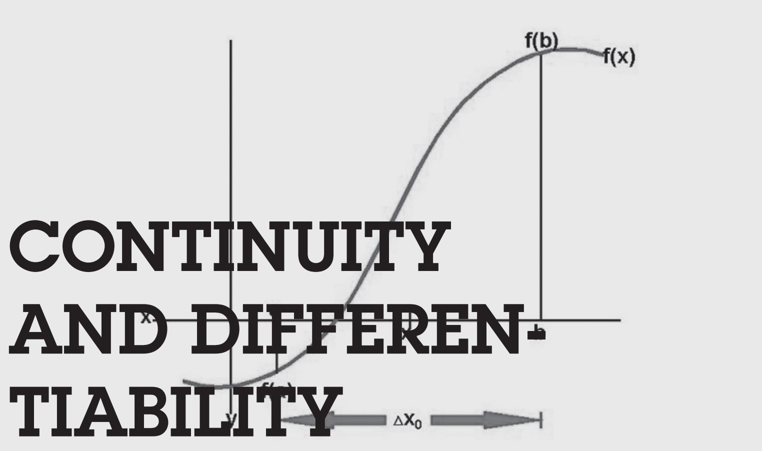
CHAPTER

5 CONTINUITY AND DIFFEREN-TIABILITY



Chapter Objectives

This chapter will help you understand :

- ***Continuity and differentiability :*** *Continuity of a function at a point; Continuity in an interval; Geometrical meaning of continuity; Discontinuity and Continuity of composite function; Differentiability and Differentiability in interval.*
- ***Derivatives :*** *Algebra of derivatives; Derivatives of composite function; Derivatives of implicit function; Derivatives of trigonometric function; Derivatives of inverse trigonometric function; Exponential function; Derivatives of exponential function; Logarithmic function; Logarithmic rules and its differentiation; Derivative of function in parametric forms and Second order of derivative.*
- ***Rolle's theorem and MVT :*** *Rolle's theorem and Mean value theorem.*

TOPIC-1
Continuity and Differentiability

Quick Review

- Continuity of a function at a point : Let f be a real function on a subset of the real numbers and let c be a point in the domain of f. Then f is continuous at c if $\lim_{x \to c} f(x) = f(c)$. In other words, if the left-hand limit, right-hand limit and the value of the function at $x = c$ exist and are equal to each other, $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$ then f is said to be continuous at $x = c$.
- Continuity in an interval :
  - The function f is said to be continuous in an open interval (a, b) if it is continuous at every point in this interval.
  - The function f is said to be continuous in closed interval $[a, b]$ if f is continuous in open interval (a, b), $\lim_{x \to a^+} f(x) = f(a)$, $\lim_{x \to b^-} f(x) = f(b)$.
- Geometrical meaning of continuity : The geometrical meaning of a continuous at c of a function f is that there is no break in the graph of the function at the point $[c, f(c)]$
- Discontinuity : The function f will be discontinuous at $x = a$ in any of the following cases :
  - $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$
  - $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a)$
  - $f(a)$ is not defined.

TIPS...

- *The constant function $f(x) = k$ is continuous for all real values of x.*
- *The identity function $f(x) = x$ is continuous for all real values of x.*
- *The modulus function $f(x) = |x|$ is continuous for all real values of x.*
- *The polynomial function $f(x) = a_n x^n + a^{n-1} + a_{n-2}x^{n-2} .... + a_2 x^2 + a_1 x^1 + a_0 x^0$ is continuous for all real values of x.*
- *The greatest integer function $f(x) = [x]$ is continuous for all real values of x except at integral values of x.*

TRICKS...

