi, n \in Z\} \to R$

(v) $\sec : R - \left\{x : x = (2n+1)\frac{\pi}{2}, n \in Z\right\} \to R - (-1,1)$

(vi) $\text{cosec} : R - \{x : x = n\pi, n \in Z\} \to R - (-1,1)$

Some important relations

(i) $y = \sin^{-1}x \Rightarrow x = \sin y$

(ii) $x = \sin y \Rightarrow y = \sin^{-1}x$

(iii) $\sin(\sin^{-1}x) = x$

(iv) $\sin^{-1}(\sin x) = x$

(v) $\sin^{-1}\frac{1}{x} = \text{cosec}^{-1}x$

(vi) $\cos^{-1}(-x) = \pi - \cos^{-1}x$

(vii) $\cos^{-1}\frac{1}{x} = \sec^{-1}x$

(viii) $\cot^{-1}(-x) = \pi - \cot^{-1}x$

(ix) $\tan^{-1}\frac{1}{x} = \cot^{-1}x$

(x) $\sec^{-1}(-x) = \pi - \sec^{-1}x$

(xi) $\sin^{-1}(-x) = -\sin^{-1}x$

(xii) $\tan^{-1}(-x) = -\tan^{-1}x$

(xiii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

(xiv) $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$

(xv) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$

(xvi) $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$

(xvii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$

(xviii) $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1+x^2}{1-x^2}$

For *eg* : to find the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$, let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = y$,

$\Rightarrow \sin y = \frac{1}{\sqrt{2}}$ The range or the principal value branch of $\sin^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

and $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ So, the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is $\frac{\pi}{4}$

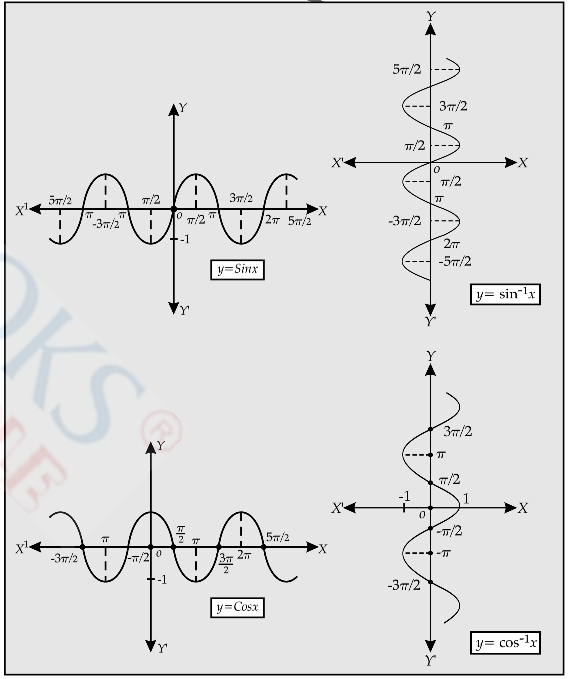
$\sin^{-1}x \neq (\sin x)^{-1}.(\sin x)^{-1}$

$= \frac{1}{\sin x}$ and same for other trigonometric functions.

Principal value branch and principal value

The range of an inverse trigonometric function is the principal value branch and those values which lies in the principal value branch is called the principal value of that inverse trigonometric functions.

Graphs of trigonometric functions and inverse trigonometric functions



MATHEMATICS (B-1)

Matrices

Definition and its types

A matrix of order m×n is an ordered rectangular array of numbers or functions having 'm' rows and 'n' columns. The matrix $A=[a_{ij}]_{m\times n}$ is given by

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ a_{m1} & a_{m2} & \dots a_{mn} \end{bmatrix}_{m\times n}$$

- Column matrix : It is of the form $\left[a_{ij}\right]_{m\times 1}$
- Row matrix : It is of the form $\left[a_{ij}\right]_{1\times n}$
- Square matrix : Here, $m = n$ (no. of rows = no. of columns)
- Diagonal matrix : All non-diagonal entries are zero i.e $a_{ij} = 0 \forall i \neq j$
- Scalar matrix : $a_{ij} = 0, i \neq j$ and $a_{ij} = k$ (Scalar), $i = j$
- Identity matrix : $a_{ij} = 0, i \neq j$ and $a_{ij} = 1, i = j$
- Zero matrix : All entries are zero.

Transpose of a Matrix

If $A=[a_{ij}]_{m\times n}$, then its transpose A' $(A^T)=[a_{ji}]_{n\times m}$ *i.e.* if $A=(2\ \ 1)$ then $A^T=\begin{pmatrix}1\\2\end{pmatrix}$.

Also, $(A')'=A$, $(kA)'=kA'$, $(A+B)'=A'+B'$, $(AB)'=B'A'$.

- A is symmetric matrix if $A=A'$ *i.e.* $A'=-A$.
- A is skew – symmetric if $A=-A'$ *i.e.* $A'=-A$.
- A is any matrix, then–

$$A=\frac{1}{2}\{(A+A')+(A-A')\}= \text{sum of a symmetric and a skew-symmetric matrix.}$$

($A+A'$ → S.M, $A-A'$ → Skew, S.M.)

For *eg* if $A=\begin{pmatrix}2&8\\6&4\end{pmatrix}$, then $A=\frac{1}{2}\left\{\begin{pmatrix}2&7\\7&4\end{pmatrix}+\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right\}$.

Equality of two matrix

$A=\left[a_{ij}\right]=\left[b_{ij}\right]=B$ if, A and B are of same order and $a_{ij}=b_{ij}\ \forall\ i$ and j.

Operations on matrices

Addition

If A, B are two matrices of same order, then $A+B=[a_{ij}+b_{ij}]$ The addition of A and B follows: $A+B=B+A$, $(A+B)+C=A+(B+C)$, $-A=(-1)A$, $k(A+B)=kA+kB$, k is scalar and $(k+I)A=kA+IA$, k and I are constants.

- If $A=\begin{pmatrix}2&3\\2&4\end{pmatrix}$, $B=\begin{pmatrix}-3&2\\-4&5\end{pmatrix}$ then $A+B=\begin{pmatrix}-1&5\\-2&9\end{pmatrix}$
- If $A=(2\ 3)_{1\times 2}$, $B=\begin{pmatrix}4\\5\end{pmatrix}_{2\times 1}$, then $AB=(2\times 4+3\times 5)=(2\ \ 3)_{1\times 1}$

Multiplication

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{jk}]_{n\times p}$, then $AB=C=[C_{ik}]_{m\times p}$, $[C_{jk}]=\sum_{j=1}^{n} a_{ij}b_{jk}$. Also, $A(BC)=(AB)C$, $A(B+C)=AB+AC$ and $(A+B)C=AC+BC$, but $AB\neq BA$ (always).

Elementary operations on a matrix

$R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

$R_i \to kR_i$ or $C_i \to kC_i$

$R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

Inverse of a matrix

If A, B are square matrices such that $AB=BA=I$ then $B=A^{-1}$ i.e., A is the inverse of B and vice-versa.

Inverse of a square matrix, if it exists, is unique

For eg: If $A=\begin{pmatrix}1&2\\3&4\\5&6\end{pmatrix}$, then after $R_1 \longleftrightarrow R$, A becomes $\begin{pmatrix}3&4\\1&2\\5&6\end{pmatrix}$

If A and B are invertible matrices of the same order, then $(AB)^{-1}=B^{-1}A^{-1}$. By elementary transformations, we can convert $A=IA$ to $I=A^{-1}A$. This is one process of finding the inverse of a given square matrix A.

MATHEMATICS (B-1)

Determinants

Minors and cofactors of a matrix

Minor of an element a_{ij} in a determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column and is denoted by M_{ij}. If M_{ij} is the minor of a_{ij} and cofactor of a_{ij} is A_{ij} given by $A_{ij} = (-1)^{i+j} M_{ij}$.

- If $A_{3\times3}$ is a matrix, then $|A| = a_{11}.A_{11} + a_{12}.A_{12} + a_{13}.A_{13}$.
- If elements of one row (or column) are multiplied with cofactors of elements of any other row (or column), then their sum is zero. For *e.g.*, $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{33} = 0$.

e.g., if $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, then $M_{11} = 4$ and $A_{11} = (-1)^{1+1}4 = 4$.

Determinant of a square matrix 'A', |A| is given by

(i) if $A = [a_{11}]_{1\times1}$, then $|A| = a_{11}$

(ii) if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2}$ then $|A| = a_{11}\,a_{22} - a_{12}\,a_{21}$

(iii) if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$, then $|A| = a_{11}(a_{22}.a_{33} - a_{23}.a_{32}) - a_{12}(a_{21}.a_{33} - a_{23}.a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

For *eg.* if $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$, then $|A| = 2 \times 4 - 3 \times 2 = 2$

Properties of |A|

(i) $|A|$ remains unchanged, if the rows and columns of A are interchanged ie., $|A| = |A'|$

(ii) if any two rows (or columns) of A are interchanged, then the sign of $|A|$ changes.

(iii) if any two rows (or columns) of A are identical, then $|A| = 0$

(iv) if each element of a row (or a column) of A is multiplied by B (const.), then $|A|$ gets multiplied by B.

(v) if $A = [a_{ij}]_{3\times3}$ then $|k.A| = k^3|A|$.

(vi) if elements of a row or a column in a determinant $|A|$ can be expressed as sum of two or more elements, then $|A|$ can be expressed as $|B| + |C|$.

(vii) if $R_i \to R_i + kR_j$ or $C_i = C_i + kC_j$ in $|A|$, then the value of $|A|$ remains same

Adjoint and inverse of a matrix

if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then adj. $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

, where A_{ij} is the cofactor of a_{ij}.

- $A(\text{adj}.A) = (\text{adj}.A).A = |A| = I$, A – square matrix of order 'n'
- if $|A| = 0$, then A is singular. Otherwise, A is non-singular.
- if $AB = BA = I$, where B is a square matrix, then B is called the inverse of A, $A^{-1} = B$ or $B^{-1} = A$, $(A^{-1})^{-1} = A$.

Inverse of a square matrix exists if A is non-singular *i.e.* $|A| \neq 0$, and is given by

$$A^{-1} = \frac{1}{|A|}(\text{adj}.A)$$

Applications of determinants & matrices

- if $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$ then we can write $AX = B$,

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

- Unique solution of $AX = B$ is $X = A^{1}B$, $|A| \neq 0$.
- $AX = B$ is consistent or inconsistent according as the solution exists or not.
- For a square matrix A in $AX = B$, if
  - (i) $|A| \neq 0$ then there exists unique solution.
  - (ii) $|A| = 0$ and (adj. A) $B \neq 0$, then no solution.
  - (iii) if $|A| = 0$ and (adj.A).$B = 0$ then system may or may not be consistent.

Area of a triangle

if (x_1, y_1), (x_2, y_2) and (x_3, y_3) $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

For eg: if (1, 2), (3, 4) and (–2, 5) are the vertices, then area of the triangle is

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ -2 & 5 & 1 \end{vmatrix} = \frac{1}{2}1(4-5) - 2(3+2) + 1(15+8) = 6 \text{ sq.units.}$$

we take positive value of the determinant.

