CHAPTER

# 6 APPLICATION OF DERIVATIVES

## Chapter Objectives

This chapter will help you understand :

➢ ***Application of derivatives :*** *Introduction, Rate of change of quantities; Increasing and decreasing functions; Tangents and normal; Approximations; Maxima and Minima.*

## Quick Review

❖ Derivatives help us to understand how larger things are made from smaller things.

❖ Derivative can tell you the instantaneous slope ($dy/dx$) of this graph.

❖ You predict by checking out the direction and speed of their movement, *i.e.*, the derivative of their movement.

❖ Derivatives are met in many engineering and science problems, especially when modelling the behaviour of moving objects.

**TIPS...**

- *The most basic way of calculating derivatives is using the definition.*
- *The chain rule is the most important rule for taking derivatives.*
- *The product rule allows you to find derivatives of functions that are products of other functions.*
- *Implicit differentiation allows you to find derivatives of functions expressed in a funny way that we call implicit.*

## Know the Links

☞ *https://www.whitman.edu/mathematics/calculus_online/chapter06.html*

☞ *https://www.slideshare.net/iramkhan66/applications-of-derivatives-56443012*

☞ *https://www.quora.com/What-are-the-applications-of-derivatives-in-real-life*

**TRICKS...**

- *Learn about derivatives of trigonometric functions.*
- *You'll take a derivative in a minute to get a relationship between the rates.*
- *Take the Derivative with Respect to Time.*
- *You need to translate the words carefully into mathematical equations.*

## Multiple Choice Questions **(1 mark each)**

**Q. 1. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. The rate at which the area increases, when side is 10 cm is :**

(a) $10\ cm^2/s$ (b) $\sqrt{3}\ cm^2/s$

(c) $10\sqrt{3}\ cm^2/s$ (d) $\frac{10}{3}\ cm^2/s$

**[NCERT Exemp. Ex. 6.3, Q. 35, Page 138]**

**Ans. Correct option :** (c)

***Explanation*** : Let the side of an equilateral triangle be $x$ cm.

$\therefore$ Area of equilateral triangle, $A = \frac{\sqrt{3}}{4}x^2$ ...(i)

Also, $\frac{dx}{dt} = 2$ cm/s

On differentiating equation (i)
with respect to $t$, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4}.2x.\frac{dx}{dt}$$

$$= \frac{\sqrt{3}}{4}.2.10.2$$

$$\left[\because x = 10 \text{ and } \frac{dx}{dt} = 2\right]$$

$$= 10\sqrt{3}\ \text{cm}^2/\text{s}$$

**Q. 2. A ladder, 5-meter long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/sec, then the rate at which the angle between the floor and the ladder is decreasing when lower end of ladder is 2 metre from the wall is :**

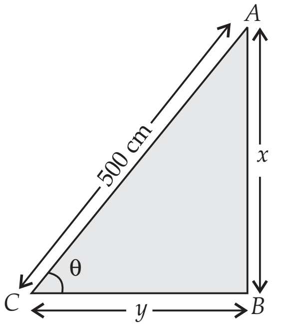
**(a)** $\frac{1}{10}$ **radian/sec** **(b)** $\frac{1}{20}$ **radian/sec**
**(c) 20 radian/sec** **(d) 10 radian/sec**

**[NCERT Exemp. Ex. 6.3, Q. 36, Page 138]**

**Ans. Correct option :** (b)

*Explanation* : Let the angle between floor and the ladder be θ.

Let $AB = x$ cm and $BC = y$ cm




$\therefore \quad \sin\theta = \frac{x}{500}$ and $\cos\theta = \frac{y}{500}$

$\Rightarrow \quad x = 500\sin\theta$ and $y = 500\cos\theta$

Also, $\frac{dx}{dt} = 10$ cm/s

$\Rightarrow 500.\cos\theta.\frac{d\theta}{dt} = 10$ cm/s

$\Rightarrow \quad \frac{d\theta}{dt} = \frac{10}{500\cos\theta} = \frac{1}{50\cos\theta}$

For $\quad y = 2$ m $= 200$ cm,

$$\frac{d\theta}{dt} = \frac{1}{50.\frac{y}{500}} = \frac{10}{y}$$

$$= \frac{10}{200} = \frac{1}{20}\ \text{rad/s}$$

**Q. 3. The curve $y = x^{1/5}$ has at (0, 0)**
**(a) a vertical tangent (parallel to $y$-axis)**
**(b) a horizontal tangent (parallel to $x$-axis)**
**(c) an oblique tangent**
**(d) no tangent**

**[NCERT Exemp. Ex. 6.3, Q. 37, Page 138]**

**Ans. Correct option :** (a)

*Explanation* : Given that, $y = x^{1/5}$

On differentiating with respect to $x$, we get

$$\frac{dy}{dx} = \frac{1}{5}x^{\frac{1}{5}-1} = \frac{1}{5}x^{-4/5}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(0,0)} = \frac{1}{5}\times(0)^{-4/5} = \infty$$

So, the curve $y = x^{1/5}$ has a vertical tangent at (0, 0), which is parallel to $y$-axis.

**Q. 4. The equation of normal to the curve $3x^2 - y^2 = 8$ which is parallel to the line $x + 3y = 8$ is**

**(a)** $3x - y = 8$ **(b)** $3x + y + 8 = 0$
**(c)** $x + 3y \pm 8 = 0$ **(d)** $x + 3y = 0$

**[NCERT Exemp. Ex. 6.3, Q. 38, Page 139]**

**Ans. Correct option :** (c)

*Explanation* : We have, the equation of the curve is

$3x^2 - y^2 = 8$ ....(i)

Also, the given equation of the line is $x + 3y = 8$.

$\Rightarrow 3y = 8 - x$

$\Rightarrow \quad y = -\frac{x}{3} + \frac{8}{3}$

Thus, slope of the line is $-\frac{1}{3}$ which should be equal to slope of the equation of normal to the curve.

On differentiating equation (i) with respect to $x$, we get

$$6x - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{6x}{2y} = \frac{3x}{y} = \text{Slope of the curve}$$

Now, slope of normal to the curve

$$= -\frac{1}{\left(\frac{dy}{dx}\right)} = -\frac{1}{\left(\frac{3x}{y}\right)} = -\frac{y}{3x}$$

$\therefore \quad -\left(\frac{y}{3x}\right) = -\frac{1}{3}$

$\Rightarrow \quad -3y = -3x$

$\Rightarrow \quad y = x$

On substituting the value of the given equation of the curve, we get

$3x^2 - x^2 = 8$

$\Rightarrow \quad 2x^2 = 8$

$\Rightarrow \quad x^2 = 4$

$\Rightarrow \quad x = \pm 2$

For $\quad x = 2$

$3(2)^2 - y^2 = 8$

$\Rightarrow \quad y^2 = 4$

$\Rightarrow \quad y = \pm 2$

and for $x = -2$,

$3(-2)^2 - y^2 = 8$

$\Rightarrow \quad y^2 = 4$

$\Rightarrow \quad y = \pm 2$

So, the points at which normal is parallel to the given line are $(\pm 2, \pm 2)$.
Hence, the equation of normal at $(\pm 2, \pm 2)$ is

$$\Rightarrow \quad y-(\pm 2) = -\frac{1}{3}[x-(\pm 2)]$$

$$\Rightarrow 3[y-(\pm 2)] = -[x-(\pm 2)]$$

$$\therefore \quad x+3y \pm 8 = 0$$

**Q. 5. If the curve $ay+x^2=7$ and $x^3=y$, cut orthogonally at (1, 1), then the value of $a$ is :**

**(a)** 1 **(b)** 0
**(c)** –6 **(d)** 6

**[NCERT Exemp. Ex. 6.3, Q. 39, Page 139]**

**Ans. Correct option :** (d)

***Explanation*** : Given that, $ay+x^2=7$ and $x^3=y$
On differentiating with respect to $x$ in both equations, we get

$$a.\frac{dy}{dx}+2x=0 \text{ and } 3x^2=\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{2x}{a} \text{ and } \frac{dy}{dx}=3x^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = \frac{-2}{a} = m_1 \text{ and } \left(\frac{dy}{dx}\right)_{(1,1)} = 3.1 = 3 = m_2$$

Since, the curve cuts orthogonally at (1, 1).

$$\therefore \quad m_1 m_2 = -1$$

$$\Rightarrow \left(\frac{-2}{a}\right).3 = -1$$

$$\therefore \quad a = 6$$

**Q. 6. If $y=x^4-10$ and if $x$ changes from 2 to 1.99, what is the change in $y$**

**(a)** 0.32 **(b)** 0.032
**(c)** 5.68 **(d)** 5.968

**[NCERT Exemp. Ex. 6.3, Q. 40, Page 139]**

**Ans. Correct option :** (a)

***Explanation*** : Given that,

$$y = x^4 - 10$$

On differentiating with respect to $x$, we get

$$\Rightarrow \quad \frac{dy}{dx} = 4x^3$$

and $\Delta x = 2.00 - 1.99 = 0.01$

$$\therefore \quad \Delta y = \frac{dy}{dx} \times \Delta x$$
$$= 4x^3 \times \Delta x$$
$$= 4 \times 2^3 \times 0.01$$
$$= 32 \times 0.01$$
$$= 0.32$$

So, the approximate change in $y$ is 0.32.

**Q. 7. The equation of tangent to the curve $y(1+x\ )=2-x$, where it crosses $x$-axis is :**

**(a)** $x+5y=2$ **(b)** $x-5y=2$
**(c)** $5x-y=2$ **(d)** $5x+y=2$

**[NCERT Exemp. Ex. 6.3, Q. 41, Page 140]**

**Ans. Correct option :** (a)

***Explanation*** : Given that the equation of curve is

$$y(1+x^2) = 2-x \qquad ....(i)$$

On differentiating with respect to $x$, we get

$$\therefore \quad y.(0+2x)+(1+x^2).\frac{dy}{dx} = 0-1$$

$$\Rightarrow \quad 2xy+(1+x^2)\frac{dy}{dx} = -1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1-2xy}{1+x^2} \qquad ...(ii)$$

Since, the given curve passes through $x$-axis, i.e., $y=0$

$$\therefore \quad 0(1+x^2) = 2-x \quad \text{[By using Eq. (i)]}$$

$$\Rightarrow \quad x = 2$$

So, the curve passes through the point (2, 0).

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(2,\,0)} = \frac{-1-2\times 0}{1+2^2} = -\frac{1}{5}$$
$$= \text{Slope of the curve}$$

$$\therefore \text{Slope of tangent to the curve} = -\frac{1}{5}$$

$\therefore$ Equation of tangent of the curve passsing through (2, 0) is

$$y-0 = -\frac{1}{5}(x-2)$$

$$\Rightarrow y+\frac{x}{5} = +\frac{2}{5}$$

$$\Rightarrow 5y+x = 2$$

**Q. 8. The points at which the tangents to the curve $y=x^3-12x+18$ are parallel to $x$-axis are :**

**(a)** (2, –2), (–2, –34) **(b)** (2, 34), (–2, 0)
**(c)** (0, 34), (–2, 0) **(d)** (2, 2), (–2, 34)

**[NCERT Exemp. Ex. 6.3, Q. 42, Page 139]**

**Ans. Correct option :** (d)

***Explanation*** : The equation of the curve is given by

$$y = x^3 - 12x + 18$$

On differentiating with respect to $x$, we get

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 12$$

So, the slope of line parallel to the $x$-axis,

$$\therefore \quad \frac{dy}{dx} = 0$$

$$\Rightarrow \quad 3x^2 - 12 = 0$$

$$\Rightarrow \quad x^2 = \frac{12}{3} = 4$$

$$\therefore \quad x = \pm 2$$

For $x = 2$,

$$y = 2^3 - 12\times 2 + 18 = 2$$

and for $x = -2$,

$$y = (-2)^3 - 12\times(-2) + 18 = 34$$

So, the points are (2, 2) and (−2, 34).

**Q. 9. The tangent to the curve $y=e^{2x}$ at the point (0, 1) meets $x$-axis at :**

**(a)** (0, 1) **(b)** $\left(-\frac{1}{2}, 0\right)$
**(c)** (2, 0) **(d)** (0, 2)

**[NCERT Exemp. Ex. 6.3, Q. 43, Page 139]**

**Ans. Correct option :** (b)

*Explanation* : The equation of the curve is given by $y = e^{2x}$

Since, it passes through the point (0, 1).

$$\therefore \quad \frac{dy}{dx} = e^{2x}.2 = 2.e^{2x}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)} = 2.e^{2.0} = 2$$

= Slope of tangent to the curve.

$\therefore$ Equation of tangent is

$$y - 1 = 2(x - 0)$$

$$\Rightarrow \quad y = 2x + 1$$

Since, tangent to the curve $y = e^{2x}$ at the point (0, 1) meets $x$-axis, *i.e.*, $y = 0$ .

$$\therefore \quad 0 = 2x + 1$$

$$\Rightarrow \quad x = -\frac{1}{2}$$

So, the required point is $\left(-\frac{1}{2}, 0\right)$.

**Q. 10. The slope of tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point (2, –1) is :**

**(a)** $\frac{22}{7}$ **(b)** $\frac{6}{7}$

**(c)** $\frac{-6}{7}$ **(d)** -6

**[NCERT Exemp. Ex. 6.3, Q. 44, Page 139]**

**Ans. Correct option :** (b)

*Explanation* : The equation of the curve is given by $x = t^2 + 3t - 8$ and $y = 2t^2 - 2t - 5$

$$\therefore \quad \frac{dx}{dt} = 2t + 3 \text{ and } \frac{dy}{dt} = 4t - 2$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t-2}{2t+3} \quad \text{...(i)}$$

Since, the curve passes through the point (2, −1).

$$2 = t^2 + 3t - 8$$

and $\quad -1 = 2t^2 - 2t - 5$

$\Rightarrow \quad t^2 + 3t - 10 = 0$

and $\quad 2t^2 - 2t - 4 = 0$

$\Rightarrow \quad t^2 + 5t - 2t - 10 = 0$

and $\quad 2t^2 + 2t - 4t - 4 = 0$

$\Rightarrow \quad t(t+5) - 2(t+5) = 0$

and $\quad 2t(t+1) - 4(t+1) = 0$

$\Rightarrow \quad (t-2)\ (t-5) = 0$

$\Rightarrow \quad t = 2,\ 5$

and $\quad (2t-4)\ (t+1) = 0$

$\Rightarrow \quad t = -1,\ 2$

At $\quad t = 2,$ we get

Slope of the tangent,

$$\left(\frac{dy}{dx}\right)_{(t=2)} = \frac{4 \times 2 - 2}{2 \times 2 + 3} = \frac{6}{7} \qquad \text{[By using Eq. (i)]}$$

**Q. 11. The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$ intersect at an angle of**

**(a)** $\frac{\pi}{4}$ **(b)** $\frac{\pi}{3}$

**(c)** $\frac{\pi}{2}$ **(d)** $\frac{\pi}{6}$

**[NCERT Exemp. Ex. 6.3, Q. 45, Page 140]**

**Ans. Correct option :** (c)

*Explanation* : Given that, equations of two curves are given by

$$x^3 - 3xy^2 + 2 = 0 \text{ and } 3x^2y - y^3 - 2 = 0$$

On differentiating with respect to $x$, we get

$$\Rightarrow \quad 3x^2 - 3\left[x.2y.\frac{dy}{dx} + y^2.1\right] + 0 = 0$$

$$\text{and } 3\left[x^2\frac{dy}{dx} + y.2x\right] - 3y^2\frac{dy}{dx} - 0 = 0$$

$$\Rightarrow \quad 3x.2y\frac{dy}{dx} + 3y^2 = 3x^2$$

$$\text{and} \quad 3y^2\frac{dy}{dx} = 3x^2\frac{dy}{dx} + 6xy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2 - 3y^2}{6xy}$$

$$\text{and} \quad \frac{dy}{dx} = \frac{6xy}{3y^2 - 3x^2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right) = \frac{3(x^2 - y^2)}{6xy}$$

$$\text{and} \quad \left(\frac{dy}{dx}\right) = \frac{-6xy}{3y^2 - 3x^2}$$

$$m_1 = \frac{x^2 - y^2}{2xy}$$

$$m_2 = \frac{-2xy}{x^2 - y^2}$$

$$\therefore \quad m_1.m_2 = \frac{x^2 - y^2}{2xy} \times \frac{-2xy}{x^2 - y^2} = -1$$

Hence, both the curves are intersecting at right angle, *i.e.*, making $\frac{\pi}{2}$ with each other.

**Q. 12. The interval on which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing is :**

**(a)** [–1, ∞) **(b)** [–2, –1]

**(c)** (–∞, –2] **(d)** [–1, 1]

**[NCERT Exemp. Ex. 6.3, Q. 46, Page 140]**

**Ans. Correct option :** (b)

*Explanation* : Given that,

$$f(x) = 2x^3 + 9x^2 + 12x - 1$$

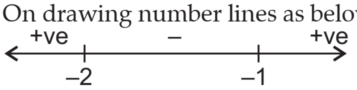
$$f'(x) = 6x^2 + 18x + 12$$

$$= 6(x^2 + 3x + 2)$$

$$= 6(x+2)(x+1)$$

So, $f'(x) \le 0$ , for decreasing.

On drawing number lines as below :




We see that $f'(x)$ is decreasing in [−2, −1].

**Q. 13. Let the $f: R \to R$ be defined by $f(x) = 2x + cosx$, then $f$ :**

**(a) has a minimum at $x = \pi$**

**(b) has a maximum, at $x = 0$**

**(c) is a decreasing function**

**(d) is an increasing function**

**[NCERT Exemp. Ex. 6.3, Q. 47, Page 140]**

**Ans. Correct option :** (d)

***Explanation*** : Given that,

$f(x) = 2x + \cos x$

Differentiating with respect to $x$, we get

$f'(x) = 2 + (-\sin x)$

$= 2 - \sin x$

Since, $f'(x) > 0, \forall x \in R$

Hence, $f(x)$ is an increasing function.

**Q. 14. $y = x(x-3)^2$ decreases for the values of $x$ given by :**

**(a) $1 < x < 3$** **(b) $x < 0$**

**(c) $x > 0$** **(d) $0 < x < \frac{3}{2}$**

**[NCERT Exemp. Ex. 6.3, Q. 48, Page 140]**

**Ans. Correct option :** (a)

***Explanation*** : Given that,

$y = x(x-3)^2$

$\therefore \frac{dy}{dx} = x.2(x-3).1 + (x-3)^2.1$

$= 2x^2 - 6x + x^2 + 9 - 6x$

$= 3x^2 - 12x + 9$

$= 3(x^2 - 3x - x + 3)$

$= 3(x-3)(x-1)$

So, $y = x(x-3)^2$ decreases for (1, 3).

[Since, $y' < 0$ for all $x \in (1, 3)$, hence $y$ is decreasing on (1, 3)].

**Q. 15. The function**

**$f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$ is strictly**

**(a) increasing in $\left(P, \frac{3P}{2}\right)$**

**(b) decreasing in $\left(\frac{P}{2}, P\right)$**

**(c) decreasing in $\left(\frac{-P}{2}, \frac{P}{2}\right)$**

**(d) decreasing in $\left(0, \frac{P}{2}\right)$**

**[NCERT Exemp. Ex. 6.3, Q. 49, Page 140]**

**Ans. Correct option :** (b)

***Explanation*** : Given that,

$f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$

On differentiating with respect to $x$, we get

$f'(x) = 12\sin^2 x.\cos x - 12\sin x.\cos x + 12\cos x$

$12\left[\sin^2 x.\cos x - \sin x.\cos x + \cos x\right]$

$12\cos x\left[\sin^2 x - \sin x + 1\right]$

$\Rightarrow f'(x) = 12\cos x\left[\sin^2 x + 1(1 - \sin x)\right]$

$1 - \sin x \geq 0$ and $\sin^2 x \geq 0$

$\sin^2 x + 1 - \sin x \geq 0$

Hence, $f'(x) > 0$, when $\cos x > 0$, *i.e.*, $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

So, $f(x)$ is increasing when $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

and $f'(x) < 0$, when $\cos x < 0$, *i.e.*, $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

Hence, $f'(x)$ is decreasing when $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Since,

$\left(\frac{\pi}{2}, \pi\right) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Hence, $f(x)$ is decreasing in $\left(\frac{\pi}{2}, \pi\right)$.

**Q. 16. Which of the following functions is decreasing on $\left(0, \frac{\pi}{2}\right)$.**

**(a) $\sin 2x$** **(b) $\tan x$**

**(c) $\cos x$** **(d) $\cos 3x$**

**[NCERT Exemp. Ex. 6.3, Q. 50, Page 140]**

**Ans. Correct option :** (c)

***Explanation*** : In the given interval $\left(0, \frac{\pi}{2}\right)$

$f(x) = \cos x$

On differentiating with respect to $x$, we get

$f'(x) = -\sin x$

which gives $f'(x) < 0$ in $\left(0, \frac{\pi}{2}\right)$

Hence, $f(x) = \cos x$ is decreasing in $\left(0, \frac{\pi}{2}\right)$.

**Q. 17. The function $f(x) = \tan x - x$**

**(a) always increases**

**(b) always decreases**

**(c) never increases**

**(d) sometimes increases and sometimes decreases**

**[NCERT Exemp. Ex. 6.3, Q. 51, Page 140]**

**An.s Correct option :** (a)

***Explanation*** : We have,

$f(x) = \tan x - x$

On differentiating with respect to $x$, we get

$f'(x) = \sec^2 x - 1$

$\Rightarrow f'(x) > 0, \forall x \in R$

So, $f(x)$ always increases.

**Q. 18. If $x$ is real, the minimum value of $x^2 - 8x + 17$ is**

**(a) –1** **(b) 0**

**(c) 1** **(d) 2**

**[NCERT Exemp. Ex. 6.3, Q. 52, Page 141]**

**Ans. Correct option :** (c)

***Explanation*** : Let,

$f(x) = x^2 - 8x + 17$

On differentiating with respect to $x$, we get

$f'(x) = 2x - 8$

So, $f'(x) = 0$

$\Rightarrow 2x - 8 = 0$

$\Rightarrow \quad 2x = 8$

$\therefore \quad x = 4$

Now, Again on differentiating with respect to $x$, we get

$f''(x) = 2 > 0, \forall x$

So, $x = 4$ is the point of local minima.

Minimum value of $f(x)$ at $x = 4$

$f(4) = 4\times4-8\times4+17 = 1$

**Q. 19. The smallest value of the polynomial $x^3 - 18x^2 + 96x$ in [0, 9] is**

**(a) 126** **(b) 0**

**(c) 135** **(d) 160**

**[NCERT Exemp. Ex. 6.3, Q. 53, Page 141]**

**Ans. Correct option :** (b)

***Explanation*** : Given that, the smallest value of polynomial is $x^3 - 18x^2 + 96x$

On differentiating with respect to $x$, we get

$f'(x) = 3x^2 - 36x + 96$

So,

$$f'(x) = 0$$

$$\Rightarrow \quad 3x^2 - 36x + 96 = 0$$

$$\Rightarrow 3(x^2 - 12x + 32) = 0$$

$$\Rightarrow \quad (x-8)(x-4) = 0$$

$$\Rightarrow \quad x = 8, 4 \in [0,9]$$

We shall now calculate the value of $f$ at these points and at the end points of the interval [0, 9], *i.e.*, at $x = 4$ and $x = 8$ and at $x = 0$ and at $x = 9$.

$$\begin{aligned} f(4) &= 4^3 - 18.4^2 + 96.4 \\ &= 64 - 288 + 384 \\ &= 160 \\ f(8) &= 8^3 - 18.8^2 + 96.8 \\ &= 128 \\ f(9) &= 9^3 - 18.9^2 + 96.9 \\ &= 729 - 1458 + 864 \\ &= 135 \\ \text{and } f(0) &= 0^3 - 18.0^2 + 96.0 \\ &= 0 \end{aligned}$$

Thus, we conclude that absolute minimum value of f on [0, 9] is 0 occurring at $x = 0$.

**Q. 20. The function $f(x) = 2x^3 - 3x^2 - 12x + 4$, has**

**(a) two points of local maximum**

**(b) two points of local minimum**

**(c) one maxima and one minima**

**(d) no maxima or minima**

**[NCERT Exemp. Ex. 6.3, Q. 54, Page 141]**

**Ans. Correct option :** (c)

***Explanation*** : We have,

$$f(x) = 2x^3 - 3x^2 - 12x + 4$$

$$f'(x) = 6x^2 - 6x - 12$$

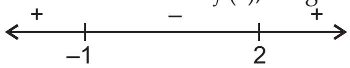
Now, $\quad f'(x) = 0$

$$6(x^2 - x - 2) = 0$$

$$6(x+1)(x-2) = 0$$

$$x = -1 \text{ and } x = +2$$

On number line for $f'(x)$, we get




Hence, $x = -1$ is point of local maxima and $x = 2$ is point of local minima.

So, $f(x)$ has one maxima and one minima.

**Q. 21. The maximum value of $\sin x . \cos x$ is**

**(a) $\frac{1}{4}$** **(b) $\frac{1}{2}$**

**(c) $\sqrt{2}$** **(d) $2\sqrt{2}$**

**[NCERT Exemp. Ex. 6.3, Q. 55, Page 141]**

**Ans. Correct option :** (b)

***Explanation*** : Let us assume that, $f(x) = \sin x . \cos x$

Now, we know that

$$\sin x . \cos x = \frac{1}{2} \sin 2x$$

$$\therefore \quad f'(x) = \frac{1}{2}.\cos 2x.2 = \cos 2x$$

Now, $\quad f'(x) = 0$

$$\Rightarrow \quad \cos 2x = 0$$

$$\Rightarrow \quad \cos 2x = \cos\frac{\pi}{2}$$

$$\Rightarrow \quad x = \frac{\pi}{4}$$

Also, $\quad f''(x) = \frac{d}{dx}.\cos 2x = -2.\sin 2x$

$$\therefore [f''(x)]_{\text{at } x=\frac{\pi}{4}} = -2\sin 2.\frac{\pi}{4} = -2\sin\frac{\pi}{2} = -2 < 0$$

At $\frac{\pi}{4}$ is maximum and $\frac{\pi}{4}$ is point of maxima.

$$f\left(\frac{\pi}{4}\right) = \frac{1}{2}.\sin .2.\frac{\pi}{4} = \frac{1}{2}$$

**Q. 22. At $x = \frac{5\pi}{6}$, $f(x) = 2\sin 3x + 3\cos 3x$ is :**

**(a) maximum**

**(b) minimum**

**(c) zero**

**(d) neither maximum nor minimum**

**[NCERT Exemp. Ex. 6.3, Q. 56, Page 141]**

**Ans. Correct option :** (d)

***Explanation*** : Given that,

$$f(x) = 2\sin 3x + 3\cos 3x$$

$$\therefore \quad f'(x) = 2.\cos 3x.3 + 3(-\sin 3x).3$$

$$f'(x) = 6\cos 3x - 9\sin 3x$$

Now, $\quad f''(x) = -18\sin 3x - 27\cos 3x$

$$= -9(2\sin 3x + 3\cos 3x)$$

$$\therefore \quad f'\left(\frac{5\pi}{6}\right) = 6\cos\left(3.\frac{5\pi}{6}\right) - 9\sin\left(3.\frac{5\pi}{6}\right)$$
$$= 6\cos\frac{5\pi}{2} - 9\sin\frac{5\pi}{2}$$
$$= 6\cos\left(2\pi + \frac{\pi}{2}\right) - 9\sin\left(2\pi + \frac{\pi}{2}\right)$$
$$= 0 - 9$$
$$\neq 0$$

So, $x = \frac{5\pi}{6}$ cannot be point of maxima or minima.

Hence, $f(x)$ at $x = \frac{5\pi}{6}$ is neither maximum nor minimum.

**Q. 23. Maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is :**

**(a)** 0 **(b)** 12

**(c)** 16 **(d)** 32

**[NCERT Exemp. Ex. 6.3, Q. 57, Page 141]**

**Ans. Correct option :** (b)

*Explanation* : Given that,

$$y = -x^3 + 3x^2 + 9x - 27$$
$$\therefore \quad \frac{dy}{dx} = -3x^2 + 6x + 9 = \text{Slope of the curve}$$
$$\text{and} \quad \frac{d^2y}{dx^2} = -6x + 6 = -6(x-1)$$
$$\therefore \quad \frac{d^2y}{dx^2} = 0$$
$$\Rightarrow -6(x-1) = 0$$
$$\Rightarrow \quad x = 1 > 0$$
$$\text{Now,} \quad \frac{d^3y}{dx^3} = -6 < 0$$

So, the maximum slope of given curve is at $x = 1$.

$$\therefore \left(\frac{dy}{dx}\right)_{(x=1)} = -3.1^2 + 6.1 + 9 = 12$$

**Q. 24. $f(x) = x^x$ has a stationary point at**

**(a)** $x = e$ **(b)** $x = \frac{1}{e}$

**(c)** $x = 1$ **(d)** $x = \sqrt{e}$

**[NCERT Exemp. Ex. 6.3, Q. 58, Page 141]**

**Ans. Correct option :** (b)

*Explanation* : We have,

$$f(x) = x^x$$

Let $y = x^x$ and $\log y = x.\log x$

$$\therefore \quad \frac{1}{y}.\frac{dy}{dx} = x.\frac{1}{x} + \log x.1$$
$$\Rightarrow \quad \frac{dy}{dx} = (1 + \log x).x^x$$
$$\therefore \quad \frac{dy}{dx} = 0$$
$$\Rightarrow (1 + \log x).x^x = 0$$
$$\Rightarrow \quad \log x = \log e^{-1}$$
$$\Rightarrow \quad x = e^{-1}$$
$$\Rightarrow \quad x = \frac{1}{e}$$

Hence, $f(x)$ has a stationary point at $x = \frac{1}{e}$.

**Q. 25. The maximum value of $\left(\frac{1}{x}\right)^x$ is :**

**(a)** $e$ **(b)** $e^e$

**(c)** $e^{1/e}$ **(d)** $\left(\frac{1}{e}\right)^{1/e}$

**[NCERT Exemp. Ex. 6.3, Q. 59, Page 141]**

**Ans.** Correct option : (c)

*Explanation* :

$$\text{Let} \quad y = \left(\frac{1}{x}\right)^x$$
$$\Rightarrow \quad \log y = x.\log\frac{1}{x}$$
$$\therefore \quad \frac{1}{y}.\frac{dy}{dx} = x.\frac{1}{\frac{1}{x}}.\left(-\frac{1}{x^2}\right) + \log\frac{1}{x}.1$$
$$= -1 + \log\frac{1}{x}$$
$$\therefore \quad \frac{dy}{dx} = \left(\log\frac{1}{x} - 1\right).\left(\frac{1}{x}\right)^x$$
$$\text{Now,} \ \frac{dy}{dx} = 0$$
$$\Rightarrow \quad \log\frac{1}{x} = 1 = \log e$$
$$\Rightarrow \quad \frac{1}{x} = e$$
$$\Rightarrow \quad x = \frac{1}{e}$$

Hence, the maximum value of $f\left(\frac{1}{e}\right) = (e)^{1/e}$.

**Q. 26. Choose the correct option.**

**The rate of change of the area of a circle with respect to its radius *r*, at *r* = 6 cm is**

**(a)** $10\pi$ **(b)** $12\pi$

**(c)** $8\pi$ **(d)** $11\pi$

**[NCERT Ex. 6.1, Q. 17, Page 198]**

**Ans. Correct option :** (b)

*Explanation* : The area of a circle ($A$) with radius ($r$) is given by, $A = \pi r^2$

Therefore, the rate of change of the area with respect to its radius, $r$ is $\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$

$\therefore$ When $r$ = 6 cm, $\frac{dA}{dr} = 2\pi \times 6 = 12\pi \text{ cm}^2/\text{s}$

Hence, the required rate of change of the area of a circle is $12\pi$ cm /s.

**Q. 27. The total revenue in Rupees received from the sale of *x* units of a product is given by $R(x) = 3x^2 + 36x + 5$. The marginal revenue, when *x* = 15 is**

**(a)** 116 **(b)** 96

**(c)** 90 **(d)** 126

**[NCERT Ex. 6.1, Q. 18, Page 199]**

**Ans. Correct option :** (d)

***Explanation*** : Marginal revenue ($MR$) is the rate of change of total revenue with respect to the number of units sold. Therefore,

$MR = \dfrac{dR}{dx} = 3(2x) + 36 = 6x + 36$

$\therefore$ When $x = 15$,

$MR = 6(15) + 36 = 90 + 36 = 126$

Hence, the required marginal revenue is ₹ 126.

**Q. 28. Which of the following functions are decreasing on $\left(0, \frac{\pi}{2}\right)$**

**(a) $\cos x$** **(b) $\cos 2x$**

**(c) $\cos 3x$** **(d) $\tan x$**

**[NCERT Ex. 6.2, Q. 12, Page 206]**

**Ans. Correct options :** (a) and (b)

***Explanation*** :

**(a)** Let $f_1(x) = \cos x$

$\therefore f_1'(x) = -\sin x$

In interval $\left(0, \frac{\pi}{2}\right)$, $f_1'(x) = -\sin x < 0$

$\therefore f_1(x) = \cos x$ is strictly decreasing in interval $\left(0, \frac{\pi}{2}\right)$.

**(b)** Let $f_2(x) = \cos 2x$.

$\therefore f_2'(x) = -2\sin 2x$.

Now, $0 < x < \frac{\pi}{2}$

$\Rightarrow 0 < 2x < \pi$

$\Rightarrow \sin 2x > 0$

$\Rightarrow -2\sin 2x < 0$

$\therefore f_2'(x) = -2\sin 2x < 0$ on $\left(0, \frac{\pi}{2}\right)$

$\therefore f_2(x) = \cos 2x$ is strictly decreasing at interval $\left(0, \frac{\pi}{2}\right)$

**(c)** Let $f_3(x) = \cos 3x$

$\therefore f_3'(x) = -3\sin 3x$

Now, $f_3'(x) = 0$

$\Rightarrow \sin 3x = 0$

$\Rightarrow 3x = \pi$

As $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow x = \frac{\pi}{3}$

The point $x = \frac{\pi}{3}$ divides the interval $x = \frac{\pi}{3}$ into two disjoint intervals, *i.e.*, $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$.

Now, in interval $\left(0, \frac{\pi}{3}\right)$, $f_3(x) = -3\sin 3x < 0$

$\left[\because 0 < x < \frac{\pi}{3} \Rightarrow 0 < 3x < \pi\right]$

$f_3$ is strictly decreasing at interval $\left(0, \frac{\pi}{3}\right)$.

However, in interval $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, $f_3(x) = -3\sin 3x > 0$

$\left[\because \frac{\pi}{3} < x < \frac{\pi}{2} \Rightarrow \pi < 3x < \frac{3\pi}{2}\right]$

$\therefore f$ is strictly increasing at interval $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$.

Hence, $f$ is neither increasing nor decreasing at interval $\left(0, \frac{\pi}{2}\right)$.

**(d)** Let $f_4(x) = \tan x$.

$\therefore f_4'(x) = \sec 2x$

In interval $\left(0, \frac{\pi}{2}\right)$, $f_4'(x) = \sec^2 x > 0$

$\therefore f$ is strictly increasing at interval $\left(0, \frac{\pi}{2}\right)$

Therefore, functions $\cos x$ and $\cos 2x$ are strictly decreasing at interval $\left(0, \frac{\pi}{2}\right)$

**Q. 29. On which of the following intervals is the function $f$ given by $f(x) = x^{100} + \sin x - 1$ decreasing?**

**(a) (0, 1)** **(b) $\left(\frac{\pi}{2}, p\right)$**

**(c) $\left(0, \frac{\pi}{2}\right)$** **(d) None of these**

**[NCERT Ex. 6.2, Q. 13, Page 206]**

**Ans. Correct option :** (d)

***Explanation*** : We have,

$f(x) = x^{100} + \sin x - 1$

$\therefore f'(x) = 100x^{99} + \cos x$

In interval (0, 1), $\cos x > 0$ and $100x^{99} > 0$.

$\therefore f'(x) > 0$

Thus, function f is strictly increasing in interval (0, 1).

In interval $\left(\frac{\pi}{2}, \pi\right)$, $\cos x < 0$ and $100x^{99} > 0$

Also, $100x^{99} > \cos x$.

$\therefore f'(x) > 0$ in $\left(\frac{\pi}{2}, \pi\right)$

This function $f$ is strictly increasing in interval $\left(\frac{\pi}{2}, \pi\right)$

In interval $\left(0, \frac{\pi}{2}\right)$, $\cos x > 0$ and $100x^{99} > 0$.

Also, $100x^{99} + \cos x > 0$.

$\Rightarrow f'(x) > 0$ in $\left(0, \frac{\pi}{2}\right)$

$\therefore f$ is strictly increasing at interval $\left(0, \frac{\pi}{2}\right)$.

Hence, function $f$ is strictly decreasing in none of the intervals.

**Q. 30. The interval in which $y = x^2 e^{-x}$ is increasing is**

**(a) $(-\infty, \infty)$** **(b) (–2, 0)**

**(c) $(2, \infty)$** **(d) (0, 2)**

**[NCERT Ex. 6.2, Q. 19, Page 206]**

**Ans. Correct option :** (d)

***Explanation*** : Given that,

$y = x^2 e^{-x}$

$\therefore \quad \frac{dy}{dx} = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2-x)$

Now, $\frac{dy}{dx} = 0$

$\Rightarrow \quad x = 0$ and $x=2$

The points $x = 0$ and $x = 2$ divide the real line into three disjoint intervals *i.e.*, $(-\infty, 0)$, $(0, 2)$ and $(2, \infty)$.

In intervals $(-\infty, 0)$ and $(2, \infty)$, $f'(x)<0$ as $e^{-x}$ is always positive.

$\therefore f$ is decreasing on $(-\infty, 0)$ and $(2, \infty)$

In interval $(0, 2)$, $f'(x)>0$

$\therefore f$ is strictly increasing on $(0, 2)$.

Hence, $f$ is strictly increasing in interval $(0, 2)$.

**Q. 31. Choose the correct answer.**
**The slope of the normal to the curve $y = 2x^2 + 3 \sin x$ at $x = 0$ is**

**(a)** 3 **(b)** $\frac{1}{3}$

**(c)** $-3$ **(d)** $-\frac{1}{3}$

**[NCERT Ex. 6.3, Q. 26, Page 213]**

**Ans. Correct option :** (d)

***Explanation*** : The equation of the given curve is $y=2x^2+3\sin x$.

Slope of the tangent to the given curve at $x = 0$ is given by,

$$\left.\frac{dy}{dx}\right]_{\text{at } x=0} = 4x + 3\cos x]_{x=0} = 0 + 3\cos 0 = 3$$

Hence, the slope of the normal to the given curve at $x = 0$ is

$$\frac{-1}{\text{Slope of the tangent at } x=0} = \frac{-1}{3}$$

**Q. 32. The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point**

**(a)** (1, 2) **(b)** (2, 1)

**(c)** (1, –2) **(d)** (–1, 2)

**[NCERT Ex. 6.3, Q. 27, Page 213]**

**Ans. Correct option :** (a)

***Explanation*** : The equation of the given curve is $y^2=4x$

Differentiating with respect to $x$, we have

$$2y\frac{dy}{dx} = 4$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{y}$$

Therefore, the slope of the tangent to the given curve at any point $(x, y)$ is given by

$$\frac{dy}{dx} = \frac{2}{y}$$

The given line is $y = x + 1$ (which is of the form, $y = mx + c$)

$\therefore$ Slope of the line = 1

The line $y = x + 1$ is a tangent to the given curve if the slope of the line is equal to the slope of the tangent. Also, the line must intersect the curve.

Thus, we must have

$$\frac{2}{y} = 1$$

$\Rightarrow \quad y = 2$

Now, $y = x + 1$

$\Rightarrow \quad x = y - 1$

$\Rightarrow \quad x = 2 - 1 = 1$

Hence, the line $y = x + 1$ is a tangent to the given curve at the point (1, 2).

**Q. 33. If $f(x) = 3x^2 + 15x + 5$, then the approximate value of $f(3.02)$ is**

**(a)** 47.66 **(b)** 57.66

**(c)** 67.66 **(d)** 77.66

**[NCERT Ex. 6.4, Q. 8, Page 216]**

**Ans. Correct option :** (d)

***Explanation*** : Let $x = 3$ and $\Delta x = 0.02$. Then, we have

$$f(3.02) = f(x + \Delta x) = 3(x + \Delta x)^2 + 15(x + \Delta x) + 5$$

Now, $\quad \Delta y = f(x + \Delta x) - f(x)$

$\Rightarrow f(x + \Delta x) = f(x) + \Delta y$

$\approx f(x) + f'(x)\Delta x \qquad (\text{As } dx = \Delta x)$

$\Rightarrow \quad f(3.02) \approx (3x^2 + 15x + 5) + (6x + 15)\Delta x$

$= [3(3)^2 + 15(3) + 5] + [6(3) + 15](0.02)$

[As $x = 3,\ \Delta x = 0.02$ ]

$= (27 + 45 + 5) + (18 + 15)(0.02)$

$= 77 + (33)(0.02)$

$= 77 + 0.66$

$= 77.66$

Therefore, the approximate value of f(3.02) is 77.66.

**Q. 34. The approximate change in the volume of a cube of side $x$ metres caused by increasing the side by 3% is**

**(a)** $0.06\, x^3\, m^3$ **(b)** $0.6\, x^3\, m^3$

**(c)** $0.09\, x^3\, m^3$ **(d)** $0.9\, x^3\, m^3$

**[NCERT Ex. 6.4, Q. 9, Page 216]**

**Ans. Correct option :** (c)

***Explanation*** : The volume of a cube ($V$) of side $x$ is given by $V = x^3$.

$$\therefore dV = \left(\frac{dV}{dx}\right)\Delta x$$

$= (3x^2)\Delta x$

$= (3x^2)(0.03x) \qquad$ [As 3% of $x$ is $0.03x$]

$= 0.09x^3\ \text{m}^3$

Hence, the approximate change in the volume of the cube is $0.09x^3\ m^3$.