- *All trigonometrical functions are continuous in their respective domains.*
- *Exponential and Logarithmic Functions are continuous for all real values of x.*

- Continuity of composite function : Let f and g be real valued functions such that (fog) is defined at a. If g is continuous at a and f is continuous at $g(a)$, then (fog) is continuous at a.
- Differentiability : If a function f is differentiable at a point c in its domain if both LHD (left-hand derivative) and RHD (right-hand derivative) are finite and equal, it means : $LHD = \lim_{h \to 0^-} \frac{f(c-h) - f(c)}{-h} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} = RHD.$

- Differentiability in an interval : The function $y = f(x)$ is said to be differentiable in an open interval (a, b) if it is differentiable at every point of (a, b). And, the function $y = f(x)$ is said to be differentiable in the closed interval $[a, b]$ if LHD and RHD exist and $f'(x)$ exists for every point of (a, b). Every differentiable function is continuous, but the converse is not true.

Know the Links

- *https://www.intmath.com/functions-and-graphs/7-continuous-discontinuous-functions.php*
- *https://www.ipracticemath.com/learn/calculus/continuous_discontinous*
- *https://www.mathsisfun.com/calculus/continuity.html*
- *https://www.mathsisfun.com/calculus/differentiable.html*
- *https://www.zweigmedia.com/RealWorld/calctopic1/contanddiffb.html*

Multiple Choice Questions **(1 mark each)**

Q. 1. If $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$ then which of the following can be a discontinuous function?

(a) $f(x) + g(x)$ **(b)** $f(x) - g(x)$

(c) $f(x).g(x)$ **(d)** $\frac{g(x)}{f(x)}$

[NCERT Exemp. Ex. 5.3, Q. 83, Page 113]

Ans. Correct option : (d)

Explanation : Since $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$ are continuous functions, then by using the algebra of continuous functions , the functions $f(x) + g(x)$, $f(x) - g(x)$, $f(x).g(x)$ are also continuous functions but $\frac{g(x)}{f(x)}$ is discontinuous function at $x = 0$.

Q. 2. The function $f(x) = \frac{4 - x^2}{4x - x^3}$

(a) discontinuous at only one point

(b) discontinuous at exactly two points

(c) discontinuous at exactly three points

(d) none of these

[NCERT Exemp. Ex. 5.3, Q. 84, Page 113]

Ans. Correct option : (c)

Explanation : Given that,

$f(x) = \frac{4 - x^2}{4x - x^3}$, then it is discontinuous if

$\Rightarrow \quad 4x - x^3 = 0$

$\Rightarrow \quad x(4 - x^2) = 0$

$\Rightarrow x(2 + x)(2 - x) = 0$

$\Rightarrow \quad x = 0, -2, 2$

Thus, the given function is discontinuous at exactly three points.

Q. 3. The set of points where the function f given by $f(x) = |2x - 1| \sin x$ is differentiable is

(a) R

(b) $R - \left\{\frac{1}{2}\right\}$

(c) $(0, \infty)$

(d) none of these

[NCERT Exemp. Ex. 5.3, Q. 85, Page 113]

Ans. Correct option : (c)

Explanation : Given that,

$f(x) = |2x - 1| \sin x$

The function $\sin x$ is differentiable.

The function $|2x - 1|$ is differentiable, except

$2x - 1 = 0$

$\Rightarrow x = \frac{1}{2}$

Thus, the given function is differentiable $R - \left\{\frac{1}{2}\right\}$.

Q. 4. The function $f(x) = \cot x$ is discontinuous on the set

(a) $\{x = n\pi : n \in Z\}$

(b) $\{x = 2n\pi : n \in Z\}$

(c) $\left\{x = (2n + 1)\frac{\pi}{2}; n \in Z\right\}$

(d) $\left\{x = \frac{n\pi}{2}; n \in Z\right\}$

[NCERT Exemp. Ex. 5.3, Q. 86, Page 114]

Ans. Correct option : (a)

Explanation : Given that,

$f(x) = \cot x = \frac{\cos x}{\sin x}$

It is discontinuous at

$\sin x = 0$

$\Rightarrow x = n\pi,\ n \in Z$

Thus, the given function is discontinuous at $\{x = n\pi : n \in Z\}$.

Q. 5. The function $f(x) = e^{|x|}$ is

(a) continuous everywhere but not differentiable at $x = 0$

(b) continuous and differentiable everywhere

(c) not continuous at $x = 0$

(d) none of these

[NCERT Exemp. Ex. 5.3, Q. 87, Page 114]

Ans. Correct option : (a)

Explanation : Given that,

$f(x) = e^{|x|}$

The functions e^x and $|x|$ are continuous functions for all real value of x.

