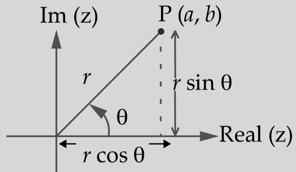
Complex Numbers & Quadratic Equations

Polar Representation

Let $a = r\cos\theta$
$b = r\sin\theta$
where,
$r = |z|$
and $\theta = \arg(z)$
$\therefore z = a + ib = r(\cos\theta + \sin\theta)$
The argument 'θ' of complex number $z = a+ib$ is called principal argument of z if
$-\pi < \theta \le \pi$.

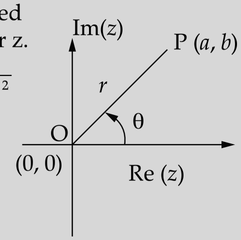


Modulus & Argument of Complex Number

If $z = a+ib$ is a complex number
(i) Distance of z from origin is called as modulus of complex number z.

It is denoted by $r = |z| = \sqrt{a^2+b^2}$

(ii) Angle θ made by OP with +ve direction of X-axis is called argument of z.



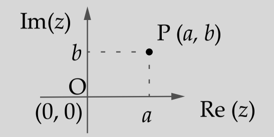
Powers of 'i'

$i = \sqrt{-1}, i^2 = -1$

In general, $i^{4k+r} = \begin{cases} 1; r = 0 \\ i; r = 1 \\ -1; r = 2 \\ -i; r = 3 \end{cases}$, where $k \in R$

Argand Plane

A complex number $z = a+ib$ can be represented by a unique point P(a, b) in the argand plane



$z = a+ib$ is represented by a point P (a, b)

Square root of Complex Number

Let $z = x+iy = \sqrt{a+ib}$, squaring both sides, we get $(x+iy)^2 = a+ib$ i.e. $x^2 - y^2 = a$, $2xy = b$ solving these equations, we get square root of z.

Multiplicative Inverse of complex Number

For a non-zero complex number $z = a+ib$ ($a \ne 0, b \ne 0$),

there exists a complex number $\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2} i$

denoted by $\frac{1}{z}$ or z^{-1}, called multiplicative inverse of z

Such that: $(a + ib)\left(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}\right) = 1 + 0\,i = 1$

Solution of Quadratic Equation

- General form of quadratic equation in x is $ax^2+bx+c=0$,
  Where $a, b, c \in R$ & $a \ne 0$
  The solutions of given quadratic equation are given by $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
  (i) If $b^2 - 4ac > 0$, roots are real and unequal
  (ii) If $b^2 - 4ac = 0$, roots are real and equal
  (iii) If $b^2 - 4ac < 0$, roots are imaginary

Note:
- A monomial equation has one root.
- A polynomial equation of degree n has n roots.

Definition of Complex Numbers

A number of the form $a+ib$, where $a, b \in R$ and $i = \sqrt{-1}$ is called a complex number and denoted by 'z'.

$z = a+ib$ → a: Real part, ib: Imaginary part

Conjugate of a complex number: For a given complex number $z = a+ib$, its conjugate is defined as $\bar{z} = a - ib$

Algebra of Complex Numbers

Let: $z_1 = a+ib$ and $z_1 = c+id$ be two complex numbers, where a, b, c, d $z_2 \in R$ and $i = \sqrt{-1}$

1. Addition: $z_1 + z_2 = (a+ib) + (c+id) = (a+c) + i(b+d)$

2. Subtraction: $z_1 + z_2 = (a+ib) - (c+id) = (a-c) + (b-d)i$

3. Multiplication: $z_1 \cdot z_2 = (a+ib)(c+id)$
$= a(c+id) + ib(c+id)$
$= (ac-bd) + (ad+bc)i$
$(\because i^2 = -1)$

4. Division: $\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{a+ib}{c+id} \cdot \frac{c-id}{c-id}$
$= \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right)i$

Note: If $a+ib = c+id$
$\Leftrightarrow a = c$ & $b = d$

Determinants

Basic Facts

(i) $A^{-1} = \frac{adj\,A}{|A|}$ where $|A| \neq 0$ (ii) $|A| = ad - bc$, If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(iii) $AA^{-1} = A^{-1}A = I$ (iv) Area of triangle, ABC $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is given by $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

(v) If matrix $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ Then adj $A = \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}$ where A_{ij} is the cofactor of a_{ij}.

(vi) Suppose AX=B be the system of n non-homogeneous linear equations in n variables then
- If $|A| \neq 0$, then the system of equations is consistent and has a unique solution which is obtained as $X = A^{-1}B$
- If $|A| = 0$ and $(adj)B = 0$, then the system of equations is consistent and has infinitely many solutions.

(vii) Suppose AX =B be the system of n homogeneous linear equations in n variables then
- If $|A| \neq 0$, then it has only one solution X=0, which is called as trivial solution.
- If $|A| = 0$, then the system has infinitely many solutions and is called non-trivial solutions.

(viii) If $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$, then $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, where $D \neq 0$

where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$, $D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

(ix) If $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$,

then $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, $z = \frac{D_3}{D}$, where $D \neq 0$

where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

(x) If $A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ then $|A| = a(ei - hf) - b(di - gf) + c(dh - ge)$

Determinant of a square matrix 'A', |A| is given by

(i) if $A = [a_{11}]_{1\times 1}$, then $|A| = a_{11}$

(ii) if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$, then $|A| = a_{11}\cdot a_{22} - a_{12}\cdot a_{21}$

(iii) if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$, then $|A|$ $a_{11}(a_{22}\cdot a_{33} - a_{23}\cdot a_{32})$ $-a_{12}(a_{21}\cdot a_{33} - a_{23}\cdot a_{31})$ $a_{13}(a_{21}a_{32} - a_{22}a_{31})$

For *eg.* if $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$, then $|A| = 2 \times 4 - 3 \times 2 = 2$

Properties of |A|

(i) $|A|$ remains unchanged, if the rows and columns of A are interchanged ie., $|A| = |A'|$

(ii) if any two rows (or columns) of A are interchanged, then the sign of $|A|$ changes.

(iii) if any two rows (or columns) of A are identical, then $|A| = 0$

(iv) if each element of a row (or a column) of A is multiplied by B (constant), then $|A|$ gets multiplied by B.

(v) if $A = [a_{ij}]_{3\times 3}$, then $|k.A| = k^3|A|$. where k is a constant.

(vi) if elements of a row or a column in a determinant $|A|$ can be expressed as sum of two or more elements, then $|A|$ can be expressed as $|B| + |C|$.

(vii) if $R_i \rightarrow R_i + kR_j$ or $C_i = C_i + kC_j$ in $|A|$, then the value of $|A|$ remains same

Area of a triangle

If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of triangle, then the area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

For eg: if (1, 2), (3, 4) and (–2, 5) are the vertices, then the area of the triangle is

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ -2 & 5 & 1 \end{vmatrix} = \frac{1}{2}\left|1(4-5) - 2(3+2) + 1(15+8)\right| = 6 \text{ sq.units.}$$

we take positive value of the determinant because area is positive.

Permutations and Combinations

Fundamental Principle of Counting (F.P.C.)

F.P.C. of Multiplication: If an event can occur in m different ways, following which another event in n different ways, then the total number of occurence of the events in the given order is $m \times n$

F.P.C. of Addition: If there are two events such that they can performed independently in m and n ways respectively, then either of the two events can be performed in $(m+n)$ ways.

Permutation

Each of the different arrangements which can be made by taking some or all of a number of distinct objects is called a **permutation**.

The number of permutations of n different things taken r at a time, where repetition is not allowed, is denoted by ^nP_r and is given by

$$^nP_r = \frac{n!}{(n-r)!}$$

Permutations under given Conditions

The number of all permutations of 'n' different objects taken r at a time:

(i) When a particular object is to be always included in each arrangement is $^{n-1}P_{r-1}$

(ii) When a particular object is never taken in each arrangement in $^{n-1}P_r$

(iii) The number of permutation of n different things taken r at a time, where repetition is allowed is $(n)^r$

Permutations of alike Objects

The number of permutations of n objects taken all at a time, where p_1 objects are of first kind, p_2 objects are of the second kind, ...p_k objects are of the k^{th} kind is $\frac{n!}{p_1!\,p_2!\,p_3!\ldots p_k!}$

Circular Permutations

1. The number of circular permutations of 'n' distinct objects is $(n-1)!$
2. If anti-clockwise and clockwise order of arrangements are not distinct then the number of circular permutations of n distinct items is $\frac{1}{2}(n-1)!$

e.g.: Arrangements of beads in a necklace, arrangements of flower in a garland etc.

Some Special Results of ^nC_r

Since,

$$^nC_r = \frac{n!}{r!(n-r)!}$$

- In particular, $r = n$
- $^nC_n = \frac{n!}{n!0!} = 1$
- $^nC_{n-r} = {}^nC_r$
- $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- $^nC_0 + {}^nC_2 + {}^nC_4 + .. = {}^nC_1 + {}^nC_3 + {}^nC_5 + ... = 2^{n-1}$

Relation between ^nP_r and ^nC_r

$^nP_r = {}^nC_r \times r!$ $\quad \because 0 < r \le n$

i.e., Corresponding to each combination of ^nC_r, we have $r!$ permutations, because r objects in every combination can be rearranged in $r!$ ways

Combinations under Certain Conditions

- The number of combinations of n things taken at a time, so that k particular objects are
  (i) Always included: $^{n-k}C_{r-k}$
  (ii) Never included: $^{n-k}C_r$
- The no. of selections from 'n' different objects, taking at least one $= {}^nC_1 + {}^nC_2 + {}^nC_3 + ...{}^nC_n = 2^n - 1$

Combinations

Each of different selections made by taking some or all of a number of distinct objects or item, irrespective of their arrangements or order in which they are placed, is called a combination.