**Q. 35. Choose the correct answer.**
**The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is**

**(a)** $(2\sqrt{2}, 4)$ **(b)** $(2\sqrt{2}, 0)$

**(c)** (0, 0) **(d)** (2, 2)

**[NCERT Ex. 6.5, Q. 27, Page 234]**

**Ans. Correct option :** (a)

***Explanation*** : The given curve is $x^2 = 2y$.

For each value of $x$, the position of the point will be $\left(x, \frac{x^2}{2}\right)$.

Let $P\left(x,\dfrac{x^2}{2}\right)$ and $A(0, 5)$ are the given points.

Now distance between the points $P$ and $A$ is given by,

$$PA=\sqrt{(x-0)^2++\left(\frac{x^2}{2}-5\right)^2}$$

$$\Rightarrow PA^2=(x-0)^2+\left(\frac{x^2}{2}-5\right)^2$$

$$\Rightarrow PA^2=x^2+\frac{x^4}{4}+25-5x^2$$

$$\Rightarrow PA^2=\frac{x^4}{4}-4x^2+25$$

$$\Rightarrow PA^2=y^2-8y+25 \qquad (\text{As, } x^2=2y)$$

Let us denote $PA^2$ by $Z$. Then,

$Z=y^2-8y+25$

Differentiating both sides with respect to $y$, we get

$$\frac{dZ}{dy}=2y-8$$

For maxima or minima, we have

$$\frac{dZ}{dy}=0$$

$$\Rightarrow\ 2y-8=0$$

$$\Rightarrow\ y=4$$

$$\frac{d^2Z}{dy^2}=2$$

Now,

$$\left[\frac{d^2Z}{dy^2}\right]_{y=4}=2>0$$

Now, $x^2=2y$

$$\Rightarrow\ x^2=2\times 4$$

$$\Rightarrow\ x^2=8$$

$$\Rightarrow\ x=2\sqrt{2} \text{ or } x=-2\sqrt{2}$$

So, $Z$ is minimum at $(2\sqrt{2}, 4)$ or $(-2\sqrt{2}, 4)$.

Or, $PA^2$ is minimum at $(2\sqrt{2}, 4)$ or $(-2\sqrt{2}, 4)$.

So, distance between the points $P\left(x, \dfrac{x^2}{2}\right)$ and $A(0, 5)$ is minimum at $(2\sqrt{2}, 4)$ or $(-2\sqrt{2}, 4)$.

**Q. 36. For all real values of $x$, the minimum value of $\left|\dfrac{1-x+x^2}{1+x+x^2}\right|$ is**

**(a)** 0 **(b) 1**

**(c)** 3 **(d)** $\dfrac{1}{3}$

**[NCERT Ex. 6.5, Q. 28, Page 234]**

**Ans. Correct option :** (d)

***Explanation*** : Let we assume that $f(x)=\left|\dfrac{1-x+x^2}{1+x+x^2}\right|$.

On differentiating with respect to $x$, we have

$$\therefore f'(x)=\frac{(1+x+x^2)(-1+2x)-(1-x+x^2)(1+2x)}{(\ +x+x\ )}$$

$$=\frac{-1+2x-x+2x^2-x^2+2x^3-1-2x+x+2x^2-x^2-2x^3}{(1+x+x^2)}$$

$$=\frac{2x^2-2}{(1+x+x^2)}$$

$$=\frac{2(x^2-1)}{(1+x+x^2)}$$

$$\therefore\ f'(x)=0$$

$$\Rightarrow\ x^2=1$$

$$\Rightarrow\ x=\pm 1$$

Now, $$f''(x)=\frac{2\left[\begin{array}{l}(1+x+x^2)^2(2x)-(x^2-1)(2)\\(1+x+x^2)(1+2x)\end{array}\right]}{(1+x+x^2)^4}$$

$$=\frac{4(1+x+x^2)\left[\begin{array}{l}(1+x+x^2)x-\\(x^2-1)(1+2x)\end{array}\right]}{(1+x+x^2)^4}$$

$$=\frac{4[x+x^2+x^3-x^2-2x^3+1+2x]}{(1+x+x^2)^3}$$

$$=\frac{4(1+3x-x^3)}{(1+x+x^2)^3}$$

And, $$f''(1)=\frac{4(1+3-1)}{(1+1+1)^3}=\frac{12}{27}=\frac{4}{9}>0$$

Also, $$f''(-1)=\frac{4(1-3+1)}{(1-1+1)^3}=4(-1)=-4<0$$

$\therefore$ By second derivative test, $f$ is the minimum at $x = 1$ and the minimum value is given by

$$f(1)=\frac{4(1-1+1)}{(1+1+1)}=\frac{1}{3}.$$

**Q. 37. The maximum value of $[x(x-1)+1]^{1/3}, 0\le x\le 1$ is**

**(a)** $\left(\dfrac{1}{3}\right)^{1/3}$ **(b)** $\left(\dfrac{1}{2}\right)$

**(c) 1** **(d) 0**

**[NCERT Ex. 6.5, Q. 29, Page 234]**

**Ans. Correct option :** (c)

***Explanation*** : Let we assume that

$$f(x)=[x(x-1)+1]^{1/3}$$

$$\therefore\ f'(x)=\frac{2x-1}{3[x(x-1)+1]^{2/3}}$$

Now, $f'(x)=0$

$$\therefore\ x=\frac{1}{2}$$

Then, we evaluate the value of f at critical point $x=\dfrac{1}{2}$ and at the end points of the interval [0, 1] {*i.e.*, at $x = 0$ and $x = 1$}.

$$f(0)=[0(0-1)+1]^{1/3}=1$$

$$f(1)=[1(1-1)+1]^{1/3}=1$$

$$f\left(\frac{1}{2}\right)=\left[\frac{1}{2}\left(\frac{-1}{2}\right)+1\right]^{1/3}=\left(\frac{3}{4}\right)^{1/3}$$

Hence, we can conclude that the maximum value of f in the interval [0, 1] is 1.

**Q. 38. Choose the correct answer.**
**A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Then the depth of the wheat is increasing at the rate of**

**(a) 1 m/h** **(b) 0.1 m/h**
**(c) 1.1 m/h** **(d) 0.5 m/h**

**[NCERT Misc Ex. Q. 19, Page 243]**

**Ans. Correct option :** (a)

***Explanation*** : Let r be the radius of the cylinder.
Then, volume ($V$) of the cylinder is given by,

$V = \pi(\text{radius})^2 \times \text{height}$
$= \pi(10^2)\, h$

Where $r = 10$ m

$V = 100\pi h$

Differentiating with respect to time $t$, we have

$$\frac{dV}{dt} = 100\pi\frac{dh}{dt}$$

The tank is being filled with wheat at the rate of 314 cubic metres per hour.
Thus, we have

$$\therefore \frac{dV}{dt} = 314 \text{ m}^3/\text{h}$$

$$\therefore 100\pi\frac{dh}{dt} = 314$$

$$\Rightarrow \frac{dh}{dt} = \frac{314}{100(3.14)} = \frac{314}{314} = 1$$

Hence, the depth of wheat is increasing at the rate of 1 m/h.

**Q. 39. The slope of the tangent to the curve $x = t^2 + 3t - 8,\ y = 2t^2 - 2t - 5$ at the point (2, –1) is**

**(a) $\frac{22}{7}$** **(b) $\frac{6}{7}$**
**(c) $\frac{7}{6}$** **(d) $\frac{-6}{7}$**

**[NCERT Misc Ex. Q. 20, Page 243]**

**Ans. Correct option :** (b)

***Explanation*** : The given curve is
$x = t^2 + 3t - 8$ and $y = 2t^2 - 2t - 5$

$$\therefore \frac{dx}{dt} = 2t + 3 \text{ and } \frac{dy}{dt} = 4t - 2$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{4t-2}{2t+3}$$

The given point is (2, −1).
At $x = 2$, we have :

$$t^2 + 3t = 8 = 2$$
$$\Rightarrow t^2 + 3t - 10 = 0$$
$$\Rightarrow (t-2)(t+5) = 0$$
$$\Rightarrow t = 2 \text{ or } t = -5$$

At $y = -1$, we have

$$\Rightarrow 2t^2 - 2t - 5 = -1$$
$$\Rightarrow 2t^2 - 2t - 4 = 0$$
$$\Rightarrow 2(t^2 - t - 2) = 0$$
$$\Rightarrow (t-2)(t+1) = 0$$
$$\Rightarrow t = 2 \text{ or } t = -1$$

The common value of $t$ is 2.

Hence, the slope of the tangent to the given curve at point (2, −1) is $\left.\frac{dy}{dx}\right|_{t=2} = \frac{4(2)-2}{2(2)+3} = \frac{6}{7}$.

**Q. 40. The line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$ if the value of m is**

**(a)** 1 **(b)** 2
**(c)** 3 **(d)** $\frac{1}{2}$

**[NCERT Misc Ex. Q. 21, Page 244]**

**Ans. Correct option :** (a)

***Explanation*** : The equation of the tangent to the given curve is $y = mx + 1$ .
Now, substituting $y = mx + 1$ in $y = 4x$ , we get :

$$\Rightarrow (mx+1)^2 = 4x$$
$$\Rightarrow m^2x^2 + 1 + 2mx - 4x = 0$$
$$\Rightarrow m^2x^2 + x(2m-4) + 1 = 0$$

Since a tangent touches the curve at one point, the roots of equation (i) must be equal.
Therefore, we have
Discriminant $= 0$

$$(2m-4)^2 - 4(m^2)(1) = 0$$
$$\Rightarrow 4m^2 + 16 - 16m - 4m^2 = 0$$
$$\Rightarrow 16 - 16m = 0$$
$$\Rightarrow m = 1$$

Hence, the required value of $m$ is 1.

**Q. 41. The normal at the point (1, 1) on the curve $2y + x^2 = 3$ is**

**(a) $x + y = 0$** **(b) $x - y = 0$**
**(c) $x + y + 1 = 0$** **(d) $x - y = 1$**

**[NCERT Misc Ex. Q. 22, Page 244]**

**Ans. Correct option :** (b)

***Explanation*** : The equation of the given curve is $2y + x^2 = 3$ .
Differentiating with respect to $x$, we have

$$\frac{2dy}{dx} + 2x = 0$$
$$\Rightarrow \frac{dy}{dx} = -x$$
$$\therefore \left.\frac{dy}{dx}\right]_{(1,1)} = -1$$

The slope of the normal to the given curve at point (1, 1) is

$$\frac{1}{\left.\frac{dy}{dx}\right]_{(1,1)}} = 1$$

Hence, the equation of the normal to the given curve at (1, 1) is given as :

$$\Rightarrow y - 1 = 1(x-1)$$
$$\Rightarrow y - 1 = x - 1$$
$$\Rightarrow x - y = 0$$

**Q. 42. The normal to the curve $x^2 = 4y$ passing (1, 2) is**

**(a) $x + y = 3$** **(b) $x - y = 3$**
**(c) $x + y = 1$** **(d) $x - y = 1$**

**[NCERT Misc Ex. Q. 23, Page 244]**

**Ans. Correct option :** (a)

***Explanation*** : The equation of the given curve is $x^2 = 4y$.

Differentiating with respect to $x$, we have

$$2x = 4.\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{2}$$

The slope of the normal to the given curve at point $(h, k)$ is given by,

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(h,k)}} = -\frac{2}{h}$$

$\therefore$Equation of the normal at point $(h, k)$ is given as :

$$y - k = \frac{-2}{h}(x - h)$$

Now, it is given that the normal passes through the point (1, 2).

Therefore, we have

$$2 - k = \frac{-2}{h}(1-h) \text{ or } k = 2 + \frac{2}{h}(1-h) \quad ...(i)$$

Since $(h, k)$ lies on the curve $x = 4y$, we have $h^2 = 4k$.

$$\Rightarrow k = \frac{h^2}{4}$$

From equation (i), we have

$$\frac{h^2}{4} = 2 + \frac{2}{h}(1-h)$$

$$\Rightarrow \frac{h^3}{4} = 2h + 2 - 2h = 2$$

$$\Rightarrow h^3 = 8$$

$$\Rightarrow h = 2$$

$$\therefore \quad k = \frac{h^3}{4}$$

$$\Rightarrow k = 1$$

Hence, the equation of the normal is given as :

$$\Rightarrow y - 1 = \frac{-2}{2}(x-2)$$

$$\Rightarrow y - 1 = -(x-2)$$

$$\Rightarrow x + y = 3$$

**Q. 43. The points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes are**

**(a)** $\left(4, \pm\frac{8}{3}\right)$ **(b)** $\left(4, -\frac{8}{3}\right)$

**(c)** $\left(4, \pm\frac{3}{8}\right)$ **(d)** $\left(\pm 4, \frac{3}{8}\right)$

**[NCERT Misc Ex. Q. 24, Page 244]**

**Ans. Correct option :** (a)

***Explanation*** : The equation of the given curve is $9y^2 = x^3$.

Differentiating with respect to $x$, we have

$$9(2y)\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{6y}$$

The slope of the normal to the given curve at point $(x_1, y_1)$ is $\frac{-1}{\left.\frac{dy}{dx}\right]_{(x_1, y_1)}} = -\frac{6y_1}{{x_1}^2}$,

Therefore, the equation of the normal to the curve at $(x_1, y_1)$ is

$$y - y_1 = -\frac{6y_1}{{x_1}^2}(x - x_1)$$

$$\Rightarrow {x_1}^2 y - {x_1}^2 y_1 = -6xy_1 + 6x_1y_1$$

$$\Rightarrow 6xy_1 + {x_1}^2 y = 6x_1y_1 + {x_1}^2 y_1$$

$$\Rightarrow \frac{6xy_1}{6x_1y_1 + {x_1}^2 y_1} + \frac{{x_1}^2 y}{6x_1y_1 + {x_1}^2 y_1} = 1$$

$$\Rightarrow \frac{x}{\frac{x_1(6+x_1)}{6}} + \frac{y}{\frac{y_1(6+x_1)}{x_1}} = 1$$

It is given that the normal makes equal intercepts with the axes.

Therefore, we have

$$\therefore \frac{x_1(6+x_1)}{6} = \frac{y_1(6+x_1)}{x_1}$$

$$\Rightarrow \frac{x_1}{6} = \frac{y_1}{x_1}$$

$$\Rightarrow {x_1}^2 = 6y_1 \quad ...(i)$$

Also, the point $(x_1, y_1)$ lies on the curve, so we have

$$9{y_1}^2 = {x_1}^3 \quad ...(ii)$$

From equations (i) and (ii), we have

$$9\left(\frac{{x_1}^2}{6}\right)^2 = {x_1}^3$$

$$\Rightarrow \frac{{x_1}^4}{4} = {x_1}^3$$

$$\Rightarrow x_1 = 4$$

From equation (ii), we have

$$9{y_1}^2 = (4)^3 = 64$$

$$\Rightarrow {y_1}^2 = \frac{64}{9}$$

$$\Rightarrow y_1 = \pm\frac{8}{3}$$

Hence, the required points are $\left(4, \pm\frac{8}{3}\right)$.

## Very Short Answer Type Questions **(1 or 2 marks each)**

**Q. 1. The curves $y = 4x^2 + 2x - 8$ and $y = x^3 - x + 13$ touch each other at the point_____.**

**[NCERT Exemp. Ex. 6.3, Q. 60, Page 142]**

**Ans.** Given that, equation of the curve is given by $y = 4x^2 + 2x - 8$ and $y = x^3 - x + 13$.

$$\therefore \quad \frac{dy}{dx} = 8x + 2$$

$$\text{and } \frac{dy}{dx} = 3x^2 - 1$$

So, the slope of both curves should be same.

$\therefore \quad 8x+2=3x^2-1$

$\Rightarrow \quad x^2-8x-3=0$

$\Rightarrow \quad 3x^2-9x+x-3=0$

$\Rightarrow \quad 3x(x-3)+1(x-3)=0$

$\Rightarrow \quad (3x+1)(x-3)=0$

$\therefore \quad x=-\frac{1}{3}$ and $x=3$

For $\quad x=-\frac{1}{3},$

$$y=4.\left(-\frac{1}{3}\right)^2+2.\left(-\frac{1}{3}\right)-8$$

$$=\frac{4}{9}-\frac{2}{3}-8$$

$$=\frac{4-6-72}{9}$$

$$=-\frac{74}{9}$$

and also for $x=3,\ y=\ 4.(3)^2+2.(3)-8$

$$=36+6-8$$

$$=34$$

So, the required points are (3, 34) and $\left(-\frac{1}{3},-\frac{74}{9}\right)$.

**[2]**

**Q. 2. The equation of normal to the curve $y=\tan x$ at (0, 0) is ________.**

**[NCERT Exemp. Ex. 6.3, Q. 61, Page 141]**

**Ans.** Given that the equation of the curve $y=\tan x$ at $(0,\ 0)$ is $x+y=0$ .

$\because \quad y=\tan x$

$\Rightarrow \quad \frac{dy}{dx}=\sec^2 x$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(0,0)}=\sec^2 0=1$

and $-\frac{1}{\left(\frac{dy}{dx}\right)}=-\frac{1}{1}=-1$

$\therefore$ Equation of normal to the curve $y=\tan x$ at (0, 0) is

$y-0=-1(x-0)$

$\Rightarrow y+x=0$ **[2]**

**Q. 3. The values of a for which the function $f(x)=\sin x-ax+b$ increases on $R$ are ______.**

**[NCERT Exemp. Ex. 6.3, Q. 62, Page 141]**

**Ans.** The value of '*a*' for which the function $f(x)=\sin x-ax+b$ increases on R are $(-\infty,\ -1)$.

$f'(x)=\cos x-a$

and $\quad f'(x)$

$\Rightarrow \quad \cos x>a$

Since, cos $\in[-1,\ 1]$

$\Rightarrow \quad <-$

$\Rightarrow \quad \in(-\infty,\ -\ )$ **[1]**

**Q. 4. The function $f(x)=\frac{2x^2-1}{x^4}, x>0.$ decreases in the interval ______.**

**[NCERT Exemp. Ex. 6.3, Q. 63, Page 141]**

**Ans.** The function $f(x)=\frac{2x^2-1}{x^4}, x>0.,$ when $x>0$, decreases in the interval $(1,\infty)$.

$$f'(x)=\frac{x^4.4x-(2x^2-1).4x^3}{x^8}$$

$$=\frac{4x^5-8x^5+4x^3}{x^8}$$

$$=\frac{-4x^5+4x^3}{x^6}=\frac{4x^3(-x^2+1)}{x^6}$$

Also, $\quad f'(x)=0$

$\Rightarrow \frac{4x^3(-x^2+1)}{x^6}<0$

$\Rightarrow \quad x^2>1$

$\Rightarrow \quad x>\pm 1$

$\therefore \quad x\in(1,\infty)$ **[2]**

**Q. 5. The least value of the function $f(x)=ax+\frac{b}{x}(a>0,b>0,x>0)$ is.........**

**[NCERT Exemp. Ex. 6.3, Q. 64, Page 141]**

**Ans.** The least value of function

$f(x)=ax+\frac{b}{x}(a>0,b>0,x>0)$ is $2\sqrt{ab}$ .

$$f'(x)=a-\frac{b}{x^2} \text{ and } f'(x)=0$$

$\Rightarrow \quad a=\frac{b}{x^2}$

$\Rightarrow \quad x^2=\frac{b}{a}$

$\Rightarrow \quad x=\pm\sqrt{\frac{b}{a}}$

Now, $f''(x)=-b.\frac{(-2)}{x^3}=+\frac{2b}{x^3}$

*At* $\quad x=+\sqrt{\frac{b}{a}}$ ,

$$f''(x)=+\frac{(-2)}{\left(\frac{b}{a}\right)^{3/2}}=\frac{+2b.a^{3/2}}{b^{3/2}}$$

$$=+2b^{-1/2}.a^{3/2}=+2\sqrt{\frac{a^3}{b}}>0\ [\because a,\ b>0]$$

Least value of $f(x)$,

$$f\left(\sqrt{\frac{b}{a}}\right)=a.\sqrt{\frac{b}{a}}+\frac{b}{\sqrt{\frac{b}{a}}}$$

$$=a.a^{-1/2}.b^{1/2}+b.b^{-1/2}.a^{1/2}$$

$$=\sqrt{ab}+\sqrt{ab}$$

$$=2\sqrt{ab}$$

**[2]**

**Q. 6. Find the rate of change of the area of a circle with respect to its radius *r* when**

**(a) *r* = 3 cm** **(b) *r* = 4 cm**

**[NCERT Ex. 6.1, Q. 1, Page 197]**

**Ans.** The area of a circle ($A$) with radius ($r$) is given by, $A=\pi r^2$

Now, the rate of change of the area with respect to its radius is given by,

$$\frac{dA}{dr}=\frac{d}{dr}(\pi r^2)=2\pi r$$

**(a)** When r = 3 cm,

$$\frac{dA}{dr}=2\pi(3)=6\pi$$

Hence, the area of the circle is changing at the rate of $6\pi$ cm when its radius is 3 cm. **[1]**

**(b)** When $r$ = 4 cm,

$$\frac{dA}{dr}=2\pi(4)=8\pi$$

Hence, the area of the circle is changing at the rate of $8\pi$ cm when its radius is 4 cm. **[1]**

**Q. 7. The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.**
**[NCERT Ex. 6.1, Q. 3, Page 197]**

**Ans.** The area of a circle ($A$) with radius ($r$) is given by, $A=\pi r^2$

Now, the rate of change of area ($A$) with respect to time ($t$) is given by,

$$\frac{dA}{dt}=\frac{d}{dt}(\pi r^2)=2\pi r\frac{dr}{dt} \qquad \text{[By Chain Rule]}$$

It is given that,

$$\frac{dr}{dt}=3 \text{ cm/s}$$

$$\frac{dA}{dt}=2\pi r(3)=6\pi \text{r}$$

Thus, when $r$ = 10 cm,

$$\frac{dA}{dt}=6\pi(10)=60\pi \text{ cm}^2/\text{s}$$

Hence, the rate at which the area of the circle is increasing when the radius is 10 cm is $60\pi$ cm2/s.
**[2]**

**Q. 8. An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?**
**[NCERT Ex. 6.1, Q. 4, Page 197]**

**Ans.** Let $x$ be the length of a side and $V$ be the volume of the cube. Then,

$$V = x^3$$

$$\frac{dV}{dt}=3x^2.\frac{dx}{dt} \quad \text{(By chain rule)}$$

It is given that,

$$\frac{dx}{dt}=3 \text{ cm/s}$$

$$\frac{dV}{dt}=3x^2.(3)\ 9x^2$$

Thus, when $x$ = 10 cm,

$$\frac{dV}{dt}=9(10)^2=900 \text{ cm}^3/\text{s}$$

Hence, the volume of the cube is increasing at the rate of 900 $\text{cm}^3$/s when the edge is 10 cm long. **[2]**

**Q. 9. A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?**
**[NCERT Ex. 6.1, Q. 5, Page 197]**

**Ans.** The area of a circle ($A$) with radius ($r$) is given by $A=\pi r^2$.

Therefore, the rate of change of area ($A$) with respect to time ($t$) is given by

$$\frac{dA}{dt}=\frac{d}{dt}(\pi r^2)=\frac{d}{dr}(\pi r^2)\frac{dr}{dt}=2\pi r.\frac{dr}{dt} \quad \text{(By chain rule)}$$

It is given that, $\frac{dr}{dt}$ cm/s

Thus, when $r$=8 cm,

$$\frac{dA}{dt}=2\pi(8)(5)=80\pi$$

Hence, when the radius of the circular wave is 8 cm, the enclosed area is increasing at the rate of $80\pi$ $\text{cm}^2$/s. **[2]**

**Q. 10. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference? [NCERT Ex. 6.1, Q. 6, Page 198]**

**Ans.** The circumference of a circle ($C$) with radius ($r$) is given by $C = 2\pi r$. Therefore, the rate of change of circumference ($C$) with respect to time ($t$) is given by,

$$\frac{dC}{dt}=\frac{dC}{dr}.\frac{dr}{dt} \qquad \textbf{[By chain rule]}$$

$$=\frac{d}{dr}(2\pi r)\frac{dr}{dt}$$

$$=2\pi.\frac{dr}{dt}$$

It is given that, $\frac{dr}{dt}=0.7$ cm/s

Hence, the rate of increase of the circumference is $2\pi(0.7)=1.4\pi$ cm/s **[2]**

**Q. 11. The length $x$ of a rectangle is decreasing at the rate of 5 cm/minute and the width $y$ is increasing at the rate of 4 cm/minute. When $x$ = 8 cm and $y$ = 6 cm, find the rates of change of**

**(a) the perimeter, and**

**(b) the area of the rectangle.**

**[NCERT Ex. 6.1, Q. 7, Page 198]**

**Ans.** The length ($x$) is decreasing at the rate of 5 cm/min and the width ($y$) is increasing at the rate of 4 cm/min, we have

$$\frac{dx}{dt}=-5 \text{ cm/min and } \frac{dy}{dt}=4 \text{ cm/min}$$

**(a)** The perimeter ($P$) of a rectangle is given by,

$P = 2(x + y)$

$$\therefore \frac{dP}{dt}=2\left(\frac{dx}{dt}+\frac{dy}{dt}\right)=2(-5+4)=-2 \text{ cm/min}$$

Hence, the perimeter is decreasing at the rate of 2 cm/min. **[1]**

**(b)** The area ($A$) of a rectangle is given by,

$A = xy$

$$\therefore \frac{dA}{dt}=\frac{dx}{dt}.y+x.\frac{dy}{dt}=-5y+4x$$

When $x$ = 8 cm and $y$ = 6 cm,

$$\therefore \frac{dA}{dt}=(-5\times6+4\times8)\ =2 \text{ cm}^2/\text{min}$$

Hence, the area of the rectangle is increasing easing at the rate of 2 $\text{cm}^2$/min. **[1]**

**Q. 12. A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.**
**[NCERT Ex. 6.1, Q. 9, Page 198]**

**Ans.** The volume of a sphere ($V$) with radius ($r$) is given by $V=\frac{4}{3}\pi r^3$.

Rate of change of volume ($V$) with respect to its radius ($r$) is given by,

$$\frac{dV}{dr}=\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)=\frac{4}{3}\pi(3r^2)=4\pi r^2$$

Therefore, when radius = 10 cm,

$$\frac{dV}{dr}=4\pi(10)^2=400\pi$$

Hence, the volume of the balloon is increasing at the rate of $400\pi$ cm$^2$. **[2]**

**Q. 13. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to $x$.**
**[NCERT Ex. 6.1, Q. 13, Page 198]**

**Ans.** The volume of a sphere ($V$) with radius ($r$) is given by, $V=\frac{4}{3}\pi r^3$

It is given that,

Diameter $=\frac{3}{2}(2x+1)$

$$\Rightarrow r=\frac{3}{4}(2x+1)$$

$$\therefore V=\frac{4}{3}\pi\left(\frac{3}{4}\right)^3(2x+1)^3=\frac{9}{16}\pi(2x+1)^3$$

Hence, the rate of change of volume with respect to $x$ is as

$$\frac{dV}{dx}=\frac{9}{16}\pi\frac{d}{dx}(2x+1)^3$$
$$=\frac{9}{16}\pi\times 3(2x+1)^2\times 2$$
$$=\frac{27}{8}\pi(2x+1)^2.$$ **[2]**

**Q. 14. The total revenue in Rupees received from the sale of $x$ units of a product is given by $R(x)=13x^2+26x+15$. Find the marginal revenue when $x = 7$.**
**[NCERT Ex. 6.1, Q. 16, Page 198]**

**Ans.** Marginal revenue ($MR$) is the rate of change of total revenue with respect to the number of units sold. Therefore,

$$MR=\frac{dR}{dx}=13(2x)+26=26x+26$$

When $x = 7$,

$$MR=26(7)+26=182+26=208$$

Hence, the required $MR$ is ₹ 208. **[2]**

**Q. 15. Show that the function given by $f(x)=3x+17$ is strictly increasing on $R$.**
**[NCERT Ex. 6.2, Q. 1, Page 205]**

**Ans.** Let $x_1$ and $x_2$ be any two numbers in $R$.

Then, we have

$$x_1 < x_2$$
$$\Rightarrow 3x_1 < 3x_2$$
$$\Rightarrow 3x_1+17<3x_2+17$$
$$\Rightarrow f(x_1)<f(x_2)$$

Hence, $f$ is strictly increasing on $R$. **[1]**

**Q. 16. Show that the function given by $f(x)=e^{2x}$ is strictly increasing on $R$.**
**[NCERT Ex. 6.2, Q. 2, Page 205]**

**Ans.** Let $x_1$ and $x_2$ be any two numbers in $R$.

Then, we have

$$x_1 < x_2$$
$$\Rightarrow 2x_1 < 2x_2$$
$$\Rightarrow e^{2x_1}<e^{2x_2}$$
$$\Rightarrow f(x_1)<f(x_2)$$

Hence, $f$ is strictly increasing on $R$. **[1]**

**Q. 17. Show that the function given by $f(x)=\sin x$ is**

**(a) strictly increasing in $\left(0,\frac{\pi}{2}\right)$**

**(b) strictly decreasing in $\left(\frac{\pi}{2},\pi\right)$**

**(c) neither increasing nor decreasing in $(0, \pi)$.**
**[NCERT Ex. 6.2, Q. 3, Page 205]**

**Ans.** The given function is $f(x)=\sin x$ ....(i)

On differentiating equation (i), we get

$$f'(x)=\cos x$$

**(a)** Since for each $x\in\left(0,\frac{\pi}{2}\right)$, $\cos x>0$ we have $f'(x)>0$

Hence, f is strictly increasing in $\left(0,\frac{\pi}{2}\right)$ **[1]**

**(b)** Since for each $x\in\left(\frac{\pi}{2},\pi\right)$, $\cos x<0$, we have $f'(x)<0$

Hence, $f$ is strictly decreasing in $\left(\frac{\pi}{2},\pi\right)$ **[1]**

**(c)** From the results obtained in options ($a$) and ($b$), it is clear that f is neither increasing nor decreasing in $(0, \pi)$. **[1]**

**Q. 18. Prove that the logarithmic function is strictly increasing on $(0, \infty)$.**
**[NCERT Ex. 6.2, Q. 10, Page 206]**

**Ans.** The given function is $f(x)=\log x$.

$$\therefore f'(x)=\frac{1}{x}$$

It is clear that for $x > 0$, $\therefore f'(x)=\frac{1}{x}>0$

Hence, $f(x)=\log x$ is strictly increasing in interval $(0, \infty)$. **[1]**

**Q. 19. Prove that the function f given by $f(x)=x^2-x+1$ is neither strictly increasing nor strictly decreasing on $(-1, 1)$.** **[NCERT Ex. 6.2, Q. 11, Page 206]**

**Ans.** The given function is $f(x)=x^2-x+1$.

$$\therefore \quad f'(x)=2x-1$$

Now, $f'(x)=0$

$$\Rightarrow \quad x=\frac{1}{2}$$

The point $\frac{1}{2}$ divides the interval $(-1, 1)$ into two disjoint intervals, *i.e.*, $\left(1,\frac{1}{2}\right)$ and $\left(\frac{1}{2},1\right)$.

Now, in interval $\left(-1,\frac{1}{2}\right)$, $f'(x)=2x-1<0$

Therefore, $f$ is strictly decreasing in interval $\left(-1, \frac{1}{2}\right)$

However, in interval $\left(\frac{1}{2}, 1\right)$, $f'(x) = 2x - 1 > 0$

Therefore, $f$ is strictly increasing in interval $\left(\frac{1}{2}, 1\right)$

Hence, $f$ is neither strictly increasing nor decreasing in interval $(-1, 1)$. **[2]**

**Q. 20. Find the least value of a such that the function $f$ given by $f(x) = x^2 + ax + 1$ is strictly increasing on (1, 2)? [NCERT Ex. 6.2, Q. 14, Page 206]**

**Ans.** We have,

$$f(x) = x^2 + ax + 1$$

$\therefore f'(x) = 2x + a$

Now, function $f$ is strictly increasing in (1, 2).

*i.e.* $f'(x) > 0$

$\therefore$ $2x + a > 0$

$a > -2x$

$\therefore$ We have to find least value of $a$, such that

$x \in (1, 2)$

$\therefore$ $a > -2$

$\Rightarrow$ $a \in (-2, \infty)$

$\therefore$ least value of a is –2 **[2]**

**Q. 21. Prove that the function $f$ given by $f(x) = \log \sin x$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$**

**[NCERT Ex. 6.2, Q. 16, Page 206]**

**Ans.** We have,

$$f(x) = \log \sin x$$

$$f'(x) = \frac{1}{\sin x.} \cos x = \cot x$$

In interval $\left(0, \frac{\pi}{2}\right)$, $f'(x) = \cot x > 0$

Therefore, $f$ is strictly increasing in $\left(0, \frac{\pi}{2}\right)$

In interval $\left(0, \frac{\pi}{2}\right)$, $f'(x) = \cot x > 0$

Therefore, $f$ is strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$ **[2]**

**Q. 22. Prove that the function f given by $f(x) = \log|\cos x|$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$ and strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$.**

**[NCERT Ex. 6.2, Q. 17, Page 206]**

**Ans.** We have,

$$f(x) = \log|\cos x|$$

$$\therefore f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x$$

In interval $\left(0, \frac{\pi}{2}\right)$, $\tan x > 0 \Rightarrow -\tan x < 0$

$\therefore f'(x) < 0$ on $\left(0, \frac{\pi}{2}\right)$

$\therefore$ f is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$

In interval $\left(\frac{\pi}{2}, \pi\right)$, $\tan x < 0 \Rightarrow -\tan x > 0$

$\therefore f'(x) > 0$ on $\left(\frac{\pi}{2}, \pi\right)$

$\therefore f$ is strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$. **[2]**

**Q. 23. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in $R$.**

**[NCERT Ex. 6.2, Q. 18, Page 206]**

**Ans.** Given that,

$$f(x) = x^3 - 3x^2 + 3x - 100$$

$$f'(x) = 3x^2 - 6x + 3$$

$$= 3(x^2 - 2x + 1)$$

$$= 3(x - 1)^2$$

For any $x \in R$, $(x - 1)^2 > 0$.

Thus, $f'(x)$ is always positive in $R$.

Hence, the given function ($f$) is increasing in $R$. **[2]**

**Q. 24. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.**

**[NCERT Ex. 6.3, Q. 1, Page 211]**

**Ans.** The given curve is $y = 3x^4 - 4x$.

Then, the slope of the tangent to the given curve at $x = 4$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=4} = 12x^3 - 4\Big]_{x=4}$$

$$= 12(4)^3 - 4$$

$$= 12(64) - 4 = 764$$ **[1]**

**Q. 25. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.**

**[NCERT Ex. 6.3, Q. 2, Page 211]**

**Ans.** Given that curve is $y = \frac{x-1}{x-2}$

$$\therefore \frac{dy}{dx} = \frac{(x-2)(1) - (x-1)(1)}{(x-2)^2}$$

$$= \frac{x - 2 - x + 1}{(x-2)^2}$$

$$= \frac{-1}{(x-2)^2}$$

Thus, the slope of the tangent at $x = 10$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=10} = \left.\frac{-1}{(x-2)^2}\right]_{x=10}$$

$$= \frac{-1}{(10-2)^2} = \frac{-1}{64}$$

Hence, the slope of the tangent at $x = 10$ is $\frac{-1}{64}$. **[2]**

**Q. 26. Find the slope of the tangent to curve $y = x^3 - x + 1$ at the point whose $x$-coordinate is 2.**

**[NCERT Ex. 6.3, Q. 3, Page 211]**

**Ans.** Given that,

$$y = x^3 - x + 1$$

Differentiating with respect to $x$, we get

$$\therefore \frac{dy}{dx} = 3x^2 - 1$$

The slope of the tangent to a curve at is $(x_0, y_0)$ is $\left.\frac{dy}{dx}\right]_{(x_0,y_0)}$

It is given that $x = 2$.

Hence, the slope of the tangent at the point where the $x$-coordinate is 2 is given by,

$$\therefore \left.\frac{dy}{dx}\right]_{x=2} = 3x^2 - 1\Big]_{x=2}$$
$$= 3(2)^2 - 1$$
$$= 12 - 1 = 11$$

Hence, the slope of the tangent at $x = 2$ is 11. **[2]**

**Q. 27. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose $x$-coordinate is 3. [NCERT Ex. 6.3, Q. 4, Page 211]**

**Ans.** As we know that,

$$y = x^3 - 3x + 2$$

On differentiating with respect to $x$, we get

$$\therefore \frac{dy}{dx} = 3x^2 - 3$$

The slope of the tangent to a curve at $(x_0, y_0)$ is $\left.\frac{dy}{dx}\right]_{(x_0,y_0)}$.

It is given that $x = 3$.

Hence, the slope of the tangent at the point where the $x$-coordinate is 3 is given by,

$$\left.\frac{dy}{dx}\right]_{x=3} = 3x^2 - 3]_{x=3}$$
$$= 3(3)^2 - 3$$
$$= 27 - 3 = 24$$

Hence, the slope of the tangent at $x = 3$ is 24. **[2]**

**Q. 28. Find a point on the curve $y = (x-2)^2$ at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4). [NCERT Ex. 6.3, Q. 8, Page 211]**

**Ans.** We know that if a tangent is parallel to the chord joining the points (2, 0) and (4, 4), then

The slope of the tangent = The slope of the chord

The slope of the chord is,

$$\frac{4-0}{4-2} = \frac{4}{2} = 2.$$

Now, the slope of the tangent to the given curve at a point $(x, y)$ is given by,

$$\frac{dy}{dx} = 2(x-2)$$

Since the slope of the tangent = Slope of the chord, we have

$$2(x-2) = 2$$
$$\Rightarrow (x-2) = 1$$
$$\Rightarrow \quad x = 3$$

When $x = 3,\ y = (3-2)^2 = 1$

Hence, the required point is (3, 1). **[2]**

**Q. 29. Find the equation of all lines having slope 2 which are tangents to the curve $y = \frac{1}{x-3}, x \neq 3$. [NCERT Ex. 6.3, Q. 11, Page 212]**

**Ans.** The equation of the given curve is $y = \frac{1}{x-3}, x \neq 3$

The slope of the tangent to the given curve at any point $(x, y)$ is given by, $\frac{dy}{dx} = \frac{-1}{(x-3)^2}$

If the slope of the tangent is 2, then we have

$$\Rightarrow \quad 2 = \frac{-1}{(x-3)^2}$$
$$\Rightarrow 2(x-3)^2 = -1$$
$$\Rightarrow \quad (x-3)^2 = -\frac{1}{2}$$

This is not possible since LHS is positive while RHS is negative.

Hence, there is no tangent to the given curve having slope 2. **[2]**

**Q. 30. Find points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangents are**
**(i) parallel to $x$-axis (ii) parallel to $y$-axis**
**[NCERT Ex. 6.3, Q. 13, Page 212]**

**Ans.** The equation of the given curve is $\frac{x^2}{9} + \frac{y^2}{16} = 1$.

On differentiating both sides with respect to $x$, we have

$$\frac{2x}{9} + \frac{2y}{16}.\frac{dy}{dx} = 0$$
$$\Rightarrow \quad \frac{dy}{dx} = -\frac{16x}{9y}$$ **[1]**

(i) The tangent is parallel to the $x$-axis if the slope of the tangent is, *i.e.*, $-\frac{16x}{9y} = 0$, which is possible if $x = 0$.

Then, $\frac{x^2}{9} + \frac{y^2}{16} = 1$ for $x = 0$

$$\Rightarrow \quad y^2 = 16$$
$$\Rightarrow \quad y = \pm 4$$

Hence, the points at which the tangents are parallel to the $x$-axis are (0, 4) and (0, −4). **[2]**

(ii) The tangent is parallel to the $y$-axis if the slope of the normal is 0, which gives

$$\frac{-1}{\left(\frac{-16x}{9y}\right)} = \frac{9y}{16x} = 0$$
$$\Rightarrow \quad y = 0$$

Then, $\frac{x^2}{9} + \frac{y^2}{16} = 1$ for $y = 0$.

$$\Rightarrow \quad x = \pm 3$$

Hence, the points at which the tangents are parallel to the $y$-axis are (3, 0) and (−3, 0). **[2]**

**Q. 31. Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ and $x = -2$ are parallel. [NCERT Ex. 6.3, Q. 16, Page 212]**

**Ans.** The equation of the given curve is $y = 7x^3 + 11$.

On differentiating with respect to $x$, we get

$$\therefore \frac{dy}{dx} = 21x^2$$

The slope of the tangent to a curve at $(x_0, y_0)$ is $\left.\frac{dy}{dx}\right]_{(x_0,y_0)}$.

Therefore, the slope of the tangent at the point where $x = 2$ is given by,

$$\left.\frac{dy}{dx}\right]_{x=2} = 21(2^2) = 84$$

It is observed that the slopes of the tangents at the points where $x = 2$ and $x = -2$ are equal. Therefore, the two tangents are parallel. **[2]**

**Q. 32. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the $y$-coordinate of the point. [NCERT Ex. 6.3, Q. 17, Page 212]**

**Ans.** The equation of the given curve is $y = x^3$.

$$\therefore \frac{dy}{dx} = 3x^2$$

The slope of the tangent at the point $(x, y)$ is given by, $\left.\frac{dy}{dx}\right]_{(x,y)} = 3x^2$

When the slope of the tangent is equal to the $y$-coordinate of the point, then $y=3x^2$.

Also, we have $y=x^3$

$$\therefore \quad 3x^2 = x^3$$

$$\Rightarrow \quad x^2(x-3) = 0$$

$$\Rightarrow \quad x = 0, x = 3$$

When $x = 0$, then $y = 0$ and when $x = 3$, then $y = 3(3)^2 = 27$.