MATHEMATICS (B-1)

Continuity and Differentiability

Continuous Function

Suppose f is a real function on a subset of the real numbers and let 'c' be a point in the domain of f.
Then f is continuous at c if $\lim_{x \to c} f(x) = f(c)$
A real function f is said to be continuous if it is continuous at every point in the domain of f. For *eg*: The function $f(x) = \frac{1}{x}, x \neq 0$ is continuous
Let 'C' be any non-zero real number, then $\lim_{x \to c} f(x) \lim_{x \to c} \frac{1}{x} = \frac{1}{c}$. For $c = 0, f(c) = \frac{1}{c}$ So $\lim_{x \to c} f(x) = f(c)$ and hence f is continuous at every point in the domain of f.

Algebra of continuous functions

Suppose f and g are two real functions continuous at a real number c, then, $f+g$, $f-g$, $f.g$ and $\frac{f}{g}$ are continuous at $x = c(g(c) \neq 0)$.

Differentiability

Suppose f is a real function and c is a point in its domain. The derivative of f at c is $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$
Every differentiable function is continuous, but the converse is not true.

Chain Rule

if $f = vou$, $t = u(x)$ and if both $\frac{dt}{dx}, \frac{dv}{dt}$ exists, then $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$.

Logarithmic differentiation

Let $y = f(x) = [u(x)]^{v(x)}$

$\log y = v(x) \log[u(x)]$

$\frac{1}{y} = v(x) \frac{1}{u(x)} u'(x) + v'(x) \log[u(x)]$

$\frac{dy}{dx} = y \left[\frac{v(x)}{u(x)} \right] u'(x) + v'(x) \log[u(x)]$

For *e.g.* : Let $y = a^x$ Then $\log y = x \log a$

$\frac{1}{y} \cdot \frac{dy}{dx} = \log a$

$\frac{dy}{dx} = y \log a = a^x \log a$.

Some Standard derivatives

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(ii) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(iv) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

(v) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{1-x^2}}$

(vi) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$

(vii) $\frac{d}{dx}(e^x) = e^x$

(viii) $\frac{d}{dx}(\log x) = \frac{1}{x}$

Mean Value Theorem

if $f : [a, b] \to R$ continuous on $[a, b]$ and differentiable on (a, b).
Then $\exists$ some c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$
e.g. Let $f(x) = x^2$ defined in the interval $[2, 4]$. Since $f(x) = x^2$ is continuous in $[2, 4]$ and differentiable in $(2, 4)$ as $f'(x) = 2x$ defined in $(2, 4)$. So,

$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{16 - 4}{4 - 2} = 6, c \in (2, 4)$.

Rolle's theorem

if $f : [a, b] \to R$ is continuous on $[a, b]$ and differentiable on (a, b). Such that $f(a) = f(b)$, then $\exists$ some c in (a, b) s.t. $f'(c) = 0$.

Second order derivative

Let $y = f(x)$ then $\frac{dy}{dx} = f'(x)$, if $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx} f'(x)$ *i.e.*,
$\frac{d^2y}{dx^2} = f''(x)$ is the second order derivative of y *w.r.t.x.*
For *eg* : if $y = 3x^2 + 2$, then $y' = 6x$ and $y'' = 6$.

Derivatives of functions in parametric form

Let $x = f(t)$, $y = g(t)$ be two functions of parameter 't'.

Then, $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ or $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \left(\frac{dx}{dt} \neq 0 \right)$

Jhus, $\frac{dy}{dx} = \frac{g'(t)}{f'(t)} \left[\text{provided } f'(t) \neq 0 \right]$

For *eg* : if $x = a\cos\theta$, $y = a\sin\theta$ then $\frac{dx}{d\theta} = -a\sin\theta$ and $\frac{dy}{d\theta} = a\cos\theta$, and so $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{a\cos\theta}{a\sin\theta} = -\cot\theta$.

MATHEMATICS (B-1)

Application of Derivatives

Rate of change of quantities

If a quantity if 'y' varies with another quantity x so that $y = f(x)$, then $\frac{dy}{dx}\left[f'(x)\right]$ represents the rate of change of y w.r.t x and $\left.\frac{dy}{dx}\right|_{x=x_o}$ $(f'(x_0))$ represents the rate of change of y w.r.t. x at $x = x_o$

If 'x' and 'y' varies with another variable 't' i.e., if $x = f(t)$ and $y = g(t)$, then by chain rule $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$, if $\frac{dx}{dt} \neq 0$.

For eg: if the radius of a circle, r = 5 cm, then the rate of change of the area of a circle per second w.r.t 'r' is – $\frac{da}{dr}|_{r=5} = \frac{d}{dr}(\pi r^2)|_{r=5} = 2\pi r|_{r=5} = 10\pi$

Increasing and decreasing functions

A function f is said to be (i) increasing on (a,b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) \leq f(x_2) \forall x_1 x_2 \in (a,b)$, and (ii) decreasing on (a, b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) > f(x_2) \forall x_1 x_2 \in (a,b)$

If $f'(x) \geq 0 \forall x \in (a,b)$ then f is increasing in (a,b) and if $f'(x) \leq 0 \forall x \in (a,b)$, then f is decreasing in (a,b) For eg: Let $f(x) = x^3 - 3x^2 + 4x, x \in R$, then $f'(x) = 3x^2 - 6x + 4 = 3(x-1)^2 + 1 > 0 \forall x \in R$. So, the function f is strictly increasing on R.

Tangents and Normals

The equation of the tangent at (x_0, y_0), to the curve $y = f(x)$ is given by $(y - y_0) = \left.\frac{dy}{dx}\right|_{(x_0, y_0)}(x - x_0)$ if $\frac{dy}{dx}$ does not exists at (x_0, y_0), then the tangent at (x_0, y_0) is parallel to the y–axis and its equation is $x = x_0$. If tangent to a curve $y = f(x)$ at $x = x_0$ is parallel to x–axis, then $\left.\frac{dy}{dx}\right|_{x=x_0} = 0$.

Equation of the normal to the curve

$y = f(x)$ at (x_0, y_0) is $y - y_0 = -\frac{1}{\left.\frac{dy}{dx}\right|_{(x_0,y_0)}}(x - x_0)$ if $\frac{dy}{dx}$ at (x_0, y_0) is zero, then equation of the normal is $x = x_0$. If $\frac{dy}{dx}$ at (x_0, y_0) does not exist, then the normal is parallel to x-axis and its equation is $y = y_0$ For eg: Let $y = x^3 - x$ be a curve, then the slope of the tangent to $y = x^3 - x$ at $x = 2$ is $\left.\frac{dy}{dx}\right|_{x=2} = 3x^2 - 1 = 3.2^2 - 1 = 11$

Approximations

Let $y = f(x) \Delta x$ be a small increment in 'x' and Δy be the small increment in y corresponding to the increment in 'x', i.e. $\Delta y = f(x + \Delta x) - f(x)$. Then, Δy is given by $dy = f'(x)dx$ or $dy = \left(\frac{dy}{dx}\right)\Delta x$, is a good approximation of Δy when $dx = \Delta x$ is relatively small and denote by $dy \approx \Delta y$. For eg: Let us approximate $\sqrt{36.6}$. To do this, we take $y = \sqrt{x}, x = 36$, $\Delta x = 0.6$ then $\Delta y = \sqrt{x + dx} - \sqrt{x}$

$= \sqrt{36.6} - \sqrt{36}$

$= \sqrt{36.6} - 6 \quad \Rightarrow \sqrt{36.6} = 6 + \bar{d}y$

Now, dy is approximately Δy and is given by $\bar{d}y$

$= \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(0.6) = \frac{1}{2\sqrt{36}}(0.6) = 0.05$. So, $\sqrt{36.6} \approx 6 + 0.05 = 6.05$.

Maxima and Minima

A point C in the domain of 'f' at which either $f'(c) = 0$ or is not differentiable is called a critical point of f.

First derivative test

Let f be continuous at a critical point C in open I. Then **(i)** if $f'(x) > 0$ at every point left of C and $f'(x) < 0$ at every point right of C, then 'C' is a point of local maxima. **(ii)** If $f'(x) < 0$ at every point left of C and $f'(x) > 0$ at every point right of C, then 'C' is a point of local minima. **(iii)** If $f'(x)$ does not change sign as 'x' increases through C, then 'C' is called the point of inflection.

Second derivative test

Let f be a function defined on I and C–I, f is twice differentiable at C. Then
(i) $x = C$ is a point of local max. If $f'(C) = 0$ and $f''(C) < 0$, $f(C)$ is local max. of f.
(ii) $x = C$ is a point of local min if $f'(C) = 0$ and $f''(C) > 0$. $f(C)$ is local min of f. **(iii)** The test fails if $f'(C) = 0$ and $f''(C) = 0$

MATHEMATICS (B-1)

Integrals

Integration

It is the inverse of differentiation. Let, $\frac{d}{dx}F(x)=f(x)$. Then $\int f(x)dx=F(x)+c$, 'c': constant of integral. These integrals are called indefinite or general integrals.

Properties of indefinite integrals are

(i) $\int[f(x)+g(x)]dx=\int f(x)dx+\int g(x)dx$, (ii) $\int kf(x)dx=k\int f(x)dx$,

For *eg* : $\int(3x^2+2x)dx=x^3+x^2+c$ where k is real.

Some standard integrals

(i) $\int x^n dx=\frac{x^{n+1}}{n+1}+c, n\neq -1$ like, $\int dx=x+c$

(ii) $\int \cos x\,dx=\sin x+c$ (iii) $\int \sin x\,dx=-\cos x+c$

(iv) $\int \sec^2 x\,dx=\tan x+c$ (v) $\int \operatorname{cosec}^2 x\,dx=-\cot x+c$

(vi) $\int \sec x\tan x\,dx=\sec x+c$ (vii) $\int \operatorname{cosec} x\cot x\,dx=-\operatorname{cosec} x+c$

(viii) $\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+c$ (ix) $\int\frac{dx}{\sqrt{1-x^2}}=-\cos^{-1}x+c$

(x) $\int\frac{dx}{1+x^2}=\tan^{-1}x+c$ (xi) $\int\frac{dx}{1+x^2}=-\cot^{-1}x+c$

(xii) $\int e^x dx=e^x+c$ (xiii) $\int a^x dx=\frac{a^x}{\log a}+c$

(xiv) $\int\frac{dx}{x\sqrt{x^2-1}}=\sec^{-1}x+c$ (xv) $\int\frac{dx}{x\sqrt{x^2-1}}=-\operatorname{cosec}^{-1}x+c$

(xvi) $\int\frac{1}{x}dx=\log|x|+c$

Integration by partial fractions

A rational function of the form $\frac{P(x)}{Q(x)}(Q(x)\neq 0)=T(x)+\frac{P_1(x)}{Q(x)}$, $P_1(x)$ has degree less than that of $Q(x)$. We can integrate $\frac{P_1(x)}{Q(x)}$ by expressing it in the following forms –

(i) $\frac{px+q}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}, a\neq b.$

(ii) $\frac{px+q}{(x+a)^2}=\frac{A}{x-a}+\frac{B}{(x-a)^2}$

(iii) $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$

(iv) $\frac{px^2+qx+r}{(x-a)^2(x-b)}=\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$

(v) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}=\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

Second fundamental theorem of integral calculus

Let f be a continuous function of x defined on $[a,b]$ and let F be another function such that $\frac{d}{dx}F(x)=f(x)\forall x\in$ domain of f_1 then $\int_a^b f(x)dx=[F(x)+c]_a^b=F(b)-F(a)$. This is called the definite integral of f over the range $[a,b]$, where a and b are called the limits of integration, a being the lower limit and b be the upper limit.