Since e^x is differentiable everywhere but $|x|$ is non-differentiable at $x = 0$.

Thus, the given functions $f(x)=e^{|x|}$ is continuous everywhere but not differentiable at $x = 0$.

Q. 6. If $f(x)=x^2\sin\frac{1}{x}$, where $x \neq 0$, then the value of the function f at $x = 0$, so that the function is continuous at $x = 0$, is

(a) 0 **(b)** –1

(c) 1 **(d) none of these**

[NCERT Exemp. Ex. 5.3, Q. 88, Page 114]

Ans. Correct option : (a)

Explanation : Given that,

$$f(x)=x^2\sin\frac{1}{x}$$

Thus,

$$f(0)=\lim_{x\to 0}\left(x^2\sin\frac{1}{x}\right)$$

$$\Rightarrow f(0)=0\times\begin{pmatrix}\text{an oscillating value}\\ \text{between }-1\text{ and }1\end{pmatrix}$$

$$\Rightarrow f(0)=0$$

Q. 7. If $f(x)=\begin{cases} mx+1 \text{ if } x\le\frac{\pi}{2}\\ \sin x+n, \text{ if } x>\frac{\pi}{2}\end{cases}$, is continuous at $x=\frac{\pi}{2}$ then

(a) $m = 1, n = 0$ **(b)** $m=\frac{n\pi}{2}+1$

(c) $n=\frac{m\pi}{2}$ **(d)** $m=n=\frac{\pi}{2}$

[NCERT Exemp. Ex. 5.3, Q. 89, Page 114]

Ans. Correct option : (c)

Explanation : Given that,

$f(x)=\begin{cases} mx+1 \text{ if } x\le\frac{\pi}{2}\\ \sin x+n, \text{ if } x>\frac{\pi}{2}\end{cases}$ is continuous function at $x=\frac{\pi}{2}$, then

$LHL = RHL$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}^-} f(x)=\lim_{x\to\frac{\pi}{2}^+} f(x)$$

$$\Rightarrow \lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)=\lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)$$

$$\Rightarrow \lim_{h\to 0} m\left(\frac{\pi}{2}-h\right)+1=\lim_{h\to 0}\sin\left(\frac{\pi}{2}+h\right)+n$$

$$\Rightarrow \lim_{h\to 0} m\left(\frac{\pi}{2}-h\right)+1=\lim_{h\to 0}\cos h+n$$

$$\Rightarrow m\left(\frac{\pi}{2}\right)+1=1+n$$

$$\Rightarrow n=\frac{m\pi}{2}$$

Q. 8. Let $f(x)=|\sin x|$, then

(a) f is everywhere differentiable

(b) f is everywhere continuous but not differentiable at $x = n\pi$, $n \in Z$.

(c) f is everywhere continuous but not differentiable at $x = (2n + 1)\frac{\pi}{2}$, $n \in Z$.

(d) none of these

[NCERT Exemp. Ex. 5.3, Q. 90, Page 114]

Ans. Correct option : (b)

Explanation : Given that,

$f(x)=|\sin x|$

The functions $|x|$ and $\sin x$ are continuous function for all real value of x.

Thus, the function $f(x)=|\sin x|$ is continuous function everywhere.

Now, $|x|$ is non-differentiable function at $x = 0$.

Since $f(x)=|\sin x|$ is non-differentiable function at $\sin x = 0$

Thus, f is everywhere continuous but not differentiable at $x = n\pi$, $n \in Z$.

Q. 9. Fill in the blanks :

An example of a function which is continuous everywhere but fails to be differentiable exactly at two points is __________

[NCERT Exemp. Ex. 5.3, Q. 97, Page 116]

Ans. $f(x)=|x^2-4|$

Explanation : The function $f(x)=|x^2-4|$ is continuous everywhere but it is non-differentiable at $x^2-4=0$

$\Rightarrow x=\pm 2$

Q. 10. State True or False for the statement :

If f is continuous on its domain D, then $|f|$ is also continuous on D.