Hence, the required points are (0, 0) and (3, 27). **[2]**

**Q. 33. Find the approximate change in the volume $V$ of a cube of side $x$ metre caused by increasing the side by 1%. [NCERT Ex. 6.4, Q. 4, Page 216]**

**Ans.** The volume of a cube (V) of side x is given by $V = x^3$.

$$dV = \left(\frac{dV}{dx}\right)\Delta x$$

$$= (3x^2)\Delta x$$

$$= (3x^2)(0.01x) \qquad [\text{As } 1\% \text{ of } x \text{ is } 0.01x]$$

$$= 0.03x^3$$

Hence, the approximate change in the volume of the cube is $0.03\, x^3$ m$^3$. **[2]**

**Q. 34. Find the approximate change in the surface area of a cube of side $x$ metres caused by decreasing the side by 1%. [NCERT Ex. 6.4, Q. 5, Page 216]**

**Ans.** The surface area of a cube ($S$) of side $x$ is given by $S = 6x^2$.

$$\therefore dS = \left(\frac{dS}{dx}\right)\Delta x$$

$$= (12x)\Delta x$$

$$= (12x)(0.01x) \qquad [\text{As } 1\% \text{ of } x \text{ is } 0.01x]$$

$$= 0.12x^2$$

Hence, the approximate change in the surface area of the cube is $0.12x^2\, m^2$. **[2]**

**Q. 35. If the radius of a sphere is measured as 7 m with an error of 0.02 m, then find the approximate error in calculating its volume. [NCERT Ex. 6.4, Q. 6, Page 216]**

**Ans.** Let $r$ be the radius of the sphere and $\Delta r$ be the error in measuring the radius.

Then,

$r = 7$ m and $\Delta r = 0.02$ m

Now, the volume ($V$) of the sphere is given by,

$$V = \frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dr} = 4\pi r^2$$

$$\therefore dV = \left(\frac{dV}{dr}\right)\Delta r$$

$$= (4\pi r^2)\Delta r$$

$$= 4\pi(7^2)(0.02) \text{ m}^3$$

$$= 3.92\pi \text{ m}^3$$

Hence, the approximate error in calculating the volume is $3.92\pi\, m^3$. **[2]**

**Q. 36. If the radius of a sphere is measured as 9 m with an error of 0.03 m, then find the approximate error in calculating in surface area. [NCERT Ex. 6.4, Q. 7, Page 216]**

**Ans.** Let $r$ be the radius of the sphere and $\Delta r$ be the error in measuring the radius.

Then,

$r = 9$ m and $\Delta r = 0.03$ m

Now, the surface area of the sphere ($S$) is given by,

$$S = 4\pi r^2$$

$$\therefore \frac{dS}{dr} = 8\pi r$$

$$\therefore dS = \left(\frac{dS}{dr}\right)\Delta r$$

$$= (8\pi r)\Delta r$$

$$= 8\pi(9)(0.03) \text{ m}^2$$

$$= 2.16\pi \text{ m}^2$$

Hence, the approximate error in calculating the surface area is $2.16\pi$ m$^2$. **[2]**

**Q. 37. Find the maximum and minimum values, if any, of the following functions given by**

**(i)** $f(x) = (2x-1)^2 + 3$

**(ii)** $f(x) = 9x^2 + 12x + 2$

**(iii)** $f(x) = -(x-1)^2 + 10$

**(iv)** $g(x) = x^3 + 1$

**[NCERT Ex. 6.5, Q. 1, Page 231]**

**Ans.** **(i)** The given function is $f(x) = (2x - 1)^2 + 3$.

It can be observed that $(2x - 1)^2 \geq 0$ for every $x \in R$.

Therefore, $f(x) = (2x - 1)^2 + 3 \geq 3$ for every $x \in R$.

The minimum value of f is attained when $2x - 1 = 0$.

$2x - 1 = 0$

$$\Rightarrow x = \frac{1}{2}$$

$$\therefore \text{Minimum value of } f = f\left(\frac{1}{2}\right) = \left(2.\frac{1}{2} - 1\right)^2 + 3 = 3$$

Hence, function $f$ does not have a maximum value. **[2]**

**(ii)** The given function is $f(x) = 9x^2 + 12x + 2 = (3x + 2)^2$.

It can be observed that $(3x + 2)^2 \geq 0$ for every $x \in R$.

Therefore, $f(x) = (3x + 2)^2 - 2 \geq -2$ for every $x \in R$.

The minimum value of f is attained when $3x + 2 = 0$.

$3x + 2 = 0$

$\Rightarrow \quad x = \frac{-2}{3}$

$\therefore$ Minimum value of $f$ =

$f\left(-\frac{2}{3}\right) = \left[3\left(\frac{-2}{3}\right) + 2\right]^2 - 2 = -2$

Hence, function f does not have a maximum value. **[2]**

**(iii)** The given function is $f(x) = -(x - 1)^2 + 10$.

It can be observed that $(x - 1)^2 \geq 0$ for every $x \in R$.

Therefore, $f(x) = -(x - 1)^2 + 10 \leq 10$ for every $x \in R$.

The maximum value of f is attained when $(x - 1) = 0$.

$(x - 1) = 0 \Rightarrow x = 1$

$\therefore$ Maximum value of $f = f(1) = -(1 - 1)2 + 10 = 10$

Hence, function f does not have a minimum value. **[2]**

**(iv)** The given function is $g(x) = x^3 + 1$.

Hence, function $g$ has neither a maximum value, nor a minimum value. **[1]**

**Q. 38. Find the maximum and minimum values, if any, of the following functions given by**

**(i)** $f(x) = |x + 2| - 1$

**(ii)** $g(x) = -|x + 1| + 3$

**(iii)** $h(x) = sin(2x) + 5$

**(iv)** $f(x) = |sin 4x + 3|$

**(v)** $h(x) = x + 1,\ x \in (-1,\ 1)$

**[NCERT Ex. 6.5, Q. 2, Page 232]**

**Ans.** **(i)** $f(x) = |x + 2| - 1$

We know that $|x + 2| \geq 0$ for every $x \in R$.

Therefore, $f(x) = |x + 2| - 1 \geq -1$ for every $x \in R$.

The minimum value of f is attained when $|x + 2| = 0$.

$|x + 2| = 0$

$\Rightarrow x = -2$

$\therefore$ Minimum value of $f = f(-2) = |-2 + 2| - 1 = -1$

Hence, function f does not have a maximum value. **[2]**

**(ii)** $g(x) = -|x + 1| + 3$

We know that $-|x + 1| \leq 0$ for every $x \in R$.

Therefore, $g(x) = -|x + 1| + 3 \leq 3$ for every $x \in R$.

The maximum value of g is attained when $|x + 1| = 0$.

$|x + 1| = 0$

$\Rightarrow x = -1$

$\therefore$ Maximum value of $g = g(-1) = -|-1 + 1| + 3 = 3$

Hence, function $g$ does not have a minimum value. **[2]**

**(iii)** $h(x) = \sin 2x + 5$

We know that, $-1 \leq \sin 2x \leq 1$.

$\Rightarrow -1 + 5 \leq \sin 2x + 5 \leq 1 + 5$

$\Rightarrow 4 \leq \sin 2x + 5 \leq 6$

Hence, the maximum and minimum values of h are 6 and 4 respectively.

**(iv)** $f(x) = |\sin 4x + 3|$

We know that $-1 \leq \sin 4x \leq 1$.

$\Rightarrow 2 \leq \sin 4x + 3 \leq 4$

$\Rightarrow 2 \leq |\sin 4x + 3| \leq 4$

Hence, the maximum and minimum values of f are 4 and 2 respectively. **[2]**

**(v)** $h(x) = x + 1, x \in (-1, 1)$

Here, if a point $x$ is closest to $-1$, then we find $\frac{x_0}{2} + 1 < x_0 + 1$ for all $x \in (-1, 1)$.

Also, if x is closest to 1, then $x_1 + 1 < \frac{x_1 + 1}{2} + 1$ for all $x \in (-1, 1)$.

Hence, function $h(x)$ has neither maximum, nor minimum value in $(-1, 1)$. **[2]**

**Q. 39. Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be :**

**(i)** $f(x) = x^2$

**(ii)** $g(x) = x^3 - 3x$

**(iii)** $h(x) = \sin x + \cos x, 0 < x < \frac{\pi}{2}$

**(iv)** $f(x) = \sin x - \cos x, 0 < x < 2\pi$

**(v)** $f(x) = x^3 - 6x^2 + 9x + 15$

**(vi)** $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$

**(vii)** $g(x) = \frac{1}{x^2 + 2}$

**(viii)** $f(x) = x\sqrt{1 - x}, x > 0$

**[NCERT Ex. 6.5, Q. 3, Page 232]**

**Ans.** **(i)** $f(x) = x^2$

$\therefore f'(x) = 2x$

Now,

$f'(x) = 0 \Rightarrow x = 0$

Thus, $x = 0$ is the only critical point which could possibly be the point of local maxima or local minima of $f$.

We have $f''(0) = 2$, which is positive.

Therefore, by second derivative test, $x = 0$ is a point of local minima and local minimum value of $f$ at $x = 0$ is $f(0) = 0$. **[2]**

**(ii)** $g(x) = x^3 - 3x$

$\therefore g'(x) = 3x^2 - 3$

Now,

$g(x) = 0 \Rightarrow 3x^2 = 3 \Rightarrow x = \pm 1$

$g''(x) = 6x$

$g''(1) = 6 > 0$

$g''(-1) = -6 < 0$

By second derivative test, $x = 1$ is a point of local minima and local minimum value of g at $x = 1$ is $g(1) = (1)^3 - 3 = 1 - 3 = -2$. However, $x = -1$ is a point of local maxima and local maximum value of g at $x = -1$ is $g(1) = (-1)^3 - 3(-1) = -1 + 3 = 2$. **[2]**

**(iii)** $h(x) = \sin x + \cos x, 0 < x < \frac{\pi}{2}$

$\therefore \; h'(x) = \cos x - \sin x$

$h'(x) = 0$

$\Rightarrow \sin x = \cos x$

$\Rightarrow \tan x = 1$

$\Rightarrow \quad x = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

$h''(x) = -\sin x - \cos x$

$= -(\sin x + \cos x)$

$h''\left(\frac{\pi}{4}\right) = -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$

$= -\frac{1}{\sqrt{2}} = -\sqrt{2} < 0$

Therefore, by second derivative test, $x = \frac{\pi}{4}$ is a point of local maxima and the local maximum value of h at $x = \frac{\pi}{4}$ is

$h\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= \sqrt{2}$ **[2]**

**(iv)** $f(x) = \sin x - \cos x, 0 < x < 2\pi$

$\therefore \; f'(x) = \cos x + \sin x$

$f'(x) = 0$

$\Rightarrow \cos x = -\sin x$

$\Rightarrow \tan x = -1$

$\Rightarrow \quad x = \frac{3x}{4}, \frac{7\pi}{4} \in (0, 2\pi)$

$f''(x) = -\sin x + \cos x$

$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4}$

$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} < 0$

$f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 0$

Therefore, by second derivative test, $x = \frac{3\pi}{4}$ is a point of local maxima and the local maximum value of f at $x = \frac{3\pi}{4}$ is

$f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4}$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= \sqrt{2}$

However, $x = \frac{7\pi}{4}$ is a point of local minima and the local minimum value of f at $x = \frac{7\pi}{4}$ is

$f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4}$

$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$

$= -\sqrt{2}$ **[2]**

**(v)** $f(x) = x^3 - 6x^2 + 9x + 15$

$\therefore \quad f'(x) = 3x^2 - 12x + 9$

$f'(x) = 0 \Rightarrow 3(x^2 - 4x + 3) = 0$

$\Rightarrow 3(x-1)(x-3) = 0$

$\Rightarrow \quad x = 1, 3$

Now, $f''(x) = 6x - 12 = 6(x-2)$

$f''(1) = 6(1-2) = -6 < 0$

$f''(3) = 6(3-2) = 6 > 0$

Therefore, by second derivative test, $x = 1$ is a point of local maxima and the local maximum value of f at $x = 1$ is $f(1) = 1 - 6 + 9 + 15 = 19$. However, $x = 3$ is a point of local minima and the local minimum value of $f$ at $x = 3$ is $f(3) = 27 - 54 + 27 + 15 = 15$. **[2]**

**(vi)** $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$

$\therefore g'(x) = \frac{1}{2} - \frac{2}{x^2}$

Now,

$g'(x) = 0$ gives $\frac{2}{x^2} = \frac{1}{2}$

$\Rightarrow x^2 = 4$

$\Rightarrow \quad x = \pm 2$

Since $x > 0$, we take $x = 2$.

Now,

$g''(x) = \frac{4}{x^3}$

$g''(2) = \frac{4}{2^3} = \frac{1}{2} > 0$

Therefore, by second derivative test, $x = 2$ is a point of local minima and the local minimum value of g at $x = 2$ is

$g(2) = \frac{2}{2} + \frac{2}{2}$

$= 1 + 1 = 2\cdot$ **[2]**

**(vii)** $g(x) = \frac{1}{x^2 + 2}$

$\therefore \quad g'(x) = \frac{-(2x)}{(x^2+2)^2}$

$g'(x) = 0$

$\Rightarrow \frac{-2x}{(x^2+2)^2} = 0$

$\Rightarrow \quad x = 0$

Now, for values close to $x = 0$ and to the left of 0, $g'(x) > 0$ Also, for values close to $x = 0$ and to the right of 0, $g'(x) < 0$.

Therefore, by first derivative test, $x = 0$ is a point of local maxima and the local maximum value of $g(0)$ is $\frac{1}{0+2} = \frac{1}{2}\cdot$ **[2]**

**(viii)** $f(x) = x\sqrt{1-x}, x > 0$

$$f'(x) = \sqrt{1-x} + x.\frac{1}{2\sqrt{1-x}}(-1)$$

$$= \sqrt{1-x} - \frac{x}{2\sqrt{1-x}}$$

$$= \frac{2(1-x)-x}{2\sqrt{1-x}} = \frac{2-3x}{2\sqrt{1-x}}$$

$$f'(x) = 0$$

$$\Rightarrow \frac{2-3x}{2\sqrt{1-x}} = 0$$

$$\Rightarrow \quad 2-3x = 0$$

$$\Rightarrow \quad x = \frac{2}{3}$$

$$f''(x) = \frac{1}{2}\left[\frac{\sqrt{1-x}(-3)-(2-3x)\left(\frac{-1}{2\sqrt{1-x}}\right)}{1-x}\right]$$

$$= \frac{\sqrt{1-x}(-3)+(2-3x)\left(\frac{1}{2\sqrt{1-x}}\right)}{2(1-x)}$$

$$= \frac{-6(1-x)+(2-3x)}{4(1-x)^{\frac{3}{2}}}$$

$$= \frac{3x-4}{4(1-x)^{\frac{3}{2}}}$$

$$f''\left(\frac{2}{3}\right) = \frac{3\left(\frac{2}{3}\right)-4}{4\left(1-\frac{2}{3}\right)^{\frac{3}{2}}} = \frac{2-4}{4\left(\frac{1}{3}\right)^{\frac{3}{2}}} = \frac{-1}{2\left(\frac{1}{3}\right)^{\frac{3}{2}}} < 0$$

Therefore, by second derivative test, $x = \frac{2}{3}$ is a point of local maxima and the local maximum value of f at $x = \frac{2}{3}$ is

$$f\left(\frac{2}{3}\right) = \frac{2}{3}\sqrt{1-\frac{2}{3}}$$

$$= \frac{2}{3}\sqrt{\frac{1}{3}} = \frac{2}{3\sqrt{3}}$$

$$= \frac{2\sqrt{3}}{9}$$ **[2]**

**Q. 40. Prove that the following functions do not have maxima or minima :**

**(i)** $f(x) = e^x$ **(ii)** $g(x) = \log x$

**(iii)** $h(x) = x^3 + x^2 + x + 1$

**[NCERT Ex. 6.5, Q. 4, Page 232]**

**Ans. (i) We have,**

$f(x) = e^x$

$\therefore f'(x) = e^x$

Now, if $f'(x) = 0$, then $e^x = 0$. But, the exponential function can never assume 0 for any value of $x$.

Therefore, there does not exist $c \in R$ such that $f'(c) = 0$

Hence, function $f$ does not have maxima or minima. **[2]**

**(ii)** We have,

$g(x) = \log x$

$\therefore g'(x) = \frac{1}{x}$

Since log $x$ is defined for a positive number $x$, $g'(x) > 0$ for any $x$.

Therefore, there does not exist $c \in R$ such that $g'(c) = 0$.

Hence, function $g$ does not have maxima or minima. **[2]**

**(iii)** We have,

$h(x) = x^3 + x^2 + x + 1$

$\therefore h'(x) = 3x^2 + 2x + 1$

Now,

$h(x) = 0 \Rightarrow 3x^2 + 2x + 1 = 0$

$\therefore h'(x) = 3x^2 + 2x + 1$

$$x = \frac{-2 \pm 2\sqrt{2i}}{6}$$

$$= \frac{-1 \pm \sqrt{2i}}{3} \notin R$$

Therefore, there does not exist $c \in R$ such that $h'(c) = 0$.

Hence, function $h$ does not have maxima or minima. **[2]**

**Q. 41. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals :**

**(i)** $f(x) = x^3, x \in [-2, 2]$

**(ii)** $f(x) = \sin x + \cos x, x \in [\ ,\ ]$

**(iii)** $f(x) = 4x - \frac{1}{2}x^2, x \in \left[-2, \frac{9}{2}\right]$

**(iv)** $f(x) = (x-1)^2 + 3, x \in [-3, 1]$

**[NCERT Ex. 6.5, Q. 5, Page 232]**

**Ans.** (i) **The given function is $f(x) = x^3$.**

$\therefore f'(x) = 3x^2$

Now,

$f'(x) = 0 \Rightarrow x = 0$

Then, we evaluate the value of f at critical point $x = 0$ and at end points of the interval $[-2, 2]$.

$f(0) = 0$

$f(-2) = (-2)^3 = -8$

$f(2) = (2)^3 = 8$

Hence, we can conclude that the absolute maximum value of f on $[-2, 2]$ is 8 occurring at $x = 2$. Also, the absolute minimum value of f on $[-2, 2]$ is $-8$ occurring at $x = -2$. **[2]**

**(ii)** The given function is $f(x) = \sin x + \cos x$.

$\therefore f'(x) = \cos x - \sin x$

Now,

$$f'(x) = 0 \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$$

Then, we evaluate the value of $f$ at critical point and at the end points of the interval $[0, \pi]$.

Hence, we can conclude that the absolute maximum value of f on $[0, \pi]$ is $\sqrt{2}$ occurring at

$x = \frac{\pi}{4}$ and the absolute minimum value of $f$ on $[0, \pi]$ is $-1$ occurring at $x = \pi$. **[2]**

**(iii)** The given function is $f(x) = 4x - \frac{1}{2}x^2$.

$\therefore f'(x) = 4 - \frac{1}{2}(2x) = 4 - x$

Now,

$f'(x) = 0 \Rightarrow x = 4$

$\therefore f'(x) = 4 - \frac{1}{2}(2x) = 4 - x$

Now,

$f'(x) = 0 \Rightarrow x = 4$

Then, we evaluate the value of $f$ at critical point $x = 4$ and at the end points of the interval $\left[-2, \frac{9}{2}\right]$.

$$f(4) = 16 - \frac{1}{2}(16) = 16 - 8 = 8$$

$$f(-2) = -8 - \frac{1}{2}(4) = -8 - 2 = -10$$

$$f\left(\frac{9}{2}\right) = 4\left(\frac{9}{2}\right) - \frac{1}{2}\left(\frac{9}{2}\right)^2$$

$$= 18 - \frac{81}{8} = 18 - 10.125 = 7.875$$

Hence, we can conclude that the absolute maximum value of f on $\left[-2, \frac{9}{2}\right]$ is 8 occurring at $x = 4$ and the absolute minimum value of f on $\left[-2, \frac{9}{2}\right]$ is $-10$ occurring at $x = -2$. **[2]**

**(iv)** The given function is $f(x) = (x - 1)^2 + 3$.

$\therefore \quad f'(x) = 2(x-1)$

Now,

$f'(x) = 0$

$\Rightarrow \quad 2(x - 1) = 0$

$\Rightarrow \quad x = 1$

Then, we evaluate the value of f at critical point $x = 1$ and at the end points of the interval $[-3, 1]$.

$(1) = (1 - 1) \; + 3 = 0 + 3 = 3$

$(-3) = (-3 - 1) \; + 3 = 16 + 3 = 19$

Hence, we can conclude that the absolute maximum value of $f$ on $[-3, 1]$ is 19 occurring at $x = -3$ and the minimum value of $f$ on $[-3, 1]$ is 3 occurring at $x = 1$. **[2]**

**Q. 42. Find the maximum profit that a company can make, if the profit function is given by $p(x) = 41 - 72x - 18x^2$.**

**[NCERT Ex. 6.5, Q. 6, Page 232]**

**Ans.** The profit function is given as $p(x) = 41 - 72x - 18x^2$.

$\therefore p'(x) = -72 - 36x$

$\Rightarrow \quad x = -\frac{72}{36} = -2$

Also,

$p''(-2) = -36 < 0$

By second derivative test, $x = -2$ is the point of local maxima of $p$.

$\therefore$ Maximum profit $= p(-2)$

$= 41 - 72(-2) - 18(-2)2 = 41 + 144 - 72 = 113$

Hence, the maximum profit that the company can make is 113 units. **[2]**

**Q. 43. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval [0, 2]. Find the value of $a$.**

**[NCERT Ex. 6.5, Q. 11, Page 233]**

**Ans.** Let $f(x) = x^4 - 62x^2 + ax + 9$.

Differentiate with respect to $x$, we get

$\therefore \quad f'(x) = 4x^3 - 124x + a$

It is given that function f attains its maximum value on the interval [0, 2] at $x = 1$.

$\therefore \quad f'(1) = 0$

$\Rightarrow 4 - 124 + a = 0$

$\Rightarrow \quad a = 120$

Hence, the value of '$a$' is 120. **[2]**

**Q. 44. A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is proportional to the surface. Prove that the radius is decreasing at a constant rate.**

**[NCERT Exemp. Ex. 6.3, Q. 1, Page 135]**

**Ans.** We have, rate of decrease of the volume of spherical ball of salt at any instant is $\alpha$ surface. Let the radius of the spherical ball of the salt be $r$.

$\therefore$ Volume of the ball $(V) = \frac{4}{3}\pi r^3$ and

surface area $(S) = 4\pi r^2$

$\because \quad \frac{dV}{dT} \propto S$

$\Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) \propto 4\pi r^2$

$\Rightarrow \quad \frac{4}{3}\pi 3r^2 \frac{dr}{dt} \propto 4\pi r^2$

$\Rightarrow \quad \frac{dr}{dt} \propto \frac{4\pi r^2}{4\pi r^2}$

$\Rightarrow \quad \frac{dr}{dt} = k.1$ [where, $k$ is the proportionality constant]

$\Rightarrow \quad \frac{dr}{dt} = k$

Hence, the radius of ball is decreasing at a constant rate. **[2]**

**Q. 45. If the area of a circle increases at a uniform rate, then prove that perimeter varies inversely as the radius. [NCERT Exemp. Ex. 6.3, Q. 2, Page 135]**

**Ans.** Let the radius of circle $= r$ and area of the circle, $A = \pi r^2$.

$$\frac{d}{dt}A = \frac{d}{dt}\pi r^2$$

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt} \quad \text{...(i)}$$

Since, the area of a circle increases at a uniform rate, then

$$\frac{dA}{dt} = k \quad \text{...(ii)}$$

Where, $k$ is a constant.

From equations (i) and (ii), we have

$$2\pi r \cdot \frac{dr}{dt} = k$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{k}{2\pi r} = \frac{k}{2\pi} \cdot \left(\frac{1}{r}\right) \quad \text{...(iii)}$$

Let the perimeter,

$$P = 2\pi r$$

$$\frac{dP}{dt} = \frac{d}{dt} \cdot 2\pi r \Rightarrow \frac{dP}{dt} = 2\pi \cdot \frac{dr}{dt}$$

$$= 2\pi \cdot \frac{k}{2\pi} \cdot \frac{1}{r} = \frac{k}{r} \quad \text{[By using Eq. (iii)]}$$

$$\Rightarrow \frac{dP}{dt} \propto \frac{1}{r}$$

**Hence proved.** **[2]**

**Q. 46. Find an angle $\theta$, $0 < \theta < \frac{\pi}{2}$, which increases twice as fast as its sine.**

**[NCERT Exemp. Ex. 6.3, Q. 5, Page 135]**

**Ans.** Let θ increases twice as fast as its sine.

$\Rightarrow \theta = 2\sin\theta$

Now, on differentiating both sides with respect to $t$, we get

$$\frac{d\theta}{dt} = 2 \cdot \cos\theta \cdot \frac{d\theta}{dt}$$

$$\Rightarrow \quad 1 = 2\cos\theta$$

$$\Rightarrow \quad \frac{1}{2} = \cos\theta$$

$$\Rightarrow \cos\theta = \cos\frac{\pi}{3}$$

$$\therefore \quad \theta = \frac{\pi}{3}$$

So, the required angle is $\frac{\pi}{3}$. **[2]**

**Q. 47. Find the approximate value of $(1.999)^5$.**

**[NCERT Exemp. Ex. 6.3, Q. 6, Page 135]**

**Ans.** Let $x = 2$

and $\Delta x = -0.001 \quad [\because 2 - 0.001 = 1.999]$

Let $y = x^5$

On differentiating both sides with respect to $x$, we get

$$\frac{dy}{dx} = 5x^4$$

Now,

$$\Delta y = \frac{dy}{dx} \cdot \Delta x = 5x^4 \times \Delta x$$

$$= 5 \times 2^4 \times [-0.001]$$

$$= -80 \times 0.001 = -0.080$$

$$(1999)^5 = y + \Delta y$$

$$= 2^5 + (-0.080)$$

$$= 32 - 0.080 = 31.920$$

**[2]**

**Q. 48. Find the approximate volume of metal in a hollow spherical shell whose internal and external radii are 3 cm and 3.0005 cm, respectively.**

**[NCERT Exemp. Ex. 6.3, Q. 7, Page 135]**

**Ans.** Let internal radius = $r$ and external radius = $R$

Volume of hollow spherical shell, $V = \frac{4}{3}\pi(R^3 - r^3)$

$$\Rightarrow \quad V = \frac{4}{3}\pi[(3.0005)^3 - (3)^3] \quad \text{...(i)}$$

Now, we shall use differentiation to get approximate value of $(3.0005)^3$.

Let $(3.0005)^3 = y + \Delta y$

and $x = 3$, $\Delta x = 0.0005$

Also, let $y = x^3$

On differentiating both sides with respect to $x$, we get

$$\frac{dy}{dx} = 3x^2$$

$$\Delta y = \frac{dy}{dx} \times \Delta x = 3x^2 \times 0.0005$$

$$= 3 \times 3^2 \times 0.0005$$

$$= 27 \times 0.0005 = 0.0135$$

Also,

$$(3.0005)^3 = y + \Delta y$$

$$= 3^3 + 0.0135 = 27.0135$$

$$V = \frac{4}{3}\pi[27.0135 - 27.000] \quad \text{[By using Eq. (i)]}$$

$$= \frac{4}{3}\pi[0.0135] = 4\pi \times (0.0045)$$

$$= 0.0180\pi \text{ cm}^3$$

**[2]**

**Q. 49. The volume of a cube increases at a constant rate. Prove that the increase in its surface area varies inversely as the length of the side.**

**[NCERT Exemp. Ex. 6.3, Q. 10, Page 136]**

**Ans.** Let the side of a cube be $x$ unit.

Volume of cube ($V$) = $x^3$

On differentiating both side with respect to $t$, we get

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt} = k \quad \text{[Constant]}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{k}{3x^2} \quad \text{...(i)}$$

Also, surface area of cube, $S = 6x^2$

On differentiating with respect to $t$, we get

$$\frac{dS}{dt} = 12x \cdot \frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dS}{dt} = 12x \cdot \frac{k}{3x^2} \quad \text{[By using Eq. (i)]}$$

$$\Rightarrow \quad \frac{dS}{dt} = \frac{12k}{3x} = 4\left(\frac{k}{x}\right)$$

$$\Rightarrow \quad \frac{dS}{dt} \propto \frac{1}{x}$$

Hence, the surface area of the cube varies inversely as the length of the side. **[2]**

**Q. 50. Prove that the curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ touch each other at the point (1, 2).**

**[NCERT Exemp. Ex. 6.3, Q. 16, Page 136]**

**Ans.** We have, $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$

Since, both the curves touch each other at (1, 2), *i.e.*, curves are passing through (1, 2).

$\therefore \quad 2y \cdot \frac{dy}{dx} = 4$

and $2x + 2y\frac{dy}{dx} = 6$

$\Rightarrow \quad \frac{dy}{dx} = \frac{4}{2y}$

and $\quad \frac{dy}{dx} = \frac{6-2x}{2y}$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,2)} = \frac{4}{4} = 1$

and $\left(\frac{dy}{dx}\right)_{(1,2)} = \frac{6-2\cdot 1}{2\cdot 2} = \frac{4}{4} = 1$

$\Rightarrow \quad m_1 = 1$ and $m_2 = 1$

Thus, we see that slope of both the curves are equal to each other, *i.e.*, $m_1 = m_2 = 1$ at the point (1, 2). Hence, both the curves touch each other. **[2]**

**Q. 51. Show that the line $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = b.e^{\frac{-x}{a}}$ at the point where the curve intersects the axis of $y$.**

**[NCERT Exemp. Ex. 6.3, Q. 19, Page 136]**

**Ans.** We have the equation of line given by $\frac{x}{a} + \frac{y}{b} = 1$, which touches the curve $y = b.e^{\frac{-x}{a}}$ at the point, where the curve intersects the axis of $y$, *i.e.*, $x = 0$.

$\therefore \quad y = b \cdot e^{-0/a} = b \quad [\because e^0 = 1]$

So, the point of intersection of the curve with $y$-axis is (0, $b$).

Now, slope of the given line at (0, $b$) is given by

$$\frac{1}{a}\cdot 1 + \frac{1}{b}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{a}\cdot b = \frac{-b}{a} = m_1 \quad \text{[say]}$$

Also, the slope of the curve at (0, $b$) is

$$\frac{dy}{dx} = b\cdot e^{-x/a}\cdot\frac{-1}{a}$$

$$\frac{dy}{dx} = \frac{-b}{a}e^{-x/a}$$

$$\left(\frac{dy}{dx}\right)_{(0,b)} = \frac{-b}{a}e^{-0} = \frac{-b}{a} = m_2 \quad \text{[say]}$$

Since, $\quad m_1 = m_2 = \frac{-b}{a}$

Hence, the line touches the curve at the point, where the curve intersects the axis of $y$. **[2]**

**Q. 52. Show that : $f(x) = 2x + \cot^{-1}x + \log\left(\sqrt{1+x^2} - x\right)$ is increasing $R$.**

**[NCERT Exemp. Ex. 6.3, Q. 20, Page 136]**

**Ans.** We have,

$$f(x) = 2x + \cot^{-1}x + \log\left(\sqrt{1+x^2} - x\right)$$

$$f'(x) = 2 + \left(\frac{-1}{1+x^2}\right) + \frac{1}{\left(\sqrt{1+x^2}-x\right)}\left(\frac{1}{2\sqrt{1+x^2}}\cdot 2x - 1\right)$$

$$= 2 - \frac{1}{1+x^2} + \frac{1}{\left(\sqrt{1+x^2}-x\right)}\cdot\frac{\left(x-\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}$$

$$= 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}$$

$$= \frac{2+2x^2-1-\sqrt{1+x^2}}{1+x^2}$$

$$= \frac{1+2x^2-\sqrt{1+x^2}}{1+x^2}$$

For increasing function,

$$f'(x) \geq 0$$

$$\Rightarrow \quad \frac{1+2x^2-\sqrt{1+x^2}}{1+x^2} \geq 0$$

$$\Rightarrow \quad 1+2x^2 \geq \sqrt{1+x^2}$$

$$\Rightarrow \quad (1+2x^2)^2 \geq \left(1+x^2\right)$$

$$\Rightarrow \quad 1+4x^4+4x^2 \geq 1+x^2$$

$$\Rightarrow \quad 4x^4+3x^2 \geq 0$$

$$\Rightarrow \quad x^2(4x^2+3) \geq 0$$

which is true for any real value of $x$.

Hence, $f(x)$ is increasing in $R$. **[2]**

**Q. 53. Show that for $a \geq 1$, $f\left(x\right) = \sqrt{3}\sin x - \cos x - 2ax + b$ is decreasing in $R$.**

**[NCERT Exemp. Ex. 6.3, Q. 21, Page 137]**

**Ans.** We have, $a \geq 1$,

$$f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$$

$$\therefore \quad f'(x) = \sqrt{3}\cos x - (-\sin x) - 2a$$

$$= \sqrt{3}\cos x + \sin x - 2a$$

$$= 2\left[\frac{\sqrt{3}}{2}\cdot\cos x + \frac{1}{2}\cdot\sin x\right] - 2a$$

$$= 2\left[\cos\frac{\pi}{6}\cdot\cos x + \sin\frac{\pi}{6}\cdot\sin x\right] - 2a$$

$$= 2\left(\cos\left(\frac{\pi}{6}-x\right)\right) - 2a$$

$$[\because \cos(A-B) = \cos A\cdot\cos B + \sin A\cdot\sin B]$$

$$= 2\left[\left(\cos\left(\frac{\pi}{6}-x\right)\right) - a\right]$$

We know that, $\quad \cos x \in [-1, 1]$

and $\quad a \geq 1$

So, $\quad 2\left[\cos\left(\frac{\pi}{6}-x\right) - a\right] \leq 0$

$\therefore \quad f'(x) \leq 0$

Hence, $f(x)$ is a decreasing function in $R$. **[2]**

**Q. 54. Show that $f\left(x\right) = \tan^{-1}\left(\sin x + \cos x\right)$ is an increasing function in $\left(0, \frac{\pi}{4}\right)$.**

**[NCERT Exemp. Ex. 6.3, Q. 22, Page 137]**

**Ans.**

We have,

$f(x) = \tan^{-1}(\sin x + \cos x)$

$f'(x) = \dfrac{1}{1+(\sin x + \cos x)^2} \cdot (\cos x - \sin x)$

$= \dfrac{1}{1+\sin^2 x + \cos^2 x + 2\sin x \cdot \cos x}(\cos x - \sin x)$

$= \dfrac{1}{(2+\sin 2x)}(\cos x - \sin x)$

$[\because \sin 2x = 2\sin x \cos x \text{ and } \sin^2 x + \cos^2 x = 1]$

For $\quad f'(x) \geq 0, \dfrac{1}{(2+\sin 2x)} \cdot (\cos x - \sin x) \geq 0$

$\Rightarrow \quad \cos x - \sin x \geq 0 \quad \left[\because (2+\sin 2x) \geq 0 \text{ in} \left(0, \dfrac{\pi}{4}\right)\right]$

$\Rightarrow \quad \cos x \geq \sin x$

which is true, if x $\in \left(0, \dfrac{\pi}{4}\right)$

Hence, f(x) is an increasing function in $\left(0, \dfrac{\pi}{4}\right)$. **[2]**

**Q. 55. At what point, the slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is maximum? Also find the maximum slope.**

**[NCERT Exemp. Ex. 6.3, Q. 23, Page 137]**

**Ans.**

We have,

$y = -x^3 + 3x^2 + 9x - 27$

$\dfrac{dy}{dx} = -3x^2 + 6x + 9$

= Slope of tangent to the curve

Now, $\dfrac{d^2y}{dx^2} = -6x + 6$

For $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = 0,$

$-6x + 6 = 0$

$\Rightarrow \quad x = \dfrac{-6}{-6} = 1$

$\therefore \quad \dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right) = -6 < 0$

So, the slope of tangent to the curve is maximum, when $x = 1$

For $x$=1, $\left(\dfrac{dy}{dx}\right)_{(x=1)} = -3 \cdot 1^2 + 6 \cdot 1 + 9 = 12,$

which is maximum slope.

Also, for $x = 1, y = -1^3 + 3 \cdot 1^2 + 9 \cdot 1 - 27$

$= -1 + 3 + 9 - 27 = -16$

So, the required point is (1, −16). **[2]**

**Q. 56. Prove that $f(x) = \sin x + \sqrt{3}\cos x$ has maximum value at $x = \dfrac{\pi}{4}$.**

**[NCERT Exemp. Ex. 6.3, Q. 24, Page 137]**

**Ans.**

We have, $f(x) = \sin x + \sqrt{3}\cos x$

$\therefore \quad f'(x) = \cos x + \sqrt{3}(-\sin x)$

$= \cos x - \sqrt{3}\sin x$

For $\quad f'(x) = 0, \cos x = \sqrt{3}\sin x$

$\Rightarrow \quad \tan x = \dfrac{1}{\sqrt{3}} = \tan\dfrac{\pi}{6}$

$\Rightarrow \quad x = \dfrac{\pi}{6}$

Again, differentiating $f'(x)$, we get

$f''(x) = -\sin x - \sqrt{3}\cos x$

At $x = \dfrac{\pi}{6}. f''(x) = -\sin\dfrac{\pi}{6} - \sqrt{3}\cos\dfrac{\pi}{6}$

$= -\dfrac{1}{2} - \sqrt{3} \cdot \dfrac{\sqrt{3}}{2}$

$= -\dfrac{1}{2} - \dfrac{3}{2} = -2 < 0$

Hence, at $x = \dfrac{\pi}{6}, f(x)$ has maximum value at $\dfrac{\pi}{6}$ **[2]**

**Q. 57. A telephone company in a town has 500 subscribers on its list and collects fixed charges of ₹ 300/- per subscriber per year. The company proposes to increase the annual subscription and it is believed that for every increase of ₹ 1/- one subscriber will discontinue the service. Find what increase will bring maximum profit?**

**[NCERT Exemp. Ex. 6.3, Q. 27, Page 137]**

**Ans.** Consider that company increases the annual subscription by ₹ $x$.

So, $x$ subscribes will discontinue the service.

$\therefore$ Total revenue of company after the increment is given by

$R(x) = (500 - x)(300 + x)$

$= 15 \times 10^4 + 500x - 300x - x^2$

$= -x^2 + 200x + 150000$

On differentiating both sides with respect to $x$, we get

$R'(x) = -2x + 200$

Now, $R'(x) = 0$

$\Rightarrow \quad 2x = 200 \Rightarrow x = 100$

$\therefore \quad R''(x) = -2 < 0$

So, $R(x)$ is maximum when $x$=100

Hence, the company should increase the subscription fee by ₹ 100, so that it has maximum profit. **[2]**

**Q. 58. If the straight line $x\cos\alpha + y\sin\alpha = p$ touches the curve $\dfrac{x^2}{a} + \dfrac{y^2}{b} = 1$. Then prove that $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$.**

**[NCERT Exemp. Ex. 6.3, Q. 28, Page 137]**

**Ans.** We know that, if a line $y = mx + c$ touches ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then the required condition is $c^2 = a^2m^2 + b^2$.

Here, given equation of the line is

$x\cos\alpha + y\sin\alpha = p$

$\Rightarrow \quad y = \dfrac{p - x\cos\alpha}{\sin\alpha}$

$= -x\cot\alpha + \dfrac{p}{\sin\alpha}$

$\Rightarrow \quad c = \dfrac{p}{\sin\alpha}$

and $\quad m = -\cot\alpha$

$\therefore \quad \left(\dfrac{p}{\sin\alpha}\right)^2 = a^2(-\cot\alpha)^2 + b^2$

$\Rightarrow \quad \dfrac{p^2}{\sin^2\alpha} = a^2\dfrac{\cos^2\alpha}{\sin^2\alpha} + b^2$

$\Rightarrow \quad p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$

Hence proved. **[2]**

**Q. 59. The total cost $C(x)$ associated with the production of $x$ units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.**

**[CBSE Board, Delhi Region, 2018]**

**Ans.**

Given that,

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$$

Marginal cost $= \left\{\dfrac{d}{dx}[c(x)]\right\}_{x=3}$

$= \left[0.015x^2 - 0.04x + 30\right]_{x=3}$

$= \left[0.015x^2 - 0.04x + 30\right]_{x=3}$ **[2]**

**Q. 60. The volume of a cube is increasing at the rate of 9 cm³/s. How fast is its surface area increasing when the length of an edge is 10 cm?**

**[CBSE Board, All India Region, 2017]**

**Ans.** Let $V$ be the volume of cube, then $\dfrac{dV}{dt} = 9\ cm^3/s$

Surface area ($S$) of cube = $6x^2$, where $x$ is the side. Then

$V = x^3$

$\Rightarrow \dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt}$

$\Rightarrow \dfrac{dx}{dt} = \dfrac{1}{3x^2}\cdot\dfrac{dV}{dt}$

$S = 6x^2$

$\Rightarrow \dfrac{dS}{dt} = 12x\dfrac{dx}{dt} = 12x\dfrac{1}{3x^2}\dfrac{dV}{dt}$

$= 4\dfrac{1}{10}\cdot 9 = 3.6\ \text{cm}^2/\text{s}$ **[2]**

**Q. 61. Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on $R$.**

**[CBSE Board, All India Region, 2017]**

**Ans.**

$f(x) = x^3 - 3x^2 + 6x - 100$

$f'(x) = 3x^2 - 6x + 6$

$= 3[x^2 - 2x + 2] = 3[(x-1)^2 + 1]$

Since $f'(x) > 0\ \forall\ x \in R$

$\therefore f(x)$ is increasing on R

**Q. 62. Find $\dfrac{dy}{dx}$ at $x = 1, y = \dfrac{\pi}{4}$ if $\sin^2 y + \cos xy = K$.**

**[CBSE Board, Delhi Region, 2017]**

**Ans.** From the given equation, we have

$$2\sin y\cos y\cdot\frac{dy}{dx} - \sin xy\cdot\left[x\cdot\frac{dy}{dx} + y\cdot 1\right] = 0$$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{y\sin xy}{\sin 2y - x\sin(xy)}$

$\therefore \left.\dfrac{dx}{dx}\right|_{x=1,y=\frac{\pi}{4}} = \dfrac{\pi}{4(\sqrt{2}-1)}$ **[2]**

**Q. 63. The volume of a sphere is increasing at the rate of 3 cubic centimeter per second. Find the rate of increase of its surface area, when the radius is 2 cm.**

**[CBSE Board, Delhi Region, 2017]**

**Ans.**

$V = \dfrac{4}{3}\pi r^3$

$\Rightarrow \quad \dfrac{dv}{dt} = 4\pi r^2\dfrac{dr}{dt}$

$\Rightarrow \quad \dfrac{dr}{dt} = \dfrac{3}{4\pi r^2}$

$S = 4\pi r^2$

$\Rightarrow \quad \dfrac{dS}{dt} = 8\pi r\cdot\dfrac{dr}{dt}$

$\Rightarrow \left.\dfrac{dS}{dt}\right|_{r=2} = 2\ \text{cm}^2/\text{s}$ **[2]**

**Q. 64. Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing on $R$.**

**[CBSE Board, Delhi Region, 2017]**

**Ans.**

$f(x) = 4x^3 - 18x^2 + 27x - 7$

$f'(x) = 12x^2 - 36x + 27$

$= 3(2x-3)^2 \geq 0\ \forall\ x \in R$

$\Rightarrow f(x)$ is increasing on $R$.