The number of combinations of n different things taken r at a time, denoted by ^nC_r is given by

$$^nC_r = \frac{n!}{r!(n-r)!};\ 0 \le r \le n$$

Matrices

Definition and its types

A matrix of order $m \times n$ is an ordered rectangular array of numbers or functions having 'm' rows and 'n' columns. The matrix $A=[a_{ij}]_{m\times n}$, $1 \le i \le m$, $1 \le j \le n$; $i, j \in N$ is given by $A=\begin{bmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ a_{m1} & a_{m2} & \dots a_{mn} \end{bmatrix}_{m\times n}$

- Column matrix : It is of the form $\left[a_{ij}\right]_{m\times 1}$
- Row matrix : It is of the form $\left[a_{ij}\right]_{1\times n}$
- Square matrix : Here, $m = n$ (no. of rows = no. of columns)
- Diagonal matrix : All non-diagonal entries are zero i.e $a_{ij} = 0 \forall i \neq j$
- Scalar matrix : $a_{ij} = 0, i \neq j$ and $a_{ij} = k$ (Scalar), $i = j$, for some constant k.
- Identity matrix : $a_{ij} = 0$, $i \neq j$ and $a_{ij} = 1$, $i = j$
- Zero matrix : All elemants are zero.

Equality of two matrices

$A=\left[a_{ij}\right]=\left[b_{ij}\right]=B$ if, A and B are of same order and $a_{ij}=b_{ij}$ i and j; $i,j \in N$

Operations on matrices

Addition

If A, B are two matrices of same order, then $A+B=\left[a_{ij}+b_{ij}\right]$. The addition of A and B follows: $A+B=B+A$, $(A+B)+C=A+(B+C)$, $-A=(-1)A$, $k(A+B)=kA+kB$, k is scalar and $(k+I)A=kA+IA$, k and I are constants.

Multiplication

If A, $[a_{ij}]_{m\times n}$ and $B=[b_{jk}]_{n\times p}$, then $AB=C=[c_{ik}]_{m\times p}$, $[C_{jk}]=\sum_{j=1}^{n} a_{ij}b_{jk}$. Also, $A(BC)=(AB)C$, $A(B+C)=AB+AC$ and $(A+B)C=AC+BC$, but $AB \neq BA$ (always).

Transpose of a Matrix

If $A=[a_{ij}]_{m\times n}$, then its transpose $A'=(A^T)=[a_{ji}]_{n\times m}$ *i.e.* if $A=[2 \quad 1]$, then $A^T=\begin{bmatrix}1\\2\end{bmatrix}$.

Also, $(A')'=A$, $(kA)'=kA'$, $(A+B)'=A'+B'$, $(AB)'=B'A'$.

- A is symmetric matrix if $A=A'$ *i.e.* $A'=A$.
- A is skew – symmetric if $A=-A'$ *i.e.* $A'=-A$.
- A is any square matrix, then– $A=\frac{1}{2}\left\{(A+A')+(A-A')\right\}$ = Half of sum of a symmetric and a skew -symmetric matrix. ($A+A'$ → S.M, $A-A'$ → Skew S.M.)

Elementary operations on a matrix

Properties for applying the operations

$R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

$R_i \to kR_i$ or $C_i \to kC_i$

$R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

Applications of determinants & matrices

- If $a_1x+b_1y+c_1z=d_1$, $a_2x+b_2y+c_2z=d_2$, $a_3x+b_3y+c_3z=d_3$ then we can write $AX=B$, where $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B=\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix}$
- Unique solution of $AX=B$ is $X=A^{-1}B$, $|A|\neq 0$
- $AX=B$ is consistent or inconsistent according as the solution exists or not.
- For a square matrix A in $AX=B$, if
  - (i) $|A|\neq 0$, then there exists unique solution.
  - (ii) $|A|=0$ and (adj. A) $B\neq 0$, then no solution.
  - (iii) If $|A|=0$ and (adj. A). $B=0$, then system may or may not be consistent.

Adjoint and inverse of a matrix

if $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then adj. $A=\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$, where A_{ij} is the cofactor of a_{ij}.

- A(adj. A)=(adj. A). $A=|A|I$, A - square matrix of order 'n'
- if $|A|=0$, then A is singular matrix. Otherwise A is non-singular matrix.
- if $AB=BA=I$, where B is a square matrix, then B is called the inverse of A, $A^{-1}=B$ or $B^{-1}=A$, $(A^{-1})^{-1}=A$.
- Inverse of a square matrix exists if A is non-singular *i.e.* $|A|\neq 0$, and is given by $A^{-1}=\frac{1}{|A|}$ (adj. A)

Minors and cofactors of a matrix

Minor of an element a_{ij} in a determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column and is denoted by M_{ij}. If M_{ij} is the minor of a_{ij} and cofactor of a_{ij} is A_{ij} then $A_{ij}=(-1)^{i+j}M_{ij}$.

- If $A_{3\times 3}$ is a matrix, then $|A|=a_{11}.A_{11}+a_{12}.A_{12}+a_{13}.A_{13}$.
- If elements of one row (or column) are multiplied with cofactors of elements of any other row (or column), then their sum is zero. For *e.g.*, $a_{11}A_{21}+a_{12}A_{22}+a_{13}A_{23}=0$.

e.g., if $A=\begin{bmatrix}1 & 2\\-3 & 4\end{bmatrix}$, then $M_{11}=4$ and $A_{11}=(-1)^{1+1}4=4$.

Binomial Theorem and its Applications

Definition

If $a, b \in R$ and $n \in N$, then

$$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots\dots + {}^nC_n a^0 b^n$$

- **Remarks:** If the index of the binomial is n then the expansion contains $n+1$ terms.
- In each term, the sum of indices of a and b is always n.
- Coefficients of the terms in binomial expansion equidistant from both the ends are equal.

$$(a-b)^n = {}^nC_0 a^n b^0 - {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots\dots\dots + (-1)^n\, {}^nC_n a^0 b^n$$

General & Middle Terms in Expansion of $(a+b)^n$

The general term of an expansion $(a+b)^n$ is

$T_{r+1} = {}^nC_r\, a^{n-r} b^r,\ 0 \le r \le n,\ r \in N$

Middle Term(s)

1. In $(a+b)^n$, if n is even, then the no. of terms in the expansion is odd. Therefore, there is only one middle term and it is $\left(\frac{n+2}{2}\right)^{th}$ term.

2. In $(a+b)^n$, if n is odd, then the no. of terms in the expansion is even. Therefore, there are two middle terms and those are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms.

Binomial Theorem for any Index

If n is a negative integer, then $n!$ is not defined. We state binomial theorem in another from as:

$$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \dots\dots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}a^{n-r}b^r + \dots\dots\dots\dots + b^n$$

Here, $T_{r+1} = \frac{n(n-1)(n-2)\dots\dots(n-r+1)}{r!}a^{n-r}b^r$

Some Special Cases of $(a+b)^n$

In the expansion of $(a+b)^n$,

(i) Taking $a = x$ and $b = -y$, we obtain

$(x-y)^n = {}^nC_0 x^n - {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \dots\dots + (-1)^n\, {}^nC_n y^n$

(ii) Taking $a=1$, $b=x$, we obtain

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \dots\dots\dots + {}^nC_n x^n$

(iii) Taking $a=1$, $b=-x$, we obtain

$(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 + \dots\dots + (-1)^n\, {}^nC_n x^n$

(iv) Taking $a=1$, $b=x$, $n=-n$

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \dots\dots$$

Binomial Coefficients

The coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \dots\dots {}^nC_n$, in the expansion of $(a+b)^n$ are called binomial coefficients and are denoted by $C_0, C_1, C_2 \dots C_n$ respectively

Properties of binomial coefficients:

(i) $C_0 + C_1 + C_2 + \dots\dots\dots + C_n = 2^n$

(ii) $C_0 - {}_1 + C_2 - \dots\dots\dots + (-1)^n C_n = 0$

(iii) ${}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 + \dots\dots + {}^nC_n^2 = {}^{2n}C_n$

(iv) ${}^nC_{r_1} = {}^nC_{r_2} \Rightarrow r_1 = r_2$ or $r_1 + r_2 = n$

(v) ${}^nC_r + {}^nC_{r-1} = {}^{1}\ {}^nC_r$

(vi) ${}^nC_r = \frac{n}{r}{}^{n-1}C_{r-1}$

(vii) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

(viii) ${}^nC_0\,{}^nC_r + {}^nC_1\,{}^nC_{r+1} + {}^nC_2\,{}^nC_{r+2} + \dots\dots\dots\ {}^nC_{n-r}\,{}^nC_n = {}^{2n}C_{n-r}$

Sequences and Series

Harmonic Mean

Harmonic Progression (HP)

For the solution of HP, we should follow below steps

Make the reciprocal of each terms of HP

Solve by AP method

Make the reciprocal of AP result

eg: if $\frac{1}{a}$, $\frac{1}{a+d}$, $\frac{1}{a+2d}$--------in HP, then

$a, (a+d), (a+2d)$--------will be in AP

Now n^{th} term of AP, $a_n = a+(n-1)d$

So, n^{th}, team of HP, $= \frac{1}{a_n} = \frac{1}{a+(n+)d}$

Some Important result

If H is the harmonic mean (H) between a&b, then $H = \frac{2ab}{a+b}$

If there are n harmonic mean between a&b, then n^{th} $HM(H_n) = \frac{ab(n+1)}{na+b}$

Geometric Progression (G.P.)