[NCERT Exemp. Ex. 5.3, Q. 103, Page 116]

Ans. True

Explanation : Let a function $f(x) = x$ which is continuous in its domain R, then the function $|f(x)| = |x|$ is also a continuous function in its domain.

Q. 11. State True or False for the statement :

The composition of two continuous functions is a continuous function.

[NCERT Exemp. Ex. 5.3, Q. 104, Page 116]

Ans. True

Explanation : The composition of two continuous functions is a continuous function.

Q. 12. State True or False for the statement :

Trigonometric and inverse-trigonometric functions are differentiable in their respective domain.

[NCERT Exemp. Ex. 5.3, Q. 105, Page 116]

Ans. True

Explanation : Trigonometric and inverse-trigonometric functions are differentiable in their respective domain.

Q. 13. State True or False for the statement :

If $f.g$ is continuous at $x = a$, then f and g are separately continuous at $x = a$.

[NCERT Exemp. Ex. 5.3, Q. 106, Page 116]

Ans. False

Explanation : Let $f(x)=\sin x$ and $g(x)=\cot x$.

Thus,

$$f(x)\times g(x)=\sin x\times\cot x=\sin x\times\frac{\cos x}{\sin x}=\cos x.$$

It is continuous function at $x = 0$ but $g(x)=\cot x$ is not continuous function at $x = 0$.

Very Short Answer Type Questions (1 or 2 marks each)

Q. 1. Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$. [NCERT Ex. 5.1, Q. 2, Page 159]

Ans. Given function is $f(x) = 2x^2 - 1$.

$$\text{LHL (at } x=3) = \lim_{x\to 3^-} f(x) = \lim_{h\to 0} f(3-h) = \lim_{h\to 0}\left[2(3-h)^2 - 1\right] = 2(3)^2 - 1 = 17$$ **[1]**

$$\text{RHL (at } x=3) = \lim_{x\to 3^+} f(x) = \lim_{h\to 0} f(3+h) = \lim_{h\to 0}\left[2(3+h)^2 - 1\right] = 2(3)^2 - 1 = 17$$

And, $f(3) = 2(3)^2 - 1 = 17$

Since LHL = RHL = $f(3) = 17$, then the given function is continuous at $x = 3$. **[1]**

Q. 2. Examine the following functions for continuity of $f(x) = x - 5$. [NCERT Ex. 5.1, Q. 3(a), Page 159]

Ans. Let k be any real number.

Given function is $f(x) = x - 5$.

$$\text{LHL (at } x=k) = \lim_{x\to k^-} f(x) = \lim_{h\to 0} f(k-h) = \lim_{h\to 0}[k-h-5] = k-5$$ **[1]**

$$\text{RHL (at } x=k) = \lim_{x\to k^+} f(x) = \lim_{h\to 0} f(k+h) = \lim_{h\to 0}[k+h-5] = k-5$$

And, $f(k) = k - 5$

Since LHL = RHL = $f(k) = k - 5$, then the given function is continuous at $x = k$.

Thus, the given function is continuous for all real numbers. **[1]**

Q. 3. Examine the following functions for continuity of $f(x) = \dfrac{1}{x-5}$. [NCERT Ex. 5.1, Q. 3(b), Page 159]

Ans. Let $k \neq 5$ be any real number.

Given function is $f(x) = \dfrac{1}{x-5}$.

$$\text{LHL (at } x=k) = \lim_{x\to k^-} f(x) = \lim_{h\to 0} f(k-h) = \lim_{h\to 0}\left(\frac{1}{k-h-5}\right) = \frac{1}{k-5}$$ **[1]**

$$\text{RHL (at } x=k) = \lim_{x\to k^+} f(x) = \lim_{h\to 0} f(k+h) = \lim_{h\to 0}\left(\frac{1}{k+h-5}\right) = \frac{1}{k-5}$$

And, $f(k) = \dfrac{1}{k-5}$

Since LHL = RHL = $f(k) = \dfrac{1}{k-5}$, then the given function is continuous at $x = k \neq 5$