**Q. 65. The radius $r$ of a right circular cylinder is decreasing at the rate of 3 cm/min. and its height $h$ is increasing at the rate of 2 cm/min. When $r$ = 7 cm and $h$ = 2 cm, find the rate of change of the volume of cylinder.**

**[CBSE Board, Foreign Scheme, 2017]**

**Ans.**

$V = \pi r^2 h$

$\dfrac{dv}{dt} = \pi\left(r^2\dfrac{dh}{dt} + 2r\dfrac{dr}{dt}h\right)$

$= \dfrac{22}{7}[49\times(+2) + 14(2)(-3)]$

$= 44\ \text{cm}^3/\text{min}$

$\therefore$ Volume is increasing at the rate of $44\ \text{cm}^3/\text{min}$.

**[2]**

## Short Answer Type Questions (3 or 4 marks each)

**Q. 1. The volume of a cube is increasing at the rate of 8 cm³/s. How fast is the surface area increasing when the length of an edge is 12 cm?**
**[NCERT Ex. 6.1, Q. 2, Page 197]**

**Ans.** Let $x$ be the length of a side, $V$ be the volume, and $S$ be the surface area of the cube.

Then, $V = x^3$ and $S = 6x^2$ where $x$ is a function of time $t$.

It is given that,

$$\frac{dV}{dt} = 8 \text{ cm}^3/\text{s}$$

Then, by using the chain rule, we have

$$\therefore \quad 8 = \frac{dV}{dt} = \frac{d}{dt}(x^3) = \frac{d}{dx}(x^3)\cdot\frac{dx}{dt} = 3x^2\cdot\frac{dx}{dt}$$

$$\therefore \frac{dx}{dt} = \frac{8}{3x^2}$$

Now,

$$\frac{dS}{dt} = \frac{d}{dt}(6x^2) = \frac{d}{dx}(6x^2)\cdot\frac{dx}{dt} \quad \text{[By chain rule]}$$

$$= 12x\cdot\frac{dx}{dt}$$

$$= 12x\cdot\left(\frac{8}{3x^2}\right)$$

$$= \frac{32}{x}$$

Thus, when $x = 12$ cm, $\frac{dS}{dt} = \frac{32}{12}\text{ cm}^2/\text{s} = \frac{8}{3}\text{cm}^2/\text{s}$.

Hence, if the length of the edge of the cube is 12 cm, then the surface area is increasing at the rate of $\frac{8}{3}$ cm²/s. **[3]**

**Q. 2. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetre of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm. [NCERT Ex. 6.1, Q. 8, Page 198]**

**Ans.** The volume of a sphere ($V$) with radius ($r$) is given by, $V = \frac{4}{3}\pi r^3$

∴ Rate of change of volume ($V$) with respect to time ($t$) is given by,

$$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} \quad \text{[By chain rule]}$$

$$= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)\cdot\frac{dr}{dt}$$

$$= 4\pi r^2\cdot\frac{dr}{dt}$$

It is given that, $\frac{dV}{dt} = 900 \text{ cm}^3/\text{s}$

$$\therefore \quad 900 = 4\pi r^2\cdot\frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{900}{4\pi r^2} = \frac{225}{\pi r^2}$$

Therefore, when radius = 15 cm,

$$\frac{dr}{dt} = \frac{225}{\pi(15)^2} = \frac{1}{\pi}$$

Hence, the rate at which the radius of the balloon increases when the radius is 15 cm is $\frac{1}{\pi}$ cm/s. **[3]**

**Q. 3. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?**
**[NCERT Ex. 6.1, Q. 10, Page 198]**

**Ans.** Let $y$ metre be the height of the wall at which the ladder touches. Also, let the foot of the ladder be $x$ metre away from the wall.

Then, by Pythagoras theorem, we have

$x^2 + y^2 = 25$ [Length of the ladder = 5 m]

$$\Rightarrow \quad y = \sqrt{25 - x^2}$$

Then, the rate of change of height ($y$) with respect to time ($t$) is given by,

$$\frac{dy}{dt} = \frac{-x}{\sqrt{25-x^2}}\cdot\frac{dx}{dt}$$

It is given that, $\frac{dx}{dt} = 2$ cm/s

$$\therefore \quad \frac{dy}{dt} = \frac{-2x}{\sqrt{25-x^2}}$$

Now, when $x = 4$ m, we have

$$\frac{dy}{dt} = \frac{-2\times 4}{\sqrt{25-4^2}} = -\frac{8}{3}$$

Hence, the height of the ladder on the wall is decreasing at the rate of $\frac{8}{3}$ cm/s. **[3]**

**Q. 4. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the $y$-coordinate is changing 8 times as fast as the $x$-coordinate.**
**[NCERT Ex. 6.1, Q. 11, Page 198]**

**Ans.** The equation of the curve is given as,

$$6y = x^3 + 2$$

The rate of change of the position of the particle with respect to time ($t$) is given by,

$$6\frac{dy}{dt} = 3x^2\frac{dx}{dt} + 0$$

$$\Rightarrow \quad 2\frac{dy}{dt} = x^2\frac{dx}{dt}$$

When the $y$-coordinate of the particle changes 8 times as fast as the $x$-coordinate, *i.e.*,

$\left(\frac{dy}{dt} = 8\frac{dx}{dt}\right)$, we have

When $x = 4, y = \frac{4^3+2}{6} = \frac{66}{6} = 11$.

When $x = -4, y = \frac{(-4)^3+2}{6} = -\frac{62}{6} = -\frac{31}{3}$.

Hence, the points required on the curve are (4, 11) and $\left(-4, \frac{-31}{3}\right)$. **[3]**

**Q. 5. The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?**

**[NCERT Ex. 6.1, Q. 12, Page 198]**

**Ans.** The air bubble is in the shape of a sphere.

Now, the volume of an air bubble (*V*) with radius (*r*) is given by, $V = \frac{4}{3}\pi r^3$

The rate of change of volume (*V*) with respect to time (*t*) is given by,

$$\frac{dV}{dt} = \frac{4}{3}\pi \frac{d}{dr}\left(r^3\right)\cdot\frac{dr}{dt} \quad \text{[By chain rule]}$$

$$= \frac{4}{3}\pi\left(3r^2\right)\frac{dr}{dt}$$

$$= 4\pi r^2 \frac{dr}{dt}$$

It is given that,

$$\frac{d_r}{d_t} = \frac{1}{2} \text{ cm/s}$$

Therefore, when r = 1 cm,

$$\frac{dV}{dt} = 4\pi(1)^2\left(\frac{1}{2}\right) = 2\pi \text{ cm}^3\text{/s}$$

Hence, the rate at which the volume of the bubble increases is $2\pi$ cm$^3$/s. **[3]**

**Q. 6. The total cost *C*(*x*) in Rupees associated with the production of x units of an item is given by $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$. Find the marginal cost when 17 units are produced.**

**[NCERT Ex. 6.1, Q. 15, Page 198]**

**Ans.** Marginal cost (*MC*) is the rate of change of total cost with respect to output. Therefore,

$$MC = \frac{dC}{dx}$$

$$= 0.007\left(3x^2\right) - 0.003(2x) + 15$$

$$= 0.021x^2 - 0.006x + 15$$

When $x = 17$, we have

$$MC = 0.021\left(17^2\right) - 0.006(17) + 15$$

$$= 0.021(289) - 0.006(17) + 15$$

$$= 6.069 - 0.102 + 15$$

$$= 20.967$$

Hence, when 17 units are produced, the marginal cost is ₹ 20.967. **[3]**

**Q. 7. Find the intervals in which the function *f* given by $f(x) = 2x^2 - 3x$ is**

**(a) Strictly increasing. (b) Strictly decreasing.**

**[NCERT Ex. 6.2, Q. 4, Page 205]**

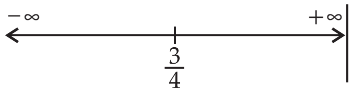
**Ans.** The given function is $f(x) = 2x^2 - 3x$.

$$f'(x) = 4x - 3$$

$$\therefore f'(x) = 0$$

$$\Rightarrow \quad x = \frac{3}{4}$$

Now, the point $\frac{3}{4}$ divides the real line into two disjoint intervals, *i.e.*, $\left(-\infty, \frac{3}{4}\right)$ and $\left(\frac{3}{4}, \infty\right)$.




In interval $\left(-\infty, \frac{3}{4}\right)$, $f'(x) = 4x - 3 < 0$.

Hence, the given function (*f*) is strictly decreasing in interval $\left(-\infty, \frac{3}{4}\right)$.

In interval $\left(\frac{3}{4}, \infty\right)$, $f'(x) = 4x - 3 > 0$.

Hence, the given function (*f*) is strictly increasing in interval $\left(\frac{3}{4}, \infty\right)$. **[3]**

**Q. 8. Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is**

**(a) Strictly increasing. (b) Strictly decreasing.**

**[NCERT Ex. 6.2, Q. 5, Page 205]**

**Ans.** The given function is $f(x) = 2x^3 - 3x^2 - 36x + 7$.

$$f'(x) = 6x^2 - 6x - 36$$

$$= 6(x^2 - x - 6)$$

$$= 6(x+2)(x-3)$$

$$\therefore \quad f'(x) = 0$$

$$\Rightarrow x = -2, 3$$

The points $x = -2$ and $x = 3$ divide the real line into three disjoint intervals, *i.e.*, $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$.


In intervals $(-\infty, -2)$ and $(3, \infty)$, $f'(x)$ is positive while in interval $(-2, 3)$, $f'(x)$ is negative.

Hence, the given function (*f*) is strictly increasing in intervals $(-\infty, -2)$ and $(3, \infty)$, while function (*f*) is strictly decreasing in interval $(-2, 3)$. **[3]**

**Q. 9. Find the intervals in which the following functions are strictly increasing or decreasing :**

**(a) $x^2 + 2x - 5$**

**(b) $10 - 6x - 2x^2$**

**(c) $-2x^3 - 9x^2 - 12x + 1$**

**(d) $6 - 9x - x^2$**

**(e) $(x+1)^3(x-3)^3$** **[NCERT Ex. 6.2, Q. 6, Page 205]**

**Ans.** (a) We have,

$$f(x) = x^2 + 2x - 5$$

$$\therefore f'(x) = 2x + 2$$

Now,

$$f'(x) = 0$$

$$\Rightarrow \quad x = -1$$

Point $x = -1$ divides the real line into two disjoint intervals, *i.e.*, $(-\infty, -1)$ and $(-1, \infty)$.

In interval $(-\infty, -1)$, $f'(x) = 2x + 2 < 0$.

$\therefore$ *f* is strictly decreasing in interval $(-\infty, -1)$.

Thus, *f* is strictly decreasing for $x < -1$.

In interval $(-1,\infty)$, $f'(x)=2x+2>0$

$\therefore f$ is strictly increasing in interval $(-1,\infty)$.

Thus, $f$ is strictly increasing for $x > -1$. **[3]**

**(b)** We have,

$f(x) = 10 - 6x - 2x^2$

$\therefore f'(x)=-6-4x$

Now,

$f'(x)=0$

$\Rightarrow x=-\frac{3}{2}$

The point $x=-\frac{3}{2}$ divides the real line into two disjoint intervals, *i.e.*, $\left(-\infty,-\frac{3}{2}\right)$ and $\left(-\frac{3}{2},\infty\right)$.

In interval, *i.e.*, $\left(-\infty,-\frac{3}{2}\right)$ when $x<-\frac{3}{2}$,.

$\therefore$ f is strictly increasing for $x<-\frac{3}{2}$.

In interval *i.e.*, when $x>-\frac{3}{2}$, $f'(x)=-6-4x<0$

$\therefore f$ is strictly decreasing for $x>-\frac{3}{2}$. **[3]**

**(c)** We have,

$f(x)=-2x^3-9x^2-12x+1$

$\therefore f'(x)=-6x^2-18x-12$

$=-6(x^2+3x+2)$

$=-6(x+1)(x+2)$

Now,

$f'(x)=0 \Rightarrow x=-1$ and $x=-2$

Points $x = -1$ and $x = -2$ divide the real line into three disjoint intervals, *i.e.*, $(-\infty,-2),(-2,-1)$ and $(-1,\infty)$.

In intervals $(-\infty,-2)$ and $(-1,\infty)$, *i.e.*, when $x < -2$ and $x > -1$,

$f'(x)=-6(x+1)(x+2)<0$

$\therefore$ f is strictly decreasing for $x < -2$ and $x > -1$.

Now, in interval $(-2, -1)$, *i.e.*, when $-2 < x < -1$, $f'(x)=-6(x+1)(x+2)>0$.

$\therefore f$ is strictly increasing for $-2<x<-1$. **[3]**

**(d)** We have,

$f(x)=6-9x-x^2$

$\therefore f'(x)=-9-2x$

Now, $f'(x)=0$ gives $x=-\frac{9}{2}$

The point $x=-\frac{9}{2}$ divides the real line into two disjoint intervals, *i.e.*, $\left(-\infty,-\frac{9}{2}\right)$ and $\left(-\frac{9}{2},\infty\right)$.

In interval $\left(-\infty,-\frac{9}{2}\right)$, *i.e.*, for, $x<-\frac{9}{2}$,

$f'(x)=-9-2x>0$.

$\therefore$ f is strictly increasing for $x<-\frac{9}{2}$.

In interval $\left(-\frac{9}{2},\infty\right)$, *i.e.*, for, $x>-\frac{9}{2}$,

$f'(x)=-9-2x<0$.

$\therefore f$ is strictly decreasing for $x>-\frac{9}{2}$. **[3]**

**(e)** We have,

$f(x)=(x+1)^3(x-3)^3$

$f'(x)=3(x+1)^2(x-3)^3+3(x-3)^2(x+1)^3$

$=3(x+1)^2(x-3)^2[x-3+x+1]$

$=3(x+1)^2(x-3)^2(2x-2)$

$=6(x+1)^2(x-3)^2(x-1)$

Now,

$f'(x)=0$

$\Rightarrow x=-1, 3, 1$

The points $x = -1$, $x = 1$, and $x = 3$ divide the real line into four disjoint intervals, *i.e.*, $(-\infty,-1)$, $(-1, 1)$, $(1, 3)$ and $(3,\infty)$.

In intervals $(-\infty,-1)$ and $(-1, 1)$,

$f'(x)=6(x+1)^2(x-3)^2(x-1)<0$.

$\therefore f$ is strictly decreasing in intervals $(-\infty,-1)$ and $(-1, 1)$.

In intervals $(1, 3)$ and $(3,\infty)$,

$f'(x)=6(x+1)^2(x-3)^2(x-1)>0$

$\therefore f$ is strictly increasing in intervals $(1, 3)$ and $(3,\infty)$. **[3]**

**Q. 10. Show that $y=\log(1+x)-\frac{2x}{2+x}, x>-1$ is an increasing function of $x$ throughout its domain.**

**[NCERT Ex. 6.2, Q. 7, Page 205]**

**Ans.** We have,

$y=\log(1+x)-\frac{2x}{2+x}$

Differentiating both sides w.r.t. $x$, we have

$$\frac{dy}{dx}=\frac{1}{1+x}-\frac{(2+x).2-(2x).1}{(2+x)^2}$$

$$=\frac{1}{1+x}-\frac{4}{(2+x)^2}$$

$$=\frac{4+x^2+4x-4-4x}{(1+x)(2+x)^2}$$

$$=\frac{x^2}{(1+x)(2+x)^2}$$

Now,

$\Rightarrow \frac{x^2}{(1+x)(2+x)^2}=0$

$\Rightarrow \quad x^2=0 \qquad [(2+x)\neq 0 \text{ as } x>-1]$

$\therefore \quad x=0$

Since $x > -1$, point $x = 0$ divides the domain $(-1, \infty)$ in two disjoint intervals, *i.e.*, $-1 < x < 0$ and $x > 0$.

When $-1 < x < 0$, we have

$$x < 0$$
$$\Rightarrow \quad x^2 > 0$$
$$x > -1$$
$$\Rightarrow (2 + x) > 0$$
$$\Rightarrow (2 + x^2) > 0$$
$$\therefore y' = \frac{x^2}{(1+x)(2+x)^2} > 0$$

Also, when $x > 0$ we have

$$x > 0$$
$$\Rightarrow x^2 > 0, (2+x)^2 > 0$$
$$\therefore \quad y' = \frac{x^2}{(1+x)(2+x)^2} > 0$$

Hence, function $f$ is increasing throughout this domain. **[3]**

**Q. 11. Find the values of $x$ for which $y = [x(x-2)]^2$ is an increasing function.**

**[NCERT Ex. 6.2, Q. 8, Page 205]**

**Ans.** We have,

$$y = [x(x-2)]^2$$
$$= [x^2 - 2x]^2$$

Therefore,

$$\frac{dy}{dx} = y'$$
$$= 2(x^2 - 2x)(2x - 2)$$
$$= 4x(x-2)(x-1)$$

Therefore,

$$\frac{dy}{dx} = 0$$
$$x = 0, x = 2, x = 1$$

The points $x = 0, x = 1$ and $x = 2$ divide the real line into four disjoint intervals, *i.e.*, $(-\infty, 0)$, $(0, 1)$, $(1, 2)$ and $(2, \infty)$.

In intervals $(-\infty, 0)$ and $(1, 2)$, $\frac{dy}{dx} < 0$.

$\therefore$ $y$ is strictly decreasing in intervals $(-\infty, 0)$ and $(1, 2)$.

However, in intervals $(0, 1)$ and $(2, \infty)$, $\frac{dy}{dx} > 0$.

$\therefore$ $y$ is strictly increasing in intervals $(0, 1)$ and $(2, \infty)$.

$y$ is strictly increasing for $0 < x < 1$ and $x > 2$. **[3]**

**Q. 12. Find the slope of the normal to the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$.**

**[NCERT Ex. 6.3, Q. 5, Page 211]**

**Ans.** It is given that,

$$x = a\cos^3\theta \text{ and } y = a\sin^3\theta$$
$$\therefore \frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta)$$
$$= -3a\cos^2\theta\sin\theta$$
$$\frac{dy}{d\theta} = 3a\sin^2\theta(\cos\theta)$$
$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$
$$= \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$
$$= -\frac{\sin\theta}{\cos\theta}$$
$$= -\tan\theta$$

Therefore, the slope of the tangent at $\theta = \frac{\pi}{4}$ is given by,

$$\left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{4}} = -\tan\theta\Big]_{\theta=\frac{\pi}{4}}$$
$$= -\tan\frac{\pi}{4}$$
$$= -1$$

Hence, the slope of the normal at $\theta = \frac{\pi}{4}$ is given by,

$$\frac{1}{\text{Slope of the tanget at } \theta = \frac{\pi}{4}} = \frac{-1}{-1} = 1$$

**[3]**

**Q. 13. Find the slope of the normal to the curve $x = 1 - a\sin\theta$, $y = b\cos^2\theta$ at $\theta = \frac{\pi}{2}$.**

**[NCERT Ex. 6.3, Q. 6, Page 211]**

**Ans.** It is given that,

$$x = 1 - a\sin\theta \text{ and } y = b\cos^2\theta$$
$$\therefore \frac{dx}{d\theta} = -a\cos\theta \text{ and}$$
$$\frac{dx}{d\theta} = -2b\cos\theta(-\sin\theta) = -2b\sin\theta\cos\theta$$
$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-2b\sin\theta\cos\theta}{-a\cos\theta} = \frac{2b}{a}\sin\theta$$

Therefore, the slope of the tangent at $\theta = \frac{\pi}{2}$ is given by,

$$\left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{2}} = \frac{2b}{a}\sin\theta\Big]_{\theta=\frac{\pi}{2}} = \frac{2b}{a}\sin\frac{\pi}{2} = \frac{2b}{a}$$

Hence, the slope of the normal at $\theta = \frac{\pi}{2}$ is given by,

$$\frac{1}{\text{Slope of the tanget at } \theta = \frac{\pi}{4}} = \frac{-1}{\left(\frac{2b}{a}\right)} = -\frac{a}{2b}$$

**[3]**

**Q. 14. Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the $x$-axis.**

**[NCERT Ex. 6.3, Q. 7, Page 211]**

**Ans.** The equation of the given curve is

$$y = x^3 - 3x^2 - 9x + 7$$
$$\therefore \quad \frac{dy}{dx} = 3x^2 - 6x - 9$$

Now, the tangent is parallel to the $x$-axis if the slope of the tangent is zero.

$\therefore\ 3x^2 - 6x - 9 = 0$

$\Rightarrow\ x^2 - 2x - 3 = 0$

$\Rightarrow (x-3)(x+1) = 0$

$\Rightarrow\ x = 3$ or $x = -1$

When $x = 3$, we have

$$y = (3)^3 - 3(3)^2 - 9(3) + 7$$
$$= 27 - 27 - 27 + 7 = -20.$$

When $x = -1$, we have

$$y = (-1) - 3(-1) - 9(-1) + 7$$
$$= -1 - 3 + 9 + 7 = 12.$$

Hence, the points at which the tangent is parallel to the $x$-axis are (3, −20) and (−1, 12). **[3]**

**Q. 15. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.**

**[NCERT Ex. 6.3, Q. 9, Page 212]**

**Ans.** The equation of the given curve is

$y = x^3 - 11x + 5$

The equation of the tangent to the given curve is given as $y = x - 11$ (which is of the form $y = mx + c$). Therefore, slope of the tangent = 1.

Now, the slope of the tangent to the given curve at the point $(x, y)$ is given by,

$$\frac{dy}{dx} = 3x^2 - 11$$

Then, we have

$3x^2 - 11 = 1$

$\Rightarrow\ 3x^2 = 12$

$\Rightarrow\ x^2 = 4$

$\therefore\ x = \pm 2$

When $x = 2$, $y = (2)^3 - 11(2) + 5 = 8 - 22 + 5 = -9$.

When $x = -2$, $y = (-2)^3 - 11(-2) + 5 = -8 + 22 + 5 = 19$.

Hence, the required points are (2, −9) and (−2, 19). But, both of these points should satisfy the equation of the tangent as there would be point of contact between tangent and the curve. Therefore, (2, −9) is the required point as (−2, 19) is not satisfying the given equation of tangent. **[3]**

**Q. 16. Find the equations of all lines having slope 0 which are tangent to the curve $y = \frac{1}{x^2 - 2x + 3}$.**

**[NCERT Ex. 6.3, Q. 12, Page 212]**

**Ans.** The equation of the given curve is $y = \frac{1}{x^2 - 2x + 3}$.

The slope of the tangent to the given curve at any point $(x, y)$ is given by,

$$\frac{dy}{dx} = \frac{-(2x-2)}{(x^2 - 2x + 3)^2}$$
$$= \frac{-2(x-1)}{(x^2 - 2x + 3)^2}$$

If the slope of the tangent is 0, then we have

$$\frac{-2(x-1)}{(x^2 - 2x + 3)^2} = 0$$

$\Rightarrow\ -2(x-1) = 0$

$\therefore\ x = 1$

When $x = 1$, $y = \frac{1}{1 - 2 + 3} = \frac{1}{2}$.

$\therefore$The equation of the tangent through $\left(1, \frac{1}{2}\right)$ is given by,

$$y - \frac{1}{2} = 0(x-1)$$

$\Rightarrow\ y - \frac{1}{2} = 0$

$\therefore\ y = \frac{1}{2}$

Hence, the equation of the required line is $y = \frac{1}{2}$. **[3]**

**Q. 17. Find the equations of the tangent and normal to the given curves at the indicated points :**

**(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5)**

**(ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (1, 3)**

**(iii) $y = x^3$ at (1, 1)**

**(iv) $y = x^2$ at (0, 0)**

**(v) $x = \cos t,\ y = \sin t$ at $t = \frac{\pi}{4}$**

**[NCERT Ex. 6.3, Q. 14, Page 212]**

**Ans.** **(i) The equation of the curve is**

$$y = x^4 - 6x^3 + 13x^2 - 10x + 5.$$

On differentiating with respect to $x$, we get

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

$$\left.\frac{dy}{dx}\right]_{(0,5)} = -10$$

Thus, the slope of the tangent at (0, 5) is −10. The equation of the tangent is given as

$y - 5 = -10(x - 0)$

$\Rightarrow\ y - 5 = -10x$

$\Rightarrow 10x + y = 5$

The slope of the normal at (0, 5) is $\frac{-1}{\text{Slope of the tangent at (0, 5)}} = \frac{1}{10}$.

Therefore, the equation of the normal at (0, 5) is given as

$$y - 5 = \frac{1}{10}(x - 0)$$

$\Rightarrow\ 10y - 50 = x$

$\Rightarrow x - 10y + 50 = 0$ **[3]**

**(ii)** The equation of the curve is $y = x^4 - 6x^3 + 13x^2 - 10x + 5$

On differentiating with respect to $x$, we get

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

$$\left.\frac{dy}{dx}\right]_{(1,3)} = 4 - 18 + 26 - 10 = 2$$

Thus, the slope of the tangent at (1, 3) is 2. The equation of the tangent is given as

$y-3=2(x-1)$

$\Rightarrow y-3=2x-2$

$\Rightarrow y-2x+1$

The slope of the normal at (1, 3) is $\dfrac{-1}{\text{Slope of the tangent at }(1,3)}=\dfrac{-1}{2}$.

Therefore, the equation of the normal at (1, 3) is given as

$$y-3=-\frac{1}{2}(x-1)$$

$\Rightarrow\quad 2y-6=-x+1$

$\Rightarrow x+2y-7=0$ **[3]**

**(iii)** The equation of the curve is $y = x^3$.

On differentiating with respect to $x$, we get :

$$\frac{dy}{dx}=3x^2$$

$$\left.\frac{dy}{dx}\right]_{(1,1)}=3(1)^2=3$$

Thus, the slope of the tangent at (1, 1) is 3 and the equation of the tangent is given as

$y-1=3(x-1)$

$\Rightarrow\quad y=3x-2$

The slope of the normal at (1, 1) is $\dfrac{-1}{\text{Slope of the tangent at }(1,1)}=\dfrac{-1}{3}$.

Therefore, the equation of the normal at (1, 1) is given as

$$y-1=\frac{-1}{3}(x-1)$$

$\Rightarrow\quad 3y-3=-x+1$

$\Rightarrow x+3y-4=0$ **[3]**

**(iv)** The equation of the curve is $y = x^2$.

On differentiating with respect to $x$, we get :

$$\frac{dy}{dx}=2x$$

$$\left.\frac{dy}{dx}\right]_{(0,0)}=0$$

Thus, the slope of the tangent at (0, 0) is 0 and the equation of the tangent is given as

$y - 0 = 0\,(x - 0)$

$\Rightarrow\quad y = 0$

The slope of the normal at (0, 0) is

$\dfrac{-1}{\text{Slope of the tangent at }(0,0)}=-\dfrac{1}{0}$ which is not defined.

Therefore, the equation of the normal at $(x, y) = (0, 0)$ is given by $x=0$. **[3]**

**(v)** The equation of the curve is $x=\cos t$ and $y=\sin t$.

$$\therefore\quad \frac{dx}{dt}=-\sin t,\frac{dy}{dt}=\cos t$$

$$\therefore\quad \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\cos t}{-\sin t}=-\cot t$$

$$\left.\frac{dy}{dx}\right]_{t=\frac{\pi}{4}}=-\cot t=-1$$

$\therefore$ The slope of the tangent at $t=\dfrac{\pi}{4}$ is $-1$.

When $t=\dfrac{\pi}{4}$, $x=\dfrac{1}{\sqrt{2}}$ and $y=\dfrac{1}{\sqrt{2}}$.

Thus, the equation of the tangent to the given curve at $t=\dfrac{\pi}{4}$, *i.e.*, at $\left[\left(\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right)\right]$ is

$$y-\frac{1}{\sqrt{2}}=-1\left(x-\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow x+y-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0$$

$$\Rightarrow\quad x+y-\sqrt{2}=0$$

The slope of the normal at $t=\dfrac{\pi}{4}$ is $\dfrac{-1}{\text{Slope of the tangent at }t=\frac{\pi}{4}}=1$.

Therefore, the equation of the normal to the given curve at $t=\dfrac{\pi}{4}$, *i.e.*, $\left[\left(\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right)\right]$ at is

$$y-\frac{1}{\sqrt{2}}=1\left(x-\frac{1}{\sqrt{2}}\right)$$

$\Rightarrow\quad x=y$ **[3]**

**Q. 18. Find the points on the curve $x^2+y^2-2x-3=0$ at which the tangents are parallel to the x-axis.**

**[NCERT Ex. 6.3, Q. 19, Page 212]**

**Ans.** The equation of the given curve is $x^2 + y^2 - 2x - 3 = 0$.

On differentiating with respect to $x$, we have

$$2x+2y\frac{dy}{dx}-2=0$$

$$\Rightarrow\quad y\frac{dy}{dx}=1-x$$

$$\therefore\quad \frac{dy}{dx}=\frac{1-x}{y}$$

Now, the tangents are parallel to the $x$-axis if the slope of the tangent is 0.

$$\because\ \frac{1-x}{y}=0$$

$\Rightarrow 1-x=0$

$\therefore\quad x=1$

But, $x^2 + y^2 - 2x - 3 = 0$ for $x = 1$.

$\Rightarrow y^2=4$

$\Rightarrow y=\pm 2$

Hence, the points at which the tangents are parallel to the $x$-axis are (1, 2) and (1, −2). **[3]**

**Q. 19. Find the equation of the normal at the point $\left(am^2,am^3\right)$ for the curve $ay^2=x^3$.**

**[NCERT Ex. 6.3, Q. 20, Page 212]**

**Ans.** The equation of the given curve is $ay^2 = x^3$.

On differentiating with respect to $x$, we have

$$2ay\frac{dy}{dx}=3x^2$$

$$\Rightarrow \frac{dy}{dx}=\frac{3x^2}{2ay}$$

The slope of the tangent to the given curve at ($am^2$, $am^3$) is

$$\left.\frac{dy}{dx}\right]_{(am^2,am^3)}=\frac{3(am^2)^2}{2a(am^3)}=\frac{3a^2m^4}{2a^2m^3}=\frac{3m}{2}.$$

$\therefore$ Slope of normal at ($am^2$, $am^3$) =

$$\frac{-1}{\text{Slope of the taangent at }(am^2,am^3)}=\frac{-2}{3m}$$

Hence, the equation of the normal at ($am^2$, $am^3$) is given by,

$$y-am^3=\frac{-2}{3m}(x-am^2)$$

$$\Rightarrow \quad 3my-3am^4=-2x+2am^2$$

$$\Rightarrow 2x+3my-am^2(2+3m^2)=0$$

**[3]**

**Q. 20. Find the equations of the tangent and normal to the parabola $y^2=4ax$ at the point ( $at^2$, $2at$ ).**

**[NCERT Ex. 6.3, Q. 22, Page 213]**

**Ans.** As we know that the equation of the given parabola is $y^2=4ax$.

On differentiating $y^2=4ax$ with respect to $x$, we have

$$2y\frac{dy}{dx}=4a$$

$$\therefore \frac{dy}{dx}=\frac{2a}{y}$$

$\therefore$ The slope of the tangent at $(at^2,2at)$ is $\left.\frac{dy}{dx}\right]_{(at^2,2at)}=\frac{2a}{2at}=\frac{1}{t}$.

Then, the equation of the tangent at $(at^2,2at)$ is given by,

$$y-2at=\frac{1}{t}(x-at^2)$$

$$\Rightarrow ty-2at^2=x-at^2$$

$$\Rightarrow \quad ty=x+at^2$$

Now, the slope of the normal at $(at^2,2at)$ is given by,

$$\frac{-1}{\text{Slope of the tangent at }(at^2,2at)}=-t$$

Thus, the equation of the normal at ($at^2$, $2at$) is given as :

$$y-2at=-t(x-at^2)$$

$$\Rightarrow y-2at=-tx+at^3$$

$$\therefore \quad y=-tx+2at+at^3$$

**[3]**

**Q. 21. Using differentials, find the approximate value of each of the following up to 3 places of decimal.**

**(i)** $\sqrt{25.3}$ **(ii)** $\sqrt{49.5}$

**(iii)** $\sqrt{0.6}$ **(iv)** $(0.009)^{\frac{1}{3}}$

**(v)** $(0.999)^{\frac{1}{10}}$ **(vi)** $(15)^{\frac{1}{4}}$

**(vii)** $(26)^{\frac{1}{3}}$ **(viii)** $(255)^{\frac{1}{4}}$

**(ix)** $(82)^{\frac{1}{4}}$ **(x)** $(401)^{\frac{1}{2}}$

**(xi)** $(0.0037)^{\frac{1}{2}}$ **(xii)** $(26.57)^{\frac{1}{3}}$

**(xiii)** $(81.5)^{\frac{1}{4}}$ **(xiv)** $(3.968)^{\frac{3}{2}}$

**(xv)** $(32.15)^{\frac{1}{5}}$

**[NCERT Ex. 6.4, Q. 1, Page 216]**

**Ans. (i)** $\sqrt{25.3}$

Consider $y=\sqrt{x}$

Let $x$ = 25 and $\Delta x$ = 0.3. Then,

$$\Delta y=\sqrt{x+\Delta x}-\sqrt{x}$$
$$=\sqrt{25.3}-\sqrt{25}$$
$$=\sqrt{25.3}-5$$

$$\sqrt{25.3}=\Delta y+5$$

Now, $dy$ is approximately equal to $\Delta y$ and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x$$
$$=\frac{1}{2\sqrt{x}}(0.3) \qquad \left[\because y=\sqrt{x}\right]$$
$$=\frac{1}{2\sqrt{25}}(0.3)$$
$$=0.03$$

Hence, the approximate value of $\sqrt{25.3}$ is 0.03 + 5 = 5.03. **[3]**

**(ii)** $\sqrt{49.5}$

Consider $y=\sqrt{x}$

Let $x$ = 49 and $\Delta x$ = 0.5, then,

$$\Delta y=\sqrt{x+\Delta x}-\sqrt{x}$$
$$=\sqrt{49.5}-\sqrt{49}$$
$$=\sqrt{49.5}-7$$

$$\sqrt{49.5}=7+\Delta y$$

Now, $dy$ is approximately equal to $\Delta y$ and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x$$
$$=\frac{1}{2\sqrt{x}}(0.5) \qquad \left[\because y=\sqrt{x}\right]$$
$$=\frac{1}{2\sqrt{49}}(0.5)$$
$$=\frac{1}{14}(0.5)$$
$$=0.035$$

Hence, the approximate value of $\sqrt{49.5}$ is 7 + 0.035 = 7.035. **[3]**

**(iii)** $\sqrt{0.6}$

Consider $y=\sqrt{x}$.

Let $x = 1$ and $\Delta x = -0.4$. Then,

$$\Delta y = \sqrt{x+\Delta x} - \sqrt{x}$$
$$= \sqrt{0.6} - 1$$
$$\sqrt{0.6} = 1 + \Delta y$$

Now, dy is approximately equal to $\Delta y$ and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$
$$= \frac{1}{2\sqrt{x}}(\Delta x) \qquad \left[\because y = \sqrt{x}\right]$$
$$= \frac{1}{2}(-0.4)$$
$$= -0.2$$

Hence, the approximate value of $\sqrt{0.6}$ is $1 + (-0.2) = 1 - 0.2 = 0.8$. **[3]**

**(iv)** $(0.009)^{\frac{1}{3}}$

Consider $y = x^{\frac{1}{3}}$.
Let $x = .008$ and $\Delta x = 0.001$. Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{3}} - (x)^{\frac{1}{3}}$$
$$= (0.009)^{\frac{1}{3}} - (0.008)^{\frac{1}{3}}$$
$$= (0.009)^{\frac{1}{3}} - 0.2$$
$$(0.009)^{\frac{1}{3}} = 0.2 + \Delta y$$

Now, dy is approximately equal to $\Delta y$ and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$
$$= \frac{1}{3(x)^{\frac{2}{3}}}(\Delta x) \qquad \left[\because y = x^{\frac{1}{3}}\right]$$
$$= \frac{1}{3 \times 0.04}(0.001)$$
$$= \frac{0.001}{0.12}$$
$$= 0.008$$

Hence, the approximate value of $(0.009)^{\frac{1}{3}}$ is $0.2 + 0.008 = 0.208$. **[3]**

**(v)** $(0.999)^{\frac{1}{10}}$

Consider $y = (x)^{\frac{1}{10}}$
Let $x = 1$ and $\Delta x = -0.001$. Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{10}} - (x)^{\frac{1}{10}}$$
$$= (0.999)^{\frac{1}{10}} - 1$$
$$(0.999)^{\frac{1}{10}} = 1 + \Delta y$$

Now, $dy$ is approximately equal to $\Delta y$ and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$
$$= \frac{1}{10(x)^{\frac{9}{10}}}(\Delta x) \qquad \left[\because y = (x)^{\frac{1}{10}}\right]$$
$$= \frac{1}{10}(-0.001)$$
$$= -0.0001$$

Hence, the approximate value of $(0.999)^{\frac{1}{10}}$ is $1 + (-0.0001) = 0.9999$. **[3]**

**(vi)** $(15)^{\frac{1}{4}}$

Consider $y = x^{\frac{1}{4}}$.
Let $x = 16$ and $\Delta x = -1$. Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}}$$
$$= (15)^{\frac{1}{4}} - (16)^{\frac{1}{4}}$$
$$= (15)^{\frac{1}{4}} - 2$$
$$(15)^{\frac{1}{4}} = 2 + \Delta y$$

Now, $dy$ is approximately equal to $\Delta y$ and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$
$$= \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[\because y = x^{\frac{1}{4}}\right]$$
$$= \frac{1}{4(16)^{\frac{3}{4}}}(-1)$$
$$= \frac{-1}{4 \times 8} = \frac{-1}{32} = -0.03125$$

Hence, the approximate value of $(15)^{\frac{1}{4}}$ is $2 + (-0.03125) = 1.96875$. **[3]**

**(vii)** $(26)^{\frac{1}{3}}$

Consider $y = (x)^{\frac{1}{3}}$. Let $x = 27$ and $\Delta x = -1$. Then,

$$\Delta y = (x+\Delta x)^{\frac{1}{3}} - (x)^{\frac{1}{3}}$$
$$= (26)^{\frac{1}{3}} - (27)^{\frac{1}{3}}$$
$$= (26)^{\frac{1}{3}} - 3$$
$$(26)^{\frac{1}{3}} = 3 + \Delta y$$

Now, $dy$ is approximately equal to $\Delta y$ and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$
$$= \frac{1}{3(x)^{\frac{2}{3}}}(\Delta x) \qquad \left[\because y = (x)^{\frac{1}{3}}\right]$$
$$= \frac{1}{3(27)^{\frac{2}{3}}}(-1)$$
$$= \frac{-1}{27} = -0.0370$$

Hence, the approximate value of $(26)^{\frac{1}{3}}$ is $3 + (-0.0370) = 2.9629$. **[3]**

**(viii)** $(255)^{\frac{1}{4}}$

Consider $y=(x)^{\frac{1}{4}}$. Let $x$ = 256 and $\Delta x$ = −1. Then,

$$\Delta y=(x+\Delta x)^{\frac{1}{4}}-(x)^{\frac{1}{4}}$$
$$=(255)^{\frac{1}{4}}-(256)^{\frac{1}{4}}$$
$$=(255)^{\frac{1}{4}}-4$$
$$(255)^{\frac{1}{4}}=4+\Delta y$$

Now, $dy$ is approximately equal to $\Delta y$ and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x$$
$$=\frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[\because\ y=x^{\frac{1}{4}}\right]$$
$$=\frac{1}{4(256)^{\frac{3}{4}}}(-1)$$
$$=\frac{-1}{4\times 4^3}=-0.0039$$

Hence, the approximate value of $(255)^{\frac{1}{4}}$ is 4 + (−0.0039) = 3.9961. **[3]**

**(ix)** $(82)^{\frac{1}{4}}$

Consider $y=x^{\frac{1}{4}}$. Let $x$ = 81 and $\Delta x$ = 1. Then,

$$\Delta y=(x+\Delta x)^{\frac{1}{4}}-(x)^{\frac{1}{4}}$$
$$=(82)^{\frac{1}{4}}-(81)^{\frac{1}{4}}$$
$$=(82)^{\frac{1}{4}}-3$$
$$(82)^{\frac{1}{4}}=\Delta y+3$$

Now, $dy$ is approximately equal to $\Delta y$ and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x$$
$$=\frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[\because\ y=x^{\frac{1}{4}}\right]$$
$$=\frac{1}{4(81)^{\frac{3}{4}}}(1)$$
$$=\frac{1}{4(3)^3}=\frac{1}{108}=0.009$$

Hence, the approximate value of $(82)^{\frac{1}{4}}$ is 3 + 0.009 = 3.009. **[3]**

**(x)** $(401)^{\frac{1}{2}}$

Consider $y=x^{\frac{1}{2}}$. Let $x$ = 400 and $\Delta x$ = 1. Then,

$$\Delta y=\sqrt{x+\Delta x}-\sqrt{x}$$
$$=\sqrt{401}-\sqrt{400}$$
$$=\sqrt{401}-20$$
$$\sqrt{401}=20+\Delta y$$