A sequence is said to be a geometric progression or G.P., if the ratio of any term to its preceding term is same throughout.This constant factor is called **'Common ratio'.** Usually we denote the first term of a G.P. by 'a' and its common ratio by 'r'.

The gereral or n^{th} term of G.P. is given by $a_n = ar^{n-1}$

The sum S_n of the first n terms of G.P. is given by

$$S_n = \frac{a(r^n-1)}{r-1}\ r<1, \text{ if } r\neq 1$$

Sum of infinite terms of G.P. is given by

$$S_\infty = \frac{a}{1-r};\ |r|<1$$

Geometric Mean (GM)

- If a, b, c are in G.P., b is the GM between a & c, $b^2 = ac$, therefore; $a > 0, c > 0$
- If a, b are two given numbers & $a, G_1, G_2, \ldots, G_n, b$ are in G.P., then $G_1, G_2, G_3 \ldots G_n$ are n GMs between a & b.

$$G_1 = a\cdot\left(\frac{b}{a}\right)^{1/n+1}\ G_2 = a\cdot\left(\frac{b}{a}\right)^{2/n+1} \ldots\ldots G_n = a\cdot\left(\frac{b}{a}\right)^{n/n+1}$$

or $G_1 = ar,\ G_2 = ar^2,\ \ldots\ G_n = ar^n$ where $r = \left(\frac{b}{a}\right)^{1/n+1}$

Arithmetico Geometric Series

In a series,where each term is formed by multiplying the corresponding term of an AP & GP is called Arithmetico-Geometric Series.

Eg: $1+3x+5x^2+7x^3+\ldots$

Here, 1, 3, 5, … are in AP and 1, x, x^2, … are in GP.

Properties of AM & GM between two quantities

- If A.M.: G.M. of two positive numbers a and b is $m : n$, then $a : b = \left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right)$
- If A and G be the AM & GM between two positive numbers, then the numbers are $A \pm \sqrt{A^2-G^2}$

Relationship between AM, GM and HM

Let A, G and H be the AM, GM and HM of two given positive real numbers a & b, respectively. Then,

$$A = \frac{a+b}{2}, G = \sqrt{ab} \text{ and } H = \frac{2}{a+b}$$

So, $G = \sqrt{AH}$ or, $G^2 = AH$

Arithmetic Progression (A.P.)

An A.P. is a sequence in which terms increase or decrease regularly by the fixed number (same constant). This fixed number is called 'common difference' of the A.P. If 'a' is the first term & 'd' is the common difference and 'l' is the last term of A.P., then general term or the n[th] term of the A.P. is given by $a_n = a+(n-1)d$ from starting and $a_n = l-(n-1)d$ from the end.

The sum S_n of the first n terms of an A.P. is given by

$$S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[a+l]$$

Arithmetic Mean (AM)

If three numbers are in A.P., then the middle term is called AM between the other two, so if a, b, c, are in A.P., b is AM of a and c.

AM for any 'n' numbers $a_1, a_2, a_3, \ldots, a_n$ is

$$AM = \frac{a_1+a_2+a_3+\ldots+a_n}{n}$$

n-Arithmetic Mean between Two Numbers: If a, b are any two given numbers & $a, A_1, A_2, \ldots, A_n, b$ are in A.P. then $A_1, A_2, \ldots, A_n$ are n AM's between a & b.

$$A_1 = a+\frac{b-a}{n+1}, A_2 = a+\frac{2(b-a)}{n+1} \ldots A_n = a+\frac{n(b-a)}{n+1}$$

or $A_1 = a+d$, $A_2 = a+2d$, …, $A_n = a+nd$,

where $d = \frac{b-a}{n+1}$

Sum upto n terms of some Special Sequences

- Sum of first 'n' natural numbers

$$\sum_{k=1}^{n} k = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

- Sum of squares of first n natural numbers

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

- Sum of cubes of first n natural numbers

$$\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\ldots+n^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n} k\right)^2$$

- Sum of first 'n' odd natural numbers

$$\sum_{k=1}^{n}(2k-1) = 1+3+5+\ldots+(2n-1) = n^2$$

Limits and Derivatives

Limit

- We say $\lim_{x \to a^-} f(x)$ is the expected value of f at $x=a$ given that the values of f near x to the left of a. This value is called the left hand limit of f at 'a'
- We say $\lim_{x \to a^+} f(x)$ is the expected value of f at $x=a$ given that the values of f near x to the right of a. This value is called the right hand limit of f at a'.
- If the right and left hand limits coincide, we call that common value as the limit of $f(x)$ at $x=a$ and denoted it by $\lim_{x \to a} f(x)$.

eg: Find limit of function $f(x)=(x-1)^2$ at $x=1$.

Sol: For $f(x)=(x-1)^2$

Left hand limit (LHL) (*at* $x=1$)$=\lim_{x \to 1^-}(x-1)^2 = 0$

and Right hand limit (RHL) (*at* $x=1$)$= \lim_{x \to 1^+}(x-1)^2 = 0$

$\therefore$ LHL = RHL

$\lim_{x \to 1}(x-1)^2 = 0$

Algebra of Limits

For functions f and g the following holds:

- $\lim_{x \to a}\left[f(x) \pm g(x)\right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
- $\lim_{x \to a}\left[f(x).g(x)\right] = \lim_{x \to a} f(x).\lim_{x \to a} g(x)$
- $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, where $\lim_{x \to a} g(x) \neq 0$

Some Standard Limits

- $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$
- $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$
- $\lim_{x \to a} \frac{\tan^p mx}{\tan^p nx} = \left(\frac{m}{n}\right)^p, x \neq 0$
- $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a, a > 0$
- $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1$
- $\lim_{x \to 0} \frac{e^{\lambda x} - 1}{x} = \lambda, \lambda \neq 0$

Derivative of a function at 'a'

The derivative of a function f at a is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

eg: Find derivative of $f(x) = \frac{1}{x}$.

Sol: We have $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{1}{h}\left[\frac{-h}{x(x+h)}\right] = \frac{-1}{x^2}$$

Derivative of polynomial function

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots\cdots + a_1 x + a_0$ be a polynomial function, where a_i s are all real numbers and $a_n \neq 0$. Then the derivative function is given by

$$\frac{d f(x)}{dx} = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots\cdots + 2a_2 x + a_1$$

Algebra of derivatives of functions

For functions u and v the following holds:

- $(u \pm v)' = u' \pm v'$
- $(uv)' = u'v + v'u$
- $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, provided all are defined and $v \neq 0$

Here, $u' = \frac{du}{dx}$ and $v' = \frac{dv}{dx}$

Some Standard Derivatives

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

Continuity and Differentiability

Continuous Function

Suppose f is a real function on a subset of the real numbers and let 'c' be a point in the domain of f.
Then, f is continuous at c if $\lim_{x\to c} f(x) = f(c)$
A real function f is said to be continuous if it is continuous at every point in the domain of f.
eg: The function $f(x) = \frac{1}{x}$, $x \neq 0$ is continuous.
Let 'c' be any non-zero real number, then $\lim_{x\to c} f(x) = \lim_{x\to c} \frac{1}{x} = \frac{1}{c}$. For $c \neq 0, f(c) = \frac{1}{c}$. So $\lim_{x\to c} f(x) = f(c)$ and hence f is continuous at every point in the domain of f.

Algebra of continuous functions

Suppose f and g are two real functions continuous at a real number c, then, $f+g$, $f-g$, $f.g$ and $\frac{f}{g}$ are continuous at $x = c, [g(c) \neq 0]$.

Differentiability

Suppose f is a real function and c is a point in its domain. The derivative of f at c is $f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$.
Every differentiable function is continuous, but the converse is not true.

Chain Rule

If $y = f(g(x))$, then $\frac{dy}{dx} = f'(g(x)).\ g'(x).\ \frac{dy}{dx}$

Logarithmic differentiation

Let $y = f(x) = [u(x)]^{v(x)}$

$\ln y = v(x) \ln [u(x)]$

$\frac{1}{y}\frac{dy}{dx} = v(x)\frac{1}{u(x)}u'(x) + v'(x) \ln [u(x)]$

$\frac{dy}{dx} = y\left[\frac{v(x)}{u(x)}\right]u'(x) + v'(x) \ln [u(x)]$

e.g. : Let $y = a^x$ then In $y = x \ln a$

$\frac{1}{y}.\frac{dy}{dx} = \ln a$

$\frac{dy}{dx} = y \ln a = a^x \ln a$.

Derivatives of Implicit functions

If two variables are expressed by some relation, then one will be the implicit function of other, is called Implicit function.
For example: Let $y = \cos x - \sin y$, then $\frac{dy}{dx} = \frac{d}{dx}\cos x - \frac{d}{dx}\sin y$
or, $\frac{dy}{dx} = -\sin x - \cos y.\frac{dy}{dx}$ or, $\frac{dy}{dx} = \frac{-\sin x}{1+\cos y}$, where $y \neq (2n+1)\pi$, $n \in I$

Some Standard derivatives

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ **(ii)** $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$ **(iii)** $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ **(iv)** $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

(v) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$ **(vi)** $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{-1}{|x|\sqrt{x^2-1}}$ **(vii)** $\frac{d}{dx}(e^x) = e^x$ **(viii)** $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$

Mean Value Theorem

If $f : [a,b] \to R$ continuous on $[a, b]$ and differentiable on (a, b).
Then ∃ some c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b-a}$

e.g. Let $f(x) = x^2$ defined in the interval [2, 4].
Since $f(x) = x^2$ is continuous in [2, 4] and differentiable in (2, 4) as $f'(x) = 2x$ defined in (2, 4).
So,

$f'(c) = \frac{f(b) - f(a)}{b-a} = \frac{16-4}{4-2} = 6, c \in (2, 4)$.