Example

$$\int_{-\pi/4}^{\pi/4}\sin^2 x\,dx$$
$$=2\int_0^{\pi/4}\sin^2 x\,dx$$
$$=2\int_0^{\pi/4}\left(\frac{1-\cos 2x}{2}\right)dx$$
$$=\int_0^{\pi/4}(1-\cos 2x)dx$$
$$=\left[x-\frac{\sin 2x}{2}\right]_0^{\pi/4}$$
$$=\frac{\pi}{4}-\frac{1}{2}$$

First fundamental theorem of integral calculus

Let the area function be defined by $A(x)=\int_a^x f(x)dx\,\forall x\geq a$, where f is continuous on $[a,b]$ then $A'(x)=f(x)\forall x\in[a,b]$.

Some special type of integrals

(i) $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+c.$

(ii) $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+c.$

(iii) $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c.$

Integration by parts

$$\int f_1(x)f_2(x)dx=f_1(x)\int f_2(x)dx-\int\left[\frac{d}{dx}f_1(x)\int f_2(x)dx\right]dx$$

Integration of some special functions

(i) $\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$ (ii) $\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$

(iii) $\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$ (iv) $\int\frac{dx}{\sqrt{x^2-a^2}}=\log\left|x+\sqrt{x^2-a^2}\right|+c$

(v) $\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}+c$ (vi) $\int\frac{dx}{\sqrt{x^2+a^2}}=\log\left|x+\sqrt{x^2+a^2}\right|+c.$

Integration by substitution

The method in which we change the variable to some other variable is called the method of substitution

$\int\tan x\,dx=\log|\sec x|+c$ $\int\cot x\,dx=\log|\sin x|+c$

$\int\sec x\,dx=\log|\sec x+\tan x|+c$ $\int\operatorname{cosec} x\,dx=\log|\operatorname{cosec} x-\cot x|+c.$

MATHEMATICS (B-1)

Application of the Integrals

Area under simple curves

The area of the region bounded by the curve $y = f(x)$, *x- axis* and the lines $x=a$ and $x = b(b > a)$ is given by

$A = \int_a^b y\, dx$ or $\int_a^b f(x) dx$.

For *eg* : the area bounded by $y=x^2$, *x- axis* in I quadrant and the lines $x=2$ and $x=3$ is -

$A = \int_2^3 y\, dx = \int_2^3 x^2\, dx = \left[\frac{x^3}{3}\right]_2^3 = \frac{1}{3}(27-8) = \frac{19}{3}$ Sq.units.

The area of the region bounded by the curve $x = f(y), y-axis$ and the lines $y=c$ and $y = d(d > c)$ is given by $A = \int_c^d x\,dy$ or $\int_c^d f(y) dy$.

For *eg* : the area bounded by $x = y^3$, $y - axis$ in the I quadrant and the lines $y=1$ and $y=2$ is

$A = \int_1^2 x\,dy = \int_1^2 y^3\,dy = \left[\frac{1}{4}y^4\right]_1^2 = \frac{1}{4}(2^4-1^4) = \frac{15}{4}$ Sq. units

Area between two curves

The area of the region enclosed between two curves $y = f(x), y = g(x)$ and the lines $x=a$, $x=b$ is given by

$A = \int_a^b [f(x) - g(x)] dx$, where $f(x) \geq g(x)$ in $[a,b]$

For eg: To find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$

(0,0) and (1,1) are points of intersection of $y = x^2$ and $y^2 - x$ and, $y^2 = x \Rightarrow y = \sqrt{x} = f(x)$, and $y = x^2 = g(x)$, where $f(x) \geq g(x)$ in $[0, 1]$.

Area, $A = \int_0^1 [f(x) - g(x)] dx$

$= \int_0^1 [\sqrt{x} - x^2] dx$

$= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1$

$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$ Sq. units.

if $f(x) \geq g(x)$ in $[a,c]$ and $f(x) \leq g(x)$ in $[c, \ b]$, $a < c < b$, then the area is

$A = \int_a^c [f(x) - g(x)] dx + \int_c^b [g(x) - f(x)] dx$

MATHEMATICS (B-1)

Differential Equations

Definition

An equation involving derivatives of the dependent variable with respect to independent variable (variables) is called a differential equation. If there is only one independent variation, then we call it as an ordinary differential equation. For eg: $2\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0$.

Order of a Differential Equation

It is the order of the highest order derivative occurring in the Differential Equation For eg: the order of $\frac{dy}{dx}=e^x$ is one and order of $\frac{d^2y}{dx^2}+x=0$ is two.

Degree of a Differential Equation

It is defined if the Differential Equations is a polynomial equation in its derivatives, and is defined as the highest power (positive integer only) of the highest order derivative.

For eg: the degree of $\left(\frac{d^2y}{dx^2}\right)^3+\frac{dy}{dx}=0$ is three

Order and degree (if defined) of a D.E. are always positive integers.

Solution of a Differential Equation

A function which satisfies the given Differential Equation is called its solution. The solution which contains as many arbitrary constants as the order of the D.E. is called a general solution and the solution free from arbitrary constants is called particular solution.

For eg: $y=e^x+1$ is a solution of $y''-y'=0$.

Since $y'=e^x$ and $y''=e^x \Rightarrow y''-y'=e^x-e^x=0$.

Formation of Differential Equations

To form a Differential Equation from a given function, we differentiate the function successively as many times as the no. of arbitrary constants in the given function, and then eliminate the arbitrary constants.

For eg: Let the function be $y=ax+b$, then we have to differentiate it two times, since there are 2 arbitrary constants a and b. $\therefore y'=a \Rightarrow y''=0$. Jhus $y''=0$ is the required Differential Equation.

Linear Differential Equations

A Differential Equation of the form $\frac{dy}{dx}+Py=Q$, where P_1 Q are constants or functions or 'x' only is called a first order linear Differential Equations its solution is $ye^{\int P.dx}=\int Q.e^{\int P.dx}dx+c$. For eg: $\frac{dy}{dx}+3y=2x$ has solution $ye^{\int 3.dx}=\int 2x.e^{\int 3.dx}dx+c \Rightarrow ye^{3x}=2\int xe^{3x}+c.$

Homogeneous Differential Equations

A Differential Equation which can be expressed in the form $\frac{dy}{dx}=f(x,y)$ or $\frac{dx}{dy}=g(x,y)$, where, $f(x,y)$ and $g(x,y)$ are homogeneous functions of degree zero is called a homogenous Differential Equation

For eg: $(x^2+xy)dy=(x^2+y^2)dx$

To solve this, we substitute $\frac{y}{x}=v \Rightarrow y=vx$.

Variable Separation Method

It is used to solve such an equation in which variables can be separated completely. For eg: $y\,dx=x\,dy$ can be solved as $\frac{dx}{x}=\frac{dy}{y}$ Integrating both sides $\log x=\log y+\log c \Rightarrow \frac{x}{y}=c \Rightarrow x=cy$,is the solution.

The order of a Differential Equations representing a family of curves is same as the number of arbitrary constants present in the equation corresponding to the family of curves. For eg: Let the family of curves be $y = mx$, m = constant, then, $y'=m$

$y=y'x \Rightarrow y=\frac{dy}{dx}x \Rightarrow x\frac{dy}{dx}-y=0.$

MATHEMATICS (B-1)

Vector Algebra

Vector

A quantity that has both magnitude and direction is called a vector. The distance between the initial and terminal points of a vector is called its magnitude. Magnitude of vector $\overline{AB}$ is $|AB|$.

Position Vector

Position vector of a point $P\ (x, y, z)$ is $x\hat{i}+y\hat{j}+z\hat{k}$ and its magnitude is $OP(r)=\sqrt{x^2+y^2+z^2}$ .For eg: Position vector of $P\ (2,3,5)$ is $2\hat{i}+3\hat{j}+5\hat{k}$ and its magnitude is $\sqrt{2^2+3^2+5^2}=\sqrt{38}$.

Direction ratios and direction cosines

The scalar components of a vector are its direction ratios, and represent its projections along the respective axes.

The magnitude (r) direction ratios (a,b,c) and direction cosines (l,m,n) of vector $a\hat{i}+b\hat{j}+c\hat{k}$ are related as:

$l=\frac{a}{r}, m=\frac{b}{r}, n=\frac{c}{r}$

For *eg* : If $\overline{AB}=\hat{i}+2\hat{j}+3\hat{k}$, then $r=\sqrt{1+4+9}=\sqrt{14}$

Direction ratios are (1,2,3) and direction cosines are $\left(\sqrt{\frac{1}{14}},\sqrt{\frac{2}{14}},\sqrt{\frac{3}{14}}\right)$

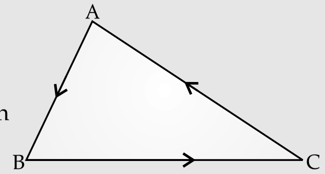
Types of Vectors

(i) Zero vector (initial and terminal points coincide)
(ii) Unit vector (magnitude is unity)
(iii) Coinitial vectors (same initial points)
(iv) Collinear vectors (parellel to the same line)
(v) Equal vectors (same magnitude and direction)
(vi) Negative of a vector (same magnitude, opp. direction)

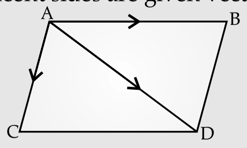
Properties of Vector

The vector sum of the three sides of a triangle taken in order is 0 .i.e

if ABC is given triangle, then $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0$.



The Vector sum of two coinitials vectors is given by the diagonal of the parallelogram whose adjacent sides are given vectors.



if $\overrightarrow{AB},\overrightarrow{AC}$ are the given vectors, then $\overrightarrow{AB}+\overrightarrow{AC}=\overrightarrow{AD}$

If we have two vectors $a=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $b=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and λ is any scalar, then-

$a+b=(a_1+b_1)\hat{i}+(a_2+b_2)\hat{j}+(a_3+b_3)\hat{k}$

$\lambda a=(\lambda a_1)\hat{i}+(\lambda a_2)\hat{j}+(\lambda a_3)\hat{k}$

$a.b=a_1b_1+a_2b_2+a_3b_3$ and

$$a\times b=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Cross product of two vectors

$a\times b=|a||b|\sin\theta\,\hat{n}$, $\hat{n}$ is a unit vector perpendicular to line joining a,b.