Now, $dy$ is approximately equal to $\Delta y$ and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x$$
$$=\frac{1}{2\sqrt{x}}(\Delta x) \qquad \left[\because\ y=x^{\frac{1}{2}}\right]$$
$$=\frac{1}{2\times 20}(1)$$
$$=\frac{1}{40}=0.025$$

Hence, the approximate value of $\sqrt{401}$ is 20 + 0.025 = 20.025. **[3]**

**(xi)** $(0.0037)^{\frac{1}{2}}$

Consider $y=x^{\frac{1}{2}}$. Let $x$ = 0.0036 and $\Delta x$ = 0.0001. Then,

$$\Delta y=(x+\Delta x)^{\frac{1}{2}}-(x)^{\frac{1}{2}}$$
$$=(0.0037)^{\frac{1}{2}}-(0.0036)^{\frac{1}{2}}$$
$$=(0.0037)^{\frac{1}{2}}-0.06$$
$$(0.0037)^{\frac{1}{2}}=0.06+\Delta y$$

Now, $dy$ is approximately equal to $\Delta y$ and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x$$
$$=\frac{1}{2\sqrt{x}}(\Delta x) \qquad \left[\because y=x^{\frac{1}{2}}\right]$$
$$=\frac{1}{2\times 0.06}(0.0001)$$
$$=\frac{0.0001}{0.12}$$
$$=0.00083$$

Thus, the approximate value of $(0.0037)^{\frac{1}{2}}$ is 0.06 + 0.00083 = 0.06083. **[3]**

**(xii)** $(26.57)^{\frac{1}{3}}$

Consider $y=x^{\frac{1}{3}}$. Let $x$ = 27 and $\Delta x$ = −0.43. Then,

$$\Delta y=(x+\Delta x)^{\frac{1}{3}}-x^{\frac{1}{3}}$$
$$=(26.57)^{\frac{1}{3}}-(27)^{\frac{1}{3}}$$
$$=(26.57)^{\frac{1}{3}}-3$$
$$(26.57)^{\frac{1}{3}}=3+\Delta y$$

Now, $dy$ is approximately equal to $\Delta y$ and is given by,

$$dy=\left(\frac{dy}{dx}\right)\Delta x$$

$$= \frac{1}{3(x)^{\frac{2}{3}}}(\Delta x) \qquad \left[\because y = x^{\frac{1}{3}}\right]$$

$$= \frac{1}{3(9)}(-0.43)$$

$$= \frac{-0.43}{27} = -0.015$$

Hence, the approximate value of $y = x^{\frac{1}{3}}$ is 3 + (−0.015) = 2.984. **[3]**

**(xiii)** $(81.5)^{\frac{1}{4}}$

Consider $y = x^{\frac{1}{4}}$. Let $x$ = 81 and $\Delta x$ = 0.5. Then,

$$\Delta y = (x + \Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}}$$

$$= (81.5)^{\frac{1}{4}} - (81)^{\frac{1}{4}}$$

$$= (81.5)^{\frac{1}{4}} - 3$$

$$(81.5)^{\frac{1}{4}} = 3 + \Delta y$$

Now, $dy$ is approximately equal to $\Delta y$ and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$

$$= \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left[\because y = x^{\frac{1}{4}}\right]$$

$$= \frac{1}{4(3)^3}(0.5)$$

$$= \frac{0.5}{108} = 0.0046$$

Hence, the approximate value of $(81.5)^{\frac{1}{4}}$ is 3 + 0.0046 = 3.0046. **[3]**

**(xiv)** $(3.968)^{\frac{3}{2}}$

Consider $y = x^{\frac{3}{2}}$. Let $x$ = 4 and $\Delta x$ = −0.032. Then,

$$\Delta y = (x + \Delta x)^{\frac{3}{2}} - x^{\frac{3}{2}}$$

$$= (3.968)^{\frac{3}{2}} - (4)^{\frac{3}{2}}$$

$$= (3.968)^{\frac{3}{2}} - 8$$

$$(3.968)^{\frac{3}{2}} = 8 + \Delta y$$

Now, $dy$ is approximately equal to $\Delta y$ and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$

$$= \frac{3}{2}(x)^{\frac{1}{2}}(\Delta x) \qquad \left[\because y = x^{\frac{3}{2}}\right]$$

$$= \frac{3}{2}(2)(-0.032)$$

$$= -0.096$$

Hence, the approximate value of $(3.968)^{\frac{3}{2}}$ is 8 + (−0.096) = 7.904. **[3]**

**(xv)** $(32.15)^{\frac{1}{5}}$

Consider $y = x^{\frac{1}{5}}$. Let $x$ = 32 and $\Delta x$ = 0.15. Then,

$$\Delta y = (x + \Delta x)^{\frac{1}{5}} - x^{\frac{1}{5}}$$

$$= (32.15)^{\frac{1}{5}} - (32)^{\frac{1}{5}}$$

$$= (32.15)^{\frac{1}{5}} - 2$$

$$(32.15)^{\frac{1}{5}} = 2 + \Delta y$$

Now, $dy$ is approximately equal to $\Delta y$ and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$

$$= \frac{1}{5(x)^{\frac{4}{5}}} \cdot (\Delta x) \qquad \left[\because y = x^{\frac{1}{5}}\right]$$

$$= \frac{1}{5 \times (2)^4}(0.15)$$

$$= \frac{0.15}{80} = 0.00187$$

Hence, the approximate value of $(32.15)^{\frac{1}{5}}$ is 2 + 0.00187 = 2.00187. **[3]**

**Q. 22. Find the approximate value of f(2.01), where $f(x) = 4x^2 + 5x + 2$.**

**[NCERT Ex. 6.4, Q. 2, Page 216]**

**Ans.** Let $x$ = 2 and $\Delta x$ = 0.01. Then, we have

$$f(2.01) = f(x + \Delta x)$$

$$= 4(x + \Delta x)^2 + 5(x + \Delta x) + 2$$

Now,

$$\Delta y = f(x + \Delta x) - f(x)$$

$$f(x + \Delta x) = f(x) + \Delta y$$

$$\approx f(x) + f'(x)\Delta x \qquad (\because dx = \Delta x)$$

$$\Rightarrow f(2.01) \approx (4x^2 + 5x + 2) + (8x + 5)\Delta x$$

$$= \left[4(2)^2 + 5(2) + 2\right] + \left[5(2) + 5\right](0.01)$$

$$[\text{as } x = 2, \Delta x = 0.01]$$

$$= (16 + 10 + 2) + (16 + 5)(0.01)$$

$$= 28 + (21)(0.01)$$

$$= 28 + 0.21$$

$$= 28.21$$

Therefore, the approximate value of f(2.01) is 28.21. **[3]**

**Q. 23. Find the approximate value of $f$(5.001), where $f(x) = x^3 - 7x^2 + 15$.**

**[NCERT Ex. 6.4, Q. 3, Page 216]**

**Ans.** Let $x$ = 5 and $\Delta x$ = 0.001. Then, we have

$$f(5.001) = f(x + \Delta x) = (x + \Delta x)^3 - 7(x + \Delta x)^2 + 15$$

Now, $\Delta y = f(x + \Delta x) - f(x)$

$$\therefore f(x + \Delta x) = f(x) + \Delta y$$

$$\approx f(x) + f'(x) \cdot \Delta x \qquad (\text{as } dx = \Delta x)$$

$$\Rightarrow f(5.001) \approx (x^3 - 7x^2 + 15) + (3x^2 - 14x)\Delta x$$
$$= \left[(5)^3 + 7(5)^2 + 15\right] + \left[3(5)^2 - 14(5)\right](0.001)$$
$$[x = 5, \Delta x = 0.001]$$
$$= (125 - 175 + 15) + (75 - 70)(0.001)$$
$$= -35 + (5)(0.001)$$
$$= -35 + 0.005$$
$$= -34.995$$

Hence, the approximate value of f(5.001) is −34.995. **[3]**

**Q. 24. Find two numbers whose sum is 24 and whose product is as large as possible.**

**[NCERT Ex. 6.5, Q. 13, Page 233]**

**Ans.** Let one number be $x$. Then, the other number is $(24 - x)$.

Let $P(x)$ denote the product of the two numbers. Thus, we have

$$P(x) = x(24 - x) = 24x - x^2$$
$$\therefore P'(x) = 24 - 2x$$
$$p''(x) = -2$$

Now,

$$P'(x) = 0$$
$$\Rightarrow \quad x = 12$$

Also,

$$P''(12) = -2 < 0$$

∴By second derivative test, $x = 12$ is the point of local maxima of $P$. Hence, the product of the numbers is the maximum when the numbers are 12 and 24 − 12 = 12. **[3]**

**Q. 25. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.**

**[NCERT Ex. 6.5, Q. 16, Page 233]**

**Ans.** Let one number be $x$. Then, the other number is $(16 - x)$.

Let the sum of the cubes of these numbers be denoted by $S(x)$. Then,

$$S(x) = x^3 + (16 - x)^3$$
$$\therefore \quad S'(x) = 3x^2 - 3(16 - x)^2,$$
$$S''(x) = 6x + 6(16 - x)$$

Now,

$$S'(x) = 0$$
$$\Rightarrow \quad 3x^2 - 3(16 - x)^2 = 0$$
$$\Rightarrow \quad x^2 - (16 - x)^2 = 0$$
$$\Rightarrow \quad x^2 - 256 - x^2 + 32x = 0$$
$$\therefore \quad x = \frac{256}{32} = 8$$

Now, $S''(8) = 6(16 - 8) = 48 + 48 = 96 > 0$

∴ By second derivative test, $x = 8$ is the point of local minima of $S$.

Hence, the sum of the cubes of the numbers is the minimum when the numbers are 8 and 16 − 8 = 8. **[3]**

**Q. 26. Using differentials, find the approximate value of each of the following :**

**(a)** $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ **(b)** $(33)^{-\frac{1}{5}}$

**[NCERT Misc Ex. Q. 1, Page 242]**

**Ans. (a)** Consider $y = x^{\frac{1}{4}}$. Let $x = \frac{16}{81}$ and $\Delta x = \frac{1}{81}$.

Then, $\Delta y = (x + \Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}}$

$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \left(\frac{16}{81}\right)^{\frac{1}{4}}$$
$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \frac{2}{3}$$
$$\therefore \left(\frac{17}{81}\right)^{\frac{1}{4}} = \frac{2}{3} + \Delta y$$

Now, $dy$ is approximately equal to $\Delta y$ and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x$$
$$= \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left(\because y = x^{\frac{1}{4}}\right)$$
$$= \frac{1}{4\left(\frac{16}{81}\right)^{\frac{3}{4}}}\left(\frac{1}{81}\right)$$
$$= \frac{27}{4 \times 8} \times \frac{1}{81}$$
$$= \frac{1}{32 \times 3} = \frac{1}{96} = 0.010$$

Hence, the approximate value of $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ is $\frac{2}{3} + 0.010$ = 0.667 + 0.010 = 0.677. **[3]**

**(b)** Consider $y = x^{\frac{-1}{5}}$

Let $x = 32$ and $\Delta x = 1$. Then,

$$\Delta y = (x + \Delta x)^{-\frac{1}{5}} - x^{-\frac{1}{5}}$$
$$= (33)^{-\frac{1}{5}} - (32)^{-\frac{1}{5}}$$
$$= (33)^{-\frac{1}{5}} - \frac{1}{2}$$
$$\therefore (33)^{-\frac{1}{5}} = \frac{1}{2} + \Delta y$$

Now, $dy$ is approximately equal to $\Delta y$ and is given by,

$$dy = \left(\frac{dy}{dx}\right)(\Delta x)$$
$$= \frac{-1}{5(x)^{\frac{6}{5}}}(\Delta x) \qquad \left(\because y = x^{-\frac{1}{5}}\right)$$
$$= \frac{1}{5(2)^6}(1)$$
$$= \frac{1}{320} = 0.003$$

Hence, the approximate value of $(33)^{-\frac{1}{5}}$ is $\frac{1}{2} + (-0.003)$ = 0.5 − 0.003 = 0.497. **[3]**

**Q. 27. Find the equation of the normal to curve $x^2 = 4y$ which passes through the point (1, 2).**

**[NCERT Misc Ex. Q. 4, Page 242]**

**Ans.** The equation of the given curve is $y^2 = 4x$.
Differentiating with respect to $x$, we have

$$2y\frac{dy}{dx} = 4$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$

$$\therefore \left.\frac{dy}{dx}\right]_{(1,2)} = \frac{2}{2} = 1$$

Now, the slope of the normal at point (1, 2) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(1,2)}} = \frac{-1}{1} = -1$$

$\therefore$Equation of the normal at (1, 2) is $y - 2 = -1(x - 1)$.

$$\Rightarrow \quad y - 2 = -x + 1$$

$$\Rightarrow x + y - 3 = 0$$ **[3]**

**Q. 28. Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}, x \neq 0$ is**
**(i) increasing (ii) decreasing**
**[NCERT Misc Ex. Q. 7, Page 242]**

**Ans.** Given that,

$$f(x) = x^3 + \frac{1}{x^3}$$

$$f'(x) = 3x^2 - \frac{3}{x^4}$$

$$= \frac{3x^6 - 3}{x^4}$$

Then,

$$f'(x) = 0$$

$$\Rightarrow 3x^6 - 3 = 0$$

$$\Rightarrow \quad x^6 = 1$$

$$\therefore \quad x = \pm 1$$

Now, the points $x = 1$ and $x = -1$ divide the real line into three disjoint intervals, *i.e.*, $(-\infty,-1),(-1,1)$ and $(1,\infty)$.

In intervals $(-\infty,-1)$ and $(1,\infty)$, *i.e.*, when $x < -1$ and $x > 1$, $f'(x) > 0$

Thus, when $x < -1$ and $x > 1$, $f$ is increasing.

In interval $(-1, 1)$ *i.e.*, when $-1 < x < 1$, $f'(x) < 0$.

Thus, when $-1 < x < 1$, f is decreasing. **[3]**

**Q. 29. Let f be a function defined on [$a$, $b$] such that $f'(x) > 0$, for all $x \in (a, b)$. Then prove that $f$ is an increasing function on ($a$, $b$).**
**[NCERT Misc. Ex. Q. 16, Page 243]**

**Ans.** Let $x_1, x_2 \in (a, b)$ | such that $x_1 < x_2$.

Consider the sub-interval $[x_1, x_2]$. Since $f(x)$ is differentiable on $(a, b)$ and $[x_1, x_2] \subset (a, b)$.

Therefore, $f(x)$ is continuous on $[x_1, x_2]$ and differentiable on $(x_1, x_2)$.

By the Lagrange's mean value theorem, there exists $c \in (x_1, x_2)$ such that $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ ...(i)

Since $f(x) > 0$ for all $x \in (a, b)$, so in particular, $f'(c) > 0$

$$f'(c) > 0$$

$$\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \qquad [\text{Using Eq. (i)}]$$

$$\Rightarrow f(x_2) - f(x_1) > 0 \qquad [\because x_2 - x_1 > 0 \text{ when } x_1 < x_2]$$

$$\Rightarrow \quad f(x_2) > f(x_1)$$

$$\Rightarrow \quad f(x_1) < f(x_2)$$

Since $x_1$, $x_2$ are arbitrary points in $(a, b)$.

Therefore, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$

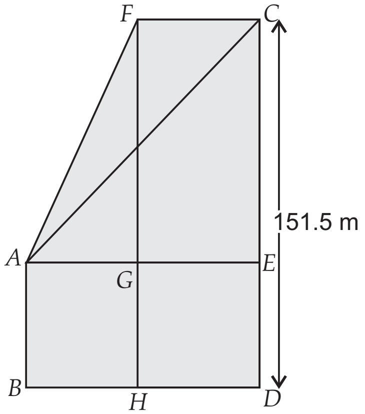
Hence, $f(x)$ is increasing on $(a, b)$. **[3]**

**Q. 30. A kite is moving horizontally at a height of 151.5 meters. If the speed of kite is 10 m/s, how fast is the string being let out; when the kite is 250 m away from the boy who is flying the kite? The height of boy is 1.5 m.**
**[NCERT Exemp. Ex. 6.3, Q. 3, Page 135]**

**Ans.** We have, height ($h$) = 151.5 m, speed of kite ($v$)=10 m/s

Let $CD$ be the height of kite and $AB$ be the height of boy.

Let $DB = x$ m $= EA$ and $AC$=250 m




$$\therefore \frac{dx}{dt} = 10 \text{ m/s}$$

From the given figure, we see that

$EC = 151.5 - 1.5 = 150$ m and $AE = x$

Also, $AC = 250$ m

In right-angled $\Delta CEA$,

$$AE^2 + EC^2 = AC^2$$

$$x^2 + (150)^2 = y^2 \qquad ...(i)$$

$$x^2 + (150)^2 = (250)^2$$

$$x^2 = (250)^2 - (150)^2$$

$$= (250+150)(250-150)$$

$$= 400 \times 100$$

$$\therefore \quad x = 20 \times 10 = 200$$

From Eq. (i) on differentiating with respect to $t$, we get

$$2x \cdot \frac{dx}{dt} + 0 = 2y\frac{dy}{dt}$$

$$\Rightarrow \quad 2y\frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$\therefore \quad \frac{dy}{dt} = \frac{x}{y} \cdot \frac{dx}{dt}$$

$$= \frac{200}{250} \cdot 10 = 8 \text{ m/s} \qquad \left[\because \frac{dx}{dt} = 10 \text{ m/s}\right]$$

So, the required rate at which the string is being let out is 8 m/s. **[3]**

**Q. 31. A swimming pool is to be drained for cleaning. If L represents the number of litres of water in the pool *t* seconds after the pool has been plugged off to drain and L = 200 (10 – *t*)2. How fast is the water running out at the end of 5 seconds? What is the average rate at which the water flows out during the first 5 seconds?**

**[NCERT Exemp. Ex. 6.3, Q. 9, Page 135]**

**Ans.** Let $L$ represents the number of litres of water in the pool $t$ seconds after the pool been plugged off to drain, then

$L = 200(10 - t)2$

$\therefore$ Rate at which the water is running out $= -\dfrac{dL}{dt}$

$$\frac{dL}{dt} = -200.2(10-t)\cdot(-1)$$
$$= 400(10-t)$$

Rate at which the water is running out at the end of 5 s,

$$= 400(10-5)$$
$$= 2000 \text{ L/s} = \text{Final rate}$$

$$\text{Since, initial rate} = \left(\frac{dL}{dt}\right)_{t=0} = 4000 \text{ L/s}$$

$$\text{Average rate during 5 s} = \frac{\text{Initial rate} + \text{Final rate}}{2}$$
$$= \frac{4000+2000}{2}$$
$$= 3000 \text{ L/s}$$

**[3]**

**Q. 32. *x* and *y* are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of second square with respect to the area of first square.**

**[NCERT Exemp. Ex. 6.3, Q. 11, Page 136]**

**Ans.** Since, $x$ and $y$ are the sides of two squares such that $y = x - x^2$

$\therefore$ Area of the first square $(A_1) = x^2$

and area of the second square $(A_2) = y^2 = (x - x^2)^2$

$$\frac{dA_2}{dt} = \frac{d}{dt}(x-x^2)^2$$
$$= 2(x-x^2)\left(\frac{dx}{dt} - 2x\cdot\frac{dx}{dt}\right)$$
$$= \frac{dx}{dt}(1-2x)2(x-x^2)$$

and $$\frac{dA_1}{dt} = \frac{d}{dt}x^2 = 2x\frac{dx}{dt}$$

$$\frac{dA_2}{dA_1} = \frac{dA_2/dt}{dA_1/dt}$$
$$= \frac{\frac{dx}{dt}(1-2x)(2x-2x^2)}{2x\cdot\frac{dx}{dt}}$$
$$= \frac{(1-2x)2x(1-x)}{2x}$$
$$= (1-2x)(1-x)$$
$$= 1 - x - 2x + 2x^2$$
$$= 2x^2 - 3x + 1$$

**[3]**

**Q. 33. Find the condition that the curves $2x = y^2$ and $2xy = k$ intersect orthogonally.**

**[NCERT Exemp. Ex. 6.3, Q. 12, Page 136]**

**Ans.** Given, equation of curves are $2x = y^2$ **(i)**

and $2xy = k$ ...(ii)

$$\Rightarrow \quad y = \frac{k}{2x} \quad \text{[From Eq. (ii)]}$$

From Eq. (i), we have

$$2x = \left(\frac{k}{2x}\right)^2$$
$$\Rightarrow \quad 8x^3 = k^2$$
$$\Rightarrow \quad x^3 = \frac{1}{8}k^2$$
$$\Rightarrow \quad x = \frac{1}{2}k^{2/3}$$
$$\therefore \quad y = \frac{k}{2x} = \frac{k}{2\cdot\frac{1}{2}k^{2/3}} = k^{1/3}$$

Thus, we get point of intersection of curves which is $\left(\frac{1}{2}k^{2/3}, k^{1/3}\right)$.

From equations (i) and (ii), we have

$$2 = 2y\frac{dy}{dx}$$
$$\text{and} \quad 2\left[x\cdot\frac{dy}{dx} + y\cdot 1\right] = 0$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{y}$$
$$\text{and} \quad \left(\frac{dy}{dx}\right) = \frac{-2y}{2x} = -\frac{y}{x}$$
$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{\left(\frac{1}{2}k^{2/3}, k^{1/3}\right)} = \frac{1}{k^{1/3}} \quad \text{[say } m_1\text{]}$$
$$\text{and} \quad \left(\frac{dy}{dx}\right)_{\left(\frac{1}{2}k^{2/3}, k^{1/3}\right)} = \frac{-k^{1/3}}{\frac{1}{2}k^{2/3}} = -2k^{-1/3} \quad \text{[say } m_2\text{]}$$

Since, the curves intersect orthogonally, *i.e.*,

$$m_1.m_2 = -1$$
$$\Rightarrow \quad \frac{1}{k^{1/3}}.(-2k^{-1/3}) = -1$$
$$\Rightarrow \quad -2k^{-2/3} = -1$$
$$\Rightarrow \quad \frac{2}{k^{2/3}} = 1$$
$$\Rightarrow \quad k^{2/3} = 2$$
$$\Rightarrow \quad k^2 = 8$$

which is required condition. **[3]**

**Q. 34. Find the co-ordinates of the point on the curve $\sqrt{x} + \sqrt{y} = 4$ at which tangent is equally inclined to the axes. [NCERT Exemp. Ex. 6.3, Q. 14, Page 136]**

**Ans.** As we know that,

We have, $\sqrt{x} + \sqrt{y} = 4$ ...(i)

$$\Rightarrow \quad x^{1/2} + y^{1/2} = 4$$
$$\Rightarrow \quad \frac{1}{2}\cdot\frac{1}{x^{1/2}} + \frac{1}{2}\cdot\frac{1}{y^{1/2}}\cdot\frac{dy}{dx} = 0$$
$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2}\cdot x^{-1/2}2\cdot y^{1/2}$$
$$= -\sqrt{\frac{y}{x}}$$

Since, tangent is equally inclined to the axes.

$$\therefore \quad \frac{dy}{dx} = \pm 1$$

$$\Rightarrow \quad -\sqrt{\frac{y}{x}} = \pm 1$$

$$\Rightarrow \quad \frac{y}{x} = 1$$

$$\Rightarrow \quad y = x$$

From Eq. (i), we have

$$\sqrt{y} + \sqrt{y} = 4$$

$$\Rightarrow \quad 2\sqrt{y} = 4$$

$$\Rightarrow \quad 4y = 16$$

$$\therefore \quad y = 4 \text{ and } x = 4$$

When $y = 4$, then $x = 4$

So, the required coordinates are (4, 4). **[3]**

**Q. 35. Find the angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$.**

**[NCERT Exemp. Ex. 6.3, Q. 15, Page 136]**

**Ans.** We have, $y = 4 - x^2$ ...(i)

and $y = x^2$ ...(ii)

$$\Rightarrow \quad \frac{dy}{dx} = -2x$$

$$\text{and} \quad \frac{dy}{dx} = 2x$$

$$\Rightarrow \quad m_1 = -2x$$

$$\text{and} \quad m_2 = 2x$$

From equations (i) and (ii), we have

$$x^2 = 4 - x^2$$

$$\Rightarrow \quad 2x^2 = 4$$

$$\Rightarrow \quad x^2 = 2$$

$$\Rightarrow \quad x = \pm\sqrt{2}$$

$$\therefore \quad y = x^2 = (\pm\sqrt{2})^2 = 2$$

So, the points of intersection are $\left(\sqrt{2},\ 2\right)$ and $\left(-\sqrt{2},\ 2\right)$.

For point $\left(\sqrt{2},\ 2\right)$, $m_1 = -2x = -2\cdot\sqrt{2} = -2\sqrt{2}$

and $m_2 = 2x = 2\sqrt{2}$

and for point $\left(-\sqrt{2},\ 2\right)$, $m_1 = 2\sqrt{2}$ and $m_2 = -2\sqrt{2}$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_3}\right| = \left|\frac{-2\sqrt{2} - 2\sqrt{2}}{1 - 2\sqrt{2}\cdot 2\sqrt{2}}\right| = \left|\frac{-4\sqrt{2}}{-7}\right|$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$$

**[3]**

**Q. 36. At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangents are parallel to the $y$-axis?**

**[NCERT Exemp. Ex. 6.3, Q. 18, Page 136]**

**Ans.** Given that the equation of curve,

$$x^2 + y^2 - 2x - 4y + 1 = 0 \quad \text{...(i)}$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} - 2 - 4\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx}(2y - 4) = 2 - 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2(1-x)}{2(y-2)}$$

Since, the tangents are parallel to the $y$-axis, *i.e.*, $\tan\theta = \tan 90° = \frac{dy}{dx}$.

$$\therefore \quad \frac{1-x}{y-2} = \frac{1}{0}$$

$$\Rightarrow \quad y - 2 = 0$$

$$\Rightarrow \quad y = 2$$

For $y = 2$ from Eq. (i), we get

$$x^2 + 2^2 - 2x - 4\times 2 + 1 = 0$$

$$\Rightarrow \quad x^2 - 2x - 3 = 0$$

$$\Rightarrow \quad x^2 - 3x + x - 3 = 0$$

$$\Rightarrow \quad x(x-3) + 1(x-3) = 0$$

$$\Rightarrow \quad (x+1)(x-3) = 0$$

$$\therefore \quad x = -1, x = 3$$

So, the required points are (–1, 2) and (3, 2). **[3]**

**Q. 37. Find the points of local maxima, local minima and the points of inflection of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$. Also find the corresponding local maximum and local minimum values.**

**[NCERT Exemp. Ex. 6.3, Q. 26, Page 137]**

**Ans.** Given that, $f(x) = x^5 - 5x^4 + 5x^3 - 1$

On differentiating with respect to $x$, we get

$$f'(x) = 5x^4 - 20x^3 + 15x^2$$

For maxima or minima, $f'(x) = 0$

$$\Rightarrow \quad 5x^4 - 20x^3 + 15x^2 = 0$$

$$\Rightarrow \quad 5x^2(x^2 - 4x + 3) = 0$$

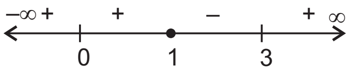
$$\Rightarrow \quad 5x^2(x^2 - 3x - x + 3) = 0$$

$$\Rightarrow \quad 5x^2[x(x-3) - 1(x-3)] = 0$$

$$\Rightarrow \quad 5x^2[(x-1)(x-3)] = 0$$

$$\therefore \quad x = 0, 1, 3$$

Sign scheme for $\frac{dy}{dx} = 5x^2(x-1)(x-3)$




So, $y$ has maximum value at $x = 1$ and minimum value at $x = 3$.

At $x = 0$, $y$ has neither maximum nor minimum value, so $x = 0$ is the point of inflection

Maximum value of $y = 1 - 5 + 5 - 1 = 0$ (at $x = 1$)

and Minimum value of $y = 35 - 5(3)4 + 5(3)3 - 1$

$= 243 - 405 + 135 - 1 = -298$ (at $x = 3$) **[3]**

**Q. 38. An open box with square base is to be made of a given quantity of card board of area $c^2$. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.** **[NCERT Exemp. Ex. 6.3, Q. 29, Page 137]**

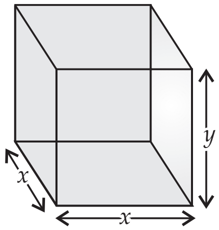
**Ans.** Let the length of side of the square base of open box be $x$ units and its height be y units.

$\therefore$ Area of the metal used $= x^2 + 4xy$

$$\Rightarrow \quad x^2 + 4xy = c^2 \quad \text{[Given]}$$

$$\Rightarrow \quad y = \frac{c^2 - x^2}{4x} \quad \text{...(i)}$$

Now, volume of the box $(V) = x^2 y$




$$\Rightarrow \quad V = x^2 \cdot \left(\frac{c^2 - x^2}{4x}\right)$$

$$= \frac{1}{4}x(c^2 - x^2)$$

$$= \frac{1}{4}(c^2x - x^3)$$

On differentiating both sides with respect to $x$, we get

$$\frac{dV}{dx} = \frac{1}{4}(c^2 - 3x^2) \quad \text{...(ii)}$$

Now, $\frac{dV}{dx} = 0 \Rightarrow c^2 = 3x^2$

$$\Rightarrow \quad x^2 = \frac{c^3}{3}$$

$$\Rightarrow \quad x = \frac{c}{\sqrt{3}} \quad \text{[Using positive sign]}$$

Again, differentiating Eq. (ii) with respect to $x$, we get

$$\frac{d^2y}{dx^2} = \frac{1}{4}(-6x) = \frac{-3}{2}x < 0$$

$$\therefore \quad \left(\frac{d^2V}{dx^2}\right)_{\text{at } x = \frac{c}{\sqrt{3}}} = -\frac{3}{2} \cdot \left(\frac{c}{\sqrt{3}}\right) < 0$$

Thus, we see that volume ($V$) is maximum at $x = \frac{c}{\sqrt{3}}$.

$\therefore$ Maximum volume of the box,

$$(V)_{x = \frac{c}{\sqrt{3}}} = \frac{1}{4}\left(c^2 \cdot \frac{c}{\sqrt{3}} - \frac{c^3}{3\sqrt{3}}\right)$$

$$= \frac{1}{4} \cdot \frac{(3c^3 - c^3)}{3\sqrt{3}}$$

$$= \frac{1}{4} \cdot \frac{2c^3}{3\sqrt{3}}$$

$$= \frac{c^3}{6\sqrt{3}} \text{ cu units}$$

**[3]**

**Q. 39. Find the equations of the tangent and the normal, to the curve $16x^2 + 9y^2 = 145$ at the point $(x_1, y_1)$ where $x_1 = 2$ and $y_1 > 0$.**

**[CBSE Board, Delhi Region, 2018]**

**Ans.** Given $16x^2 + 9y^2 = 145$

Substitute $x = 2$ in $16x^2 + 9y^2 = 145$

$16(2)^2 + 9y^2 = 145$

$\Rightarrow \quad 64 + 9y^2 = 145$

$\Rightarrow \quad y = 3$ (Neglecting $y = -3$ as $y > 0$)

Now, differentiate $16x^2+9y^2=145$ with respect to $x$ which will give us the slope of the tangent.

$$32x + 18y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{32x}{18y}$$

Substitute $x = 2$ and $y = 3$ in $\frac{dy}{dx} = -\frac{32x}{18y}$

$$\frac{dy}{dx} = -\frac{64}{54} = -\frac{32}{27}$$

The equation of tangent is given by

$$y - 3 = -\frac{32}{27}(x - 2)$$

$$\Rightarrow \quad 27y - 81 = -32x + 64$$

$$\Rightarrow 32x + 27y - 145 = 0$$

The slope of the normal is $\frac{-1}{\text{Slope of tangent}} = \frac{27}{32}$

The equation of normal is given by

$$y - 3 = \frac{27}{32}(x - 2)$$

$$\Rightarrow \quad 32y - 96 = 27x - 54$$

$$\Rightarrow \quad 27x - 32y + 42 = 0$$

**[4]**

**Q. 40. Find the intervals in which the function $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ is**

**(a) strictly increasing, (b) strictly decreasing.**

**[CBSE Board, Delhi Region, 2018]**

**Ans.** Given, $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$

$$\Rightarrow \quad f'(x) = x^3 - 3x^2 - 10x + 24$$

Find the critical points by equation $f'(x)$ to zero.

$$\therefore \quad x^3 - 3x^2 - 10x + 24 = 0$$

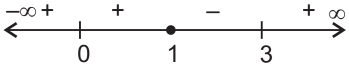
$$\Rightarrow x^2(x - 2) - x(x - 2) - 12(x - 2) = 0$$

$$\Rightarrow \quad (x - 2)(x^2 - x - 12) = 0$$

$$\Rightarrow \quad (x - 2)(x - 4)(x + 3) = 0$$

$$\Rightarrow \quad x = -3, 2 \text{ and } 4$$

Therefore, we have the intervals $(-\infty, -3)$, $(-3, 2)$, $(2, 4)$ and $(4, \infty)$.




**(a)** For $f(x)$ to be strictly increasing, we should have $f'(x) > 0$

$\Rightarrow (x - 2)(x - 4)(x + 3) > 0$

Therefore, $x \in (-3, 2) \cup (4, \infty)$

**(b)** For $f(x)$ to be strictly decreasing, we should have $f'(x) < 0$

$\Rightarrow (x - 2)(x - 4)(x + 3) > 0$

Therefore, $x \in (-\infty, -3) \cup (2, 4)$ **[4]**

**Q. 41. The equation of tangent at (2, 3) on the curve $y^2 = ax^3 + b$ is $y = 4x - 5$. Find the values of $a$ and $b$.**

**[CBSE Board, All India Region, 2016]**

**Ans.**

$$y^2 = ax^3 + b$$

$$\Rightarrow 2y\frac{dy}{dx} = 3ax^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{3a}{2}\frac{x^2}{y} \quad \textbf{[1]}$$

Slope of tangent at (2, 3) $= \left.\frac{dy}{dx}\right]_{(2,3)} = \frac{3a}{2} \cdot \frac{4}{3} = 2a$ **[1]**

Comparing with slope of tangent $y = 4x - 5$, we get, $2a = 4 \therefore \boxed{a = 2}$ **[1]**

Also (2, 3) lies on the curve.

Therefore, $9 = 8a + b$, put $a = 2$, we get $b = -7$ **[1]**

## Long Answer Type Questions (5 or 6 marks each)

**Q. 1. Sand is pouring from a pipe at the rate of 12 cm$^3$/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?**
**[NCERT Ex. 6.1, Q. 14, Page 198]**

**Ans.** The volume of a cone ($V$) with radius ($r$) and height ($h$) is given by, $V = \frac{1}{3}\pi r^2 h$

It is given that,

$$h = \frac{1}{6}r \Rightarrow r = 6h$$

$$\therefore V = \frac{1}{3}\pi(6h)^2 h = 12\pi h^3$$

The rate of change of volume with respect to time ($t$) is given by,

$$\frac{dV}{dt} = 12\pi\frac{d}{dh}\left(h^3\right)\cdot\frac{dh}{dt}$$

[By chain rule]

$$= 12\pi\left(3h^2\right)\frac{dh}{dt}$$

$$= 36\pi h^2\frac{dh}{dt}$$

It is also given that, $\frac{dV}{dt} = 12 \text{ cm}^3/\text{s}$ .

Therefore, when $h = 4$ cm, we have

$$12 = 36\pi\left(4\right)^2\frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{12}{36\pi(16)} = \frac{1}{48\pi}$$

Hence, when the height of the sand cone is 4 cm, its height is increasing at the rate of $\frac{1}{48\pi}$ cm/s . **[5]**

**Q. 2. Prove that $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.**
**[NCERT Ex. 6.2, Q. 9, Page 205]**

**Ans.** We have,

$$y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$$

$$\therefore \frac{dy}{dx} = \frac{(2+\cos\theta)(4\cos\theta) - 4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4}{(2+\cos\theta)^2} - 1$$

Now, $\frac{dy}{dx} = 0$ .

$$\Rightarrow \frac{8\cos\theta + 4}{(2+\cos\theta)^2} = 1$$

$$\Rightarrow 8\cos\theta + 4 = 4 + \cos^2\theta + 4\cos\theta$$

$$\Rightarrow \cos^2\theta - 4\cos\theta = 0$$

$$\Rightarrow \cos\theta(\cos\theta - 4) = 0$$

$$\Rightarrow \cos\theta = 0 \text{ or } \cos\theta = 4$$

Since cos θ ≠ 4, cos θ = 0.

$$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

Now,

$$\frac{dy}{dx} = \frac{8\cos\theta + 4 - \left(4 + \cos^2\theta + 4\cos\theta\right)}{(2+\cos\theta)^2}$$

$$= \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$$= \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

In interval $\left(0, \frac{\pi}{2}\right)$, we have cos θ > 0. Also, 4 > cos θ ⇒ 4 − cos θ > 0.

$$\therefore \cos\theta(4-\cos\theta) > 0 \text{ and also } (2+\cos\theta)^2 > 0$$

$$\Rightarrow \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} > 0$$

$$\Rightarrow \frac{dy}{dx} > 0$$

Therefore, $y$ is strictly increasing in interval $\left(0, \frac{\pi}{2}\right)$.

Also, the given function is continuous at $x = 0$ and $x = \frac{\pi}{2}$.