Thus, the given function is continuous for all real numbers except 5. **[1]**

Q. 4. Examine the following functions for continuity of $f(x) = |x - 5|$. [NCERT Ex. 5.1, Q. 3(d), Page 159]

Ans. Given function is $f(x) = |x-5| = \begin{cases} (x-5), & x > 5 \\ 0, & x = 5 \\ -(x-5), & x < 5 \end{cases}$.

$$\text{LHL (at } x=5) = \lim_{x\to 5^-} f(x) = \lim_{h\to 0} f(5-h) = \lim_{h\to 0}\left[-(5-h-5)\right] = 0$$

$$\text{RHL (at } x=5) = \lim_{x\to 5^+} f(x) = \lim_{h\to 0} f(5+h) = \lim_{h\to 0}\left[(5+h-5)\right] = 0$$ **[1]**

And, $f(5) = 0$

Since LHL = RHL = $f(5) = 0$, then the given function is continuous at $x = 5$.

Thus, the given function is continuous for all real numbers. **[1]**

Q. 5. Prove that the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer. [NCERT Ex. 5.1, Q. 4, Page 159]

Ans. Given function is $f(x) = x^n$.

$$\text{LHL (at } x=n) = \lim_{x\to n^-} f(x) = \lim_{h\to 0} f(n-h) = \lim_{h\to 0}(n-h)^n = n^n$$ **[1]**

$$\text{RHL (at } x=n) = \lim_{x\to n^+} f(x) = \lim_{h\to 0} f(n+h) = \lim_{h\to 0}(n+h)^n = n^n$$

And, $f(n) = n^n$

Since LHL = RHL = $f(n) = n^n$, then the given function is continuous at $x = n$. **[1]**

Q. 6. Find all points of discontinuity of *f*, where *f* is defined by $f(x)=\begin{cases}2x+3, & \text{if } x\le 2\\ 2x-3, & \text{if } x>2\end{cases}$.

[NCERT Ex. 5.1, Q. 6, Page 159]

Ans. Given function is $f(x)=\begin{cases}2x+3, & \text{if } x\le 2\\ 2x-3, & \text{if } x>2\end{cases}$.

From the function, $x = 2$

$$\text{LHL (at } x=2) = \lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0}\left[2(2-h)+3\right] = 7$$

[1]

$$\text{RHL (at } x=2) = \lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0}\left[2(2+h)-3\right] = 1$$

And, $f(2)=2(2)+3=7$

Since $\text{LHL}\ne\text{RHL}\ne f(2)=7$, then the given function is discontinuous at $x = 2$. **[1]**

Q. 7. Find all points of discontinuity of *f*, where *f* is defined by $f(x)=\begin{cases}\frac{|x|}{x}, & \text{if } x\ne 0\\ 0, & \text{if } x=0\end{cases}$.

[NCERT Ex. 5.1, Q. 8, Page 159]

Ans. Given function is

$$f(x)=\begin{cases}\frac{|x|}{x}, & \text{if } x\ne 0\\ 0, & \text{if } x=0\end{cases}$$

$$f(x)=\begin{cases}\frac{-x}{x}=-1, & \text{if } x<0\\ 0, & \text{if } x=0\\ \frac{x}{x}=1, & \text{if } x>0\end{cases}$$

From the function, $x = 0$

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0}[-1] = -1$$

[1]

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0}[1] = 1$$

And, $f(0)=0$

Since $\text{LHL}\ne\text{RHL}\ne f(0)=0$, then the given function is discontinuous at $x = 0$. **[1]**

Q. 8. Find all points of discontinuity of *f*, where *f* is defined by

$$f(x)=\begin{cases}\frac{x}{|x|}, & \text{if } x<0\\ -1, & \text{if } x\ge 0\end{cases}.$$

[NCERT Ex. 5.1, Q. 9, Page 159]

Ans. Given function is

$$f(x)=\begin{cases}\frac{x}{|x|}, & \text{if } x<0\\ -1, & \text{if } x\ge 0\end{cases}$$

$$f(x)=\begin{cases}\frac{x}{-x}=-1, & \text{if } x<0\\ -1, & \text{if } x\ge 0\end{cases}$$

From the function, $x = 0$

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0}[-1] = -1$$

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0}[-1] = -1$$

[1]

And, $f(0)=-1$

Since $\text{LHL}=\text{RHL}=f(0)=-1$, then the given function is continuous at $x = 0$.