Scalar product of two vectors

If a, b are the vectors and 'θ', angle between them, then their scalar product $a.b=|a||b|\cos.\theta$

$\Rightarrow \cos.\theta=\frac{a.b}{|a||b|}$

Position of vectors

The Position vector of a point R dividing a line segment joining P,Q whose position vectors are a, b resp., in the ratio $m:n$
(i) internally is $\frac{na+mb}{m+n}$, (ii) externally is $\frac{mb-na}{m-n}$

Unit vector

For a given vector a, the vector $\hat{a}=\frac{a}{|a|}$ gives the unit vector in the direction of a. for *eg* , if $a=5i$, then $\hat{a}=\frac{5i}{|5|}=i$,which is a unit vector.

Multiplication of vector by a scalar

If λ Multiplied to vector AB, then the magnitude is multiplied by |λ| and direction remain same (or opp.) according as λ is the+ *ve* or , –*ve*.

MATHEMATICS (B-1)

Three Dimensional Geometry

Shortest distance between

(i) two skew lines is the line segment perpendicular to both the lines

(ii) $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ is $\left|\frac{(\vec{b}_1\times\vec{b}_2).(\vec{a}_2-\vec{a}_1)}{|\vec{b}_1\times\vec{b}_2|}\right|$

(iii) $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is $\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$

(iv) Parallel line $\vec{r}=\vec{a}_1+\lambda\vec{b}$ and $\vec{r}=\vec{a}_2+\mu\vec{b}$ is $\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}$

Direction ratios and direction cosines of a line

D. Cs of a line are the cosines of the angles made by the line with the positive direction of the co-ordinate axes. If l, m, n are the D. Cs of a line, then $l^2+m^2+n^2=1$. D. Cs of a line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_1}{PQ}$, where $PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

D.Rs of a line are the no.s which are proportional to the D.Cs of the line if l, m, n are the D.Cs and a, b, c are D.Rs of a line, then

$l=\frac{a}{\sqrt{a^2+b^2+c^2}}$, $m=\frac{b}{\sqrt{a^2+b^2+c^2}}$, $n=\frac{c}{\sqrt{a^2+b^2+c^2}}$

Skew lines

These are the lines in space which are neither parallel nor intersecting. They lie in different planes. Angle between skew lines is the angle between two intersecting lines drawn from any point (origin) parallel to each of the skew lines.

Angle between the two lines

if $l_1, m_1, n_1, l_2, m_2, n_2$ are the D.Cs and $a_1, b_1, c_1, a_2, b_2, c_2$ are the D.Rs of the two lines and 'θ' is the acute angle between them, then

$\cos\theta=|l_1l_2+m_1m_2+n_1n_2|=\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$

Line of Vector equations in 3D

Vector equation of a line passing through the given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r}=\vec{a}+\lambda\vec{b}$

Vector equation of a line which passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$

Equation of a line through point (x_1, y_1, z_1) and having D.Cs l, m, n is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ Also, equation of a line that passes through two points.

Angle between two lines

If 'θ' is the acute angle between $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$, $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$

then, $\cos\theta=\left|\frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1|.|\vec{b}_2|}\right|$

if $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ are the equations of two lines, then acute angle between them is $\cos\theta=|l_1.l_2+m_1.m_2+n_1.n_2|$

Equation of a plane

(i) which is at distance 'd' from origin and D.C.s of the normal to the plane as l,m,n is $lx+my+nz=d$.

(ii) $\perp r$ to a given line with D.Rs. A,B,C and passing through (x_1,y_1,z_1) is $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$

(iii) Passing through three non-collinear points (x_1,y_1,z_1) (x_2,y_2,z_2), (x_3,y_3,z_3) is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0.$

Vector equation of a plane

(i) which contains three non-collinear points having position vectors $\vec{a}, \vec{b}, \vec{c}$ is $(\vec{r}-\vec{a}).[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})]=0$.

(ii) That passes through the intersection of planes $\vec{r}.\vec{n}_1=d_1$ & $\vec{r}.\vec{n}_2=d_2$ is $\vec{r}(\vec{n}_1+\lambda\vec{n}_2)=d_1+\lambda d_2$, λ – non-zero constant.

Characteristics of planes

Two lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$, $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ are coplanar if $(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)=0$. Equation of a plane that cuts co-ordinate axes at (a,0,0), (0,b,0), (0,0,c) is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

The distance of a point with postion vector $\vec{a}$ from the plane $\vec{r}.\hat{n}=d$ is $|d-\vec{a}.\hat{n}|$. The distance from a point (x_1, y_1, z_1) to the plane $Ax+By+Cz+D=0$ is $\left|\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}\right|$

MATHEMATICS (B-1)

Linear Programming

Definition

A. L.P.P is one that is concerned with finding the optimal value (max. or min.) of a linear function of several variables (called objective function) subject to the conditions that the variables are non-negative and satisty a set of linear inequalities (called linear constraints). Variables are sometimes called decision variables and are non-negative.

Types of L.P.P

(i) Diet problems
(ii) Manufacturing Problem
(iii) Transportation Problems

Solution of a L.P.P

The common region determined by all the constraints including the non-negative constraint $x \geq 0, y \geq 0$ of a L.P.P is called the feasible region (or solution region) for the problem. Points within and on the boundary of the feasible region represent feasible solutions of the constraints. Any point outside the feasible region is an infeasible solution. Any point in the feasible region that gives the optimal value (max. or min.) of the objective function is called an optimal solution.

For *eg* : Max Z = $250x + 75y$, subject to the

Constraints: $5x + y \leq 100$
$x + y \leq 60$
$x \geq 0, y \geq 0$ is an L.P.P.

Fundamental Theorems

Theorem I : Let R be the feasible region (convex polygon) for a L.P. and let $Z=ax+by$ be the objective function. When Z has an optimal value (max. or min.), where the variables x,y are subject to the constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region, **Theorem 2**: Let R be the feasible region for a L.P.P, and let $Z=ax+by$ be the objective function. If R is bounded then the O.F Z has both a max. and a min. value on R and each of these occurs at a corner point (vertex) of R. If the feasible region is unbounded, then a max. or a min. may not exist. If it exists, it must occur at a corner point of R.

Corner point method

(i)Find the feasible region of the L.P.P and determine its corner points (vertices). (ii) Evaluate the O.F. $Z = ax+by$ at each corner point. Let M and m be the largest and smallest values respectively at these points. If the feasible region is unbounded, M and m are the maximum and minimum values of the O.F. If the feasible region is unbounded , then (i) 'M' is the max. value of the O.F., if the open half plane determined by $ax+by>M$ has no point in common with the feasible region. Otherwise, the O.F. has no maximum value. (ii) 'm' is the minimum value of the O.F., if the open half plane determined by $ax+by<m$ has no point in common with the feasible region. Otherwise, the O.F. has no minimum value.

MATHEMATICS (B-1)

Probability

Random Variable

Real valued function whose domain is the sample space of a random experiment.

Conditional Probability

The probability of the event E is called the conditional probability of E given that F has already occurred, and is denoted by $P(E/F)$. Also, $P(E/F)=\dfrac{P(E\cap F)}{P(F)}, P(F)\neq 0.$

Properties

(i) $0\le P(E/F)\le 1$, $P(E'/F)=1-P(E/F)$

(ii) $P((E\cup F)/G)=P(E/G)+P(F/G)-P((E\cap F)/G)$

(iii) $P(E\cap F)=P(E)P(F/E), P(E)\neq 0$

(iv) $P(E\cap F)=P(F)P(E/F), P(F)\neq 0$

For *eg* : if $P(A)=\dfrac{7}{13}$, $P(B)=\dfrac{9}{13}$ and $P(A\cap B)=\dfrac{4}{13}$, then $P(A/B)=\dfrac{p(A\cap B)}{P(B)}=\dfrac{\frac{4}{13}}{\frac{9}{13}}=\dfrac{4}{9}.$

If E and F are independent, then $P(E\cap F)=P(E)P(F)$, $P(E|F)=P(E), P(F)\neq 0$ and $P(F|E)=P(F), P(E)\neq 0$.

Theorem of total probability

Let, $\{E_1, E_2 \ldots E_n\}$ be a partition of a sample space 'S' and suppose that each of $E_1, E_2, \ldots E_n$ has non-zero probability. Let 'A' be any event associated with S, then $P(A) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + \ldots + P(E_n)P(A|E_n)$.

Bayes' Theorem

If $E_1, E_2, \ldots, E_n$ are events which constitute a partition of sample space S, *i.e.*, $E_1, E_2, \ldots E_n$ are pair wise dis joint and $E_1\cup E_2\cup\ldots\cup E_n=S$ and A be any event with non-zero probability, then $P(E_i|A)=\dfrac{P(E_i)P(A|E_i)}{\sum_{j=1}^{n}P(E_j)P(A|E_j)}$

Mean of a random variable

The probability distribution of a random variable x is the system of numbers x: $x_1 \quad x_2 \quad \ldots \quad x_n$
$P(x)$: $p_1 \quad p_2 \quad \ldots \quad p_n$ where, $pi>0$, $\sum_{i=1}^{n}p_i=1$, $i=1.2,\ldots, n$.

Let x be *a* R.V. whose possible values $x_1, x_2, \ldots, x_n$ occur with probabilities $p_1, p_2, \ldots, p_n$ resp. Then, mean of $x, \mu=\sum_{i=1}^{n}x_ip_i$ It is also called the expectation of x, denoted by $E(x)$

Variance and standard deviation

Let x be a R.V. whose possible values $x_1, x_2, \ldots, x_n$ occurs with probabilities $p(x_1), p(x_2), \ldots, p(x_n)$ respectively. Let, $\mu=E(x)$ be the mean of x. The variance of x, var (x) or $\sigma_x^2=\sum_{i=1}^{n}(x_i-\mu)^2p(x_i)$ or $E(x-\mu)^2$

The non-negative number

$6x=\sqrt{\text{var}(x)}=\sqrt{\sum_{i=1}^{n}(x_i-\mu)^2p(x_i)}$ is called the standard deviation of the R.V. 'X'. Also, $\text{var}(x)=E(x^2)-[E(x)]^2$ For *eg*: $E(x)=3$ and $E(x^2)=10$, then var $x= 10-9 = 1$ and SD$=\sqrt{1}=1$.

Bernoulli's Trials

Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions :

(i)There should be a finite no. of trials.

(ii)The trials should be independent.

(iii)Each trial has exactly two outcomes: success or failure.

(iv)The probability of success remains the same in each trial.