Rolle's theorem

If $f : [a,b] \to R$ is continuous on $[a, b]$ and differentiable on (a, b), such that $(a) = f(b)$, then ∃ some c in (a, b), such that $'(c) = 0$.

Second order derivative

Let $y = f(x)$ then $\frac{dy}{dx} = f'(x)$, if $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}f'(x)$ *i.e.*,
$\frac{d^2y}{dx^2} = f''(x)$, is the second order derivative of y *w.r.t.x*.
For *eg* : If $y = 3x^2+2$, then $y' = 6x$ and $y'' = 6$.

Derivatives of functions in parametric form

Let $x = f(t)$, $y = g(t)$ be two functions of parameter 't'.

Then, $\frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt}$ or $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}\left(\frac{dx}{dt} \neq 0\right)$

Thus, $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$, (provided $f'(t) \neq 0$)

eg : If $x = a\cos\theta$, $y = a\sin\theta$ then $\frac{dx}{d\theta} = -a\sin\theta$ and $\frac{dy}{d\theta}$ $a\cos$, and so $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{a\cos\theta}{a\sin\theta} = -\cot\theta$.

Applications of Derivatives

Rate of change of quantities

If a quantity 'y' varies with another quantity x so that $y = f(x)$, then $\frac{dy}{dx} = [f'(x)]$ represents the rate of change of y w.r.t x and $\left.\frac{dy}{dx}\right|_{x=x_0} (f'(x_0))$ represents the rate of change of y w.r.t. x at $x = x_0$.

If 'x' and 'y' varies with another variable 't' i.e., if $x = f(t)$ and $y = g(t)$, then by chain rule $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$, if $\frac{dx}{dt} \neq 0$.

E.g: If the radius of a circle, $r = 5$ cm, then the rate of change of the area of a circle per second w.r.t 'r' is – $\left.\frac{dA}{dr}\right|_{r=5} = \frac{d}{dr}(\pi r^2)\Big|_{r=5} = 2\pi r\Big|_{r=5} = 10\pi$

Increasing and decreasing functions

A function f is said to be (i) increasing on (a, b), if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) \leq f(x_2) \forall x_1 x_2 \in (a, b)$, and (ii) decreasing on (a, b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) > f(x_2) \forall x_1 x_2 \in (a, b)$

If $f'(x) \geq 0 \forall x \in (a, b)$, then f is increasing in (a,b), and if $f'(x) \leq 0 \forall x \in (a,b)$, then f is decreasing in (a,b)

e.g: Let $f(x) = x^3 - 3x^2 + 4x, x \in R$, then $f'(x) = 3x^2 - 6x + 4 = 3(x-1)^2 + 1 > 0 \forall x \in$ R. So, the function f is strictly increasing on R.

Tangents Equation on curve

The equation of the tangent at (x_0, y_0), to the curve $y = f(x)$ is given by $(y - y_0) = \left.\frac{dy}{dx}\right|_{(x_0, y_0)} (x - x_0)$, if $\frac{dy}{dx}$ does not exists at (x_0, y_0), then the tangent at (x_0, y_0) is parallel to the y–axis and its equation is $x = x_0$.

If tangent to a curve $y = f(x)$ at $x = x_0$ is parallel to x–axis, then $\left.\frac{dy}{dx}\right|_{x=x_0} = 0$.

Equation of the normal to the curve

The equation of normal at (x_0, y_0) to the curve $y = f(x)$ is $y - y_0 = -\frac{1}{\left.\frac{dy}{dx}\right|_{(x_0, y_0)}} (x - x_0)$.

If $\frac{dy}{dx}$ at (x_0, y_0) is zero, then equation of the normal is $x = x_0$. If $\frac{dy}{dx}$ at (x_0, y_0) does not exist, then the normal is parallel to x-axis and its equation is $y = y_0$ eg: Let $y = x^3 - x$ be a curve, then the slope of the tangent to $y = x^3 - x$ at $x = 2$ is $\left.\frac{dy}{dx}\right|_{x=2} = 3x^2 - 1\Big|_{x=2} = 3.2^2 - 1 = 11$.
The equation of normal will be $x + 11y - 68 = 0$

Approximations

Let $y = f(x)$; Δx be a small increment in 'x' and Δy be the small increment in y corresponding to the increment in 'x', i.e. $\Delta y = f(x + \Delta x) - f(x)$. Then, Δy is given by $dy = f'(x)dx$ or $dy = \left(\frac{dy}{dx}\right)\Delta x$, is approximation of Δy, when $dx = \Delta x$ is relatively small and denoted by $dy \approx \Delta y$.

e.g., Let us approximate $\sqrt{36.6}$. To do this, we take $y = \sqrt{x}$, $x = 36$, $\Delta x = 0.6$, then $\Delta y = \sqrt{x + dx} - \sqrt{x}$

$$= \sqrt{36.6} - \sqrt{36}$$
$$= \sqrt{36.6} - 6 \Rightarrow \sqrt{36.6} = 6 + dy$$

Now, dy is approximately Δy and is given by

$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(0.6) = \frac{1}{2\sqrt{36}}(0.6) = 0.05$. So, $\sqrt{36.6} \approx 6 + 0.05 = 6.05$.

Maxima and Minima

A point C in the domain of 'f' at which either $f'(C) = 0$ or is not differentiable is called a critical point of f.

First derivative test

Let f be continuous at a critical point C in open interval. Then **(i)** if $f'(x) > 0$ at every point left of C and $f'(x) < 0$ at every point right of C, then 'C' is a point of local maxima. **(ii)** If $f'(x) < 0$ at every point left of C and $f'(x) > 0$ at every point right of C, then 'C' is a point of local minima.
(iii) If $f'(x)$ does not change sign as 'x' increases through C, then 'C' is called the point of inflection.

Second derivative test

Let f be a function defined on given interval, f is twice differentiable at C. Then
(i) $x = C$ is a point of local maxima, if $f'(C) = 0$ and $f''(C) < 0$, $f(C)$ is local maxima, of f.
(ii) $x = C$ is a point of local minima, if $f'(C) = 0$ and $f''(C) > 0$. $f(C)$ is local minima of f. **(iii)** The test fails if $f'(C) = 0$ and $f''(C) = 0$

Applications of the Integrals

Area bounded by two curves

The area of the region enclosed between two curves $y = f(x), y = g(x)$ and the lines $x=a$, $x=b$ is given by $A = \int_a^b [f(x) - g(x)]dx$, where $f(x) \geq g(x)$ in $[a, b]$

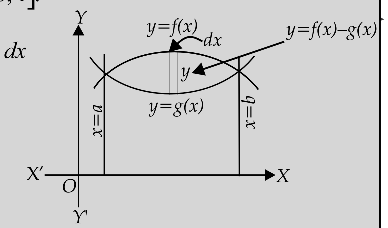
e.g., To find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$. (0,0) and (1,1) are points of intersection of $y = x^2$ and $y^2 = x$ and $y^2 = x \Rightarrow y = \sqrt{x} = f(x)$, and $y = x^2 = g(x)$, where $f(x) \geq g(x)$ in $[0, 1]$.

Area, $A = \int_0^1 [f(x) - g(x)]dx$

$= \int_0^1 [\sqrt{x} - x^2]dx$

$= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1$

$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$ Sq. units.



If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$, $a < c < b$, then the area is

$A = \int_a^c [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx$

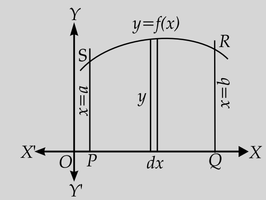
Area under simple curves

The area of the region bounded by the curve $y = f(x)$, x- *axis* and the lines $x=a$ and $x = b(b > a)$ is given by

$A = \int_a^b y\,dx$ or $\int_a^b f(x)dx$.

eg : the area bounded by $y=x^2$, x- *axis* in I quadrant and the lines $x=2$ and $x=3$ is -

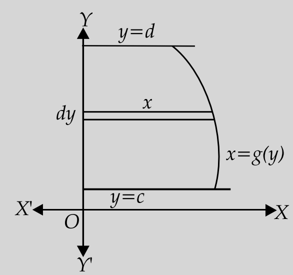
$A = \int_2^3 y\,dx = \int_2^3 x^2\,dx = \left[\frac{x^3}{3}\right]_2^3 = \frac{1}{3}(27-8) = \frac{19}{3}$ sq.units.



The area of the region bounded by the curve $x = f(y)$, $y - axis$ and the lines $y=c$ and $y = d(d > c)$ is given by $A = \int_c^d x\,dy$ or $\int_c^d f(y)dy$.

eg : the area bounded by $x = y^3$, $y - axis$ in the I quadrant and the lines $y=1$ and $y=2$ is

$\int_1^2 x dy = \int_1^2 y^3\,dy = \left[\frac{1}{4}y^4\right]_1^2 = \frac{1}{4}(2^4 - 1^4) = \frac{15}{4}$ sq. units



Properties of Definite integrals

Some fundamental properties of definite integral are ;

- Value of integration is independent of change of variable $\int_a^b f(x)dx = \int_a^b f(t)dt$
- If the limits of definite integral are interchanged then, its Value changes only by minus sign i.e.