Thus, the given function is continuous at all real numbers. There is no point of discontinuity. **[1]**

Q. 9. Find all points of discontinuity of *f*, where *f* is defined by $f(x)=\begin{cases}x+1, & \text{if } x\ge 1\\ x^2+1, & \text{if } x>1\end{cases}$.

[NCERT Ex. 5.1, Q. 10, Page 159]

Ans. Given function is $f(x)=\begin{cases}x+1, & \text{if } x\ge 1\\ x^2+1, & \text{if } x<1\end{cases}$.

From the function, $x = 1$

$$\text{LHL (at } x=1) = \lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}\left[(1-h)^2+1\right] = 2$$

$$\text{RHL (at } x=1) = \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}\left[(1+h)+1\right] = 2$$

[1]

And, $f(1)=1+1=2$

Since $\text{LHL}=\text{RHL}=f(1)=2$, then the given function is continuous at $x = 1$.

Thus, the given function is continuous at all real numbers. There is no point of discontinuity. **[1]**

Q. 10. Find all points of discontinuity of f, where f is defined by $f(x)=\begin{cases} x^3-3, & \text{if } x\le 2 \\ x^2+1, & \text{if } x>2 \end{cases}$.

[NCERT Ex. 5.1, Q. 11, Page 159]

Ans. Given function is $f(x)=\begin{cases} x^3-3, & \text{if } x\le 2 \\ x^2+1, & \text{if } x>2 \end{cases}$.

From the function, $x = 2$

$$\text{LHL (at } x=2) = \lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0}\left[(2-h)^3-3\right] = 5$$

$$\text{RHL (at } x=2) = \lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0}\left[(2+h)^2+1\right] = 5$$ **[1]**

And, $f(2)=(2)^3-3=5$

Since LHL = RHL = $f(2)=5$, then the given function is continuous at $x = 2$.

Thus, the given function is continuous at all real numbers. There is no point of discontinuity. **[1]**

Q. 11. Find all points of discontinuity of f, where f is defined by $f(x)=\begin{cases} x^{10}-1, & \text{if } x\le 1 \\ x^2, & \text{if } x>1 \end{cases}$.

[NCERT Ex. 5.1, Q. 12, Page 159]

Ans. Given function is $f(x)=\begin{cases} x^{10}-1, & \text{if } x\le 1 \\ x^2, & \text{if } x>1 \end{cases}$.

From the function, $x = 1$

$$\text{LHL (at } x=1) = \lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}\left[(1-h)^{10}-1\right] = 0$$ **[1]**

$$\text{RHL (at } x=1) = \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}\left[(1+h)^2\right] = 1$$

And, $f(1)=(1)^{10}-1=0$

Since LHL ≠ RHL ≠ $f(1)=0$, then the given function is discontinuous at $x = 1$. **[1]**

Q. 12. Is the function defined by $f(x)=\begin{cases} x+5, & \text{if } x\le 1 \\ x-5, & \text{if } x>1 \end{cases}$ a continuous function?

[NCERT Ex. 5.1, Q. 13, Page 159]

Ans. Given function is $f(x)=\begin{cases} x+5, & \text{if } x\le 1 \\ x-5, & \text{if } x>1 \end{cases}$.

From the function, $x = 1$

$$\text{LHL (at } x=1) = \lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}\left[(1-h)+5\right] = 6$$

$$\text{RHL (at } x=1) = \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}\left[(1+h)-5\right] = -4$$ **[1]**

And, $f(1)=1+5=6$

Since LHL ≠ RHL ≠ $f(1)=6$, then the given function is discontinuous at $x = 1$.