For Binomial distribution, B (n,p), $P(X=x)=({}^nC_x q^{n-x}p^x, x=0,1,\ldots,n)$ $(q=1-p)$

APPLIED MATHEMATICS (B-2)

Numbers and Quantification

Arithmetic function

An arithmetic function is a function whose domain is positive integer. *i.e.*, $f : c \to Z^+$

Congruence modulo

Let $a, b \in Z$, $m \in Z^+$
Then,
a is congruent to b modulo m written as '$a \cong b$ (mod)m' iff
1. $a \bmod m = b \bmod m$
2. $m \mid (a - b)$ *i.e.*, $(a - b) \bmod m = 0$

Properties of congruence modulo

- If $a \cong b \pmod{m}$, then $b \cong a \pmod{m}$
- If $a \cong b \pmod{m}$, then $(a \pm c) \cong (b \pm c) \pmod{m}$
- If $a \cong b \pmod{m}$, then $ac \cong bc \pmod{m}$
- If $a \cong b \pmod{m}$ and $c \cong d \pmod{m}$, then $(a \pm c) \cong (b \pm d) \pmod{m}$
- If $a \cong b \pmod{m}$ and $c \cong d \pmod{m}$, then $ac \cong bd \pmod{m}$
- If x_1 is a solution of $ax \cong b \pmod{m}$, then any other integer $x_2 \cong x_1$ is also a solution.

Types of arithmetic functions

Divisor function

$\tau(n)$ or $d(n)$ = number of divisors of $n = \sum_{d/n} 1$

e.g., $\tau(4)$ = number of divisors of 4
Divisors of 4 = 1, 2, 4
So, $\tau(4) = 3$

Sigma function

$\sigma(n)$ = sum of number of divisors of $n = \sum_{d/n} d$

e.g., $\sigma(4)$ = sum of divisors of 4
= 1+2+4=7

Sigma k function

$\sigma_k(n)$ = Sum of kth powers of divisors of $n = \sum_{d/n} d^k$

e.g., $\sigma_2(4)$ = Sum of divisors of 2nd powers of 4.
$= 1^2 + 2^2 + 4^2$
$= 1 + 4 + 16 = 21$

Omega function

$\omega(n)$ = Number of prime divisors of $n = \sum_{p/n} 1$ where is prime

e.g., $\omega(4)$ = Number of prime divisor of 4 = 1, as 1 is the only prime divisors of 4.

Mobius function

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } p^2/n, \text{ for some prime } p \\ (-1)^k & \text{if } n = p_1, p_2, \ldots, p_k \text{ where } p_1, p_2, \ldots, p_k \text{ are distinct prime} \end{cases}$$

when $n = 1$, $\mu(1) = 1$
when $n = 2$, $m(2) = (-1)^1 = -1$
when $n = 4$, $m(4) = 2^2 / 2^2 = 0$

Euler's totient f (n) function

If n is a positive integer, then f function counts the positive integers less than or equal to n that are relatively prime to n.
$\phi(n)$ = number of integers $\leq n$ coprime to n.
e.g., f (10) = 4
(as 1, 3, 7, 9 are coprime to 10)
f (7) = 6 (as 1, 2, 3, 4, 5, 6 are coprime to 7)

Trace the Mind Map
- First Level
- Second Level
- Third Level

APPLIED MATHEMATICS (B-2)

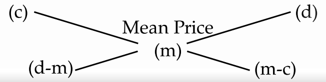
Numerical Applications

Alligation or mixture

If ingredients are mixed in a ratio, then

$$\frac{\text{Quantity of cheaper}}{\text{Quantity of dearer}} = \frac{\text{C.P. of the dearer- Mean price}}{\text{Mean price - C.P. of the cheaper}}$$

We represent it as writer:
C.P. of unit quantity of cheaper C.P. of unit quantity of dearer



Races and games

- A race is a contest which can measure the speed in driving running, sailing, rowing, jumping etc. This is calculated over a centre distance.
- In every race end point is known as winning point
- In a dead heat race, every participant reach at the winning point
- The person that comes first wins the race and called winner.
- A game of leo in a race is the game where participants agree that whosoever scores the 100 point first wins the race.

Pipes and gistern

- If a pipe can fill a tank in x hours and another pipe can empty the full tank in y hours (where $y>x$), then on opening both pipes, the net part filled in 1 hour $= \left(\frac{1}{x}-\frac{1}{y}\right)$
- If a pipe can fill a tank in x hours and another pipe can empty the full tank in y hours (where $x>y$), then on opening both pipes, the net part emptied in 1 hour $= \left(\frac{1}{y}-\frac{1}{x}\right)$

Boat and streams

Boat (u km/h) →, Stream (v km/h) → : Down stream $(u+v)$km/h

Boat (u km/h) →, Stream (v km/h) ← : Up stream $(u-v)$km/h

If the speads of downstream and upstream are a km/h and b km/h, respectively, then

Relative speed of boat in up stream $= \frac{1}{2}(a+b)$km/h

Relative speed of boat in down stream $= \frac{1}{2}(a-b)$km/h

Partnership

There are basically 2 types of partners in a business. One in the active partner who participates actively in the business and the other the **sleeping partners** who invests his capital and stays away from the day to day running of the business.

Amount and Time period same : When people invest the same amount of money for the same period then the profit is shared in 1 : 1 : 1 ratio (*i.e.*) equally among the partners.

Amount same and Time Period Different : When the money invested is the same but the time period for which it is invested is different, then the profit is shared in the ratio of the time period for which the amount is invested.

Time Period same and Amount Different : When the time period of the amounts invested is the same but the amounts are different then the profit is shared in the ratio of the amounts invested respectively.

Inequalities

Types of Inequalities

Numerical Inequalities : Relation between numbers e.g.: $3<5, 7>4$

Literal or Variable Ineqalities : Relation between two variables or variable and numbers.
e.g.:$y<8$

Double Inequalities: Relationship from two sides
e.g.:$2<x<5$

Strict Inequalities: An inequality that uses the symbol <or> e.g.: $x<5, 3<5$

Slack Inequality: An inequality that uses the symbols $\le$ or $\ge$. e.g.: $y \le 5$

Linear Inequalities in One Variable: An inequality which involves a linear function in one variable
e.g.: $x<5$

Linear Inequalities in two variables: An inequality which involves a linear function in two variables,
e.g.: $3x+2y<5$

Quadratic Inequalities: An inequality which involves a quadratic function, eg.: $x^2+2x \le 5$.

Rules of inqualities

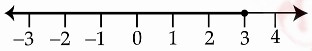
- Equal numbers may be added to (or subtracted from) both the sides of an inequality without affecting the sign of inequality:
  e.g.: $x<7$ is same as $x+2 < 7+2$
- Both sides of an inequality can be multiplied (or divided) by the same positive number without affecting the sign of inequality.
  e.g.: $x+y<7$ is same as $(x+y) \times 3 < 7 \times 3$
- When both sides are multiplied or divided by a – ve number, then sign of inequality is reversed.
  e.g.: $x+y<7$ is same as
  $(x+7) \times (-3) > 7 \times (-3)$

Representation

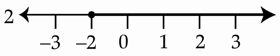
To reptriest $x<a$ or $x>-a$ on a number line, put a circle on the number 'a' and dark line to the left (or right) of the number 'a'.

e.g.:

$x<3$



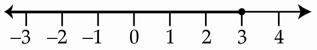
$x>-2$



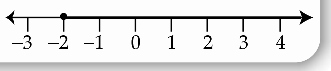
To represent $x\le a$ (or $x\ge a$) on a number line, put a dark circle on the number 'a' and dark the line to the left (or right) of the number 'a'.

e.g.:

$x \le 3$



$x \ge -2$



Definition

- An inequality is a relation that holds between two values, when they are different eg.: $x<5$, here, there is a relation between x & 5.
- Two real numbers or algebraic expressions related by symbol '<', '>', '≤' or '≥' form inequality.

e.g.: $y\le4$, $2x+3y\ge5$, $3>x$.

Trace the Mind Map
First Level
Second Level

APPLIED MATHEMATICS (B-2)

Matrices

What is Matrix ?

A matrix of order $m\times n$ is an ordered rectangular array of numbers or functions having 'm' rows and 'n' columns. The matrix $A = [a_{ij}]_{m\times n}$ is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ a_{m1} & a_{m2} & \dots a_{mn} \end{bmatrix}_{m\times n}$$

Symmetric and Skew-Symmetric Matrix

Symmetric

If $A^T = A$, e.g. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$A^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = A$

Skew-Symmetric

If $A^T = -A$, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,

$A^T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = A$

Operation on Matrices

Addition

Let $A = [a_{ij}]_{m\times n}$ –A and $B = [b_{ij}]_{m\times n}$, than $A+B = [a_{ij} + b_{ij}]$

e.g. $A = \begin{bmatrix} 1 & 0 \\ 2 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}$

$A+B = \begin{bmatrix} 2 & -2 \\ 2 & 10 \end{bmatrix}$

Multiplication

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$, then $C = AB = [C_{ik}]_{m\times p}$

when $C_{ik} = \sum_{f=1}^{n} a_{ij} \cdot b_{jk}$

e.g. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

then, $AB = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$

Multiplication of a Matrix by a Scalar

If $A = [a_{ij}]_{m\times n}$, then $kA = [ka_{ij}]_{m\times n}$

e.g. $A = \begin{bmatrix} 2 & -4 \\ 5 & 6 \end{bmatrix}$, $3A = \begin{bmatrix} 6 & -12 \\ 15 & 18 \end{bmatrix}$

Existence of non - zero matrices whose product is the zero matrix

Product of two matrices is zero matrix, as it is not necessary that one of the matrices is a zero matrix

e.g. $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$, $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Transpose of matrix

A new matrix whose rows are columns of the original, it is denoted by A^T or A'

e.g. $A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$,

then $A^T = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 5 \\ 5 & 4 & 7 \end{bmatrix}$

Equality of two matrix

$A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are said to be equal, if $a_{ij} = b_{ij}\ \forall\ i, j$

e.g. $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 4/2 & 2-1 \\ \sqrt{9} & 0 \end{bmatrix}$

but $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$

Types of Matrices

Row matrix

It is of the form $A = [a_{ij}]_{1\times n}$

e.g. $[2\ 5\ -4]_{1\times 3}$, $[\sqrt{2}\ \ 4]_{1\times 2}$

Column matrix

It is of the form $A = [a_{ij}]_{m\times 1}$

e.g. $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}_{3\times 1}$, $\begin{bmatrix} 4 \\ 5 \end{bmatrix}_{2\times 1}$

Square matrix

Here $m = n$ (no. of rows = no.of columns)

$A = [a_{ij}]_{n\times n}$

e.g. $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & -4 \\ 0 & -1 & -2 \end{bmatrix}_{3\times 3}$

Diagonal matrix

All non-diagonal entries are zero i.e.

$a_{ij} = 0\ \forall\ i \neq j$

e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3}$

Scalar matrix

$a_{ij} = 0$, $i \neq j$ and $a_{ij} = k$ (scalar), $i = j$

e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Identity matrix

$a_{ij} = 0$ when $i \neq j$ and $a_{ij} = 1$ when $i = j$

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

Zero matrix

All entries are zero

e.g. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

Trace the Mind Map

First Level · Second Level · Third Level

APPLIED MATHEMATICS (B-2)

Determinants

Determinant of Square Matrix

A unique number (real or complex) can be associated, to every square matrix $A=[a_{ij}]$ of order m, is called determinant of matrix A. It is denoted by $|A|$ or det (A)

Expansion of a Determinant

of Order 2

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}.a_{22} - a_{21}.a_{12}$$

of Order 3

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}.a_{33} - a_{32}.a_{23}) - a_{12}(a_{21}.a_{33} - a_{31}.a_{23}) + a_{13}(a_{21}.a_{32} - a_{31}.a_{22})$$

Minors and Cofactors

Minors

If $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then minor of $a_{11} = M_{11}$

$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, Minor of $a_{12} = M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and so on.