  $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- $\int_o^a f(x)dx = \int_o^a f(x-a)dx$
- If f(t) is an odd function, then

  $\phi(x) = \int_a^x f(t)dt$ is an even function.
- If f(t) is an even function, then

  $\phi(x) = \int_0^x f(t)dt$ is an odd function.
- If f(x) is continous on [a, ∞] then $\int_0^\infty \boldsymbol{f(x)dx}$ is called an improper integral and is defined as

  $\int_0^\infty \boldsymbol{f(x)dx} = \lim_{b\to\infty} \int_0^b \boldsymbol{f(x)dx}$
- $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, if\ f(2a-x) = f(x) \\ 0, if\ f(2a-x) = -f(x) \end{cases}$
- $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
- If f(x) is a periodic function with period T, then- $\int_0^T f(x)dx = n\int_0^{nT} f(x)dx$
- If f(x)≥ 0 on the interval [a, b], then $\int_a^b f(x)dx \geq 0$
- If f(x) ≤ g(x)on the interval [a, b], then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$

Differential Equations

Variable Separable Method

It is used to solve such an equation in which variables can be separated completely. eg : $y\,dx = x\,dy$ can be solved as $\frac{dx}{x} = \frac{dy}{y}$; Integrating both sides $\ln x = \ln y + \ln c \Rightarrow \frac{x}{y} = c \Rightarrow x = cy$, is the solution.

Differential equation of family of curves

The order of a differential equations representing a family of curves is same as the number of arbitrary constants present in the equation corresponding to the family of curves. eg: Let the family of curves be $y = mx$, $m =$ constant, then, $y' = m$

$y = y'x \Rightarrow y = \frac{dy}{dx}x \Rightarrow x\frac{dy}{dx} - y = 0.$

Homogeneous Differential Equations

A differential equation which can be expressed in the form $\frac{dy}{dx} = f(x, y)$ or $\frac{dx}{dy} = g(x, y)$, where, $f(x, y)$ and $g(x, y)$ are homogeneous functions of degree zero is called a homogenous differential equation.

$eg : (x^2 + xy)dy = (x^2 + y^2)dx$

To solve this, we substitute $y = vx$. and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Linear Differential Equations

Let us take the differential equations of the form $\frac{dy}{dx} + Py = Q$, where P and Q are constants or the function of 'x' is called a first order linear differential equations. Its solution is given as $ye^{\int P.dx} = \int Q.e^{\int P.dx}\,dx + c$.

$eg : \frac{dy}{dx} + 3y = 2x$ has solution

$ye^{\int 3.dx} = \int 2x.e^{\int 3.dx}\,dx + c \Rightarrow ye^{3x} = 2\int xe^{3x}dx + c.$

Reducible to Homogeneous Equation

Type 1. Let $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c}$, here $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

Reduce the equation by substituting $x = X + h$ and $y = Y + k$

$\frac{dY}{dX} = \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)}$ --------(i)

Equate $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$ and find h & k

So, equation (i) can be reduced to $\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$

Now, Solved by homogeneous way

Type 2 : Let $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$, here $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Put, $a_2x + b_2y = v$ then $a_1x + b_1y = m(a_2x + b_2y) = mv$, here $m = \frac{a_1}{a_2} = \frac{b_1}{b_2}$

So, $\frac{d}{dx}(a_2x + b_2y) = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{b_2}\left(\frac{dv}{dx} - a_2\right)$

Now, substitute above values in given equation and reduce by variable separable type.

Order & Degree of a Differential Equation

It is defined if the differential equation is a polynomial equation in its derivatives, and is defined as the highest power (positive integer only) of the highest order derivative.

eg: the degree of $\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} = 0$ is three

Order and degree (if defined) of a differential equation are always positive integers.

Formation of Differential Equations

To form a differential equation from a given function, we differentiate the function successively as many times as the number of arbitrary constants in the given function, and then eliminate the arbitrary constants.

eg: Let the function be $y = ax + b$, then we have to differentiate it two times, since there are 2 arbitrary constants a and b. $\therefore y' = a \Rightarrow y'' = 0$. Thus, $y'' = 0$ is the required differential equation.

Solution of a Differential Equation

A function which satisfies the given differential equation is called its solution. The solution which contains as many arbitrary constants as the order of the differential equation is called a general solution and the solution free from arbitrary constants is called particular solution.

eg: $y = e^x + 1$ is a solution of $y'' - y' = 0$.

Since $y' = e^x$ and $y' = e^x \Rightarrow y'' - y' = e^x - e^x = 0$.

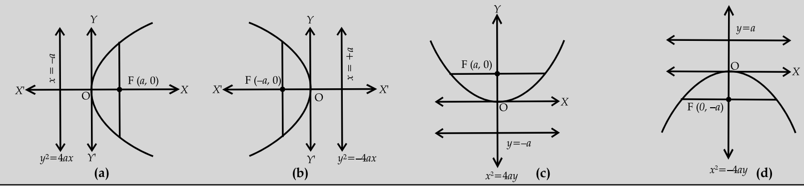
Conic Sections

Parabola

- A parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point in the plane. Fixed line is called 'directrix' of parabola. Fixed point F is called the 'focus'. A line through focus & perpendicular to directrix is called 'axis'. Point of intersection of parabola with axis is called 'vertex'.

Main facts about the parabola

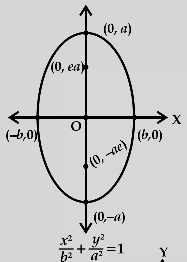
Forms of Parabolas	$y^2=4ax$	$y^2=-4ax$	$x^2=4ay$	$x^2=-4ay$
Axis	$y=0$	$y=0$	$x=0$	$x=0$
Directix	$x=-a$	$x=a$	$y=-a$	$y=a$
Vertex	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Lenght of latus rectum	$4a$	$4a$	$4a$	$4a$
Equations of latus rectum	$x=a$	$x=-a$	$y=a$	$y=-a$

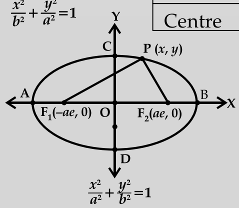


Ellipse

- An ellipse is the set of all points in a plane, that the sum of their distances from two fixed points in the plane is constant.
- The two fixed points are called the 'foci' of the ellipse.
- The midpoint of line segment joining foci is called the 'centre' of the ellipse.
- The line segment through the foci of the ellipse is called' major axis'.
- The line segment through centre & perpendicular to major axis is called minor axis.

Forms of ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$
	$a>b$	$a>b$
Equation of major axis	$y=0$	$x=0$
Length of major axis	$2a$	$2a$
Equation of Minor axis	$x=0$	$y=0$
Lenght of Minor axis	$2b$	$2b$
Directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{a}{e}$
Equation of latus rectum	$x=\pm ae$	$y=\pm ae$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Centre	(0, 0)	(0, 0)



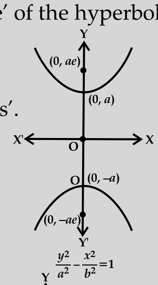


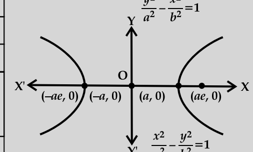
Here, $a>b$ and $b^2 = a^2(1-e^2)$, $e<1$

Hyperbola

- A hyperbola is the set of all points in a plane, that the difference of whose distances from two fixed points in the plane is a constant.
- The two fixed points are called the 'foci' of the hyperbola.
- The mid-point of the line segment joining the foci is called the 'centre' of the hyperbola.
- The line through the foci is called 'transverse axis'.
- Line through centre and perpendicular to transverse axis is called 'conjugate axis'.
- Points at which hyperbola intersects transverse axis are called 'vertices'.

Forms of the hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Equation of transverse axis	$y=0$	$x=0$
Equation of conjugate axis	$x=0$	$y=0$
Length of transverse axis	$2a$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm ae)$
Equation of latus rectum	$x=\pm ae$	$y=\pm ae$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Centre	(0, 0)	(0, 0)





Here, $b^2 = a^2(e^2-1)$, $e>1$

Definition

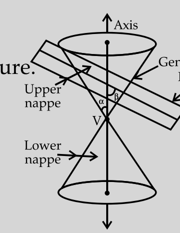
Circles, ellipses, parabolas and hyperbolas are known as conic sections because they can be obtained as intersections of plane with a double napped right circular cone α. From the given figure.

(i) Section will represent circle, if β =90°

(ii) Section will represent an Ellipse, if $\alpha<\beta<\pi/2$

(iii) Section will represent a parabola if if $\alpha=\beta$

(iv) Section will represent a hyperbola if $0 \le \beta<\alpha$



Circle

A circle is a set of all points in a plane that are equidistant from a fixed point in the plane. The fixed point is called the 'centre' of the circle and the distance from the centre to a point on the circle is called the 'radius' of the circle.