Hence, $y$ is increasing in interval $\left(0, \frac{\pi}{2}\right)$. **[5]**

**Q. 3. Let $I$ be any interval disjoint from (−1, 1). Prove that the function $f$ given by $f(x) = x + \frac{1}{x}$ is strictly increasing on $I$.** **[NCERT Ex. 6.2, Q. 15, Page 206]**

**Ans.** We have,

$$f(x) = x + \frac{1}{x}$$

$$f'(x) = 1 - \frac{1}{x^2}$$

Now,

$$f'(x) = 0$$

$$\Rightarrow \frac{1}{x^2} = 1$$

$$\therefore x = \pm 1$$

The points $x = 1$ and $x = -1$ divide the real line in three disjoint intervals, *i.e.*, $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$.
In interval $(-1, 1)$, it is observed that

$$-1 < x < 1$$
$$\Rightarrow \quad x^2 < 1$$
$$\Rightarrow \quad 1 < \frac{1}{x^2}, x \neq 0$$
$$\Rightarrow 1 - \frac{1}{x^2} < 0, x \neq 0$$
$$\therefore f'(x) = 1 - \frac{1}{x^2} < 0 \text{ on } (-1,1) \sim \{0\}$$

$\therefore f$ is strictly decreasing on $(-1, 1) \sim \{0\}$.
In intervals $(-\infty, -1)$ and $(1, \infty)$, it is observed that

$$x < -1 \text{ or } 1 < x$$
$$\Rightarrow \quad x^2 > 1$$
$$\Rightarrow \quad 1 > \frac{1}{x^2}$$
$$\Rightarrow 1 - \frac{1}{x^2} > 0$$
$$\therefore f'(x) = 1 - \frac{1}{x^2} > 0 \text{ on } (-\infty, -1) \text{ and } (1, \infty)$$

$\therefore f$ is strictly increasing on $(-\infty, 1)$ and $(1, \infty)$.
Hence, function $f$ is strictly increasing in interval $I$ disjoint from $(-1, 1)$. Hence, the given result is proved. **[5]**

**Q. 4. Find the equation of all lines having slope $-1$ that are tangents to the curve $y = \frac{1}{x-1}, x \neq 1$.**
**[NCERT Ex. 6.3, Q. 10, Page 212]**

**Ans.** The equation of the given curve is $y = \frac{1}{x-1}, x \neq 1$
The slope of the tangents to the given curve at any point $(x, y)$ is given by,

$$\frac{dy}{dx} = \frac{-1}{(x-1)^2}$$

If the slope of the tangent is $-1$, then we have

$$\frac{-1}{(x-1)^2} = -1$$
$$\Rightarrow \quad (x-1)^2 = 1$$
$$\Rightarrow \quad x - 1 = \pm 1$$
$$\Rightarrow \quad x = 2, 0$$

When $x = 0$, $y = -1$ and when $x = 2$, $y = 1$.
Thus, there are two tangents to the given curve having slope $-1$. These are passing through the points $(0, -1)$ and $(2, 1)$.
$\therefore$ The equation of the tangent through $(2, 1)$ is given by,

$$y - 1 = -1(x - 2)$$
$$\Rightarrow \quad y - 1 = -x + 2$$
$$\Rightarrow y + x - 3 = 0$$

Hence, the equations of the required lines are $y + x + 1 = 0$ and $y + x - 3 = 0$. **[5]**

**Q. 5. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is**
**(a) parallel to the line $2x - y + 9 = 0$**
**(b) perpendicular to the line $5y - 15x = 13$.**
**[NCERT Ex. 6.3, Q. 15, Page 212]**

**Ans.** The equation of the given curve is $y = x^2 - 2x + 7$
On differentiating with respect to $x$, we get :

$$\frac{dy}{dx} = 2x - 2$$

**(a)** The equation of the line is $2x - y + 9 = 0$.
$2x - y + 9 = 0$
$\Rightarrow \quad y = 2x + 9$
This is of the form $y = mx + c$.
$\therefore$ Slope of the line $= 2$
If a tangent is parallel to the line $2x - y + 9 = 0$, then the slope of the tangent is equal to the slope of the line.
Therefore, we have

$$2 = 2x - 2$$
$$\Rightarrow \quad 2x = 4$$
$$\Rightarrow \quad x = 2$$

Now, $x = 2$
$\Rightarrow \quad y = 4 - 4 + 7 = 7$
Thus, the equation of the tangent passing through $(2, 7)$ is given by,

$$y - 7 = 2(x - 2)$$
$$\Rightarrow y - 2x - 3 = 0$$

Hence, the equation of the tangent line to the given curve (which is parallel to line $2x - y + 9 = 0$) is
$y - 2x - 3 = 0$. **[2½]**

**(b)** The equation of the line is $5y - 15x = 13$.
$5y - 15x = 13$

$$\Rightarrow \quad y = 3x + \frac{13}{5}$$

This is of the form $y = mx + c$.
$\therefore$ Slope of the line $= 3$.
If a tangent is perpendicular to the line $5y - 15x = 13$, then the slope of the tangent is

$$\frac{-1}{\text{Slope of the line}} = \frac{-1}{3}.$$
$$\Rightarrow 2x - 2 = \frac{-1}{3}$$
$$\Rightarrow \quad 2x = \frac{-1}{3} + 2$$
$$\Rightarrow \quad 2x = \frac{5}{3}$$
$$\Rightarrow \quad x = \frac{5}{6}$$

Now, $x = \frac{5}{6}$

$$\Rightarrow \quad y = \frac{25}{36} - \frac{10}{6} + 7 = \frac{25 - 60 + 252}{36} = \frac{217}{36}$$

Thus, the equation of the tangent passing through $\left(\frac{5}{6}, \frac{217}{36}\right)$ is given by,

$$y - \frac{217}{36} = -\frac{1}{3}\left(x - \frac{5}{6}\right)$$
$$\Rightarrow \quad \frac{36y - 217}{36} = \frac{-1}{18}(6x - 5)$$
$$\Rightarrow \quad 36y - 217 = -2(6x - 5)$$
$$\Rightarrow \quad 36y - 217 = -12x + 10$$
$$\Rightarrow 36y + 12x - 227 = 0$$

Hence, the equation of the tangent line to the given curve (which is perpendicular to line $5y-15x=13$) is $36y+12x-227=0$. **[2½]**

**Q. 6. For the curve $y=4x^3-2x^5$, find all the points at which the tangent passes through the origin.**
**[NCERT Ex. 6.3, Q. 18, Page 212]**

**Ans.** The equation of the given curve is $y=4x^3-2x^5$.
On differentiating with respect to $x$, we get

$$\therefore \quad \frac{dy}{dx}=12x^2-10x^4$$

Therefore, the slope of the tangent at a point $(x, y)$ is $12x^2 - 10x^4$.
The equation of the tangent at $(x, y)$ is given by,

$$Y-y=(12x^2-10x^4)(X-x) \quad \text{...(i)}$$

When the tangent passes through the origin (0, 0), then X = Y = 0.
Therefore, equation (i) reduces to :

$$-y=(12x^2-10x^4)(-x)$$
$$y=12x^3-10x^5$$

Also, we have

$$y=4x^3-2x^5$$
$$\therefore 12x^3-10x^5=4x^3-2x^5$$
$$\Rightarrow 8x^5-8x^3=0$$
$$\Rightarrow x^5-x^3=0$$
$$\Rightarrow x^3\left(x^2-1\right)=0$$
$$\Rightarrow x=0,\pm1$$

When $x = 0$, $y = 4(0)^3 - 2(0)^5 = 0$.
When $x = 1$, $y = 4(1)^3 - 2(1)^5 = 2$.
When $x = -1$, $y = 4(-1)^3 - 2(-1)^5 = -2$.
Hence, the required points are (0, 0), (1, 2) and (−1, −2). **[5]**

**Q. 7. Find the equation of the normals to the curve $y=x^3+2x+6$ which are parallel to the line $x+14y+4=0$. [NCERT Ex. 6.3, Q. 21, Page 213]**

**Ans.** The equation of the given curve is $y=x^3+2x+6$.
The slope of the tangent to the given curve at any point $(x, y)$ is given by,

$$\frac{dy}{dx}=3x^2+2$$

$\therefore$ Slope of the normal to the given curve at any point $(x, y)$ =

$$\frac{-1}{\text{Slope of the tangent at the point }(x,y)}=\frac{-1}{3x^2+2}$$

The equation of the given line is $x + 14y + 4 = 0$.
$x + 14y + 4 = 0$

$$\Rightarrow y=-\frac{1}{14}x-\frac{4}{14} \quad \text{(which is of the form } y=mx+c\text{)}$$

$\therefore$ Slope of the given line $=\frac{-1}{14}$

If the normal is parallel to the line, then we must have the slope of the normal being equal to the slope of the line.

$$\therefore \frac{-1}{3x^2+2}=\frac{-1}{14}$$
$$\Rightarrow 3x^2+2=14$$
$$\Rightarrow 3x^2=12$$
$$\Rightarrow x^2=4$$
$$\Rightarrow x=\pm2$$

When $x = 2$, $y = 8 + 4 + 6 = 18$.
When $x = -2$, $y = -8 - 4 + 6 = -6$.
Therefore, there are two normals to the given curve with slope $\frac{-1}{14}$ and passing through the points (2, 18) and (−2, −6).
Thus, the equation of the normal through (2, 18) is given by,

$$y-18=\frac{-1}{14}(x-2)$$
$$\Rightarrow 14y-252=-x+2$$
$$\Rightarrow x+14y-254=0$$

and, the equation of the normal through (−2, −6) is given by,

$$y-(-6)=\frac{-1}{14}\left[x-(-2)\right]$$
$$\Rightarrow y+6=\frac{-1}{14}(x+2)$$
$$\Rightarrow 14y+84=-x-2$$
$$\Rightarrow x+14y+86=0$$

Hence, the equations of the normals to the given curve (which are parallel to the given line) are $x+14y-254=0$ and $x+14y+86=0$ **[5]**

**Q. 8. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles* if $8k^2 = 1$.**
**[NOTE : *Two curves intersect at right angle if the tangents to the curves at the point of intersection are perpendicular to each other.]**
**[NCERT Ex. 6.3, Q. 23, Page 213]**

**Ans.** The equations of the given curves are given as $x = y^2$ and $xy = k$.
Putting $x = y^2$ in $xy = k$, we get :

$$y^3=k \Rightarrow y=k^{\frac{1}{3}}$$
$$\therefore \quad x=k^{\frac{2}{3}}$$

Thus, the point of intersection of the given curves is $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$.

Differentiating $x = y^2$ with respect to $x$, we have

$$1=2y\frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{dx}=\frac{1}{2y}$$

Therefore, the slope of the tangent to the curve $x = y^2$ at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ is $\left.\frac{dy}{dx}\right]_{\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)}=\frac{1}{2k^{\frac{1}{3}}}$.

On differentiating $xy = k$ with respect to $x$, we have

$$x\frac{dy}{dx}+y=0$$
$$\Rightarrow \frac{dy}{dx}=\frac{-y}{x}$$

$\therefore$ Slope of the tangent to the curve $xy = k$ at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ is given by,

$$\left.\frac{dy}{dx}\right]_{\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)} = \left.\frac{-y}{x}\right]_{\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)} = -\frac{k^{\frac{1}{3}}}{k^{\frac{2}{3}}} = \frac{-1}{k^{\frac{1}{3}}}$$

We know that two curves intersect at right angles if the tangents to the curves at the point of intersection, *i.e.*, at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ are perpendicular to each other.

This implies that we should have the product of the tangents as −1.

Thus, the given two curves cut at right angles if the product of the slope of their respective tangents at $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ is −1, *i.e.*,

$$\left(\frac{1}{2k^{\frac{1}{3}}}\right)\left(\frac{-1}{k^{\frac{1}{3}}}\right) = -1$$

$$\Rightarrow \quad 2k^{\frac{2}{3}} = 1$$

$$\Rightarrow \quad \left(2k^{\frac{2}{3}}\right) = (1)^3$$

$$\Rightarrow \quad 8k^2 = 1$$

Hence, the given two curves cut at right angles if $8k^2 = 1$. **[5]**

**Q. 9. Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(x_0, y_0)$.**

**[NCERT Ex. 6.3, Q. 24, Page 213]**

**Ans.** On Differentiating $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with respect to $x$, we have

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{2y}{b^2}\frac{dy}{dx} = \frac{2x}{a^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

Therefore, the slope of the tangent at $(x_0, y_0)$ is $\left.\frac{dy}{dx}\right]_{(x_0, y_0)} = \frac{b^2 x_0}{a^2 y_0}$

Then, the equation of the tangent at $(x_0, y_0)$ is given by,

$$y - y_0 = \frac{b^2 x_0}{a^2 y_0}(x - x_0)$$

$$\Rightarrow \quad a^2 y y_0 - a^2 y_0^2 = b^2 x x_0 - b^2 x_0^2$$

$$\Rightarrow b^2 x x_0 - a^2 y y_0 - b^2 x_0^2 + a^2 y_0^2 = 0$$

$$\Rightarrow \quad \frac{x x_0}{a^2} - \frac{y y_0}{b^2} - \left(\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}\right) = 0$$

$$\Rightarrow \quad \frac{x x_0}{a^2} - \frac{y y_0}{b^2} - \left(\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}\right) = 0$$

[On dividing both sides by $a^2 b^2$]

$$\Rightarrow \quad \frac{x x_0}{a^2} - \frac{y y_0}{b^2} - 1 = 0$$

$\left[(x_0, y_0) \text{lies on the hyperbola } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right]$

$\left[(x_0, y_0) \text{lies on the hyperbola } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right]$

Now, the slope of the normal at $(x_0, y_0)$ is given by,

$$\frac{-1}{\text{Slope of the tangent at } (x_0, y_0)} = \frac{-a^2 y_0}{b^2 x_0}$$

Hence, the equation of the normal at $(x_0, y_0)$ is given by,

$$y - y_0 = \frac{-a^2 y_0}{b^2 x_0}(x - x_0)$$

$$\Rightarrow \quad \frac{y - y_0}{a^2 y_0} = \frac{-(x - x_0)}{b^2 x_0}$$

$$\Rightarrow \quad \frac{y - y_0}{a^2 y_0} + \frac{(x - x_0)}{b^2 x_0} = 0$$

**[5]**

**Q. 10. Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - 2y + 5 = 0$.** **[NCERT Ex. 6.3, Q. 25, Page 213]**

**Ans.** The equation of the given line is $4x - 2y + 5 = 0$.

$4x - 2y + 5 = 0$

$\Rightarrow y = 2x + \frac{5}{2}$ (which is of the form $y = mx + c$)

$\therefore$ Slope of the line = 2

Now, the tangent to the given curve is parallel to the line $4x - 2y - 5 = 0$ if the slope of the tangent is equal to the slope of the line.

$$\frac{3}{2\sqrt{3x - 2}} = 2$$

$$\Rightarrow \sqrt{3x - 2} = \frac{3}{4}$$

$$\Rightarrow \quad 3x - 2 = \frac{9}{16}$$

$$\Rightarrow \quad 3x = \frac{9}{16} + 2 = \frac{41}{16}$$

$$\Rightarrow \quad x = \frac{41}{48}$$

When $x = \frac{41}{48}$, we have

$$y = \sqrt{3\left(\frac{41}{48}\right) - 2}$$

$$= \sqrt{\frac{41}{16} - 2}$$

$$= \sqrt{\frac{41 - 32}{16}}$$

$$= \sqrt{\frac{9}{16}} = \frac{3}{4}$$

$\therefore$ Equation of the tangent passing through the point $\left(\frac{41}{48}, \frac{3}{4}\right)$ is given by,

$$y - \frac{3}{4} = 2\left(x - \frac{41}{48}\right)$$

$$\Rightarrow \quad \frac{4y - 3}{4} = 2\left(\frac{48x - 41}{48}\right)$$

$\Rightarrow \quad 4y - 3 = \dfrac{48x - 41}{6}$

$\Rightarrow \; 24y - 18 = 48x - 41$

$\Rightarrow 48x - 24y = 23$

Hence, the equation of the required tangent is $48x - 24y = 23$. **[5]**

**Q. 11. Find both the maximum value and the minimum value of $3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval [0, 3]. [NCERT Ex. 6.5, Q. 7, Page 232]**

**Ans.** Let $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$.

$$\therefore \; f'(x) = 12x^3 - 24x^2 + 24x - 48$$
$$= 12(x^3 - 2x^2 + 2x - 4)$$
$$= 12[x^2(x-2) + 2(x-2)]$$
$$= 12(x-2)(x^2+2)$$

Now, $f'(x) = 0$ gives $x = 2$ or $x^2 + 2 = 0$ for which there are no real roots.

Therefore, we consider only $x = 2 \in [0, 3]$.

Now, we evaluate the value of $f$ at critical point $x = 2$ and at the end points of the interval [0, 3].

$$f(2) = 3(16) - 8(8) + 12(4) - 48(2) + 25$$
$$= 48 - 64 + 48 - 96 + 25$$
$$= -39$$
$$f(0) = 3(0) - 8(0) + 12(0) - 48(0) + 25$$
$$= 25$$
$$f(3) = 3(81) - 8(27) + 12(9) - 48(3) + 25$$
$$= 243 - 216 + 108 - 144 + 25 = 16$$

Hence, we can conclude that the absolute maximum value of $f$ on [0, 3] is 25 occurring at $x = 0$ and the absolute minimum value of $f$ at [0, 3] is $-39$ occurring at $x = 2$. **[5]**

**Q. 12. At what points in the interval $[0, 2\pi]$, does the function sin $2x$ attain its maximum value? [NCERT Ex. 6.5, Q. 8, Page 232]**

**Ans.** Let $f(x) = \sin 2x$.

$$\therefore f'(x) = 2\cos 2x$$

Now,

$$f'(x) = 0$$
$$\Rightarrow \cos 2x = 0$$
$$\Rightarrow \quad 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$
$$\Rightarrow \quad x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Then, we evaluate the values of $f$ at critical points $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ and at the end points of the interval $[0, 2\pi]$.

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{2} = 1,$$
$$f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{2} = -1$$
$$f\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{2} = 1,$$
$$f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{2} = -1$$
$$f(0) = \sin 0 = 0,$$
$$f(2\pi) = \sin 2\pi = 0$$

Hence, we can conclude that the absolute maximum value of $f$ on $[0, 2\pi]$ is occurring at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$. **[5]**

**Q. 13. What is the maximum value of the function $\sin x + \cos x$? [NCERT Ex. 6.5, Q. 9, Page 232]**

**Ans.** Let $f(x) = \sin x + \cos x$.

$$\therefore \quad f'(x) = \cos x - \sin x$$
$$f'(x) = 0$$
$$\Rightarrow \quad \sin x = \cos x$$
$$\Rightarrow \quad \tan x = 1$$
$$\Rightarrow \quad x = \frac{\pi}{4}, \frac{5\pi}{4} \ldots,$$
$$f''(x) = -\sin x - \cos x$$
$$= -(\sin x + \cos x)$$

Now, $f''(x)$ will be negative when ($\sin x + \cos x$) is positive, *i.e.*, when $\sin x$ and $\cos x$ are both positive. Also, we know that $\sin x$ and $\cos x$ both are positive in the first quadrant. Then, $f''(x)$ will be negative when $x \in \left(0, \frac{\pi}{2}\right)$.

Thus, we consider $x = \frac{\pi}{4}$.

$$f''\left(\frac{\pi}{4}\right) = -\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right)$$
$$= -\left(\frac{2}{\sqrt{2}}\right)$$
$$= -\sqrt{2} < 0$$

$\therefore$ By second derivative test, $f$ will be the maximum at $x = \frac{\pi}{4}$ and the maximum value of $f$ is

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$$
$$= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$$
$$= \frac{2}{\sqrt{2}} = \sqrt{2}$$

**[5]**

**Q. 14. Find the maximum value of $2x^3 - 24x + 107$ in the interval [1, 3]. Find the maximum value of the same function in [–3, –1]. [NCERT Ex. 6.5, Q. 10, Page 232]**

**Ans.** Let $f(x) = 2x^3 - 24x + 107$.

$$\therefore \quad f'(x) = 6x^2 - 24 = 6(x^2 - 4)$$

Now,

$$f'(x) = 0$$
$$\Rightarrow 6(x^2 - 4) = 0$$
$$\Rightarrow \quad x^2 = 4$$
$$\therefore \quad x = \pm 2$$

We first consider the interval [1, 3]

Then, we evaluate the value of $f$ at the critical point $x = 2 \in [1, 3]$ and at the end points of the interval [1, 3].

$$f(2) = 2(8) - 24(2) + 107$$
$$= 16 - 48 + 107 = 75$$
$$f(1) = 2(1) - 24(1) + 107$$
$$= 2 - 24 + 107 = 85$$
$$f(3) = 2(27) - 24(3) + 107$$
$$= 54 - 72 + 107 = 89$$

Hence, the absolute maximum value of $f(x)$ in the interval [1, 3] is 89 occurring at $x = 3$.

Next, we consider the interval $[-3, -1]$.

Evaluate the value of $f$ at the critical point $x = -2 \in [-3, -1]$ and at the end points of the interval [1, 3].

$$f(-3) = 2(-27) - 24(-3) + 107$$
$$= -54 + 72 + 107 = 125$$
$$f(-1) = 2(-1) - 24(-1) + 107$$
$$= -2 + 24 + 107 = 129$$
$$f(-2) = 2(-8) - 24(-2) + 107$$
$$= -16 + 48 + 107 = 139$$

Hence, the absolute maximum value of $f(x)$ in the interval $[-3, -1]$ is 139 occurring at $x = -2$. **[5]**

**Q. 15. Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$.**

**[NCERT Ex. 6.5, Q. 12, Page 233]**

**Ans.** Let $f(x) = x + \sin 2x$.

$\therefore f'(x) = 1 + 2\cos 2x$

Now,

$$f'(x) = 0$$

$$\Rightarrow \cos 2x = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$$

$$2x = 2n\pi \pm \frac{2\pi}{3}, n \in \mathbf{Z}$$

$$\Rightarrow \quad x = n\pi \pm \frac{\pi}{3}, n \in \mathbf{Z}$$

$$\Rightarrow \quad x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \in [0, 2\pi]$$

Then, we evaluate the value of $f$ at critical points $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ and at the end points of the interval $[0, 2\pi]$.

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$
$$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\frac{4\pi}{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$
$$f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + \sin\frac{8\pi}{3} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$$
$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sin\frac{10\pi}{3} = \frac{5\pi}{3} - \frac{\sqrt{3}}{2}$$
$$f(0) = 0 + \sin 0 = 0$$
$$f(2\pi) = 2\pi + \sin 4\pi = 2\pi + 0 = 2\pi$$

Hence, we can conclude that the absolute maximum value of $f(x)$ in the interval $[0, 2\pi]$ is $2\pi$ occurring at $x = 2\pi$ and the absolute minimum value of $f(x)$ in the interval $[0, 2\pi]$ is 0 occurring at $x = 0$. **[5]**

**Q. 16. Find two positive numbers $x$ and $y$ such that $x + y = 60$ and $xy^3$ is maximum.**

**[NCERT Ex. 6.5, Q. 14, Page 233]**

**Ans.** The two numbers are $x$ and $y$ such that $x + y = 60$.

$\Rightarrow \quad y = 60 - x$

Let $f(x) = xy^3$

$$\Rightarrow f(x) = x(60 - x)^3$$
$$\therefore \quad f'(x) = (60 - x)^3 - 3x(60 - x)^2$$
$$= (60 - x)^2[60 - x - 3x]$$
$$= (60 - x)^2(60 - 4x)$$

$$\text{And,} \quad f''(x) = -2(60 - x)(60 - 4x) - 4(60 - x)^2$$
$$= -2(60 - x)[60 - 4x + 2(60 - x)]$$
$$= -2(60 - x)(180 - 6x)$$
$$= -12(60 - x)(30 - x)$$

$$\text{Now,} \quad f'(x) = 0 \Rightarrow x = 60 \text{ or } x = 15$$

When $x = 60$, $f''(x) = 0$.

When $x = 15$, $f''(x) = -12(60 - 15)(30 - 15) = -12 \times 45 \times 15 < 0$.

$\therefore$ By second derivative test, $x = 15$ is a point of local maxima of $f$. Thus, function $xy^3$ is maximum when $x = 15$ and $y = 60 - 15 = 45$.

Hence, the required numbers are 15 and 45. **[5]**

**Q. 17. Find two positive numbers $x$ and $y$ such that their sum is 35 and the product $x^2 y^5$ is a maximum.**

**[NCERT Ex. 6.5, Q. 15, Page 233]**

**Ans.** Let one number be $x$. Then, the other number is $y = (35 - x)$.

Let $\quad P(x) = x^2 y^5$. Then, we have :

$$P(x) = x^2(35 - x)^5$$
$$\therefore P'(x) = 2x(35 - x)^5 - 5x^2(35 - x)^4$$
$$= x(35 - x)^4[2(35 - x) - 5x]$$
$$= x(35 - x)^4(70 - 7x)$$
$$= 7x(35 - x)^4(10 - x)$$

and,

$$P''(x) = 7(35 - x)^4(10 - x) + 7x\left[-(35 - x)^4 - 4(35 - x)^3(10 - x)\right]$$
$$= 7(35 - x)^4(10 - x) - 7x(35 - x)^4 - 28x(35 - x)^3(10 - x)$$
$$= 7(35 - x)^3[(35 - x)(10 - x) - x(35 - x) - 4x(10 - x)]$$
$$= 7(35 - x)^3\left[350 - 45x + x^2 - 35x + x^2 - 40x + 4x^2\right]$$
$$= 7(35 - x)^3(6x^2 - 120x + 350)$$

Now, $P'(x) = 0 \Rightarrow x = 0, x = 35, x = 10$

When $x = 35$, $f'(x) = f(x) = 0$ and $y = 35 - 35 = 0$. This will make the product $xy$ equal to 0.

When $x = 0$, $y = 35 - 0 = 35$ and the product $x^2 y^2$ will be 0.

$\therefore x = 0$ and $x = 35$ cannot be the possible values of $x$.

When $x = 10$, we have

$$P''(x) = 7(35-10)^3(6\times100-120\times10+350)$$
$$= 7(25)^3(-250) < 0$$

$\therefore$ By second derivative test, $P(x)$ will be the maximum when $x = 10$ and $y = 35 - 10 = 25$.

Hence, the required numbers are 10 and 25. **[5]**

**Q. 18. A square piece of tin of side 18 cm is to made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible? [NCERT Ex. 6.5, Q. 17, Page 233]**

**Ans.** Let the side of the square to be cut off be $x$ cm. Then, the length and the breadth of the box will be $(18 - 2x)$ cm each and the height of the box is $x$ cm.

$\therefore \quad V = x(18-2x)^2$

$$\therefore V'(x) = (18-2x)^2 - 4x(18-2x)$$
$$= (18-2x)[18-2x-4x]$$
$$= (18-2x)(18-6x)$$
$$= 6\times2(9-x)(3-x)$$
$$= 12(9-x)(3-x)$$

And, $V''(x) = 12[-(9-x)-(3-x)]$
$$= -12(9-x+3-x)$$
$$= -12(12-2x)$$
$$= -24(6-x)$$

Now, $V'(x) = 0 \Rightarrow x = 9$ or $x = 3$

If $x = 9$, then the length and the breadth will become 0.

So, $x \neq 9$.

$\Rightarrow x = 3$.

Now, $V''(3) = -24(6-3) = -72 < 0$

By second derivative test, $x = 3$ is the point of maxima of $V$.

Hence, if we remove a square of side 3 cm from each corner of the square tin and make a box from the remaining sheet, then the volume of the box obtained is the largest possible. **[5]**

**Q. 19. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum? [NCERT Ex. 6.5, Q. 18, Page 233]**

**Ans.** Let the side of the square to be cut off be $x$ cm. Then, the height of the box is $x$, the length is $45 - 2x$, and the breadth is $24 - 2x$.

Therefore, the volume $V(x)$ of the box is given by,

$$V(x) = x(45-2x)(24-2x)$$
$$= x(1080-90x-48x+4x^2)$$
$$= 4x^3-138x^2+1080x$$
$$\therefore V'(x) = 12x^2-276x+1080$$
$$= 12(x^2-23x+90)$$
$$= 12(x-18)(x-5)$$
$$V''(x) = 24x-276 = 12(2x-23)$$

Now, $V'(x) = 0$

$\Rightarrow x = 18$ and $x = 5$

It is not possible to cut off a square of side 18 cm from each corner of the rectangular sheet. Thus, $x$ cannot be equal to 18.

$\therefore x = 5$

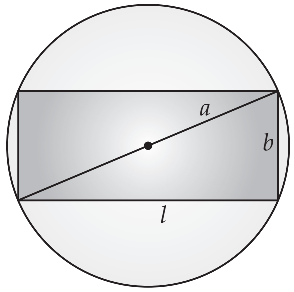
Now, $V'(5) = 12(10-23) = 12(-13) = -156 < 0$

By second derivative test, $x = 5$ is the point of maxima.

Hence, the side of the square to be cut off to make the volume of the box maximum possible is 5 cm. **[5]**

**Q. 20. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area. [NCERT Ex. 6.5, Q. 19, Page 233]**

**Ans.** Let a rectangle of length $l$ and breadth $b$ be inscribed in the given circle of radius $a$. Then, the diagonal passes through the center and is of length $2a$ cm.




Now, by applying the Pythagoras theorem, we have

$$(2a)^2 = l^2 + b^2$$
$$\Rightarrow b^2 = 4a^2 - l^2$$
$$\Rightarrow b = \sqrt{4a^2-l^2}$$

$\therefore$ Area of the rectangle, $A = l\sqrt{4a^2-l^2}$

$$\frac{dA}{dl} = \sqrt{4a^2-l^2} + l\frac{1}{2\sqrt{4a^2-l^2}}(-2l)$$
$$= \sqrt{4a^2-l^2} - \frac{l^2}{\sqrt{4a^2-l^2}}$$
$$= \frac{4a^2-2l^2}{\sqrt{4a^2-l^2}}$$
$$\frac{d^2A}{dl} = \frac{\sqrt{4a^2-l^2}(-4l)-(4a^2-2l^2)\frac{(-2l)}{2\sqrt{4a^2-l^2}}}{(4a^2-l^2)}$$
$$= \frac{(4a^2-l^2)(-4l)+l(4a^2-2l^2)}{(4a^2-l^2)^{\frac{3}{2}}}$$
$$= \frac{-12a^2l+2l^3}{(4a^2-l^2)^{\frac{3}{2}}}$$
$$= \frac{-2l(6a^2-2l^2)}{(4a^2-l^2)^{\frac{3}{2}}}$$

Now, $\frac{dA}{dl} = 0$ gives $4a^2 = 2l^2$

$\Rightarrow \quad l = \sqrt{2}a$

$\therefore \quad b = \sqrt{4a^2 - 2a^2}$

$= \sqrt{2a^2}$

$= \sqrt{2}a$

Now, when $l = \sqrt{2}a$,

$$\frac{d^2A}{dl^2} = \frac{-2\left(\sqrt{2}a\right)\left(6a^2 - 2a^2\right)}{2\sqrt{2}a^3}$$

$$= \frac{-8\sqrt{2}a^3}{2\sqrt{2}a^3} = -4 < 0$$

By the second derivative test, when $l = \sqrt{2}a$, then the area of the rectangle is the maximum.

Since $l = b = \sqrt{2}a$, the rectangle is a square.

Hence, it has been proved that of all the rectangles inscribed in the given fixed circle, the square has the maximum area. **[5]**

**Q. 21. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.**

**[NCERT Ex. 6.5, Q. 20, Page 233]**

**Ans.** Let $r$ and $h$ be the radius and height of the cylinder respectively.

Then, the surface area ($S$) of the cylinder is given by,

$$S = 2\pi r^2 + 2\pi rh$$

$$\Rightarrow h = \frac{S - 2\pi r^2}{2\pi r}$$

$$= \frac{S}{2\pi}\left(\frac{1}{r}\right) - r$$

Let $V$ be the volume of the cylinder. Then,

$$V = \pi r^2 h$$

$$= \pi r^2\left[\frac{S}{2\pi}\left(\frac{1}{r}\right) - r\right]$$

$$= \frac{Sr}{2} - \pi r^3$$

Then, $\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2, \frac{d^2V}{dr^2} = -6\pi r$

Now, $\frac{dV}{dr} = 0 \Rightarrow \frac{S}{2} - 3\pi r^2 \Rightarrow r^2 = \frac{S}{6\pi}$

When $r^2 = \frac{S}{6\pi}$, then $\frac{d^2V}{dr^2} = -6\pi\left(\sqrt{\frac{S}{6\pi}}\right) < 0.$

$\therefore$ By second derivative test, the volume is the maximum when $r^2 = \frac{S}{6\pi}$.

Now, when

$$r^2 = \frac{S}{6\pi}, \text{ then } h = \frac{6\pi r^2}{2\pi}\left(\frac{1}{r}\right) - r = 3r - r = 2r.$$

Hence, the volume is the maximum when the height is twice the radius, *i.e.*, when the height is equal to the diameter. **[5]**

**Q. 22. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area? [NCERT Ex. 6.5, Q. 21, Page 233]**

**Ans.** Let $r$ and $h$ be the radius and height of the cylinder respectively.

Then, volume ($V$) of the cylinder is given by,

$$V = \pi r^2 h = 100 \quad \text{(Given)}$$

$$\therefore \quad h = \frac{100}{\pi r^2}$$

Surface area ($S$) of the cylinder is given by,

$$S = 2\pi r^2 + 2\pi rh$$

$$= 2\pi r^2 + \frac{200}{r}$$

$$\therefore \quad \frac{dS}{dr} = 4\pi r - \frac{200}{r^2},$$

$$\frac{d^2S}{dr^2} = 4\pi + \frac{400}{r^3}$$

$$\frac{dS}{dr} = 0$$

$$\Rightarrow 4\pi r = \frac{200}{r^2}$$

$$\Rightarrow \quad r^3 = \frac{200}{4\pi} = \frac{50}{\pi}$$

$$\therefore \quad r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

Now, it is observed that when $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}, \frac{d^2S}{dr^2} > 0$.

$\therefore$ By second derivative test, the surface area is the minimum when the radius of the cylinder is $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

When $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$, we have

$$h = \frac{100}{\pi\left(\frac{50}{\pi}\right)^{\frac{2}{3}}} = \frac{2 \times 50}{(50)^{\frac{2}{3}}(\pi)^{1-\frac{2}{3}}} = 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}} \text{ cm}$$

Hence, the required dimensions of the can which has the minimum surface area are given by

Radius $= \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm and height $= 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm **[5]**

**Q. 23. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?**

**[NCERT Ex. 6.5, Q. 22, Page 233]**

**Ans.** Let $r$ be the radius of the circle. Then,

$$2\pi r = 28 - l$$

$$\Rightarrow r = \frac{1}{2\pi}(28 - l).$$

The combined areas of the square and the circle ($A$) is given by,

$$A = (\text{Side of the square})^2 + \pi r^2$$

$$= \frac{l^2}{16} + \pi\left[\frac{1}{2\pi}(28 - l)\right]^2$$

$$= \frac{l^2}{16} + \frac{1}{4\pi}(28-l)^2$$

$$\frac{dA}{dl} = \frac{2l}{16} + \frac{2}{4\pi}(28-l)(-1)$$

$$= \frac{l}{8} - \frac{1}{2\pi}(28-l)$$

$$\frac{d^2A}{dl^2} = \frac{1}{8} + \frac{1}{2\pi} > 0$$

Now,

$$\frac{dA}{dl} = 0$$

$$\frac{l}{8} - \frac{1}{2\pi}(28-l) = 0$$

$$\Rightarrow \frac{\pi l - 4(28-l)}{8\pi} = 0$$

$$\Rightarrow (\pi+4)l - 112 = 0$$

$$\Rightarrow \quad l = \frac{112}{\pi+4}$$

Thus, when $l = \frac{112}{\pi+4}, \frac{d^2A}{dl^2} > 0.$

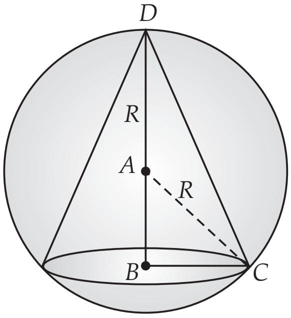
By second derivative test, the area ($A$) is the minimum when $l = \frac{112}{\pi+4}$.

Hence, the combined area is the minimum when the length of the wire in making the square is $\frac{112}{\pi+4}$ cm while the length of the wire in making the circle is $28 - \frac{112}{\pi+4} = \frac{28\pi}{\pi+4}$ cm. **[5]**

**Q. 24. Prove that the volume of the largest cone that can be inscribed in a sphere of radius $R$ is $\frac{8}{27}$ of the volume of the sphere.**

**[NCERT Ex. 6.5, Q. 23, Page 233]**

**Ans.** Let $r$ and $h$ be the radius and height of the cone respectively inscribed in a sphere of radius $R$.




Let $V$ be the volume of the cone.

Then, $V = \frac{1}{3}\pi r^2 h$

Height of the cone is given by,

$h = R + AB = R + \sqrt{R^2 - r^3}$ **[ABC is a right triangle]**

$$\therefore\ V = \frac{1}{3}\pi r^2\left(R + \sqrt{R^2 - r^2}\right)$$

$$= \frac{1}{3}\pi r^2 R + \frac{1}{3}\pi r^2\sqrt{R^2 - r^2}$$

$$\therefore \frac{dV}{dr} = \frac{2}{3}\pi rR + \frac{2}{3}\pi r\sqrt{R^2 - r^2} + \frac{1}{3}\pi r^2 \cdot \frac{(-2r)}{2\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi R + \frac{2}{3}\pi r\sqrt{R^2 - r^2} - \frac{1}{3}\pi\frac{r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi rR + \frac{2\pi r\left(R^2 - r^2\right) - \pi r^3}{3\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi rR + \frac{2\pi R^2 - 3\pi r^3}{3\sqrt{R^2 - r^2}}$$

$$\frac{d^2V}{dr^2} = \frac{2\pi R}{3} + \frac{3\sqrt{R^2 - r^2}\left(2\pi R^2 - 9\pi r^2\right) - \left(2\pi r R^2 - 3\pi r^3\right)\cdot\frac{(-2r)}{6\sqrt{R^2 - r^2}}}{9\left(R^2 - r^2\right)}$$

$$= \frac{2}{3}\pi R + \frac{9\left(R^2 - r^2\right)\left(2\pi R^2 - 9\pi r^2\right) + 2\pi r^2 R^2 + 3\pi r^4}{27\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

Now,

$$\frac{dV}{dr} = 0$$

$$\pi\frac{2}{3}rR = \frac{3\pi r^3 - 2\pi r R^2}{3\sqrt{R^2 - r^2}}$$

$$\Rightarrow \quad 2R = \frac{3r^2 - 2R^2}{\sqrt{R^2 - r^2}}$$

$$\Rightarrow\ 2R\sqrt{R^2 - r^2} = 3r^2 - 2R^2$$

$$\Rightarrow 4R^2\left(R^2 - r^2\right) = \left(3r^2 - 2R^2\right)^2$$

$$\Rightarrow\ 4R^4 - 4R^2r^2 = 9r^4 + 4R^4 - 12r^2R^2$$

$$\Rightarrow \quad 9r^4 = 8R^2r^2$$

$$\Rightarrow \quad r^2 = \frac{8}{9}R^2$$

When $r^2 = \frac{8}{9}R^2$, then $\frac{d^2V}{dr^2} < 0$.

$\therefore$ By second derivative test, the volume of the cone is the maximum when $r^2 = \frac{8}{9}R^2$.

When $r^2 = \frac{8}{9}R^2$, we have

$$h = R + \sqrt{R^2 - \frac{8}{9}R^2}$$

$$= R + \sqrt{\frac{1}{9}R^2}$$

$$= R + \frac{R}{3} = \frac{4}{3}R.$$

Therefore,

$$= \frac{1}{3}\pi\left(\frac{8}{9}R^2\right)\left(\frac{4}{3}R\right)$$

$$= \frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$$

$$= \frac{8}{27} \times \text{ (Volume of the sphere)}$$

Hence, the volume of the largest cone that can be inscribed in the sphere is $\frac{8}{27}$ of the volume of the sphere. **[5]**

**Q. 25. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.**
**[NCERT Ex. 6.5, Q. 24, Page 233]**

**Ans.** Let $r$ and $h$ be the radius and the height (altitude) of the cone respectively.

Then, the volume ($V$) of the cone is given as :

$V = 1/3(\pi r^2 h)$

$\Rightarrow h = 3V/\pi r^2$

The surface area ($S$) of the cone is given by,

$S = \pi r l$ (where $l$ is the slant height)

$$= \pi r\sqrt{r^2+h^2}$$

$$= \pi r\sqrt{r^2+\frac{9V^2}{\pi^2 r^4}} = \frac{\pi r\sqrt{\pi^2 r^6+9V^2}}{\pi r^2}$$

$$= \frac{1}{r}\sqrt{\pi^2 r^6+9V^2}$$

$$\therefore \frac{dS}{dr} = \frac{r\cdot\dfrac{6\pi^2 r^5}{2\sqrt{\pi^2 r^6+9V^2}} - \sqrt{\pi^2 r^6+9V^2}}{r^2}$$

$$= \frac{3\pi^2 r^6 - \pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6+9V^2}}$$

$$= \frac{2\pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6+9V^2}}$$

$$= \frac{2\pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6+9V^2}}$$

Now, $\frac{dS}{dr} = 0 \Rightarrow 2\pi^2 r^6 = 9V^2 \Rightarrow r^6 = \frac{9V^2}{2\pi^2}$

Thus, it can be easily verified that when $r^6 = \frac{9V^2}{2\pi^2}, \frac{d^2S}{dr^2} > 0.$

$\therefore$ By second derivative test, the surface area of the cone is the least when $r^6 = \frac{9V^2}{2\pi^2}$.

When

$$r^6 = \frac{9V^2}{2\pi^2}, h = \frac{3V}{\pi r^2} = \frac{3}{\pi r^2}\left(\frac{2\pi^2 r^6}{9}\right)^{\frac{1}{2}} = \frac{3}{\pi r^2}\cdot\frac{\sqrt{2}\pi r^3}{3} = \sqrt{2}r.$$

Hence, for a given volume, the right circular cone of the least curved surface has an altitude equal to $\sqrt{2}$ times the radius of the base. **[5]**

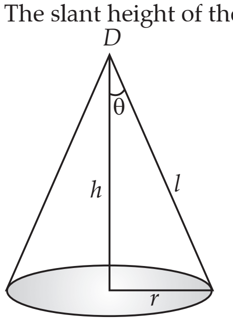
**Q. 26. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.** **[NCERT Ex. 6.5, Q. 25, Page 233]**

**Ans.** Let θ be the semi-vertical angle of the cone.

It is clear that $\theta \in \left[0, \frac{\pi}{2}\right]$.

Let $r$, $h$ and $l$ be the radius, height and the slant height of the cone, respectively.

The slant height of the cone is given as constant.




Now, $r = l\sin\theta$ and $h = l\cos\theta$

The volume ($V$) of the cone is given by,

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(l^2\sin^2\theta\right)(l\cos\theta)$$

$$= \frac{1}{3}\pi l^3 \sin^2\theta\cos\theta$$

$$\therefore \quad \frac{dV}{d\theta} = \frac{l^3\pi}{3}\left[\sin^2\theta(-\sin\theta) + \cos\theta(2\sin\theta\cos\theta)\right]$$

$$= \frac{l^3\pi}{3}\left[-\sin^3 + 2\sin\theta\cos^2\theta\right]$$

$$\frac{d^2V}{d\theta} = \frac{l^3\pi}{3}\left[-3\sin^2\theta\cos\theta + 2\cos^3\theta - 4\sin^2\theta\cos\theta\right]$$

$$= \frac{l^3\pi}{3}\left[2\cos^3\theta - 7\sin^2\theta\cos\theta\right]$$

Now, $\frac{dV}{d\theta} = 0$

$\Rightarrow \sin^3\theta = 2\sin\theta\cos^2\theta$

$\Rightarrow \tan^2\theta = 2$

$\Rightarrow \tan\theta = \sqrt{2}$

$\Rightarrow \quad \theta = \tan^{-1}\sqrt{2}$

Now, when $\theta = \tan^{-1}\sqrt{2}$, then $\tan^2\theta = 2$ or $\sin^2\theta = 2\cos^2\theta$.

Then, we have

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}\left[2\cos^3\theta - 14\cos^3\theta\right]$$

$$= -4\pi l^3\cos^3\theta < 0 \text{ for } \theta \in \left[0, \frac{\pi}{2}\right]$$

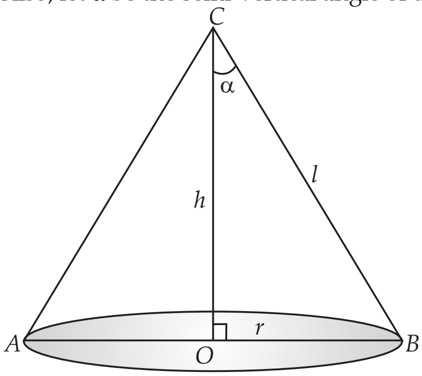
$\therefore$ By second derivative test, the volume ($V$) is the maximum when $\theta = \tan^{-1}\sqrt{2}$.

Hence, for a given slant height, the semi-vertical angle of the cone of the maximum volume is $\tan^{-1}\sqrt{2}$. **[5]**

**Q. 27. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$** **[NCERT Ex. 6.5, Q. 26, Page 233]**

**Ans.** Let $r$ be the radius, $l$ be the slant height and $h$ be the height of the cone of given surface area, $S$.

Also, let α be the semi-vertical angle of the cone.




Then $S = \pi r l + \pi r^2$

$$\Rightarrow \quad l = \frac{S - \pi r^2}{\pi r} \quad \text{...(i)}$$

Let $V$ be the volume of the cone.