Cofactors

If the minor M_{ij} is multiplied by $(-1)^{i+j}$, then we obtain the factor of a_{ij}. It is denoted by A_{ij} or C_{ij}, i.e., $A_{ij} = (-1)^{i+j}$ M_{ij}, cofactor of $a_{11} = (-1)^{1+1} M_{11}$ or $A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

Properties of Determinants

- If each element of a row or column is zero, then determinant becomes zero. *e.g.* $A = \begin{bmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{bmatrix} \therefore |A| = 0$
- The value of a determinant remains unchanged if its row and columns are interchanged. *e.g.* $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$
- If any two adjacent rows or columns are interchanged then the determinant retains its absolute value, but its sign is changed. *e.g.* $\Delta = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \Delta_1 = \begin{bmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $\Rightarrow \Delta_1 = -1\Delta$
- If each element of a row or a column of a determinant is multiplied by a constant k, then the value of new determinant are k times the value of the original determinant. *e.g.* $\Delta = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \Delta_1 = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = k\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \Rightarrow \Delta_1 = k\Delta$
- If each element of a row (or column of determinant is the sum of the two more terms, then the determinants can be expressed as the sum of the two or more determinants. *e.g.* $\begin{bmatrix} a_1+a_2 & b & c \\ p_1+p_2 & q & r \\ u_1+u_2 & v & w \end{bmatrix} = \begin{bmatrix} a_1 & b & c \\ p_1 & q & r \\ u_1 & v & w \end{bmatrix} + \begin{bmatrix} a_2 & b & c \\ p_2 & q & r \\ u_2 & v & w \end{bmatrix}$

Leontief Model

$(I - B)x = D$
x = variable vector
D = demand vector
I – B = Leontief Matrix
I – B is non singular
$x = (I - B)^{-1} D$

Adjoint of a Matrix

Statement

Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order n and C_{ij} (or A_{ij}) be the cofactor of a_{ij} in the determinant $|A|$. Then, the adjoint of A is defined as the transpose of the cofactor matrix and it is denoted by adj (A).

Example

If $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then adj $A = \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}$,

where A_{ij}'s are coafactors of A.

Singular and Non-Singular Matrix

Non-Singular: If A is a square matrix and $|A| \neq 0$, then A is known as non - singular matrix.

Singular: If A is a square matrix, and $|A| = 0$, then A is known as singular matrix.

Properties of Adjoint of a Matrix

If A is a square matrix of order $n\times n$, then
(i) $A(\text{adj } A) = (\text{adj } A)A = |A|I_n$
(ii) adj $(A^T) = (\text{adj } A)^T$
(iii) $|\text{adj } A| = |A|^{n-1}$, provided $|A| \neq 0$

Inverse of Square Matrix

Statement

Let A be a non-singular (*i.e.* $|A| \neq 0$) square matrix. Then, a square matrix B, such that $AB = BA = I$ is called of inverse of A and it is denoted by A^{-1}

i.e. $A^{-1} = \frac{1}{|A|}$ adj (A).

Properties

Let A and B be square matrices of same order, then
(i) A square matrix is invertible if and only if it is non-singular
(ii) $(A^{-1})^{-1} = A$
(iii) $(A')^{-1} = (A^{-1})'$
(iv) $(AB)^{-1} = B^{-1} A^{-1}$
(v) $|A^{-1}| = |A|^{-1}$
(vi) $AA^{-1} = A^{-1} A = I$
(vii) $(kA)^{-1} = \frac{1}{k}A^{-1}$, if $k \neq 0$

Cramer's Rule

The solution of the system $a_1x + b_1y + c_1z + d$, $a_2x + b_2y + c_2z + d_2$ and $a_3x + b_3y + c_3z + d_3$ is given by $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$ and $z = \frac{D_z}{D}$, where

If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Provided that $D \neq 0$

Trace the Mind Map: First Level, Second Level, Third Level

APPLIED MATHEMATICS (B-2)

Differentiation and its Applications

Higher Order Derivatives

Because the derivative of a function $y = f(x)$ is itself a function $y' = f'(x)$, we can take the derivative of $f'(x)$, which is generally referred to as the *second derivative of f(x)* and written $f''(x)$ or $f^{2}(x)$. This differentiation process can be continued to find the third, fourth, and successive derivatives of $f(x)$, which are called higher order derivatives of $f(x)$.

e. g. : Find the first, second and third derivatives of $f(x) = 5x^4 - 3x^3 + 7x^2 - 9x + 2$.

Solution:

$$f'(x) = 20x^3 - 9x^2 + 14x - 9$$
$$f''(x) = 60x^2 - 18x + 14$$
$$f'''(x) = 120x - 18$$

Increasing and decreasing functions

A function f is said to be **(i)** increasing on (a,b) if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) \le f(x_2) \forall x_1, x_2 \in (a, b)$,
and **(ii)** decreasing on (a, b) if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) \ge f'(x_2)$ $\forall x_1, x_2 \in (a, b)$

If $f'(x) \ge 0 \forall x \in (a, b)$ then f is increasing in (a,b) and if $f'(x) \le 0 \forall x \in (a,b)$, then f is decreasing in (a,b)

eg: Let $f(x) = x^3 - 3x^2 + 4x, x \in R$, then
$f'(x) = 3x^2 - 6x + 4 =$
$3(x-1)^2 + 1 > 0 \forall x \in R$.
So, the function f is strictly increasing on R.

Maxima and Minima

A point C in the domain of 'f' at which either $f'(C)=0$ or is not differentiable is called a critical point of f.

Second derivative test

Let f be a function defined on given interval, f is twice differentiable at C. Then

(i) $x=C$ is a point of local maxima. If $f'(C)=0$ and $f''(C) < 0$, $f(C)$ is local maxima. of f.

(ii) $x=C$ is a point of local minima if $f'(C)=0$ and $f''(C)>0$. $f(C)$ is local minima of f.

(iii) The test fails if $f'(C)=0$ and $f''(C)=0$

First derivative test

Let f be continuous at a critical point C in open interval. Then **(i)** if $f'(x) > 0$ at every point left of C and $f'(x) < 0$ at every point right of C, then 'C' is a point of local maxima.

(ii) If $f'(x) < 0$ at every point left of C and $f'(x) > 0$ at every point right of C, then 'C' is a point of local minima.

(iii) If $f'(x)$ does not change sign as 'x' increases through C, then 'C' is called the point of inflection.

Marginal Cost and Marginal Revenue

Marginal refers to an instantaneous rate of change, i.e., a derivative.
If x is the number of units of a product produced in some time interval, then
If Total Cost = C(x), then Marginal Cost = C′(x)
Similarly, marginal revenue is the rate of change of total revenue w.r.t. total number of unit sold.
If Total Revenue = R(x), where x is the number of unit sold then Marginal Revenue = R′(x).
e.g. : If $C(x) = 9.5x + 4500$ $R(x) = 36x - .03x^2$

- Find the marginal cost function.
  - $C'(x) = 9.5$
- Find the marginal revenue function.
  - $R'(x) = 36 - .06x$
- Find the marginal profit function.
  - $P'(x) = R'(x) - C'(x)$
    $= 36 - .06x - 9.5$
    $= 26.5 - .06x$

Trace the Mind Map

- First Level
- Second Level
- Third Level

APPLIED MATHEMATICS (B-2)

Probability Distributions

Probability Distribution

The probability distribution of a random variable x is the system of numbers x: $x_1\ x_2 \ldots\ldots, x_n$

P (x): $p_1\ p_2 \ldots\ldots p_n$ where, $p_i > 0$,

$\sum_{i=1}^{n} p_i = 1, i = 1, 2 \ldots\ldots n.$

Expectation

Let x be a random variable whose possible values $x_1 x_2, \ldots\ldots\ldots, x_n$ occurs with probabilities $p_1\ p_2 \ldots\ldots p_n$ resp. Then, mean of x, $\mu = \sum_{i=1}^{n} x_i p_i$. It is also called the expectation of x, denoted by E (x)

Variance

Let x be a random variable whose possible values $x_1 x_2, \ldots\ldots\ldots, x_n$ occurs with probabilities $p(x_1), p(x_2), \ldots\ldots\ldots p(x_n)$, respectively, Let, $\mu = E(x)$ be the mean of x. The variance of x, var (x) or

$\sigma_x^2 = \sum_{i=1}^{n} (x_i - \mu)^2\ p\ (x_i)$ or $E\ [(x - \mu)^2]$

Binomial Theorem

$P\ (x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{(n-x)!x!} p^x q^{n-x}$

where

n = the number of trials (or the number being sampled)

x = the number of successes desired

p = probability of getting a success in one trial

$q = 1 - p$ = the probability of getting a failure in one trial

Let

x = the number of successes = 15

n = the number of trials = 18

p = the probability of success = 0.85

$P\ (15) = {}^{18}C_{15}(0.85)^{15}\ (1 - 0.85)^{18-15}$

$P\ (15) = (\frac{18!}{15!\ (18-15)!})\ (0.85)^{15}\ [(1 - 0.85)^3]$

$P\ (15) = (816)\ (0.87)\ (.0036)$

$P\ (15) = .239$

Normal Distribution

$f\ (x) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

μ = mean of x

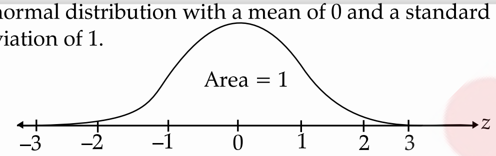
σ = Standard deviation of x

$\pi \approx 3.14159\ldots.$

$e \approx 2.71828\ldots.$

Standard Normal Curve

- A normal distribution with a mean of 0 and a standard deviation of 1.