The equation of a circle with centre (h, k) and the radius r is $(x-h)^2+(y-k)^2=r^2$

The general equation of circle is $x^2+y^2+2gx+2fy+c=0$

its centre is $(-g, -f)$ and radius $r=\sqrt{g^2+f^2-c}$

Straight Lines

Slope Formula

- If θ is the angle at which a straight line is inclined to be +ve direction of x-axis and $0° \le \theta < 180°$, $\theta \ne 90°$, then the slope of the line, denoted by m, is defined by $m = \tan\theta$. If θ is 90°, m doesn't exist, but the line is parallel to y-axis. If $\theta = 0°$, then $m = 0$ and the line is parallel to x-axis.
- If $A(x_1, y_1)$ and $B(x_2, y_2)$, $x_1 \ne x_2$ are points on straight line, then the slope m of the line is given by $m = \frac{(y_1 - y_2)}{(x_1 - x_2)}$

Area of Triangle

Area of triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is $\frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$

Distance Formula

The distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Section Formula

The $P(x, y)$ divided the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m:n$, then $x = \frac{mx_2 \pm nx_1}{m \pm n}$; $y_1 = \frac{my_2 \pm ny_1}{m \pm n}$

Note:
- (+ve) sign tells the division is internal, but (–ve) signs tells, the division is external.
- If $m = n$, then P is the mid-point of the line segment joining A & B.

Equation of Straight Line in various forms

1. **POINT-SLOPE FORM**: $y - y_1 = m(x - x_1)$ is the equation of a straight line whose slope is 'm' and passes through the point (x_1, y_1).
2. **SLOPE INTERCEPT FORM**: $y = mx + c$ is the equation of a straight line whose slope is 'm' and makes an intercept 'c' on the y-axis.
3. **TWO POINT FORM**: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ is the equation of a straight line which passes through (x_1, y_1) & (x_2, y_2).
4. **INTERCEPT FORM**: $\frac{x}{a} + \frac{y}{b} = 1$ is the equation of a straight line which makes intercepts a & b on x and y axis respectively.
5. **NORMAL / PERPENDICULAR FORM**: $x\cos\alpha + y\sin\alpha = p$ (where $p > 0$, $0 \le \alpha < 2\pi$) is the equation of a straight line where the length of the perpendicular from origin O on the line is p and this perpendicular makes an angle α with +ve x-axis.
6. **GENERAL FORM**: $ax + by + c = 0$ is the equation of a straight line in general form. In this case, slope of line $= -\frac{a}{b}$

Intersection Point

The point of intersection between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$

Length of the perpendicular from a point on a line

The length of the perpendicular from $P(x_1, y_1)$ on $ax + by + c = 0$ is:

$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

Reflection & foot of perpendicular a point about a line

1. The image of a point (x_1, y_1) about a line $ax + by + c = 0$ is: $[(ar + x_1), (br + y_1)]$ where, $r = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$
2. Perpendicular distance from a point (x_1, y_1) on the line $ax + by + c = 0$ is: $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$
3. The foot of $\perp^{rd}$ from a point (x_1, y_1) on the line $ax + by + c = 0$ is: $[(ar + x_1), (br + y_1)]$, where $r = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2}$

Perpendicular lines

1. When two lines of the slope m_1 & m_2 are at right angles, the Product of their slope is −1, i.e., $m_1m_2 = -1$. Thus, any line perpendicular to $y = mx + c$ is of the form, $y = \frac{-1}{m}x + d$ where d is any parameter.
2. Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are perpendicular if $aa' + bb' = 0$. Thus, any line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$, where k is any parameter.

Equations of bisectors

The equation of bisectors of the angles between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{+a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

Parallel lines

1. When two lines are parallel their slopes are equal. Thus, any line parallel to $y = mx + c$ is of the type $y = mx + d$, where d is any parameter
2. Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are parallel if $\frac{a}{a'} = \frac{b}{b'} \ne \frac{c}{c'}$
3. The distance between two parallel lines with equations $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $d = \left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$

Angle between two straight lines in terms of their slopes

If m_1 and m_2 are the slopes of two intersecting lines ($m_1m_2 \ne -1$) and θ be the acute angle between them then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$

Three Dimensional Geometry-I

Introduction

- In three dimensions, the coordinate axes of a rectangular cartesian coordinate system are three mutually perpendicular lines. The axes are called x, y and z-axes.
- The three planes determined by the pair of axes are the coordinate planes, called xy, yz and zx-planes.
- The three coordinate planes divide the space into eight parts known as octants.
- The coordinates of a point P in 3D Geometry is always written in the form of triplet like (x,y,z). Here, x, y and z are the distances from yz, zx and yx planes, respectively.

e.g.:
- Any point on x-axis is : $(x, 0, 0)$
- Any point on y-axis is : $(0, y, 0)$
- Any point on z-axis is : $(0, 0, z)$

Coordinates of a Midpoint

The coordinates of the midpoint of the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

e.g.: Find the midpoint of the line joining two points P(1, –3, 4) and Q(–4, 1, 2).

Sol: Coordinates of the midpoint of the line joining the points P & Q are

$$\left(\frac{1-4}{2}, \frac{-3+1}{2}, \frac{4+2}{2}\right) \text{ i.e. } \left(\frac{-3}{2}, -1, 3\right)$$

Coordinates of the Centroid of a Triangle

The coordinates of the centroid of the triangle, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$

e.g.: The centroid of a triangle ABC is at the point (1, 1, 1). If the coordinates of A and B are (3, –5, 7) and (–1, 7, –6), respectively, find the coordinates of the point C.

Sol: Let the coordinates of C be (x, y, z) and the coordinates of the centroid G be (1, 1, 1). Then $\frac{x+3-1}{3} = 1$, *i.e.*, $x = 1$;

$$\frac{y-5+7}{3} = 1, \text{ i.e., } y = 1;$$

$$\frac{z+7-6}{3} = 1, \text{ i.e., } z = 2. \text{ So, } C(x, y, z) = (1, 1, 2)$$

Distance between Two Points

Distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

e.g.: Find the distance between the points P(1, –3, 4) and (–4, 1, 2).

Sol: The distance PQ between the points P & Q is given by

$$PQ = \sqrt{(-4-1)^2 + (1+3)^2 + (2-4)^2}$$

$$= \sqrt{25+16+4} = \sqrt{45} = 3\sqrt{5} \text{ units}$$

Section Formula

The coordinates of the point R which divides the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally and externally in the ratio m : n are given by

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right) \;\&\; \left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}, \frac{mz_2-nz_1}{m-n}\right)$$

respectively.

e.g.: Find the coordinates of the point which divides the line segment joining the points (1, –2, 3)and (3, 4, –5) in the ratio 2 : 3 internally.

Sol : Let $P(x, y, z)$ be the point which divides line segment joining A (1,–2, 3) and B (3, 4, –5) internally in the ratio 2 : 3. Therefore,

$$x = \frac{2(3)+3(1)}{2+3} = \frac{9}{5} \quad y = \frac{2(4)+3(-2)}{2+3} = \frac{2}{5} \quad z = \frac{2(-5)+3(3)}{2+3} = \frac{-1}{5}$$

Thus, the required point is $\left(\frac{9}{5}, \frac{2}{5}, \frac{-1}{5}\right)$.

Three Dimensional Geometry-II

Direction ratios and direction cosines of a line

Direction cosines of a line are the cosines of the angles made by the line with the positive direction of the co-ordinate axes. If l, m, n are the direction cosines of a line, then $l^2 + m^2 + n^2 = 1$. direction cosines of a line joining P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are $\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$, where $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Direction ratios of a line are the numbers which are proportional to the direction cosines of the line. If l, m, n are the direction cosines and a, b, c are direction ratios of a line, then $l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$, $m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$, $n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$

Skew lines

These are the lines in space which are neither parallel nor intersecting. They lie in different planes. Angle between skew lines is the angle between two intersecting lines drawn from any point (origin) parallel to each of the skew lines.

Angle between the two lines

If l_1, m_1, n_1, and l_2, m_2, n_2 are the direction cosines and a_1, b_1, c_1, and a_2, b_2, c_2 are the direction ratios of the two lines and 'θ' is the acute angle between them, then

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2| = \left|\frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

Vector equation of line equations in 3D

Vector equation of a line passing through the given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda \vec{b}$

Vector equation of a line which passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda (\vec{b} - \vec{a})$

Equation of a line through point (x_1, y_1, z_1) and having direction cosines l, m, n is $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ Also, equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

Angle between two lines

If 'θ' is the acute angle between $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$, $\vec{r} = \vec{a_2} + \lambda \vec{b_2}$

then, $\cos\theta = \left|\frac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}| \cdot |\vec{b_2}|}\right|$

If $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ are the equations of two lines, then acute angle between them is $\cos q = |l_1 . l_2 + m_1 . m_2 + n_1 . n_2|$

Characteristics of planes

Two lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$, $\vec{r} = \vec{a_2} + m\vec{b_2}$ are coplanar if $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0$.

Equation of a plane that cuts co-ordinate axes at $(a, 0, 0), (0, b, 0), (0, 0, c)$ is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

The distance of a point with position vector $\vec{a}$ from the plane $\vec{r} \cdot \hat{n} = d$ is $|d - \vec{a} \cdot \hat{n}|$.

The distance from a point (x_1, y_1, z_1) to the plane is $Ax + By + Cz + D = 0$ $\left|\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right|$

Vector equation of a plane

(i) which contains three non-collinear points having position vectors $\vec{a}, \vec{b}, \vec{c}$ is $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$.

(ii) that passes through the intersection of planes $\vec{r} \cdot \vec{n_1} = d_1$ & $\vec{r} \cdot \vec{n_2} = d_2$ is $\vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2$, $\lambda =$ non-zero constant.

Equation of a plane

(i) which is at distance 'd' from the origin and direction cosines of the normal to the plane as l, m, n is $lx + my + nz = d$.

(ii) ⊥ to a given line with direction ratios A, B, C and passing through (x_1, y_1, z_1) is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

(iii) passing through three non-collinear points (x_1, y_1, z_1) (x_2, y_2, z_2), (x_3, y_3, z_3) is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$.