Then $V = \frac{1}{3}\pi r^2 h$

$$\Rightarrow \quad V^2 = \frac{1}{9}\pi^2 r^4 h^2$$

$$= \frac{1}{9}\pi^2 r^4 \left(l^2 - r^2\right) \qquad \left[\because l^2 = r^2 + h^2\right]$$

$$= \frac{1}{9}\pi^2 r^4 \left[\left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2\right]$$

$$= \frac{1}{9}\pi^2 r^4 \left[\frac{\left(S - \pi r^2\right)^2 - \pi^2 r^4}{\pi^2 r^2}\right]$$

$$= \frac{1}{9}\pi^2 \left(S^2 - 2S\pi r^2\right)$$

$$\Rightarrow \quad V^2 = \frac{1}{9} S r^2 \left(S - 2\pi r^2\right) \qquad \text{...(ii)}$$

Differentiating equation (ii) with respect to $r$, we get

$$2V\frac{dV}{dr} = \frac{1}{9}S\left(2Sr - 8\pi r^3\right)$$

For maximum or minimum, put $\frac{dV}{dr} = 0$

$$\Rightarrow \frac{1}{9}S\left(2Sr - 8\pi r^3\right) = 0$$

$$\Rightarrow \quad 2Sr - 8\pi r^3 = 0 \quad (\text{as } S \neq 0)$$

$$\Rightarrow \quad S = 4\pi r^2 \quad (\text{as } r \neq 0)$$

$$\Rightarrow \quad r^2 = \frac{S}{4\pi}$$

Differentiating again with respect to $r$, we get

$$2V\frac{d^2V}{dr^2} + 2\left(\frac{dV}{dr}\right)^2 = \frac{1}{9}S\left(2S - 24\pi r^2\right)$$

$$2V\frac{d^2V}{dr^2} = \frac{1}{9}S\left(2S - 24\pi \times \frac{S}{4\pi}\right)$$

$$\left(\because \frac{dV}{dr} = 0 \text{ and } r^2 = \frac{S}{4\pi}\right)$$

$$= \frac{1}{9}S(2S - 6S)$$

$$= -\frac{4}{9}S^2 < 0$$

Thus, $V$ is maximum when $S = 4\pi r^2$

$$As \quad S = \pi r l + \pi r^2$$

$$\Rightarrow \quad 4\pi r^2 = \pi r l + \pi r^2$$

$$\Rightarrow \quad 3\pi r^2 = \pi r l$$

$$\Rightarrow \quad l = 3r$$

Now, in $\Delta COB$,

$$\text{Sin } \alpha = \frac{OB}{BC} = \frac{r}{l} = \frac{r}{3r} = \frac{1}{3}$$

$$\Rightarrow \quad \alpha = \text{Sin}^{-1}\left(\frac{1}{3}\right)$$

**[5]**

**Q. 28. Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = e$.**

**[NCERT Misc Ex. Q. 2, Page 242]**

**Ans.** The given function is $f(x) = \frac{\log x}{x}$.

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \log x}{x^2}$$

$$= \frac{1 - \log x}{x^2}$$

Now, $f'(x) = 0$

$\Rightarrow 1 - \log x = 0$

$\Rightarrow \log x = 1$

$\Rightarrow \log x = \log e$

$\Rightarrow x = e$

$$\text{Now, } f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \log x)(2x)}{x^4}$$

$$= \frac{-x - 2x(1 - \log x)}{x^4}$$

$$= \frac{-3 + 2\log x}{x^3}$$

$$\text{Now, } f''(e) = \frac{-3 + 2\log e}{e^3}$$

$$= \frac{-3 + 2}{e^3}$$

$$= \frac{-1}{e^3} < 0$$

Therefore, by second derivative test, $f$ is the maximum at $x = e$. **[5]**

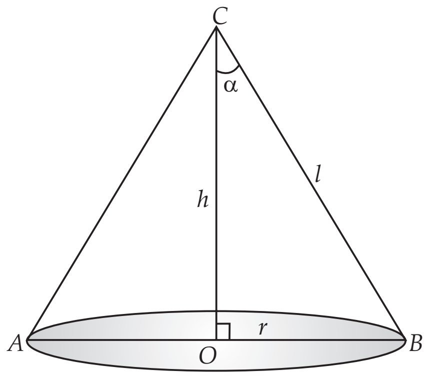
**Q. 29. The two equal sides of an isosceles triangle with fixed base $b$ are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?**

**[NCERT Misc Ex. Q. 3, Page 242]**

**Ans.** Let $\Delta ABC$ be isosceles where $BC$ is the base of fixed length $b$.

Let the length of the two equal sides of $\Delta ABC$ be $a$.

Draw $AD \perp BC$.




Now, in $\Delta ADC$, by applying the Pythagoras theorem, we have

$$AD = \sqrt{a^2 - \frac{b^2}{4}}$$

$$\therefore \text{ Area of triangle } (A) = \frac{1}{2}b\sqrt{a^2 - \frac{b^2}{4}}$$

The rate of change of the area with respect to time ($t$) is given by,

$$\frac{dA}{dt}=\frac{1}{2}b\cdot\frac{2a}{2\sqrt{a^2-\frac{b^2}{4}}}\frac{da}{dt}$$

$$=\frac{ab}{\sqrt{4a^2-b^2}}\frac{da}{dt}$$

It is given that the two equal sides of the triangle are decreasing at the rate of 3 cm per second.

$$\therefore \qquad \frac{da}{dt}=-3 \text{ cm/s}$$

Then, when $a=b$, we have

$$\frac{dA}{dt}=\frac{-3b^2}{\sqrt{4b^2-b^2}}$$

$$=\frac{-3b^2}{\sqrt{3b^2}}$$

$$=-\sqrt{3}b$$

Hence, if the two equal sides are equal to the base, then the area of the triangle is decreasing at the rate of $\sqrt{3}\,b\text{ cm}^2/\text{s}$. **[5]**

**Q. 30. Show that the normal at any point θ to the curve $x=a\cos\theta+a\theta\sin\theta, y=a\sin\theta-a\theta\cos\theta$ is at a constant distance from the origin.**

**[NCERT Misc Ex. Q. 5, Page 242]**

**Ans.** We have $x=a\cos\theta+a\theta\sin\theta$.

$$\frac{dx}{d\theta}=-a\sin\theta+a\sin\theta+a\theta\cos\theta$$

$$=a\theta\cos\theta$$

$$y=a\sin\theta-a\theta\cos\theta$$

$$\frac{dy}{d\theta}=a\cos\theta-a\cos\theta+a\theta\sin\theta$$

$$=a\theta\sin\theta$$

$$\frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$=\frac{a\theta\sin\theta}{a\theta\cos\theta}=\tan\theta$$

∴ Slope of the normal at any point θ is $-\frac{1}{\tan\theta}$.

The equation of the normal at a given point $(x, y)$ is given by,

$$y-a\sin\theta+a\theta\cos\theta=\frac{-1}{\tan\theta}(x-a\cos\theta-a\theta\sin\theta)$$

$$\Rightarrow y\sin\theta-a\sin^2\theta+a\theta\sin\theta\cos\theta=$$

$$-x\cos\theta+a\cos^2\theta+a\theta\sin\theta\cos\theta$$

$$\Rightarrow x\cos\theta+y\sin\theta-a\left(\sin^2\theta+\cos^2\theta\right)=0$$

$$\Rightarrow x\cos\theta+y\sin\theta-a=0$$

Now, the perpendicular distance of the normal from the origin is

$\frac{|-a|}{\sqrt{\cos^2\theta+\sin^2\theta}}=\frac{|-a|}{\sqrt{1}}=|-a|$, which is independent of θ.

Hence, the perpendicular distance of the normal from the origin is constant. **[5]**

**Q. 31. Find the intervals in which the function f given by $f(x)=\frac{4\sin x-2x-x\cos x}{2+\cos x}$ is**

**(i) increasing (ii) decreasing**

**[NCERT Misc Ex. Q. 6, Page 242]**

**Ans.** Let we assume that,

$$f(x)=\frac{4\sin x-2x-x\cos x}{2+\cos x}$$

$$\therefore f'(x)=\frac{(2+\cos x)(4\cos x-2-\cos x+x\sin x)-(4\sin x-2x-x\cos x)(-\sin x)}{(2+\cos x)^2}$$

$$=\frac{(2+\cos x)(3\cos x-2+x\sin x)+\sin x(4\sin x-2x-x\cos x)}{(2+\cos x)^2}$$

$$=\frac{6\cos x-4+2x\sin x+3\cos^2 x-2\cos x+x\sin x\cos x+4\sin^2 x-2x\sin x-x\sin x\cos x}{(2+\cos x)^2}$$

$$=\frac{4\cos x-4+3\cos^2 x+4\sin^2 x}{(2+\cos x)^2}$$

$$=\frac{4\cos x-4+3\cos^2 x+4-4\cos^2 x}{(2+\cos x)^2}$$

$$=\frac{4\cos x-\cos^2 x}{(2+\cos x)^2}$$

$$=\frac{\cos x(4-\cos x)}{(2+\cos x)^2}$$

$$f'(x)=0$$

$\Rightarrow \quad \cos x=0$ or $\cos x=4$

But, $\cos x\neq 4$

$\therefore \quad \cos x=0$

$$\Rightarrow \qquad x=\frac{\pi}{2},\frac{3\pi}{2}$$

Now, $x=\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$ divides $(0, 2\pi)$ into three disjoint intervals, *i.e.*,

In intervals $\left(0,\frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2},2\pi\right)$, $f'(x)>0$.

Thus, $f(x)$ is increasing for $0<x<\frac{x}{2}$ and $\frac{3\pi}{2}<x<2\pi$.

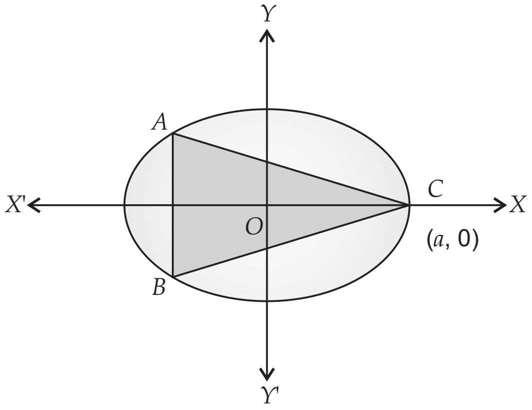
In the interval $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, $f'(x)<0$.

Thus, $f(x)$ is decreasing for $\frac{\pi}{2}<x<\frac{3\pi}{2}$. **[5]**

**Q. 32. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with its vertex at one end of the major axis.**

**[NCERT Misc Ex. Q. 8, Page 242]**

**Ans.**




The given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let the major axis be along the $x$-axis.

Let ABC be the triangle inscribed in the ellipse where vertex $C$ is at (a, 0).

Since the ellipse is symmetrical with respect to the $x$-axis and $y$-axis, we can assume the coordinates of $A$ to be $(-x_1, y_1)$ and the coordinates of $B$ to be $(-x_1, -y_1)$.

Now, we have

$$y_1=\pm\frac{b}{a}\sqrt{a^2-x_1^2}.$$

$\therefore$ Coordinates of $A$ are $\left(-x_1,\frac{b}{a}\sqrt{a^2-x_1^2}\right)$ and the coordinates of $B$ are $\left(x_1,-\frac{b}{a}\sqrt{a^2-x_1^2}\right)$.

As the point $(x, y)$ lies on the ellipse, the area of triangle ABC ($A$) is given by,

$$A=\frac{1}{2}a\left(\frac{2b}{a}\sqrt{a^2-x_1^2}\right)+(x_1)\left(-\frac{b}{a}\sqrt{a^2-x_1^2}\right)+(-x_1)\left(-\frac{b}{a}\sqrt{a^2-x_1^2}\right)$$

$$\Rightarrow \quad A=b\sqrt{a^2-x_1^2}+x_1\frac{b}{a}\sqrt{a^2-x_1^2}$$

$$\therefore \quad \frac{dA}{dx_1}=\frac{-2x_1b}{2\sqrt{a^2-x_1^2}}+\frac{b}{a}\sqrt{a^2-x_1^2}-\frac{2bx_1^2}{a2\sqrt{a^2-x_1^2}}$$

$$=\frac{b}{a\sqrt{a^2-x_1^2}}\left[-x_1a+\left(a^2-x_1^2\right)-x_1^2\right]$$

$$=\frac{b\left(-2x_1^2-x_1a+a^2\right)}{a\sqrt{a^2-x_1^2}}$$

Now, $\frac{dA}{dx_1}=0$

$$\Rightarrow -2x_1^2-x_1a+a^2=0$$

$$\Rightarrow \quad x_1=\frac{a\pm\sqrt{a^2-4(-2)(a^2)}}{2(-2)}$$

$$=\frac{a\pm\sqrt{9a^2}}{-4}$$

$$=\frac{a\pm 3a}{-4}$$

$$\Rightarrow \quad x_1=-a,\frac{a}{2}$$

But, $x$ cannot be equal to $a$.

$$\therefore \quad x_1=\frac{a}{2}$$

$$y_1=\frac{b}{a}\sqrt{a^2-\frac{a^2}{4}}$$

$$=\frac{ba}{2a}\sqrt{3}$$

$$=\frac{\sqrt{3}b}{2}$$

Now,

$$\frac{d^2A}{dx_1^2}=\frac{b}{a}\left\{\frac{\sqrt{a^2-x_1^2}\,(-4x_1-a)-\left(-2x_1^2-x_1a+a^2\right)\frac{(-2x_1)}{2\sqrt{a^2-x_1^2}}}{a^2-x_1^2}\right\}$$

$$=\frac{b}{a}\left\{\frac{\left(a^2-x_1^2\right)\left(-4x_1-a\right)+x_1\left(-2x_1^2-x_1a+a^2\right)}{\left(a^2-x_1^2\right)^{\frac{3}{2}}}\right\}$$

$$=\frac{b}{a}\left\{\frac{2x^3-3x^2x-a^3}{\left(a^2-x_1^2\right)^{\frac{3}{2}}}\right\}$$

Also, when $x_1=\frac{a}{2}$, then

$$\frac{d^2A}{dx_1^2}=\frac{b}{a}\left\{\frac{2\frac{a^3}{8}-3\frac{a^3}{2}-a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\}=\frac{b}{a}\left\{\frac{\frac{a^3}{4}-\frac{3}{2}a^3-a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\}$$

$$=-\frac{b}{a}\left\{\frac{\frac{9}{4}a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\}<0$$

Thus, the area is the maximum when $x_1=\frac{a}{2}$.

$\therefore$ Maximum area of the triangle is given by,

$$A=b\sqrt{a^2-\frac{a^2}{4}}+\left(\frac{a}{2}\right)\frac{b}{a}\sqrt{a^2-\frac{a^2}{4}}$$

$$=ab\frac{\sqrt{3}}{2}+\left(\frac{a}{2}\right)\frac{b}{a}\times\frac{a\sqrt{3}}{2}$$

$$= \frac{ab\sqrt{3}}{2} + \frac{ab\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}ab$$ **[5]**

**Q. 33. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m$^3$. If building of tank costs Rs 70 per sq meters for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?**

**[NCERT Misc Ex. Q. 9, Page 242]**

**Ans.** Let $l$, $b$ and $h$ represent the length, breadth and height of the tank, respectively.

Then, we have height ($h$) = 2 m

Volume of the tank = 8 m$^3$

Volume of the tank = $l \times b \times h$

$\therefore \quad 8 = l \times b \times 2$

$\Rightarrow \quad lb = 4$

$\therefore \quad b = \frac{4}{l}$

Now, area of the base = $lb$ = 4

Area of the 4 walls ($A$) = $2h(l + b)$

$\therefore \quad A = 4\left(l + \frac{4}{l}\right)$

$\Rightarrow \quad \frac{dA}{dl} = 4\left(1 - \frac{4}{l^2}\right)$

Now, $\frac{dA}{dl} = 0$

$\Rightarrow \quad 1 - \frac{4}{l^2} = 0$

$\Rightarrow \quad l^2 = 4$

$\Rightarrow \quad l = \pm 2$

However, the length cannot be negative.

Therefore, we have $l = 4$.

$\therefore \quad b = \frac{4}{l} = \frac{4}{2} = 2$

Now, $\frac{d^2A}{dl^2} = \frac{32}{l^3}$

When $l = 2, \frac{d^2A}{dl^2} = \frac{32}{8} = 4 > 0.$

Thus, by second derivative test, the area is the minimum when $l = 2$.

We have $l = b = h = 2$.

$\therefore$ Cost of building the base = ₹ 70 × ($lb$) = ₹ 70 (4) = ₹ 280

Cost of building the walls = ₹ $2h$ ($l + b$) × 45

= ₹ 90 (2) (2 + 2) = ₹ 8 (90) = ₹ 720.

Required total cost= ₹ (280+720) = ₹ 1000

Hence, the total cost of the tank will be ₹ 1000. **[5]**

**Q. 34. The sum of the perimeter of a circle and square is $k$, where $k$ is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.**

**[NCERT Misc Ex. Q. 10, Page 242]**

**Ans.** Let $r$ be the radius of the circle and $a$ be the side of the square.

Then, we have

$2\pi r + 4a = k$ (where $k$ is constant)

$\therefore \quad a = \frac{k - 2\pi r}{4}$

The sum of the areas of the circle and the square ($A$) is given by,

$$A = \pi r^2 + a^2$$

$$= \pi r^2 + \frac{(k - 2\pi r)^2}{16}$$

$$\frac{dA}{dr} = 2\pi r + \frac{2(k - 2\pi r)(-2\pi)}{16}$$

$$= 2\pi r - \frac{\pi(k - 2\pi r)}{4}$$

Now, $\frac{dA}{dr} = 0$

$\Rightarrow \quad 2\pi r = \frac{\pi(k - 2\pi r)}{4}$

$\Rightarrow \quad 8r = k - 2\pi r$

$\Rightarrow (8 + 2\pi)r = k$

$\Rightarrow \quad r = \frac{k}{8 + 2\pi}$

$= \frac{k}{2(4 + \pi)}$

Now, $\frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$

$\therefore$ When $\quad r = \frac{k}{2(4+\pi)}, \frac{d^2A}{dr^2} > 0.$

$\therefore$ The sum of the areas is least when $r = \frac{k}{2(4+\pi)}$.

When $\quad r = \frac{k}{2(4+\pi)}$, we have

$$a = \frac{k - 2\pi\left[\frac{k}{2(4+\pi)}\right]}{4}$$

$$= \frac{k(4+\pi) - \pi k}{4(4+\pi)}$$

$$= \frac{4k}{4(4+\pi)}$$

$$= \frac{k}{4+\pi} = 2r$$

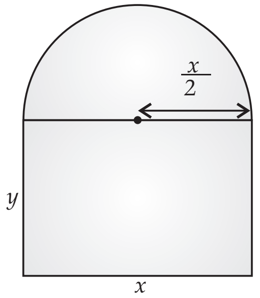
Hence, it has been proved that the sum of their areas is least when the side of the square is double the radius of the circle. **[5]**

**Q. 35. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.**

**[NCERT Misc Ex. Q. 11, Page 243]**

**Ans.** Let $x$ and $y$ be the length and breadth of the rectangular window.

Radius of the semicircular opening $= \frac{x}{2}$




It is given that the perimeter of the window is 10 m.

$$\therefore \quad x + 2y + \frac{\pi x}{2} = 10$$

$$\Rightarrow x\left(1+\frac{\pi}{2}\right) + 2y = 10$$

$$\Rightarrow \quad 2y = 10 - x\left(1+\frac{\pi}{2}\right)$$

$$\Rightarrow \quad y = 5 - x\left(\frac{1}{2}+\frac{\pi}{4}\right)$$

$\therefore$ Area of the window ($A$) is given by,

$$A = xy + \frac{\pi}{2}\left(\frac{\pi}{2}\right)^2$$

$$= x\left[5 - x\left(\frac{1}{2}+\frac{\pi}{4}\right)\right] + \frac{\pi}{8}x^2$$

$$= 5x - x^2\left(\frac{1}{2}+\frac{\pi}{4}\right) + \frac{\pi}{8}x^2$$

$$\therefore \frac{dA}{dx} = 5 - 2x\left(\frac{1}{2}+\frac{\pi}{4}\right) + \frac{\pi}{4}x$$

$$= 5 - x\left(1+\frac{\pi}{2}\right) + \frac{\pi}{4}x$$

$$\therefore \frac{d^2A}{dx^2} = -\left(1+\frac{\pi}{2}\right) + \frac{\pi}{4}$$

$$= -1 - \frac{\pi}{4}$$

Now, $\frac{dA}{dx} = 0$

$$\Rightarrow 5 - x\left(1+\frac{\pi}{2}\right) + \frac{\pi}{4}x = 0$$

$$\Rightarrow \quad 5 - x - \frac{\pi}{4}x = 0$$

$$\Rightarrow \quad x\left(1+\frac{\pi}{4}\right) = 5$$

$$\Rightarrow \quad x = \frac{5}{\left(1+\frac{\pi}{4}\right)} = \frac{20}{\pi+4}$$

Thus, when $x = \frac{20}{\pi+4}$ then $\frac{d^2A}{dx^2} < 0.$

Therefore, by second derivative test, the area is the maximum when length $x = \frac{20}{\pi+4}$ m

Now,

$$y = 5 - \frac{20}{\pi+4}\left(\frac{2+\pi}{4}\right)$$

$$= 5 - \frac{5(2+\pi)}{\pi+4}$$

$$= \frac{10}{\pi+4}\text{ m}$$

Hence, the required dimensions of the window to admit maximum light is given by

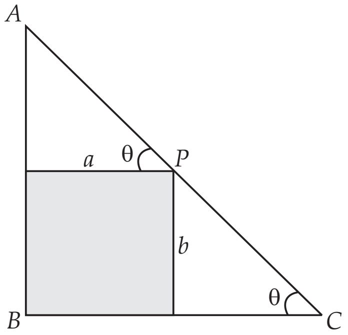
Length $= \frac{20}{\pi+4}$ m and breadth $= \frac{10}{\pi+4}$ m **[5]**

**Q. 36. A point on the hypotenuse of a triangle is at distance $a$ and $b$ from the sides of the triangle. Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$. [NCERT Misc Ex. Q. 12, Page 243]**

**Ans.** Let ΔABC be right-angled at $B$. Let AB = $x$ and BC = $y$.

Let $P$ be a point on the hypotenuse of the triangle such that $P$ is at a distance of $a$ and $b$ from the sides AB and BC respectively.

Let ∠$C$ = θ.




We have,

$$\text{AC} = \sqrt{x^2 + y^2}$$

Now,

$$PC = b\operatorname{cosec}\theta$$

and, $$AP = a\sec\theta$$

$$\therefore \quad AC = AP + PC$$

$$\Rightarrow \quad AC = b\operatorname{cosec}\theta + a\sec\theta \quad \text{...(i)}$$

$$\therefore \quad \frac{d(AC)}{d\theta} = -b\operatorname{cosec}\theta\cot\theta + a\sec\theta\tan$$

$$\therefore \quad \frac{d(AC)}{d\theta} = 0$$

$$\Rightarrow a\sec\theta\tan\theta = b\operatorname{cosec}\theta\cot\theta$$

$$\Rightarrow \frac{a}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} = \frac{b}{\sin\theta}\frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow \quad a\sin^3\theta = b\cos^3\theta$$

$$\Rightarrow \quad (a)^{\frac{1}{3}}\sin\theta = (b)^{\frac{1}{3}}\cos\theta$$

$$\Rightarrow \quad \tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$$

$\therefore \sin\theta = \dfrac{(b)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}$ and $\cos\theta = \dfrac{(a)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}$ ...(ii)

It can be clearly shown that $\dfrac{d^2(AC)}{d\theta^2} < 0$ when $\tan\theta = \left(\dfrac{b}{a}\right)^{\frac{1}{3}}$.

Therefore, by second derivative test, the length of the hypotenuse is the maximum when $\tan\theta = \left(\dfrac{b}{a}\right)^{\frac{1}{3}}$.

Now, when $\tan\theta = \left(\dfrac{b}{a}\right)^{\frac{1}{3}}$, we have

$$AC = \frac{b\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{b^{\frac{1}{3}}} + \frac{a\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{a^{\frac{1}{3}}}$$

[By using Eqs. (i) and (ii)]

$$= \sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}\left(b^{\frac{2}{3}} + a^{\frac{2}{3}}\right)$$

$$= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

Therefore, the maximum length of the hypotenuses is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$. **[5]**

**Q. 37. Find the points at which the function *f* given by $f(x) = (x-2)^4(x+1)^3$ has**

**(i) local maxima (ii) local minima**

**(iii) point of inflexion**

**[NCERT Misc Ex. Q. 13, Page 243]**

**Ans.** **(i)** The given function is $f(x) = (x-2)^4(x+1)^3$

$$\therefore f'(x) = 4(x-2)^3(x+1)^3 + 3(x+1)^2(x-2)^4$$
$$= (x-2)^3(x+1)^2[4(x+1) + 3(x-2)]$$
$$= (x-2)^3(x+1)^2(7x-2)$$

Now, $f'(x) = 0 \Rightarrow x = -1$ and $x = \dfrac{2}{7}$ or $x = 2$

Now, for values of $x$ close to and to the left of $x = \dfrac{2}{7} f'(x) > 0$. Also, for values of $x$ close to $\dfrac{2}{7}$ and to the right of $\dfrac{2}{7}, f'(x) < 0$.

Thus, $x = \dfrac{2}{7}$ is the point of local maxima.

**(ii)** Now, for values of $x$ close to 2 and to the left of $2, f'(x) < 0$. Also, for values of $x$ close to 2 and to the right of 2, $f'(x) < 0$.

Thus, $x = 2$ is the point of local minima.

**(iii)** Now, as the value of $x$ varies through $-1$, $f'(x)$ does not change its sign.

Thus, $x = -1$ is the point of inflexion. **[5]**

**Q. 38. Find the absolute maximum and minimum values of the function *f* given by $f(x) = \cos^2 x + \sin x, x \in [0, \pi]$.**

**[NCERT Misc Ex. Q. 14, Page 243]**

**Ans.** Given that,

$$f(x) = \cos^2 x + \sin x$$
$$f'(x) = 2\cos x(-\sin x) + \cos x$$
$$= -2\sin x\cos x + \cos x$$

Now, $f'(x) = 0$

$\Rightarrow 2\sin x\cos x = \cos x$

$\Rightarrow \cos x(2\sin x - 1) = 0$

$\Rightarrow \sin x = \dfrac{1}{2}$ or $\cos x = 0$

$\Rightarrow x = \dfrac{\pi}{6}$, or $\dfrac{\pi}{2}$ as $x \in [0, \pi]$

Now, evaluating the value of $f$ at critical points $x = \dfrac{\pi}{2}$ and $x = \dfrac{\pi}{6}$ and at the end points of the interval $[0, \pi]$ (*i.e.*, at $x = 0$ and $x = \pi$), we have

$$f\left(\frac{\pi}{6}\right) = \cos^2\frac{\pi}{6} + \sin\frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{5}{4}$$
$$f(0) = \cos^2 0 + \sin 0 = 1 + 0 = 1$$
$$f(\pi) = \cos^2\pi + \sin\pi = (-1)^2 + 0 = 1$$
$$f\left(\frac{\pi}{2}\right) = \cos^2\frac{\pi}{2} + \sin\frac{\pi}{2} = 0 + 1 = 1$$

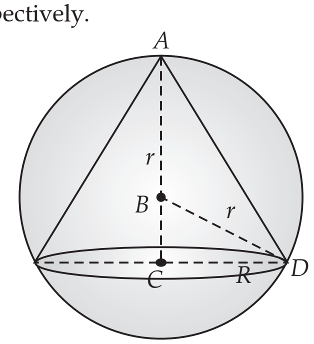
Hence, the absolute maximum value of $f$ is $\dfrac{5}{4}$ occurring at $x = \dfrac{\pi}{6}$ and the absolute minimum value of $f$ is 1 occurring at $x = 0, \dfrac{\pi}{2}$ and $\pi$. **[5]**

**Q. 39. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius *r* is $\dfrac{4r}{3}$.**

**[NCERT Misc Ex. Q. 15, Page 243]**

**Ans.** A sphere of fixed radius ($r$) is given.

Let $R$ and $h$ be the radius and the height of the cone, respectively.




The volume ($V$) of the cone is given by,

$$V = \frac{1}{3}\pi R^2 h$$

Now, from the right triangle BCD, we have

$$BC = \sqrt{r^2 - R^2}$$

$$\therefore \quad h = r + \sqrt{r^2 - R^2}$$

$$V = \frac{1}{3}\pi R^2\left(r + \sqrt{r^2 - R^2}\right)$$

$$= \frac{1}{3}\pi R^2 r + \frac{1}{3}\pi R^2\sqrt{r^2 - R^2}$$

$$\frac{dV}{dR} = \frac{2}{3}\pi Rr + \frac{2}{3}\pi R\sqrt{r^2 - R^2} + \frac{\pi R^2}{3}\cdot\frac{(-2R)}{2\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi Rr + \frac{2}{3}\pi R\sqrt{r^2 - R^2} - \frac{\pi R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi Rr + \frac{2\pi R\left(r^2 - R^2\right) - \pi R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi Rr + \frac{2\pi Rr^2 - 3\pi R^3}{3\sqrt{r^2 - R^2}}$$

Now, $\dfrac{dV}{dR^2} = 0$

$$\Rightarrow \quad \frac{2\pi rR}{3} = \frac{3\pi R^3 - 2\pi Rr^2}{3\sqrt{r^2 - R^2}}$$

$$\Rightarrow \quad 2r\sqrt{r^2 - R^2} = 3R^2 - 2r^2$$

$$\Rightarrow 4r^2\left(r^2 - R^2\right) = \left(3R^2 - 2r^2\right)^2$$

$$\Rightarrow \quad 4r^4 - 4r^2R^2 = 9R^4 + 4r^4 - 12R^2r^2$$

$$\Rightarrow 9R^4 - 8r^2R^2 = 0$$

$$\Rightarrow \quad 9R^2 = 8r^2$$

$$\Rightarrow \quad R^2 = \frac{8r^2}{9}$$

Now,

$$\frac{d^2V}{dR^2} = \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2}\left(2\pi r^2 - 9\pi R^2\right) - \left(2\pi Rr^2 - 3\pi R^3\right)(-6R)\dfrac{1}{2\sqrt{r^2 - R^2}}}{9\left(r^2 - R^2\right)}$$

$$= \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2}\left(2\pi r^2 - 9\pi R^2\right) + \left(2\pi Rr^2 - 3\pi R^3\right)(3R)\dfrac{1}{2\sqrt{r^2 - R^2}}}{9\left(r^2 - R^2\right)}$$

Now, when $R^2 = \dfrac{8r^2}{9}$, it can be shown that $\dfrac{d^2V}{dR^2} < 0$.

$\therefore$ The volume is the maximum when $R^2 = \dfrac{8r^2}{9}$.

When $R^2 = \dfrac{8r^2}{9}$, height of the cone

$$= r + \sqrt{r^2 - \frac{8r^2}{9}} = r + \sqrt{\frac{r^2}{9}} = r + \frac{r}{3} = \frac{4r}{3}.$$

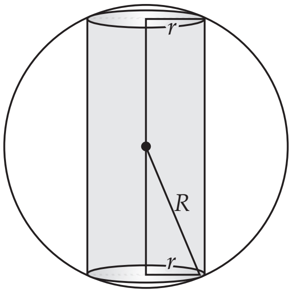
Hence, it can be seen that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\dfrac{4r}{3}$. **[5]**

**Q. 40. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius $R$ is $\dfrac{2R}{\sqrt{3}}$. Also find the maximum volume.**

**[NCERT Misc Ex. Q. 17, Page 243]**

**Ans.** A sphere of fixed radius ($R$) is given.

Let $r$ and $h$ be the radius and the height of the cylinder, respectively.




From the given figure, we have $h = 2\sqrt{R^2 - r^2}$.

The volume ($V$) of the cylinder is given by,

$$V = \pi r^2 h$$

$$= 2\pi r^2\sqrt{R^2 - r^2}$$

$$\frac{dV}{dr} = 4\pi r\sqrt{R^2 - r^2} + \frac{2\pi r^2(-2r)}{2\sqrt{R^2 - r^2}}$$

$$= 4\pi r\sqrt{R^2 - r^2} - \frac{2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r\left(R^2 - r^2\right) - 2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi R^2 - 6\pi r^3}{\sqrt{R^2 - r^2}}$$

Now, $\dfrac{dV}{dr} = 0 \Rightarrow 4\pi rR^2 - 6\pi r^3 = 0$

$$\Rightarrow \quad r^2 = \frac{2R^2}{3}$$

Now,

$$\frac{d^2V}{dr^2} = \frac{\sqrt{R^2 - r^2}\left(4\pi R^2 - 18\pi r^2\right) - \left(4\pi rR^2 - 6\pi r^3\right)\dfrac{(-2r)}{2\sqrt{R^2 - r^2}}}{\left(R^2 - r^2\right)}$$

$$= \frac{\left(R^2 - r^2\right)\left(4\pi R^2 - 18\pi r^2\right) + r\left(4\pi rR^2 - 6\pi r^3\right)}{\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

$$= \frac{4\pi R^4 - 22\pi r^2R^2 + 12\pi r^4 + 4\pi r^2R^2}{\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

Now, it can be observed that at $r^2 = \dfrac{2R^2}{3}, \dfrac{d^2V}{dr^2} < 0$.

$\therefore$ The volume is the maximum when $r^2 = \dfrac{2R^2}{3}$.

When $r^2 = \dfrac{2R^2}{3}$, the height of the cylinder is

$$2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}.$$

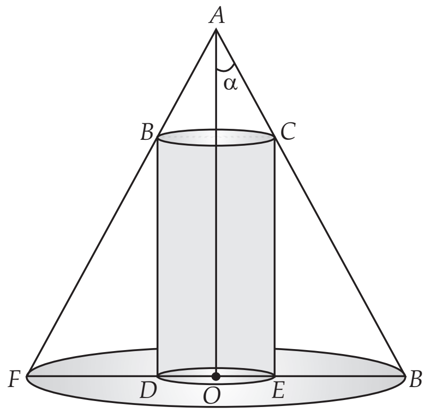
Hence, the volume of the cylinder is the maximum when the height of the cylinder is $\dfrac{2R}{\sqrt{3}}$. **[5]**

**Q. 41. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height $h$ and semi vertical angle $\alpha$ is one-**

**third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.**

**[NCERT Misc Ex. Q. 18, Page 243]**

**Ans.** The given right circular cone of fixed height ($h$) and semi-vertical angle ($\alpha$) can be drawn as :




Here, a cylinder of radius $R$ and height $H$ is inscribed in the cone.

Then, $\angle GAO = \alpha$, $OG = r$, $OA = h$, $OE = R$, and $CE = H$.

We have,

$r = h \tan \alpha$

Now, since $\Delta AOG$ is similar to $\Delta CEG$, we have :

$$\frac{AO}{OG} = \frac{CE}{EG}$$

$$\frac{h}{r} = \frac{H}{r-R} \quad [EG = 0G - OE]$$

$$H = \frac{h}{r}(r-R)$$

$$= \frac{h}{h\tan\alpha}(h\tan\alpha - R)$$

$$= \frac{1}{\tan\alpha}(h\tan\alpha - R)$$

$$V = \pi R^2 H$$

$$= \frac{\pi R^2}{\tan\alpha}(h\tan\alpha - R)$$

$$= \pi R^2 h - \frac{\pi R^3}{\tan\alpha}$$

$$\therefore \quad \frac{dV}{dR} = 2\pi Rh - \frac{3\pi R^2}{\tan\alpha}$$

Now, $\frac{dV}{dR} = 0$

$$\Rightarrow \quad 2\pi Rh = \frac{3\pi R^2}{\tan\alpha}$$

$$\Rightarrow 2h\tan\alpha = 3R$$

$$\Rightarrow \quad R = \frac{2h}{3}\tan\alpha$$

Now, $\frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi R}{\tan\theta}$

And, for $R = \frac{2h}{3}\tan\alpha$, we have

$$\frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi}{\tan\alpha}\left(\frac{2h}{3}\tan\alpha\right)$$

$$= 2\pi h - 4\pi h = -2\pi h < 0$$

$\therefore$ By second derivative test, the volume of the cylinder is the greatest when $R = \frac{2h}{3}\tan\alpha$.

When $R = \frac{2h}{3}\tan\alpha, H = \frac{1}{\tan\alpha}\left(h\tan\alpha - \frac{2h}{3}\tan\alpha\right)$

$$= \frac{1}{\tan\alpha}\left(\frac{h\tan\alpha}{3}\right) = \frac{h}{3}.$$

Thus, the height of the cylinder is one-third the height of the cone when the volume of the cylinder is the greatest.

Now, the maximum volume of the cylinder can be obtained as :

$$\pi\left(\frac{2h}{3}\tan\alpha\right)^2\left(\frac{h}{3}\right) = \pi\left(\frac{4h^2}{9}\tan^2\alpha\right)\left(\frac{h}{3}\right)$$

$$= \frac{4}{27}\pi h^3 \tan^2\alpha$$

Hence, the given result is proved. **[5]**

**Q. 42. Two men A and B start with velocities $v$ at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, find the rate at which they are being separated.**

**[NCERT Exemp. Ex. 6.3, Q. 4, Page 135]**

**Ans.** Let two men start from the point $C$ with velocity $v$ each at the same time.

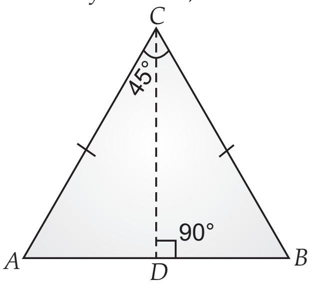
Also, $\angle BCA = 45°$

Since, $A$ and $B$ are moving with same velocity $v$, so they will cover same distance in same time.

Therefore, $\Delta ABC$ is an isosceles triangle with $AC = BC$.

Now, draw $CD \perp AB$.

Let at any instant $t$, the distance between them is $AB$




Let $AC = BC = x$ and $AB = y$

In $\Delta ACD$ and $\Delta DCB$,

$$\angle CAD = \angle CBD \qquad [\because AC = BC]$$

$$\angle CAD = \angle CDB = 90°$$

$$\therefore \quad \angle ACD = \angle DCB$$

$$\text{or} \quad \angle ACD = \frac{1}{2} \times \angle ACB$$

$$\Rightarrow \quad \angle ACD = \frac{1}{2} \times 45°$$

$$\Rightarrow \quad \angle ACD = \frac{\pi}{8}$$

$$\therefore \quad \sin\frac{\pi}{8} = \frac{AD}{AC}$$

$\Rightarrow \quad \sin\frac{\pi}{8} = \frac{y/2}{x} \quad [\because AD = y/2]$

$\Rightarrow \quad \frac{y}{2} = x\sin\frac{\pi}{8}$

$\Rightarrow \quad y = 2x \cdot \sin\frac{\pi}{8}$

Now, differentiating both sides with respect to $t$, we get

$$\frac{dy}{dt} = 2 \cdot \sin\frac{\pi}{8} \cdot \frac{dx}{dt}$$

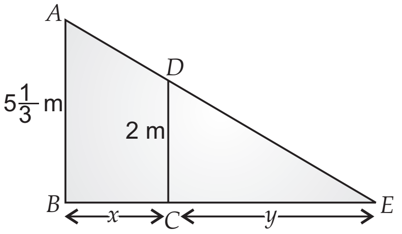
$$= 2 \cdot \sin\frac{\pi}{8} \cdot v \quad \left[\because v = \frac{dx}{dt}\right]$$

$$= 2v \cdot \frac{\sqrt{2-\sqrt{2}}}{2} \quad \left[\because \sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}\right]$$

This is the rate at which $A$ and $B$ are being separated. **[5]**

**Q. 43. A man, 2 m tall, walks at the rate of $1\frac{2}{3}$ m/s towards a street light which is $5\frac{1}{3}$ m above the ground. At what rate is the tip of his shadow moving? At what rate is the length of the shadow changing when he is $3\frac{1}{3}$ m from the base of the light? [NCERT Exemp. Ex. 6.3, Q. 8, Page 135]**

**Ans.** Let $AB$ be the street light post and $CD$ be the height of man, *i.e.*, $CD = 2$ m.




Let $BC = x$ m, $CE = y$ m and $\frac{dx}{dt} = \frac{-5}{3}$ m/s

From ΔABE and ΔDCE, we see that

$\Delta ABE \sim \Delta DCE$ [By AAA similarity]

$\therefore \quad \frac{AB}{DC} = \frac{BE}{CE}$

$\Rightarrow \quad \frac{\frac{16}{3}}{2} = \frac{x+y}{y}$

$\Rightarrow \quad \frac{16}{6} = \frac{x+y}{y}$

$\Rightarrow 16y = 6x + 6y$

$\Rightarrow 10y = 6x$

$\therefore \quad y = \frac{3}{5}x$

On differentiating both sides with respect to $t$, we get

$$\frac{dy}{dt} = \frac{3}{5} \cdot \frac{dx}{dt} = \frac{3}{5} \cdot \left(-1\frac{2}{3}\right)$$

[Since, man is moving towards the light post.]

$$= \frac{3}{5} \cdot \left(\frac{-5}{3}\right) = -1 \text{ m/s}$$

Let $z = x + y$

Now, differentiating both sides with respect to $t$, we get

$$\frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = -\left(\frac{5}{3} + 1\right)$$

$$= -\frac{8}{3} = -2\frac{2}{3} \text{ m/s}$$

Hence, the tip of shadow is moving at the rate of $2\frac{2}{3}$ m/s towards the light source and length of the shadow is decreasing at the rate of 1 m/s. **[5]**

**Q. 44. Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.**

**[NCERT Exemp. Ex. 6.3, Q. 13, Page 136]**

**Ans.** Given equation of curves are

$xy = 4$ ...(i)

and $x^2 + y^2 = 8$ ...(ii)

$\Rightarrow \quad x \cdot \frac{dy}{dx} + y = 0$

and $2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-y}{x}$

and $\quad \frac{dy}{dx} = \frac{-2x}{2y}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-y}{x} = m_1(\text{say})$

and $\quad \frac{dy}{dx} = \frac{-x}{y} = m_2(\text{say})$

Since, both the curves should have same slope.

$\therefore \quad \frac{-y}{x} = \frac{-x}{y}$

$\Rightarrow \quad -y^2 = -x^2$

$\Rightarrow \quad x^2 = y^2$ ...(iii)

Using the value of $x^2$ in Eq. (ii), we get

$y^2 + y^2 = 8$

$\Rightarrow \quad y^2 = 4$

$\Rightarrow \quad y = \pm 2$

For $y = 2, x = \frac{4}{2} = 2$ and for $y = -2, x = \frac{4}{-2} = -2$.