- Any x-value can be transformed into a z-score by using the formula

$$z = \frac{\text{Value} - \text{Mean}}{\text{Standard deviation}} = \frac{x - \mu}{\sigma}$$

Poisson Distribution

The mean number of power outages in the city or Brunswick is 4 per year. Find the probability that in a given year, there are exactly 3 outages,

$\mu = 4, x = 3$

$P(3) = \frac{4^3(2.71828)^{-4}}{3!}$

≈ 0.195

$P\ (X = x) = \frac{\mu^x e^{-\mu}}{x!}$

where

$x = 0, 1, 2, 3, \ldots.$

μ = mean number of occurrences in the interval

e = Euler's constant ≈ 2.71828

Trace the Mind Map

First Level · Second Level · Third Level

APPLIED MATHEMATICS (B-2)

Index numbers and Time based Data

Time Series

Its componants:-
A time series is a set of observations taken at specified times, usually at equal time intervals

It has 4 components:-
(i) Secular (simple) Trend
(ii) Seasonal variations
(iii) cyclic variations
(iv) Irregular or random variations

Time Reversal Test

Index number satisfies the relationship

$$P_{01}^{F} \times P_{10}^{F} = 1$$

where P_{01}^{F} is Fisher's price index number

Index Number

Index numbers are used as an indicator to indicate the changes in economic activity. They also provide frame work for decision making and to predict future events. There are three types of index numbers (i) Price index, (ii) quantity index and (iii) value index

Methods of Constructing Index Numbers

1. Unweighted (simple) Index Number
   - Simple aggregative method
   - Simple average of price relative method
2. Weighted Index Number
   - Weighted average of price relative method
   - Weighted aggregative method

Formulae

1. Simple aggregative method

$$P_{01} = \frac{\sum p_1}{\sum p_0} \times 100$$

2. Simple average of price relative method

$$P_{01} = \frac{\sum\left(\frac{p_1}{p_0} \times 100\right)}{N}$$

3. Weighted aggregative method

$$P_{01} = \frac{\sum p_1 w}{\sum p_0 w} \times 100$$

4. Weighted average of price relative method

$$P_{01} = \frac{\sum\left(\frac{p_1}{p_2} \times 100\right) \times 100}{\sum w}$$

where, p_0 = price of base year,
p_1 = price of current year,
N = number of items,
w = weight related to item

Some other important formulae

- Laspeyre's price Index Number

$$P_{01}^{L} = \frac{\sum p_1 q_0}{\sum p_0 q_0} \times 100$$

- Passche's price Index Number

$$P_{01}^{P} = \frac{\sum p_1 q_1}{\sum p_0 q_1} \times 100$$

- Fisher's price Index Number

$$P_{01}^{F} = \sqrt{P_{01}^{L} \times P_{01}^{P}} = \sqrt{\frac{\sum p_1 q_0 \times \sum p_1 q_1}{\sum p_0 q_0 \times \sum p_0 q_1}} \times 100$$

where p_0 = base year price
p_1 = current year price
q_0 = base year quantity
q_1 = current year quantity

Mathematical models for time series

Additive model:
$Y = T + S + C + I$
Multiplicative modes:-
$Y = T \times S \times C \times I$
where, Y = original value
T = Trend value
S = Seasonal variations
C = Cyclic variations
I = Irregular variations

Measurement of Trends

Method of moving averages

It helps to make forecast when all statistical information we take into account the last few data points, e.g: data for last three, four, five and six periods.
Three period moving average is defined as a value by adding numerical data for three periods and then dividing by 3.
Four period moving average is defined as a value by adding numerical data for four periods and then dividing by 4

Method of least square

The straight line trend in represented by
$Y = aX + b$
where, Y = Trend value to be computed
X = Unit of time (Independent variable)
a = constant to be calculated
b = constant to be calculated
Also, $a = \frac{\sum Y}{n}$ and $b = \frac{\sum xy}{\sum x^2}$
n = number of items

How to understand Mind Map?
First Level · Second Level · Third Level

APPLIED MATHEMATICS (B-2)

Integrals and It's Applications-1

Integration by Parts

For functions $f(x)$ and $g(x)$, we have

$\int [f(x).\, g(x)]\, dx = f(x).\int g(x).\, dx - \int \{f'(x).\int g(x).\, dx\}.\, dx$

Here, we can choose first function according to its position in ILATE, where

I = Inverse trigonometric function

L = Logarithmic function

A = Algebraic function

T = Trigonometric function

E = Exponential function

Integration by Substitution

$\int f(x).\, dx = \int f\{g(t)\}.\, g'(t)\, dt$, if we substitute $x = g(t)$, such that $dx = g'(t)\, dt$.

(i) $\int \{f(x)\}^n.\, f'(x).\, dx = \dfrac{\{f(x)\}^{n+1}}{n+1} + C,\ n \neq -1$

(ii) $\int \dfrac{f'(x)}{f(x)}.\, dx = \log |f(x)| + C,\ f(x) \neq 0$

Some Examples

Integration by parts

Evaluate: $\int e^{-2x} \sin x\, dx$

Method:-

(i) Use integration by part method i.e. consider $\sin x$ as 1st function and e^{-2x} as 2nd function.

(ii) Use the formula: $\int \underset{\text{I}}{u}.\underset{\text{II}}{v}\, dx = u \int v.dx - \int \left[\dfrac{d}{dx}(u).\int v\, dx\right] dx$

(iii) If necessary use again integration by parts and simplify it.

Integration by substitution

Evalute : $\int \dfrac{1}{x+\sqrt{x}}\, dx$

Method:- To solve this problem, we use the concept of method of substitution. Here first of all put $x = t^2$, then apply the formula i.e.

$\int \dfrac{1}{x} dx = \log |x| + C$

Integration as process of differentiation

$\dfrac{d}{dx}[F(x)] = f(x)$, then $\int f(x).dx = F(x) + C$

Basic Formulae of Integrals

1. $\int x^n\, dx = \dfrac{x^{n+1}}{n+1} + C,\ n \neq -1$
2. $\int e^x\, dx = e^x + C$
3. $\int e^{ax}\, dx = \dfrac{e^{ax}}{a} + C$
4. $\int a^x\, dx = \dfrac{a^x}{\log a} + C$
5. $\int \dfrac{1}{x} dx = \log |x| + C$
6. $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \log \left|x + \sqrt{x^2 - a^2}\right| + C$
7. $\int \dfrac{dx}{\sqrt{x^2 + a^2}} = \log \left|x + \sqrt{x^2 + a^2}\right| + C$
8. $\int \dfrac{1}{a^2 - x^2} dx = \dfrac{1}{2a} \log \left|\dfrac{a+x}{a-x}\right| + C$
9. $\int \dfrac{dx}{x^2 - a^2} = \dfrac{1}{2a} \log \left|\dfrac{x-a}{x+a}\right| + C$
10. $\int \sqrt{x^2 - a^2}\, dx = \dfrac{x}{2}\sqrt{x^2 - a^2} - \dfrac{a^2}{2} \log \left|x + \sqrt{x^2 - a^2}\right| + C$
11. $\int \sqrt{x^2 + a^2}\, dx = \dfrac{x}{2}\sqrt{x^2 + a^2} + \dfrac{a^2}{2} \log \left|x + \sqrt{x^2 + a^2}\right| + C$
12. $\int (ax+b)^n\, dx = \dfrac{1}{a}\dfrac{(ax+b)^{n+1}}{n+1} + C,\ n \neq -1$

Trace the Mind Map

- First Level
- Second Level
- Third Level

APPLIED MATHEMATICS (B-2)

Integrals and It's Applications-2

Evaluation of Definite Integrals by Substitution

- **Step-I:** Consider a definite integral: $\int_a^b f.[g(x)].g'(x)$ If Substitute $g(x) = t$ $\Rightarrow g'(x)\,dx = dt$
- **Step-II:** Find the limits of integration in new system of variable i.e. the lower limit $g(a)$ and the upper limit $g(b)$ and the given integral becomes $\int_{g(a)}^{g(b)} f(t).dt$.
- **Step-III:** Evaluate the integral, so obtained by usual method.
- **Example:**
  $\int_0^1 x e^{x^2}\,dx$
  $let\ x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow x\,dx = \frac{dt}{2}$
  Substituting, when $x = 0, t = 0$
  $x = 1, t = 1$
  $I = \int_0^1 e^t \frac{dt}{2} = \frac{1}{2}\int_0^1 e^t dt$
  $= \frac{e^t}{2}\Big|_0^1 = \frac{1}{2}\left[e^1 - e^0\right] = \frac{1}{2}\left[e - 1\right] = \frac{e}{2} - \frac{1}{2}$

Steps for calculating Definite Integral

- **Step-I:** There is no need to keep integration constant C.
  i.e., $\int_a^b f(x).dx = [F(x) + C]_a^b$
  $= [F(b) + C] - [F(a) + C]$
  $= F(b) - F(a)$
- **Step-II:** Evaluate: $\int_a^b f(x).dx = [F(x)]_a^b$
  $= F(b) - F(a)$
- **Step-III:** Calculate: $F(b) - F(a)$

Properties of Definite Integral

(i) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

(ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$

(iv) $\int_a^a f(x)\,dx = 0$

(v) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(vi) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dt$

(vii) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

(viii) $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

(ix) $\int_{-a}^{a} f(x)\,dx = \int_0^a [f(x) + f(-x)]\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even i.e. } f(-x) = f(x) \\ 0, & \text{if } f(x) \text{ is odd i.e. } f(-x) = -f(x) \end{cases}$

Trace the Mind Map
- First Level
- Second Level
- Third Level

APPLIED MATHEMATICS (B-2)

Integrals and It's Applications-3

Consumer's Surplus

The consumers surplus is the difference between what the consumers would be willing to pay for a commodity and what they actually pay for them.
The consumers' surplus is given by

$$CS = \int_0^{Q_e} D(x)\,dx - Q_e P_e$$

where D is the demand function, Q_e is the equilibrium quantity and P_e is equilibrium price.

Example

Example : Suppose that the demand function for producing a can of tennis balls is $P(x) = 20 - 0.05x$ and that the current price level is P_e = Rs. 8. Find the consumers surplus.
Sol.: First, we need to find the value of Q_e that corresponds to P_e = Rs. 8. Setting P = 8 and solving for x gives
$8 = 20 - 0.05x \Rightarrow 0.05x = 12 \Rightarrow x = 240$ or $Q_e = 240$
Using the integral formula for consumer surplus, we find that

$$CS = \int_0^{240} (20 - 0.05x)\,dx - 240.8$$

$$= 20x\Big|_0^{240} - 0.025x^2\Big|_0^{240} - 1920 = \text{Rs.}1,440.$$

Producer's Surplus

Producer's surplus is the difference between what producers are willing and able to supply goods for and the price they actually receive.
The producers surplus is given by

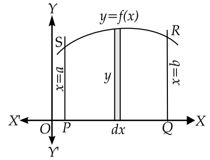
$$PS = Q_e P_e - \int_0^{Q_e} S(x)\,dx$$

Where S is the supply function, Q_e is the equilibrium quantity and P_e is equilibrium price.