Shortest distance between

(i) Two skew lines is the line segment perpendicular to both the lines

(ii) $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ is $\left|\frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|}\right|$

(iii) $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is given by

$$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2 + (a_1 b_2 - a_2 b_1)^2}}$$

(iv) Distance between parallel lines $\vec{r} = \vec{a_1} + \lambda \vec{b}$ and $\vec{r} = \vec{a_2} + \mu \vec{b}$ is $\left|\frac{\vec{b} \times (\vec{a_2} - \vec{a_1})}{|\vec{b}|}\right|$

Vector Algebra

Vector

A quantity that has both magnitude and direction is called a vector. The distance between the initial and terminal points of a vector is called its magnitude. Magnitude of vector $\vec{AB}$ is $|\vec{AB}|$.

Position Vector

Position vector of a point P (x, y, z) is $x\hat{i} + y\hat{j} + z\hat{k}$ and its magnitude is $OP(r) = \sqrt{x^2 + y^2 + z^2}$. eg: Position vector of P (2, 3, 5) is $2\hat{i} + 3\hat{j} + 5\hat{k}$ and its magnitude is $\sqrt{2^2 + 3^2 + 5^2} = \sqrt{38}$.

Direction ratios and direction cosines

The scalar components of a vector are its direction ratios, and represent its projections along the respective axes.

The magnitude (r), direction ratios (a,b,c) and direction cosines (l,m,n) of vector $a\hat{i} + b\hat{j} + c\hat{k}$ are related as:

$l = \frac{a}{r}, m = \frac{b}{r}, n = \frac{c}{r}$

e.g.: If $\overrightarrow{AB} = \hat{i} + 2\hat{j} + 3\hat{k}$, then $r = \sqrt{1+4+9} = \sqrt{14}$

Direction ratios are (1, 2, 3)

and direction cosines are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

Types of Vectors

(I) Zero vector (initial and terminal points coincide)

(ii) Unit vector (magnitude is unity)

(iii) Coinitial vectors (same initial points)

(iv) Collinear vectors (parellel to the same line)

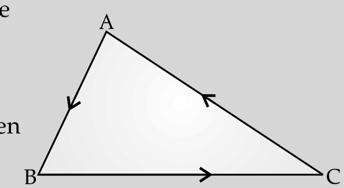
(v) Equal vectors (same magnitude and direction)

(vi) Negative of a vector (same magnitude, opposite direction)

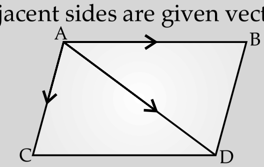
Properties of Vector

The vector sum of the three sides of a triangle taken in order is $\vec{0}$. *i.e*

if ABC is given triangle, then $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$.



The vector sum of two coinitial vectors is given by the diagonal of the parallelogram whose adjacent sides are given vectors.



If $\overrightarrow{AB}, \overrightarrow{AC}$ are the given vectors, then $\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}$

If we have two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and λ is any scalar, then-

(i) $\vec{a} \pm \vec{b} = (a_1 \pm b_1)\hat{i} + (a_2 \pm b_2)\hat{j} + (a_3 \pm b_3)\hat{k}$

(ii) $\lambda\vec{a} = (\lambda a_1)\hat{j} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$

(iii) $\vec{a}\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ and

(iv) $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Cross product of two vectors

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, $\hat{n}$ is a unit vector perpendicular to line joining $\vec{a}, \vec{b}$.

Properties

- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $k(\vec{a} \times \vec{b}) = \vec{a} \times (k\vec{b})$
- $\vec{a} \times \vec{b} = \vec{0}$, if $\vec{a} \,||\, \vec{b}$

Scalar product of two vectors

If $\vec{a}, \vec{b}$ are the vectors and 'θ' is the angle between them, then their scalar product $\vec{a}\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

Section Formulas

The Position vector of a point R dividing a line segment joining P, Q whose position vectors are $\vec{a}, \vec{b}$ respectively, in the ratio $m : n$

(i) internally is $\frac{n\vec{a} + m\vec{b}}{m+n}$, (ii) externally is $\frac{m\vec{b} - n\vec{a}}{m-n}$

Unit vector

For a given vector $\vec{a}$, the vector $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$ gives the unit vector in the direction of $\vec{a}$. *eg.*, If $\vec{a} = 5\hat{i}$, then $\hat{a} = \frac{5\hat{i}}{|5|} = \hat{i}$, which is a unit vector.

Probability

Random Experiment

An Experiment is called random experiment if it satisfies the following two conditions:

- It has more than one possible outcome.
- It is not possible to predict the outcome in advance.

Outcome: A possible result of a random experiment is called its outcome.

Sample Space: Set of all possible outcomes of a random experiment is called sample space. It is denoted by symbol 'S'.

e.g.: In a toss of a coin, sample space is Head & Tail. i.e., S= {H,T}

Sample Point: Each element of the Sample Space is called a sample point.

e.g.: In a toss of a coin, head is a sample point

Equally Likely Outcomes: All outcomes with equal probability.

Definition

Probability is the measure of uncertainty of various phenomenon, numerically. It can have positive value from 0 to 1.

$$\text{Probability} = \frac{\text{No. of favourable outcome}}{\text{Total no. of outcomes}}$$

e.g.: Probability of getting an even no. in a throw of a die.

Sol. Here, favourable outcomes = {2, 4, 6}
Total no. of outcomes = {1, 2, 3, 4, 5, 6}

$$\text{Probability} = \frac{3}{6} = \frac{1}{2}$$

Event

It is the set of favourable outcomes. Any subset E of a sample space S is called an event.

e.g.: Event of getting an even number (outcome) in a throw of a die.

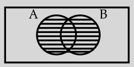
Occurrence of event: The event E of a sample space 'S' is said to have occurred if the outcome w of the experiment is such that $w \in E$. If the outcome w is such that $w \notin E$, we say that event E has not occurred.

Types of Events

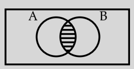
- **Impossible and Sure Event:** The empty set φ is called an Impossible event, whereas the whole sample space 'S' is called 'Sure event'.
  e.g.: In a rolling of a die, impossible event is getting number more than 6 and event of getting number less than or equal to 6 is sure event.
- **Simple Event:** If an event has only one sample point of a sample space, it is called a 'simple event'.
  e.g.: In rolling of a die, simple event could be the event of getting number 4.
- **Compound Event:** If an event has more than one sample point, it is called a 'compound event'.
  e.g.: In rolling of a die, compound event could be event of getting an even number.
- **Complementary Event:** Complement event to A = 'not A'
  e.g.: If an event A= Event of getting odd number in a throw of a die i.e., {1, 3, 5} then , complementary event to A = Event of getting an even number in a throw of a die, i.e. {2, 4, 6}
  $A' = \{w : w \in S \text{ and } w \notin A\} = S - A$ (where S is the sample space)

Algebra of Events

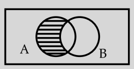
- Event A or B or $(A \cup B)$
  $A \cup B = \{w : w \in A \text{ or } w \in B\}$



- Event A and B or $(A \cap B)$
  $A \cap B = \{w : w \in A \text{ and } w \in B\}$



- Event A but not B or (A–B)
  $A - B = A \cap B'$



Exhaustive Events

Many events that together form sample space are called exhaustive events.

e.g.: A die is thrown. Event A = All even outcomes and event B = All odd outcomes.
Event A & B together forms exhaustive events as it forms sample space.

Mutually Exclusive Events

Events A & B are called mutually exclusive events if occorrence of any one of them excludes occurrence of other event, i.e. they cannot occur simultaneously.

e.g.: A die is thrown. Event A= All even outcomes & event B = All odd outcomes. then,
A & B are mutually exclusive events, they cannot occur simultaneously.

Note: Simple events of a sample space are always mutually exclusive.

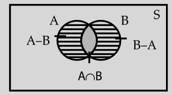
Probability of AUB, A∩B and P(notA)

If A and B are any two events, then

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

If A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$

(As $P(A \cap B) = \varphi$)



- Probability of the event 'not A'

$P(A') = P(\text{not } A) = 1 - P(A)$

Trigonometric Functions

Important Trigonometric Ratios values

- $\sin 15°$ or $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75°$ or $\cos\frac{5\pi}{12}$
- $\cos 15°$ or $\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75°$ or $\sin\frac{5\pi}{12}$
- $\tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3} = \cot 75°$
- $\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15°$
- $\sin\frac{\pi}{10}$ or $\sin 18° = \frac{\sqrt{5}-1}{4}$
- $\cos\frac{\pi}{5}$ or $\cos 36° = \frac{\sqrt{5}+1}{4}$

Factorisation of the Sum or Difference of sine and cosine with two variables

- $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
- $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$
- $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
- $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

Transformation of Product into Sum or Difference of Sines or Cosines

- $2\sin A\cos B = \sin(A+B)+\sin(A-B)$
- $2\cos A\sin B = \sin(A+B) - \sin(A-B)$
- $2\cos A\cos B = \cos(A+B)+\cos(A-B)$
- $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Trigonometrical Function sum and Difference of Two Angles

- $\sin(A\pm B) = \sin A\cos B \pm \cos A\sin B$
- $\cos(A\pm B) = \cos A\cos B \mp \sin A\sin B$
- $\tan(A\pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
- $\cot(A\pm B) = \frac{\cot A\cot B \mp 1}{\cot A \pm \cot B}$
- $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B)\sin(A-B)$
- $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B)\cos(A-B)$

Degree & Radian Measures

If in a circle of radius r, an arc of length l subtends an angle of θ radians, then $l = r\theta$.