Thus, the required points of intersection are (2, 2) and (–2, –2).

For $(2,2)$, $m_1 = \frac{-y}{x} = \frac{-2}{2} = -1$ and

$m_2 = \frac{-x}{y} = \frac{-2}{2} = -1$

$\because \quad m_1 = m_2$

For $(-2,-2), m_1 = \frac{-y}{x} = \frac{-(-2)}{-2} = -1$ and

$m_2 = \frac{-x}{y} = \frac{-(-2)}{-2} = -1$

Thus, for both the intersection points, we see that slope of both the curves are same.

Hence, the curves touch each other. **[5]**

**Q. 45. Find the equation of the normal lines to the curve $3x^2 - y^2 = 8$ which are parallel to the line $x + 3y = 4$.**
**[NCERT Exemp. Ex. 6.3, Q. 17, Page 136]**

**Ans.** Given equation of the curve is

$3x^2 - y^2 = 8$ ...(i)

On differentiating both sides with respect to $x$, we get

$$6x - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{6x}{2y} = \frac{3x}{y}$$

$$\Rightarrow \quad m_1 = \frac{3x}{y} \quad [\text{say}]$$

and slope of normal $(m_2) = \frac{-1}{m_1} = \frac{-y}{3x}$ ...(ii)

Since, slope of normal to the curve should be equal to the slope of line $x + 3y = 4$, which is parallel to curve.

For line, $y = \frac{4-x}{3} = \frac{-x}{3} + \frac{4}{3}$

$\Rightarrow$ Slope of the line, $(m_3) = \frac{-1}{3}$

$$\therefore \quad m_2 = m_3$$

$$\Rightarrow \quad \frac{-y}{3x} = -\frac{1}{3}$$

$$\Rightarrow \quad -3y = -3x$$

$$\Rightarrow \quad y = x \quad \text{...(iii)}$$

On substituting the value of $y$ in Eq. (i), we get

$$3x^2 - x^2 = 8$$

$$\Rightarrow \quad x^2 = 4$$

$$\Rightarrow \quad x = \pm 2$$

For $x = 2, y = 2$ [Using Eq. (iii)]

and for $x = -2, y = -2$ [Using Eq. (iii)]

Thus, the points at which normal to the curve are parallel to the line $x + 4y = 4$ are (2, 2) and (–2, –2).

Required equations of normal are

$$y - 2 = m_2(x-2) \text{ and } y + 2 = m_2(x+2)$$

$$\Rightarrow \quad y - 2 = \frac{-2}{6}(x-2) \text{ and } y + 2 = \frac{-2}{6}(x+2)$$

$$\Rightarrow \quad 3y - 6 = -x + 2 \quad \text{and} \quad 3y + 6 = -x - 2$$

$$\Rightarrow \quad 3y + x = +8 \quad \text{and} \quad 3y + x = -8$$

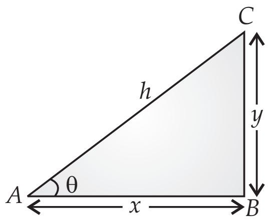
So, the required equations are $3y + x = \pm 8$. **[5]**

**Q. 46. If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.**
**[NCERT Exemp. Ex. 6.3, Q. 25, Page 137]**

**Ans.** Let $ABC$ be a triangle with $AC = h$, $AB = x$ and $BC = y$.

Also, $\angle CAB = \theta$

Let $h + x = k$ **...(i)**




$$\therefore \quad \cos\theta = \frac{x}{h}$$

$$\Rightarrow \quad x = h\cos\theta \quad [\text{Using Eq. (i)}]$$

$$\Rightarrow \quad h + h\cos\theta = k$$

$$\Rightarrow h(1+\cos\theta) = k$$

$$\Rightarrow \quad h = \frac{k}{(1+\cos\theta)} \quad \text{...(ii)}$$

also, area of $\Delta ABC = \frac{1}{2}(AB \cdot BC)$

$$A = \frac{1}{2}\cdot x \cdot y$$

$$= \frac{1}{2}h\cos\theta \cdot h\sin\theta \quad \left[\because \sin\theta = \frac{y}{h}\right]$$

$$= \frac{1}{2}h^2\sin\theta\cdot\cos\theta$$

$$= \frac{2h^2}{4}\sin\theta\cdot\cos\theta$$

$$= \frac{1}{4}h^2\sin 2\theta \quad \text{...(iii)}$$

Since, $h = \frac{k}{1+\cos\theta}$

$$\therefore \quad A = \frac{1}{4}\left(\frac{k}{1+\cos\theta}\right)\cdot\sin 2\theta$$

$$\Rightarrow \quad A = \frac{k^2}{4}\cdot\frac{\sin 2\theta}{(1+\cos\theta)^2} \quad \text{...(iv)}$$

$$\therefore \frac{dA}{d\theta} = \frac{k^2}{4}\left[\frac{\begin{array}{l}(1+\cos\theta)^2\cdot\cos 2\theta\cdot 2 \\ \qquad -\sin 2\theta\cdot 2(1+\cos\theta)\cdot(0-\sin\theta)\end{array}}{(1+\cos\theta)^4}\right]$$

$$= \frac{k^2}{4}\left\{\frac{2(1+\cos\theta)(1+\cos\theta)\cdot\cos 2\theta + \sin 2\theta(\sin\theta)}{(1+\cos\theta)^4}\right\}$$

$$= \frac{k^2}{4}\cdot\frac{2}{(1+\cos\theta)^3}\left[(1+\cos\theta)\cdot\cos 2\theta + 2\sin^2\theta\cdot\cos\theta\right]$$

$$= \frac{k^2}{2(1+\cos\theta)^3}\left[(1+\cos\theta)(1-2\sin^2\theta) + 2\sin^2\theta\cdot\cos\theta\right]$$

$$= \frac{k^2}{2(1+\cos\theta)^3}\left[\begin{array}{r}1+\cos\theta - 2\sin^2\theta - 2\sin^2\theta\cdot\cos \\ +2\sin^2\theta\cdot\cos\theta\end{array}\right]$$

$$= \frac{k^2}{2(1+\cos\theta)^3}\left[(1+\cos\theta) - 2\sin^2\theta\right]$$

$$= \frac{k^2}{2(1+\cos\theta)^3}\left[1+\cos\theta - 2 + 2\cos^2\theta\right]$$

$$= \frac{k^2}{2(1+\cos\theta)^3}(2\cos^2\theta + \cos\theta - 1) \quad \text{...(v)}$$

For $\frac{dA}{d\theta}=0,$ we have

$$\frac{k^2}{2(1+\cos\theta)^3}\left(2\cos^2\theta+\cos\theta-1\right)=0$$

$\Rightarrow \quad 2\cos^2\theta+\cos\theta-1=0$

$\Rightarrow \quad 2\cos^2\theta+2\cos\theta-\cos\theta-1=0$

$\Rightarrow \quad 2\cos\theta(\cos\theta+1)-1(\cos\theta+1)=0$

$\Rightarrow \quad (2\cos\theta-1)(\cos\theta+1)=0$

$\Rightarrow \quad \cos\theta=\frac{1}{2}$ or $\cos\theta=-1$

$\Rightarrow \quad \theta=\frac{\pi}{3}$ [Possible]

or $\quad \theta=2n\pi\pm\pi$ [not possible]

$\therefore \quad \theta=\frac{\pi}{3}$

Again, differentiating with respect to θ in Eq. (v), we get

$$\frac{d}{d\theta}\left(\frac{dA}{d\theta}\right)=\frac{d}{d\theta}\left[\frac{k^2}{2(1+\cos\theta)^3}\left(2\cos^2\theta+\cos\theta-1\right)\right]$$

$$\therefore \frac{d^2A}{d\theta^2}=\frac{d}{d\theta}\left[\frac{k^2(2\cos\theta-1)(1+\cos\theta)}{2(1+\cos\theta)^3}\right]=\frac{d}{d\theta}\left[\frac{k^2}{2}\cdot\frac{(2\cos\theta-1)}{(1+\cos\theta)^2}\right]$$

$$=\frac{k^2}{2}\left[\frac{(1+\cos\theta)^2\cdot(-2\sin\theta)-2(1+\cos\theta)\cdot(-\sin\theta)(2\cos\theta-1)}{(1+\cos\theta)^4}\right]$$

$$=\frac{k^2}{2}\left[\frac{(1+\cos\theta)\cdot[1+\cos\theta](-2\sin\theta)+2\sin\theta(2\cos\theta-1)}{(1+\cos\theta)^4}\right]$$

$$=\frac{k^2}{2}\left[\frac{-2\sin\theta-2\sin\theta\cdot\cos\theta+4\sin\theta\cdot\cos\theta-2\sin\theta}{(1+\cos\theta)^3}\right]$$

$$=\frac{k^2}{2}\left[\frac{-4\sin\theta-\sin2\theta+2\sin2\theta}{(1+\cos\theta)^3}\right]$$

$$=\frac{k^2}{2}\left[\frac{\sin2\theta-4\sin\theta}{(1+\cos\theta)^3}\right]$$

$$\therefore \left(\frac{d^2A}{d\theta^2}\right)_{\text{at }\theta=\frac{\pi}{3}}=\frac{k^2}{2}\left[\frac{\sin\frac{2\pi}{3}-4\sin\frac{\pi}{3}}{\left(1+\cos\frac{\pi}{3}\right)}\right]$$

$$=\frac{k^2}{2}\left[\frac{\frac{\sqrt{3}}{2}-\frac{4\sqrt{3}}{2}}{\left(1+\frac{1}{2}\right)^3}\right]$$

$$=\frac{k^2}{2}\left[\frac{-3\sqrt{3}\cdot8}{2.27}\right]$$

$$=-k^2\left(\frac{2\sqrt{3}}{9}\right)$$

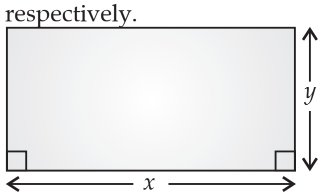
This is less than zero.

Hence, area of the right-angled triangle is maximum, when the angle between them is $\frac{\pi}{3}$. **[5]**

**Q. 47. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its sides. Also find the maximum volume.**

**[NCERT Exemp. Ex. 6.3, Q. 30, Page 137]**

**Ans.** Let breadth and length of the rectangle be $x$ and $y$, respectively.




$\because$ Perimeter of the rectangle = 36 cm

$\Rightarrow 2x + 2y = 36$

$\Rightarrow \quad x + y = 18$

$\Rightarrow \quad y = 18 - x$ ...(i)

Let the rectangle is being revolved about its length $y$. Then, volume ($V$) of resultant cylinder = $\pi x^2 y$

$\Rightarrow V=\pi x^2\cdot(18-x) \quad [\because V=\pi r^2h$ and using Eq. (i)$]$

$=18\pi x^2-\pi x^3$

$=\pi\left[18x^2-x^3\right]$

On differentiating both sides with respect to $x$, we get

$$\frac{dV}{dx}=\pi\left(36x-3x^2\right)$$

Now, $\quad \frac{dV}{dx}=0$

$\Rightarrow \quad 36x=3x^2$

$\Rightarrow \quad 36x=3x^2$

$\Rightarrow \quad 3x^2-36x=0$

$\Rightarrow 3\left(x^2-12x\right)=0$

$\Rightarrow \quad 3x(x-12)=0$

$\Rightarrow \quad x=0, x=12$

$\therefore \quad x=12 \quad [\because x,\neq 0]$

Again, differentiating with respect to $x$, we get

$$\frac{d^2V}{dx^2}=\pi(36-6x)$$

$$\Rightarrow \left(\frac{d^2V}{dx^2}\right)_{x=12}=\pi(36-6\times12)=-36\pi<0$$

At $x$ = 12, volume of the resultant cylinder is the maximum.

So, the dimensions of rectangle are 12 cm and 6 cm, respectively [using Eq. (i)].

Maximum volume of resultant cylinder,

$$(V)_{x=12}=\pi\left[18\cdot(12)^2-(12)^3\right]$$

$$=\pi\left[12^2(18-12)\right]$$

$$=\pi\times144\times6$$

$$=864\pi \text{ cm}^3$$ **[5]**

**Q. 48. If the sum of the surface areas of cube and a sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere, when the sum of their volumes is minimum?**

**[NCERT Exemp. Ex. 6.3, Q. 31, Page 138]**

**Ans.** Let length of one cube be $x$ units and radius of sphere be $r$ units.

Surface area of cube, $A = 6x^2$ and surface area of sphere, $S = 4\pi r^2$

Also, $6x^2 + 4\pi r^2 = k$ **[Constant, given]**

$$\Rightarrow 6x^2 = k - 4\pi r^2$$

$$\Rightarrow \quad x^2 = \frac{k - 4\pi r^2}{6}$$

$$\Rightarrow \quad x = \left[\frac{k - 4\pi r^2}{6}\right]^{1/2} \quad \text{...(i)}$$

Now, volume of cube $= x^3$

And volume of sphere $= \frac{4}{3}\pi r^3$

Let sum of volume of the cube and volume of the sphere be given by

$$S = x^3 + \frac{4}{3}\pi r^3$$

$$= \left[\frac{k - 4\pi r^2}{6}\right]^{3/2} + \frac{4}{3}\pi r^3$$

On differentiating both sides with respect to $r$, we get

$$\frac{dS}{dr} = \frac{3}{2}\left[\frac{k - 4\pi r^2}{6}\right]^{1/2} \cdot \left(\frac{-8\pi r}{6}\right) + \frac{12}{3}\pi r^2$$

$$= -2\pi r\left[\frac{k - 4\pi r^2}{6}\right]^{1/2} + 4\pi r^2 \quad \text{...(ii)}$$

$$= 2\pi r\left[\left\{\frac{k - 4\pi r^2}{6}\right\}^{1/2} - 2r\right]$$

Now, $\quad \frac{dS}{dr} = 0$

$$\Rightarrow \quad r = 0 \text{ or } 2t = \left(\frac{k - 4\pi r^2}{6}\right)^{1/2}$$

$$\Rightarrow \quad 4r^2 = \frac{k - 4\pi r^2}{6} \Rightarrow 24r^2 = k - 4\pi r^2$$

$$\Rightarrow \quad 24r^2 + 4\pi r^2 = k \Rightarrow r^2[24 + 4\pi] = k$$

$$\therefore \quad r = 0 \text{ or } r = \sqrt{\frac{k}{24 + 4\pi}} = \frac{1}{2}\sqrt{\frac{k}{6 + \pi}}$$

We know that, $r \neq 0$

$$\therefore \quad r = \frac{1}{2}\sqrt{\frac{k}{6 + \pi}}$$

Again, differentiating with respect to $t$ in Eq. (ii), we get

$$\frac{d^2S}{dr^2} = \frac{d}{dr}\left[-2\pi r\left\{\left(\frac{k - 4\pi r^2}{6}\right)^{1/2} + 4\pi r^2\right\}\right]$$

$$= -2\pi\left[r \cdot \frac{1}{2}\left(\frac{k - 4\pi r^2}{6}\right)^{-1/2} \cdot \left(\frac{-8\pi r}{6}\right) + \left(\frac{k - 4\pi r^2}{6}\right)^{1/2} \cdot 1\right]$$

$$+ 4\pi \cdot 2r$$

$$= -2\pi\left[t \cdot \frac{1}{2\sqrt{\frac{k - 4\pi r^2}{6}}} \cdot \left(\frac{-8\pi r}{6}\right) + \sqrt{\frac{k - 4\pi r^2}{6}}\right] + 8\pi r$$

$$= -2\pi\left[\frac{-8\pi r^2 + 12\left(k - \frac{4\pi r^2}{6}\right)}{12\sqrt{\frac{k - 4\pi r^2}{6}}}\right] + 8\pi r$$

$$= -2\pi\left[\frac{-48\pi r^2 + 72k - 48\pi r^2}{72\sqrt{\frac{k - 4\pi r^2}{6}}}\right] + 8\pi r$$

$$= -2\pi\left[\frac{-96\pi r^2 + 72k}{72\sqrt{\frac{k - 4\pi r^2}{6}}}\right] + 8\pi r > 0$$

For $t = \frac{1}{2}\sqrt{\frac{k}{6 + \pi}}$, then the sum of their volume is minimum.

For $t = \frac{1}{2}\sqrt{\frac{k}{6 + \pi}}$, $x = \left[\frac{k - 4\pi \cdot \frac{1}{4}\frac{k}{(6 + \pi)}}{6}\right]^{1/2}$

$$= \left[\frac{(6 + \pi)k - \pi k}{6(6 + \pi)}\right]^{1/2} = \left[\frac{k}{6 + \pi}\right]^{1/2} = 2r$$

Since, the sum of their volume is minimum when $x = 2r$

Hence, the ratio of an edge of cube to the diameter of the sphere is 1 :1. **[5]**

**Q. 49. AB is a diameter of a circle and C is any point on the circle. Show that the area of ΔABC is maximum, when it is isosceles.**

**[NCERT Exemp. Ex. 6.3, Q. 32, Page 138]**

**Ans.** We have, $AB = 2r$

and $\angle ACB = 90°$ [Since, angle in the semi-circle is always 90°.]

Let $AC = x$ and $BC = y$

$\therefore \ (2r)^2 = x^2 + y^2$

$\Rightarrow \quad y^2 = 4r^2 - x^2$

$$y = \sqrt{4r^2 - x^2} \quad \text{...(i)}$$

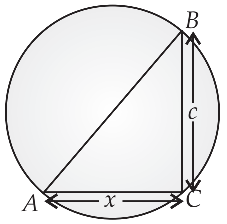
Now, area of $\Delta ABC$, $A = \frac{1}{2} \times x \times y$

$$= \frac{1}{2} \times x \times \left(4r^2 - x^2\right)^{1/2} \quad \text{[Using Eq. (i)]}$$

Now, differentiating both sides with respect to $x$, we get

$$\frac{dA}{dx} = \frac{1}{2}\left[x \cdot \frac{1}{2}\left(4r^2 - x^2\right)^{-1/2} \cdot (0 - 2x) + \left(4r^2 - x^2\right)^{1/2} \cdot 1\right]$$

$$= \frac{1}{2}\left[\frac{-2x^2}{2\sqrt{4r^2 - x^2}} + \left(4r^2 - x^2\right)^{1/2}\right]$$




$$= \frac{1}{2}\left[\frac{-x^2}{\sqrt{4r^2 - x^2}} + \sqrt{4r^2 - x^2}\right]$$

$$= \frac{1}{2}\left[\frac{-x^2 + 4r^2 - x^2}{\sqrt{4r^2 - x^2}}\right]$$

$$=\frac{1}{2}\left[\frac{-2x^2+4r^2}{\sqrt{4r^2-x^2}}\right]$$

$$\therefore \frac{dA}{dx}=\left[\frac{\left(-x^2+2r^2\right)}{\sqrt{4r^2-x^2}}\right]$$

Now, $\frac{dA}{dx}=0$

$$\Rightarrow -x^2+2r^2=0$$

$$\Rightarrow \quad r^2=\frac{1}{2}x^2$$

$$\Rightarrow \quad r=\frac{1}{\sqrt{2}}x$$

$$\therefore \quad x=r\sqrt{2}$$

Again, differentiating both sides with respect to $x$, we get

$$\frac{d^2A}{dx^2}=\frac{\sqrt{4r^2-x^2}\cdot(-2x)+\left(2r^2-x^2\right)\cdot\frac{1}{2}\left(4r^2-x^2\right)^{-1/2}(-2x)}{\left(\sqrt{4r^2-x^2}\right)^2}$$

$$=\frac{-2x\left[\sqrt{4r^2-x^2}+\left(2r^2-x^2\right)\cdot\frac{1}{2\sqrt{4r^2-x^2}}\right]}{\left(\sqrt{4r^2-x^2}\right)^2}$$

$$=\frac{-4x\cdot\left(\sqrt{4r^2-x^2}\right)^2+\left(2r^2-x^2\right)(-2x)}{2\cdot\left(4r^2-x^2\right)^{3/2}}$$

$$=\frac{-4x\left(4r^2-x^2\right)+\left(2r^2-x^2\right)\cdot(-2x)}{2\cdot\left(4r^2-x^2\right)^{3/2}}$$

$$=\frac{-16xr^2+4x^3+\left(2r^2-x^2\right)(-2x)}{2\cdot\left(4r^2-x^2\right)^{3/2}}$$

$$\left(\frac{d^2A}{dx^2}\right)_{x=r\sqrt{2}}=\frac{-16\cdot r\sqrt{2}\cdot r^2+4\cdot\left(r\sqrt{2}\right)^3+\left[2r^2-\left(r\sqrt{2}\right)^2\right]\cdot\left(-2\cdot r\sqrt{2}\right)}{2\cdot\left(4r^2-2r^2\right)^{3/2}} \quad \left[\because x=r\sqrt{2}\right]$$

$$=\frac{-16\sqrt{2}\cdot r^3+8\sqrt{2}r^3}{2\left(2r^2\right)^{3/2}}$$

$$=\frac{8\sqrt{2}r^2[r-2r]}{4r^3}$$

$$=\frac{-8\sqrt{2}r^3}{4r^3}$$

$$=-2\sqrt{2}<0$$

For $x=r\sqrt{2}$, the area of triangle is maximum.

For $x=r\sqrt{2}$, $y=\sqrt{4r^2-\left(r\sqrt{2}\right)^2}=\sqrt{2r^2}=r\sqrt{2}$

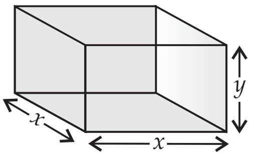
Since, $x=r\sqrt{2}=y$

Hence, the triangle is isosceles. **[5]**

**Q. 50. A metal box with a square base and vertical sides is to contain 1024 cm$^3$. The material for the top and bottom costs ₹ 5/cm$^2$ and the material for the sides costs ₹ 2.50/cm$^2$. Find the least cost of the box.**
**[NCERT Exemp. Ex. 6.3, Q. 33, Page 138]**

**Ans.** Since, volume of the box = 1024 cm$^3$

Let length of the side of square base be $x$ cm and height of the box $y$ cm.




$\therefore$ Volume of the box $(V)$ = $x^2y$ = 1024

Since, $x^2y=1024\Rightarrow y=\frac{1024}{x^2}$

Let $C$ denotes the cost of the box. Therefore,

$$C=2x^2\times5+4xy\times2.50$$

$$=10x^2+10xy$$

$$=10x(x+y)$$

$$=10x\left(x+\frac{1024}{x^2}\right)$$

$$=\frac{10x}{x^2}\left(x^3+1024\right)$$

$$\Rightarrow \quad C=10x^2+\frac{10240}{x} \quad \text{...(i)}$$

On differentiating both sides with respect to $x$, we get

$$\frac{dC}{dx}=20x+10240(-x)^{-2}$$

$$=20x-\frac{10240}{x^2} \quad \text{...(ii)}$$

Now, $\frac{dC}{dx}=0$

$$\Rightarrow \quad 20x=\frac{10240}{x^2}$$

$$\Rightarrow \quad 20x^3=10240$$

$$\Rightarrow \quad x^3=512=8^3$$

$$\therefore \quad x=8$$

Again, differentiating Eq. (ii) with respect to $x$, we get

$$\frac{d^2C}{dx^2}=20-10240(-2),\frac{1}{x^3}$$

$$=20+\frac{20480}{x^3}>0$$

$$\therefore \left(\frac{d^2C}{dx^2}\right)_{x=8}=20+\frac{20480}{512}=60>0$$

For $x$ = 8, cost is minimum and the corresponding least cost of the box.

$$C(8)=10.8^2+\frac{10240}{8}$$

$$=640+1280$$

$$=1920$$

$\therefore$ Least cost = ₹ 1920 **[5]**

**Q. 51. The sum of the surface areas of a rectangular parallel piped with sides $x$, $2x$ and $\frac{x}{3}$ and a sphere**

**is given to be constant. Prove that the sum of their volumes is minimum, if $x$ is equal to three times the radius of the sphere. Also find the minimum value of the sum of their volumes.**

**[NCERT Exemp. Ex. 6.3, Q. 34, Page 138]**

**Ans.** We have given that the sum of the surface areas of a rectangular parallel piped with sides $x$, $2x$ and $\frac{x}{3}$ and a sphere is constant.

Let $S$ be the sum of both the surface area.

$$\therefore \quad S = 2\left(x \cdot 2x + 2x \cdot \frac{x}{3} + \frac{x}{3} \cdot x\right) + 4\pi r^2 = k$$

$$k = 2\left[2x^2 + \frac{2x^2}{3} + \frac{x^2}{3}\right] + 4\pi r^2$$

$$= 2\left[3x^2\right] + 4\pi r^2$$

$$= 6x^2 + 4\pi r^2$$

$$\Rightarrow 4\pi r^2 = k - 6x^2$$

$$\therefore \quad r^2 = \frac{k - 6x^2}{4\pi}$$

$$\Rightarrow \quad r = \sqrt{\frac{k - 6x^2}{4\pi}} \qquad \text{...(i)}$$

Let $V$ denotes the volume of both the parallelepiped and the sphere.

Then, $V = 2x \cdot x \cdot \frac{x}{3} + \frac{4}{3}\pi r^3 = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi\left(\frac{k - 6x^2}{4\pi}\right)^{3/2}$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi \cdot \frac{1}{8\pi^{3/2}}\left(k - 6x^2\right)^{3/2}$$

$$= \frac{2}{3}x^3 + \frac{1}{6\sqrt{\pi}}\left(k - 6x^2\right)^{3/2} \qquad \text{...(ii)}$$

On differentiating both sides with respect to $x$, we get

$$\frac{dV}{dx} = \frac{2}{3} \cdot 3x^2 + \frac{1}{6\sqrt{\pi}} \cdot \frac{3}{2}\left(k - 6x^2\right)^{1/2} .(-12x)$$

$$= 2x^2 - \frac{12x}{4\sqrt{\pi}}\sqrt{k - 6x^2}$$

$$= 2x^2 - \frac{3x}{\sqrt{\pi}}\left(k - 6x^2\right)^{1/2} \qquad \text{...(iii)}$$

$$\because \quad \frac{dV}{dx} = 0$$

$$\Rightarrow \quad 2x^2 = \frac{3x}{\sqrt{\pi}}\left(k - 6x^2\right)^{1/2}$$

$$\Rightarrow \quad 4x^4 = \frac{9x^2}{\pi}\left(k - 6x^2\right)$$

$$\Rightarrow \quad 4\pi x^4 = 9kx^2 - 54x^4$$

$$\Rightarrow \quad 4\pi x^4 = 9kx^2$$

$$\Rightarrow x^4[4\pi + 54] = 9 \cdot k \cdot x^2$$

$$\Rightarrow \quad x^2 = \frac{9k}{4\pi + 54}$$

$$\Rightarrow \quad x = 3 \cdot \sqrt{\frac{k}{4\pi + 54}} \qquad \text{...(iv)}$$

Again, differentiating Eq. (iii) with respect to $x$, we get

$$\frac{d^2V}{dx^2} = 4x - \frac{3}{\sqrt{\pi}}\left[x \cdot \frac{1}{2}\left(k - 6x^2\right)^{-1/2} \cdot (-12x) + \left(k - 6x^2\right)^{1/2} \cdot 1\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[-6x^2 \cdot \left(k - 6x^2\right)^{-1/2} + \left(k - 6x^2\right)^{1/2}\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[\frac{-6x^2 + k - 6x^2}{\sqrt{k - 6x^2}}\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[\frac{k - 12x^2}{\sqrt{k - 6x^2}}\right]$$

Now,

$$\left(\frac{d^2V}{dx^2}\right)_{x = 3\cdot\sqrt{\frac{k}{4\pi + 54}}} = 4 \cdot 3\sqrt{\frac{k}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{k - 12 \cdot 9 \cdot \frac{k}{4\pi + 54}}{\sqrt{k - \frac{6 \cdot 9 \cdot k}{4\pi + 54}}}\right]$$

$$= 12\sqrt{\frac{k}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{k - \frac{108k}{4\pi + 54}}{\sqrt{k - \frac{54k}{4\pi + 54}}}\right]$$

$$= 12\sqrt{\frac{k}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{4k\pi + 54k - 108k / 4\pi + 54}{\sqrt{4k\pi + 54k - 54k / 4\pi + 54}}\right]$$

$$= 12\sqrt{\frac{k}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{4k\pi - 54k}{\sqrt{4k\pi}\sqrt{4\pi + 54}}\right]$$

$$= 12\sqrt{\frac{k}{4\pi + 54}} - \frac{6}{\sqrt{\pi}}\left[\frac{k(2\pi - 27)}{\sqrt{k}\sqrt{16\pi^2 + 216\pi}}\right]$$

$$\left[\because (2\pi - 27) < 0 \Rightarrow \frac{d^2V}{dx^2} >; k > 0\right]$$

For $x = 3\sqrt{\frac{k}{4\pi + 54}}$, the sum of volume is minimum.

For $x = 3\sqrt{\frac{k}{4\pi + 54}}$, then $r = \sqrt{\frac{k - 6x^2}{4\pi}}$ **[Using Eq. (i)]**

$$= \frac{2}{2\sqrt{\pi}}\sqrt{k - 6 \cdot \frac{9k}{4\pi + 54}}$$

$$= \frac{1}{2\sqrt{\pi}} \cdot \sqrt{\frac{4k\pi + 54k - 54k}{4\pi + 54}}$$

$$= \frac{1}{2\sqrt{\pi}}\sqrt{\frac{4k\pi}{4\pi + 54}}$$

$$= \frac{\sqrt{k}}{\sqrt{4\pi + 54}} = \frac{1}{3}x$$

$$\Rightarrow x = 3r$$

Hence proved.

$\therefore$ Minimum sum of volume,

$$V_{\left(x = 3\cdot\sqrt{\frac{k}{4\pi + 54}}\right)} = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi \cdot \left(\frac{1}{3}x\right)^3$$

$$=\frac{2}{3}x^3+\frac{4}{3}\pi\cdot\frac{x^3}{27}$$

$$=\frac{2}{3}x^3\left(1+\frac{2\pi}{27}\right) \quad \textbf{[5]}$$

**Q. 52. Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube.**

**[CBSE Board, All India Region, 2017]**

**Ans.** Let the sides of cuboid be $x$, $x$ and $y$.

$\Rightarrow x^2y=k$ and $S=2\left(x^2+xy+xy\right)=2\left(x^2+2xy\right)$ [½+1]

$$\therefore \quad S=2\left[x^2+2x\frac{k}{x^2}\right]=2\left[x^2+\frac{2k}{x}\right] \quad \textbf{[1]}$$

$$\frac{ds}{dx}=2\left[2x-\frac{2k}{x^2}\right] \quad \textbf{[1]}$$

$$\therefore \quad \frac{ds}{dx}=0$$

$$\Rightarrow \quad x^3=k=x^2y$$

$$\Rightarrow \quad x=y \quad \textbf{[1]}$$

$$\frac{d^2s}{dx^2}=2\left[2+\frac{4k}{x^3}\right]>0$$

$\therefore \quad x=y$ will give minimum surface area **[1]**

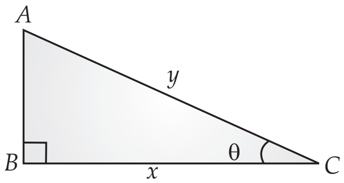
and $x = y$, means sides are equal

$\therefore$ Cube will have minimum surface area [½]

**Q. 53. If the sum of lengths of the hypotenuse and a side of a right-angled triangle is given, show that the area of the triangle is maximum, when the angle between them is $\frac{\pi}{3}$.**

**[CBSE Board, Delhi Region, 2017]**

**Ans.**




Given that, $x+y=k$ **[1]**

Area of $\quad \Delta=\frac{1}{2}x\sqrt{y^2-x^2}$

Let $\quad Z=\frac{1}{4}x^2\left(y^2-x^2\right)$

$$=\frac{1}{4}x^2\left[(k-x)^2-x^2\right]$$

$$=\frac{1}{4}\left[k^2x^2-2kx^3\right] \quad \textbf{[1]}$$

$$\frac{dz}{dx}=\frac{1}{4}\left[2k^2x-6kx^2\right]=0$$

$$\Rightarrow \quad k-3x=0$$

$$\Rightarrow \quad x=\frac{k}{3} \quad \textbf{[1½]}$$

$$\Rightarrow \quad x+y-3x=0 \text{ or } y=2x$$

$$\frac{d^2z}{dx^2}=\frac{1}{4}\left[2k^2-12kx\right] \quad \textbf{[1]}$$

$$\left.\frac{d^2z}{dx^2}\right|_{x=\frac{k}{3}}=\frac{1}{4}\left[2k^2-4k^2\right]=-\frac{k^2}{2}<0$$

$\therefore$ Area will be maximum for $2x = y$ **[1]**

But,

$$\frac{x}{y}=\cos\theta$$

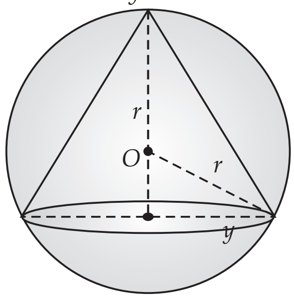
$$\Rightarrow \quad \cos\theta=\frac{x}{2x}=\frac{1}{2}$$

$$\therefore \quad \theta=\frac{\pi}{3} \quad \textbf{[½]}$$

**Q. 54. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius $r$ is $4r/3$. Also find maximum volume in terms of volume of the sphere.**

**[CBSE Board, Delhi Region, 2016]**

**Ans.** Let radius of cone be $y$ and the altitude be $r + x$




**[1]**

$$\therefore x^2+y^2=r^2 \quad \text{...(i)} \quad \textbf{[½]}$$

Volume, $V=\frac{1}{3}\pi y^2(r+x)$

$$=\frac{1}{3}\pi\left(r^2-x^2\right)(r+x) \quad \textbf{[1]}$$

$$\frac{dV}{dx}=\frac{\pi}{3}\left[\left(r^2-x^2\right)1+(r+x)(-2x)\right]$$

$$=\frac{\pi}{3}(r+x)(r-3x) \quad \textbf{[1]}$$

$$\frac{dV}{dx}=0\Rightarrow x=\frac{r}{3} \quad \textbf{[½]}$$

$\therefore$ Altitude $=r+\frac{r}{3}=\frac{4r}{3}$ **[½]**

And

$$\frac{d^2V}{dx^2}=\frac{\pi}{3}\left[(r+x)(-3)+(r-3x)\right]$$

$$=\frac{\pi}{3}\left[-2r-6x\right]<0 \quad \textbf{[1]}$$

$\therefore$ Maximum volume

$$=\frac{\pi}{3}\left(r^2-\frac{r^2}{9}\right)\left(r+\frac{r}{3}\right)=\frac{8}{27}\left(\frac{4}{3}\pi r^3\right) \quad \textbf{[½]}$$

$$=\frac{8}{27}\text{ (Volume of sphere)}$$

**Q. 55. Find the intervals in which $f(x) = \sin 3x - \cos 3x, 0 < x < p$, is strictly increasing or strictly decreasing.**

**[CBSE Board, Delhi Region, 2016]**

**Ans.** $f(x) = \sin 3x - \cos 3x, 0 < x < \pi$

$f'(x) = 3\cos 3x + 3\sin 3x$ **[1]**

$f'(x) = 0 \Rightarrow \tan 3x = -1$ [½]

$\Rightarrow\ x = \frac{n\pi}{3} + \frac{\pi}{4}, n \in Z$

$\Rightarrow\ x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}$ **[1½]**

Intervals are : $\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{7\pi}{12}\right), \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right), \left(\frac{11\pi}{12}, \pi\right)$ **[1]**

$f(x)$ is strictly increasing in $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$ **[1]**

and strictly decreasing in $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right) \cup \left(\frac{11\pi}{12}, \pi\right)$. **[1]**

**Q. 56. Find the equation of tangents to the curve $y = cos(x + y), -2p < x < 2p$ that are parallel to the line $x + 2y = 0$.**

**[CBSE Board, Foreign Scheme, 2016]**

**Ans.** Equation of given curve,

$$y = \cos(x + y) \quad ...(i)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\sin(x + y)\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-\sin(x + y)}{1 + \sin(x + y)}$$ **[2]**

Given that,

Line $x + 2y = 0$, its slope $= -\frac{1}{2}$ [½]

Condition of || lines

$$\frac{-\sin(x + y)}{1 + \sin(x + y)} = \frac{1}{2}$$ **[1]**

$\Rightarrow \sin(x + y) = 1$

$\Rightarrow cos(x + y) = 0$

[$\because$ $y = \cos(x + y) = 0$ and using Eq. (i)]

$$\Rightarrow \quad \cos x = 0 \Rightarrow x = (2n + 1)\frac{\pi}{2}, n \in I$$

$$\therefore \quad x = \frac{-3\pi}{2}, \frac{\pi}{2} \in [-2\pi, 2\pi]$$ **[1]**

Thus tangents are || to the line $x + 2y = 0$

Only at points $\left(-\frac{3\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 0\right)$ [½]

$\therefore$ Required equation of tangents are

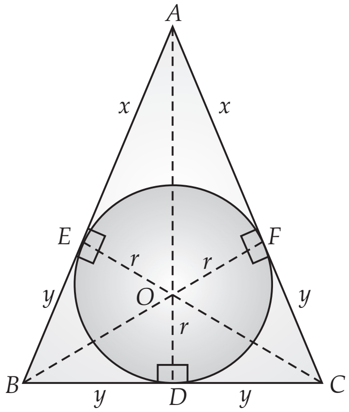
$$y - 0 = -\frac{1}{2}\left(x + \frac{3\pi}{2}\right) \Rightarrow 2x + 4y + 3\pi = 0$$ [½]

$$y - 0 = -\frac{1}{2}\left(x - \frac{1}{2}\right) \Rightarrow 2x + 4y - \pi = 0$$ [½]

**Q. 57. Prove that the least perimeter of an isosceles triangle in which a circle of radius $r$ can be inscribed is $6\sqrt{3}\, r$.**

**[CBSE Board, All India Region, 2016]**

**Ans.**




**[1]**

Let $\Delta ABC$ be isosceles with inscribed circle of radius '$r$' touching sides $AB$, $AC$ and $BC$ at $E$, and $D$ respectively.

Let $AE = AF = x$, $BE = BD = y$, $CF = CD = y$ then,

Area ($\Delta ABC$) = Ar ($\Delta AOB$) + Ar ($\Delta AOC$) + Ar ($\Delta BOC$)

$$\Rightarrow \quad \frac{1}{2} \cdot 2y\left(r + \sqrt{r^2 + x^2}\right) = \frac{1}{2}\{2yr + 2(x + y)r$$

$$\Rightarrow \quad x = \frac{2r^2 y}{y^2 - r^2}$$ **[1]**

Then,

$P$ (perimeter of $\Delta ABC$) $= 2x + 4y = \frac{4r^2 y}{y^2 - r^2} + 4y$ **[1]**

$$\frac{dP}{dy} = \frac{-4r^2(r^2 + y^2)}{(y^2 - r^2)^2} + 4 \text{ and } \frac{dP}{dy} = 0 \Rightarrow y = \sqrt{3}r$$ **[1+½]**

$$\left.\frac{d^2P}{dy^2}\right]_{y=\sqrt{3}r} = \frac{4r^2 y(2y^2 + 6r^2)}{(y^2 - r^2)^3} = \frac{6\sqrt{3}}{r} > 0$$ [½]

$\therefore$ Perimeter is least if $y = \sqrt{3}r$ and least perimeter is

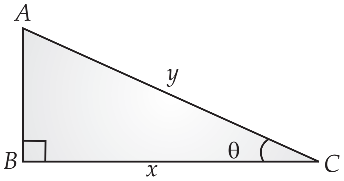
$$P = 4y + \frac{4r^2 y}{y^2 - r^2}$$

$$= 4\sqrt{3}r + \frac{4r^2\sqrt{3}r}{2r^2}$$

$$= 6\sqrt{3}r$$ **[1]**

**Q. 58. If the sum of lengths of hypotenuse and a side of a right angled triangle is given, show that area of triangle is maximum, when the angle between them is $\frac{\pi}{3}$. [CBSE Board, All India Region, 2016]**

**Ans.**



Let ABC be the right triangle with $\angle B = 90°$

$\angle ACB = \theta$, $AC = y$, $BC = x, x + y = k$ (Constant)

$A$ (Area of triangle) $= \frac{1}{2} \cdot BC \cdot AB = \frac{1}{2} \cdot x\sqrt{y^2 - x^2}$

Let,

$$z = A^2 = \frac{1}{4}x^2\left(y^2 - x^2\right)$$

$$= \frac{1}{4}x^2\left\{(k-x)^2 - x^2\right\}$$

$$= \frac{1}{4}\left(x^2k^2 - 2kx^3\right)$$

$$\frac{dz}{dx} = \frac{1}{4}\left(2xk^2 - 6kx^2\right) \text{ and } \frac{dz}{dx} = 0 \Rightarrow x = \frac{k}{3},$$

$$y = k - x = \frac{2k}{3}$$

$$\left.\frac{d^2z}{dx^2}\right]_{x=\frac{k}{3}} = \frac{1}{4}\left(2k^2 - 12kx\right)\Big]_{x=\frac{k}{3}} = \frac{k^2}{2} < 0$$

$\therefore$ z and area of $\Delta$ABC is max at $x = \frac{k}{3}$

and, $\cos\theta = \frac{x}{y} = \frac{k}{3} \cdot \frac{3}{2k} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

## Some Commonly Made Errors

- Generally, students confuse in negative and Fractional Exponents.
- Conceptual errors occur because Students have misunderstood the underlying concepts.
- Students confuse in terms of area, volume and surface area of the given shape to find the derivative.
- Students do not apply the right rule of the derivatives.

## EXPERT ADVICE

- *Derivative by First Principle is simply a measure of the rate of change.*
- *One-sided Derivative means consider a function* $f : [a, b] \to R$, where $a, b \in R$
- *Learn all the properties of differentiation.*
- *Students try to make solutions in detailed form.*

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