Area under simple curves

The area of the region bounded by the curve $y = f(x)$, X- *axis* and the lines $x = a$ and $x = b\,(b > a)$ is given by

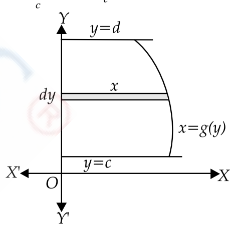
$$A = \int_a^b y\,dx \text{ or } \int_a^b f(x)dx$$



eg : The area bounded by $y = x^2$, X- *axis* in Iquadrant and the lines $x = 2$ and $x = 3$ is:

$$A = \int_2^3 y\,dx = \int_2^3 x^2\,dx = \left[\frac{x^3}{3}\right]_2^3 = \frac{1}{3}(27-8) = \frac{19}{3} \text{ sq.units.}$$

The area of the region bounded by the curve $x = f(y)$, $y-axis$ and the lines $y = c$ and $y = d\,(d > c)$ is given by $A = \int_c^d x\,dy$ or $\int_c^d f(x)dy$



eg : The area bounded by $x = y^3$, $Y-axis$ in the I quadrant and the lines $y = 1$ and $y = 2$ is:

$$\int_a^b f(x)dx = \int_a^b y^3\,dy = \left[\frac{1}{4}y^4\right]_1^2 = \frac{1}{4}(2^4 - 1^4) = \frac{15}{4} \text{ sq. units}$$

Trace the Mind Map
- First Level
- Second Level
- Third Level

APPLIED MATHEMATICS (B-2)

Differential Equations and Modeling

Ordinary Differential Equations

An equation involving derivatives of the dependent variable with respect to independent variable (variables) and constant *e.g.*: $xy\frac{d^2y}{dx^2}+x\frac{dy}{dx}+k=0$. If there is only one independent variable, then we call it as an ordinary differential equation. *e.g.*: $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2=0$

Order of Differential Equations

It is the order of the highest order derivative of the dependent variable with respect to the independent variable involved in the given differential equation.

e.g.: The order of $\frac{dy}{dx}=e^x$ is one and order of $\frac{d^2y}{dx^2}+x=0$ is two.

Degree of Differential Equations

The highest exponent of the highest order derivative.

e.g.: The degree of $\left(\frac{d^2y}{dx^2}\right)^3+\frac{dy}{dx}=0$ is three.

Order and degree (if defined) of a differential equation are always positive integers.

Solution of Differential Equations

- A function which satisfies the given differential equation is called its solution.
- The solution which contains as many arbitrary constants as the order of the differential equations is called a general solution.
- The solution free from arbitrary constants is called particular solution.
  e.g.: $y=e^x+1$ is a solution of $y''-y'=0$.
  Since, $y'=e^x$ and $y''=e^x \Rightarrow y''-y'=e^x-e^x=0$

Growth and Decay Model

The differential equation for growth and decay is

$\frac{dx}{dt}=kx,\ x(t_0)=x_0$

If $k>0$, it models growth. If $k<0$, it models decay.

The above equation is solved using variable separation method as:

$$\int\frac{dx}{x}=\int k\,dt \Rightarrow \log x = kt+\log c.$$

$$\Rightarrow \log\left(\frac{x}{c}\right)=kt \Rightarrow x=ce^{kt}$$

Variable Separable Method

It is used to solve such an equation in which variables can be separated completely .

e.g.: $y\,dx = x\,dy$ can be solved as $\frac{dx}{x}=\frac{dy}{y}$

Integrating both sides,

$\log x = \log y + \log c$

$\Rightarrow \frac{x}{y}=c \Rightarrow x = cy$ is the solution.

Trace the Mind Map

- First Level
- Second Level
- Third Level

APPLIED MATHEMATICS (B-2)

Inferential Statistics

Sampling Errors

- **Sampling Errors–**
  Errors caused by the act of taking a sample.
  Make sample results inaccurate.
- **Random Sampling Error**
  Errors caused by the chance in selecting a random sample.
- **Nonsampling Error**
  Errors not related to the act of selecting a sample from the population. They can even be present in a census.

Sampling

Sampling is the process of selecting observations (a sample) to provide an adequate description and robust inferences of the population.
Sampling can be broadly grouped into two :
- Random (Probability) Sampling
- Non Random Sampling.

Inferential statistics

It use measurements from the sample of subjects in the experiment to compare the treatment groups and make generalizations about the larger population of subjects.

Population

A **population** is a group of phenomena that have something in common. The term often refers to a group of people, as in the following examples:
- All registered voters in India.
- All members of the International Cricket Team.

Sample

A **sample** is a smaller group of members of a population selected to represent the population. A **random sample** is one in which every member of a population has an equal chance of being selected.

Parameter & Statistics

Parameter: The statistical constants of the population like mean (μ), variance (σ^2) are referred as population parameters.
Statistic : Any statistical measure computed from sample is known as statistic.

Standard Error

S.No.	Statistic	Standard Error
1.	Sample mean ($\bar{x}$)	$\sigma/\sqrt{n}$
2.	Observed sample proportion (p)	$\sqrt{PQ/n}$
3.	Sample standard deviation (s)	$\sqrt{\sigma^2/2n}$
4.	Sample variance (s^2)	$\sigma^2\sqrt{2/n}$

where, $\sigma = \sqrt{\dfrac{\Sigma(x_i-\mu)^2}{N}}$

σ = population standard deviation
N = the size of the population
x_i = each value from the population
μ = population mean
P = Population Proportion
Q = 1-P

Null Hypothesis

The null hypothesis for a 1-sample t-test is :
$H_0 : \mu = \mu_0$ where :
- μ = the population mean
- μ_0 = the hypothesized mean

You can choose any one of three alternative hypotheses :

$H_1 : \mu > \mu_0$	One-tailed test
$H_1 : \mu < \mu_0$	One-tailed test
$H_1 : \mu \neq \mu_0$	Two-tailed test

Two Sample t-test

Let the two independent samples be $x_1, x_2, x_3, \ldots.. x_{n_1}$ and $y_1, y_2, y_3, \ldots.., y_{n_2}$ with means $\bar{x} = \dfrac{\Sigma x}{n_1}$ and $\bar{y} = \dfrac{\Sigma y}{n_2}$ from two normal population with means μ_1 and μ_2 and common variance σ^2 (unknown).

Let $s_1^2 = \dfrac{1}{n_1-1}\Sigma(x-\bar{x})^2$ and $s_2^2 = \dfrac{1}{n_2-1}\Sigma(y-\bar{y})^2$

Thus, standard error can be given by $s = \sqrt{\dfrac{\Sigma(x-\bar{x})^2 + \Sigma(y-\bar{y})^2}{n_1+n_2-2}}$

The t-test formula is given as : $t = \dfrac{\bar{x}-\bar{y}}{s}\sqrt{\dfrac{n_1 n_2}{n_1+n_2}}$

Example

A healthcare consultant wants to compare the patient satisfaction ratings of two hospitals. The consultant collects ratings from 20 patients for each of the hospitals. The consultant performs a 2-sample t-test to determine whether there is a difference in the patient ratings between the hospitals.

One Sample t-test

The One-Sample t-Test procedure tests whether the mean of a single variable differs from a specified constant.
Example : A researcher might want to test whether the average IQ score for a group of students differs from 100.
Formula for One Sample t-test $t = \dfrac{\bar{x}-\mu}{s/\sqrt{n}}$

where, t = the t statistic
$\bar{x}$ = the mean of the sample
μ = the comparison mean
s = the sample standard deviation
n = the sample size

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APPLIED MATHEMATICS (B-2)

Time Series

Time Series

A time series is a set of observations taken at specified times, usually at equal time intervals.

It has 4 components:-
(i) Secular (simple) Trend
(ii) Seasonal variations
(iii) Cyclic variations
(iv) Irregular or Random Variations

Mathematical models for time series

Additive model:
$Y = T + S + C + I$
Multiplicative modes:-
$Y = T \times S \times C \times I$
where, Y = original value
T = Trend value
S = Seasonal variations
C = Cyclic variations
I = Irregular variations

Measurement of Trends

Method of moving averages

It helps to make forecast when all statistical information we take into account the last few data points, *e.g.*: Data for last three, four, five and six periods.

Three period moving average is defined as a value by adding numerical data for three periods and then dividing by 3.

Four period moving average is defined as a value by adding numerical data for four periods and then dividing by 4.

Method of Least Square

The straight line trend in represented by:
$Y = aX + b$
where, Y = Trend value to be computed
X = Unit of time (Independent variable)
a = constant to be calculated
b = constant to be calculated

Also, $a = \frac{\sum Y}{n}$ and $b = \frac{\sum XY}{\sum X^2}$
n = number of items

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APPLIED MATHEMATICS (B-2)

Financial Mathematics

Perpetuity

A perpetuity is a type of annuity that lasts forever.
e.g.: In a real state sector when an owner purchases a property and then rents it out. This owner is entitled to an infinite stream of cash flow from the renter as long as property continues to exist.
It is of 2 types :

1. Flat perpetuity $= \dfrac{\text{Cash flow}}{\text{Interest rate or yield}}$

2. Growing perpetuity $= \dfrac{\text{Payment}}{\text{Interest rate - Growth rate}}$

Sinking Fund

A sinking fund is a fund containing money set aside or saved to pay off a debt or bond.

$$\text{Sinking fund A} = \frac{\left[\left(1+\frac{r}{m}\right)^{n\times m}\right]-1}{\left(\frac{r}{m}\right)}\times P$$

where P = periodic payments
r = annualized rate of interest
n = No. of year
m = No. of payments per year

Bond Valuation

Bond valuation is a process of calculating its fair price.

$$\text{Present value of Bond} = \sum_{t=1}^{N}\frac{C}{(1+r)^t}+\frac{P}{(1+r)^N}$$

where C = periodic coupon payment
r = Market interest rate
P = Par value
N = Years of maturity

Effective Interest Rate

Effective Interest rate, $E=(1+i)^n-1$
where i=rate of interest/nominal or actual rate of interest
e.g.: UCO bank offers nominal rate of interest 12% on its certificate of deposit to its client. The client initially invested ₹ 1,000 and agreed to have the interest rate compounded monthly for a full year.

$$\text{Effective rate} = \left(1+\frac{0.12}{12}\right)^{12}-1 = 12.683\%$$

Compounded Annual Growth Rate

Compounded Annual Growth Rate (C.A.G.R.) indicates the annual growth rate of you investments over a number of years.

$$C.A.G.R. = \left(\frac{\textit{End value}}{\textit{begining value}}\right)^{\frac{1}{n}}-1$$

where n= investment period

e.g.:

	Date	Value
Begining value	3-01-2010	2,000/-
End value	3-01-2017	5,000/-

$$C.A.G.R. = \left(\frac{5000}{2000}\right)^{\frac{1}{7}}-1 = 1.1398-1$$
$$=0.1398$$
$$=13.98\%$$

Nominal Rate of Return

The nominal rate of return is the total rate of return earned on investment before adjusting for any deductions or remiums.

$$\text{Nominal rate of return} = \frac{(\text{Ending value of investment} - \text{Begning value of investment})}{\text{Begning value of investment}}\times 100$$

e.g.: Suppose Ram has invested ₹10,000 from 1st Feb. 2020. on 31 Jan. 2021, calculate NROR, if investment grown to ₹ 20,000.

Sol. $NROR = \dfrac{₹20,000 - ₹10,000}{₹10,000}\times 100$
$=100\%$

Depreciation and Linear Method

Depreciation is an accounting practice of spreading the cost of an asset purchased across its useful lifetime.
Linear Method for Calculating Depreciation:
Annual Depreciation Expense=
(Asset cost – Residual value)/Useful life of the asset
e.g.: Seema wants to buy a mobile phone for ₹ 5000. Based on the past experience she estimates a 3 year life span and a salvage value of ₹ 200, then

$$\text{Annual depreciation expenses} = \frac{₹(5000-200)}{3}$$
$$=₹1600$$

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