- Radian Measure $= \frac{\pi}{180} \times$ Degree Measure
- Degree Measure $= \frac{180}{\pi} \times$ Radian Measure

Trigonometric Identities

- $\sin^2\theta + \cos^2\theta = 1,\ -1 \le \sin\theta \le 1,\ -1 \le \cos\theta \le 1,\ \forall\theta\in R$
- $\sec^2\theta - \tan^2\theta = 1,\ |\sec\theta| \ge 1,\ \forall\theta\in R$
- $\mathrm{cosec}^2\theta - \cot^2\theta = 1,\ |\mathrm{cosec}\,\theta| \ge 1,\ \theta\in R$

Trigonometrical Ratio of allied Angles

- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = \cos\theta$
- $\tan(-\theta) = -\tan\theta$
- $\cot(-\theta) = -\cot\theta$
- $\sec(-\theta) = \sec\theta$
- $\mathrm{cosec}(-\theta) = -\mathrm{cosec}\theta$

- $\sin\left(\frac{\pi}{2}\mp\theta\right) = \cos\theta$
- $\cos\left(\frac{\pi}{2}\mp\theta\right) = \pm\sin\theta$
- $\tan\left(\frac{\pi}{2}\mp\theta\right) = \pm\cot\theta$
- $\cot\left(\frac{\pi}{2}\mp\theta\right) = \pm\tan\theta$
- $\sec\left(\frac{\pi}{2}\mp\theta\right) = \pm\,\mathrm{cosec}\theta$
- $\mathrm{cosec}\left(\frac{\pi}{2}\mp\theta\right) = \sec\theta$

- $\cos(\pi\mp\theta) = -\cos\theta$
- $\sin(\pi\mp\theta) = \pm\sin\theta$
- $\tan(\pi\mp\theta) = \mp\tan\theta$
- $\cot(\pi\mp\theta) = \mp\cot\theta$
- $\sec(\pi\mp\theta) = -\sec\theta$
- $\mathrm{cosec}(\pi\mp\theta) = \pm\mathrm{cosec}\theta$

- $\sin(2\pi\mp\theta) = \mp\sin\theta$
- $\cos(2\pi\mp\theta) = \cos\theta$
- $\tan(2\pi\mp\theta) = \mp\tan\theta$
- $\cot(2\pi\mp\theta) = \mp\cot\theta$
- $\sec(2\pi\mp\theta) = \sec\theta$
- $\mathrm{cosec}(2\pi\mp\theta) = +$

- $\sin\left(\frac{3\pi}{2}\mp\theta\right) = -\cos\theta$
- $\cos\left(\frac{3\pi}{2}\mp\theta\right) = \mp\sin\theta$
- $\tan\left(\frac{3\pi}{2}\mp\theta\right) = \pm\cot\theta$
- $\cot\left(\frac{3\pi}{2}\mp\theta\right) = \pm\tan\theta$
- $\sec\left(\frac{3\pi}{2}\mp\theta\right) = \mp\mathrm{cosec}\theta$
- $\mathrm{cosec}\left(\frac{3\pi}{2}\mp\theta\right) = \sec\theta$

Multiple and Half Angles

- $\sin 2A = 2\sin A\cos A$
- $\cos 2A = \cos^2 A - \sin^2 A$
  $= 2\cos^2 A - 1$
  $= 1 - 2\sin^2 A$
- $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$
- $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$
- $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$
- $\sin 3A = 3\sin A - 4\sin^3 A$
- $\cos 3A = 4\cos^3 A - 3\cos A$
- $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$
- $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$
- $\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}$
- $\tan\theta = \left(2\tan\frac{\theta}{2}\right)/\left(1 - \tan^2\frac{\theta}{2}\right)$

Standard General Solution of Trigonometric Equation

General solution: The solution consisting of all possible solutions of a trigonometric equation is called its General Solution.

For $n \in I$

- $\sin\theta = 0 \Leftrightarrow \theta = n\pi$, $\cos\theta = 0 \Leftrightarrow \theta = (2n+1)\frac{\pi}{2}$
- $\tan\theta = 0 \Leftrightarrow \theta = n\pi$, $\sin\theta = \sin\alpha \Leftrightarrow \theta = n\pi + (-1)^n\alpha$, where $\alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
- $\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi \pm \alpha$, where $\alpha \in [0,\pi]$
- $\tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha$, where $\alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
- $\sin^2\theta = \sin^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha$, $\cos^2\theta = \cos^2\alpha \Leftrightarrow n\pi \pm \alpha$
- $\tan^2\theta = \tan^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha$, $\sin\theta = 1 \Leftrightarrow \theta = (4n+1)\frac{\pi}{2}$
- $\cos\theta = 1 \Leftrightarrow \theta = 2n\pi$, $\cos = -1 \Leftrightarrow \theta = (2n+1)\pi$

Trigonometric Equations

The equation involving trigonometric functions of unknown angles are known as Trigonometric equations. e.g., $\cos\theta = 0$, $\cos^2\theta - 4\cos\theta = 1$

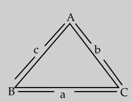
A solution of trigonometric equation is the value of the unknown angle that satisfies the equation.

e.g., $\sin\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \ldots$

Solutions of Triangle

In, ΔABC,



(i) Sine Law

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

(ii) Projection formula

(a) $a\cos C + c\cos A = b$

(b) $b\cos A + a\cos B = c$

(c) $c\cos B + b\cos C = a$

(iii) Cosine Law

(a) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

(b) $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

(c) $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Area of Triangle

Area of triangle ABC

$= \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$

$= \sqrt{s(s-a)(s-b)(s-c)}$; here, $s = \frac{a+b+c}{2}$

I nerve Trigonoetric Function

Domain and range of inverse trigonometric functions

(i) $y = \sin^{-1}x$. Domain = $[-1, 1]$, Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $y = \cos^{-1}x$. Domain = $[-1, 1]$ Range = $[0, \pi]$

(iii) $y = \text{cosec}^{-1}x$. Domain = $R-(-1, 1)$, Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

(iv) $y = \sec^{-1}x$. Domain = $R-(-1, 1)$, Range = $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

(v) $y = \tan^{-1}x$. Domain = R, Range = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(vi) $y = \cot^{-1}x$. Domain = R, Range = $(0, \pi)$.

Trigonometric functions

(i) $\sin : R \to [-1, 1]$

(ii) $\cos : R \to [-1, 1]$

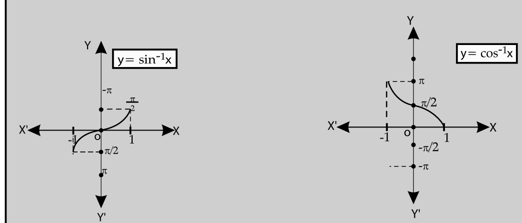
(iii) $\tan : R - \left\{x : x = (2n+1)\frac{\pi}{2}, n \in Z\right\} \to R$

(iv) $\cot : R - \{x : x = n\pi, n \in Z\} \to R$

(v) $\sec : R - \left\{x : x = (2n+1)\frac{\pi}{2}, n \in Z\right\} \to R - (-1, 1)$

(vi) $\text{cosec} : R - \{x : x = n\pi, n \in Z\} \to R - (-1, 1)$

Graphs of trigonometric functions and inverse trigonometric functions



Some important relations

(i) $y = \sin^{-1}x \Rightarrow x = \sin y$

(ii) $x = \sin y \Rightarrow y = \sin^{-1}x$

(iii) $\sin(\sin^{-1}x) = x$

(iv) $\sin^{-1}(\sin x) = x$

(v) $\sin^{-1}\frac{1}{x} = \text{cosec}^{-1}x$

(vi) $\cos^{-1}(-x) = \pi - \cos^{-1}x$

(vii) $\cos^{-1}\frac{1}{x} = \sec^{-1}x$

(viii) $\cot^{-1}(-x) = \pi - \cot^{-1}x$

(ix) $\tan^{-1}\frac{1}{x} = \cot^{-1}x$

(x) $\sec^{-1}(-x) = \pi - \sec^{-1}x$

(xi) $\sin^{-1}(-x) = -\sin^{-1}x$

(xii) $\tan^{-1}(-x) = -\tan^{-1}x$

(xiii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

(xiv) $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}, |x| \ge 1$

(xv) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}, y < 1, x > 0, y > 0$

(xvi) $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}, -1<x<1$

(xvii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}, x > 0, y > 0$

(xviii) $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2}$

$\sin^{-1}x \neq (\sin x)^{-1} = \frac{1}{\sin x}$ and same for other trigonometric functions.

Principal value branch and principal value

The range of an inverse trigonometric function is the principal value branch and those values which lies in the principal value branch is called the principal value of that inverse trigonometric functions.

$0 \le \sin^{-1}x \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le \sin^{-1}x < 0$
$0 \le \cos^{-1}x \le \frac{\pi}{2}$	$\frac{\pi}{2} < \cos^{-1}x \le \pi$
$0 \le \tan^{-1}x < \frac{\pi}{2}$	$-\frac{\pi}{2} < \tan^{-1}x < 0$
$0 < \cot^{-1}x \le \frac{\pi}{2}$	$\frac{\pi}{2} < \cot^{-1}x < \pi$
$0 \le \sec^{-1}x < \pi/2$	$\frac{\pi}{2} < \sec^{-1}x \le \pi$
$0 < \text{cosec}^{-1}x \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le \text{cosec}^{-1}x < 0$