Thus, the given function is not continuous at $x = 1$. **[1]**

Q. 13. Discuss the continuity of the following functions $f(x) = \sin x + \cos x$.

[NCERT Ex. 5.1, Q. 21(a), Page 160]

Ans. We know that $p(x) = \sin x$ and $q(x) = \cos x$ both are continuous function for all real value of x. **[1]**

By the algebra of continuous functions, we get that $f(x) = p(x) + q(x) = \sin x + \cos x$ is also a continuous function for all real value of x. **[1]**

Q. 14. Discuss the continuity of the following functions $f(x) = \sin x - \cos x$.

[NCERT Ex. 5.1, Q. 21(b), Page 160]

Ans. We know that $p(x) = \sin x$ and $q(x) = \cos x$ both are continuous function for all real value of x.

By the algebra of continuous functions, we get that $f(x) = p(x) - q(x) = \sin x - \cos x$ is also a continuous function for all real value of x. **[1]**

Q. 15. Discuss the continuity of the following functions $f(x) = \sin x \cos x$. [NCERT Ex. 5.1, Q. 21(c), Page 160]

Ans. We know that $p(x) = \sin x$ and $q(x) = \cos x$ both are continuous function for all real value of x.

By the algebra of continuous functions, we get that $f(x) = p(x) \times q(x) = \sin x \cos x$ is also a continuous function for all real value of x. **[1]**

Q. 16. Find the values of k so that the function f is continuous at the indicated point in

$$f(x)=\begin{cases} kx^2, & \text{if } x\le 2 \\ 3, & \text{if } x>2 \end{cases} \text{ at } x=2.$$

[NCERT Ex. 5.1, Q. 27, Page 161]

Ans. Given function is $f(x)=\begin{cases} kx^2, & \text{if } x\le 2 \\ 3, & \text{if } x>2 \end{cases}$.

From the function, $x = 2$

$$\text{LHL (at } x=2) = \lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0}\left[k(2-h)^2\right] = 4k$$ **[1]**

$$\text{RHL (at } x=2) = \lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0}[3] = 3$$

And, $f(2)=4k$

Since the given function is continuous function at $x=2$, then LHL = RHL = $f(2)=4k$.

Thus,

$4k = 3$

$\Rightarrow k = \frac{3}{4}$ **[1]**

Q. 17. Find the values of k so that the function f is continuous at the indicated point in

$$f(x) = \begin{cases} kx + 1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases} \text{ at } x = \pi.$$

[NCERT Ex. 5.1, Q. 28, Page 161]

Ans. Given function is $f(x) = \begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$.

From the function, $x = \pi$

$$\text{LHL (at } x = \pi) = \lim_{x \to \pi^-} f(x)$$
$$= \lim_{h \to 0} f(\pi - h)$$
$$= \lim_{h \to 0} \left[k(\pi - h) + 1\right]$$
$$= k(\pi) + 1$$

[1]

$$\text{RHL (at } x = \pi) = \lim_{x \to \pi^+} f(x)$$
$$= \lim_{h \to 0} f(\pi + h)$$
$$= \lim_{h \to 0} \left[\cos(\pi + h)\right]$$
$$= \lim_{h \to 0} \left[-\cos(h)\right]$$
$$= -1$$

[1]

And, $f(\pi) = k\pi + 1$

Since the given function is continuous function at $x = \pi$, then LHL = RHL = $f(\pi) = k\pi + 1$. Thus,

$k\pi + 1 = -1$

$\Rightarrow \quad k = -\frac{2}{\pi}$

[1]

Q. 18. Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

[NCERT Ex. 5.1, Q. 31, Page 161]

Ans. Let $g(x) = \cos x$ and $h(x) = x^2$, then $goh(x) = g(h(x)) = f(x) = \cos(x^2)$.

We know that if $g(x)$ and $h(x)$ are continuous function then $goh(x) = g(h(x))$ is a continuous function. **[1]**

Since $g(x) = \cos x$ and $h(x) = x^2$ is continuous function for all real value of x, then $goh(x) = g(h(x)) = f(x) = \cos(x^2)$ is also a continuous function. **[1]**

Q. 19. Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

[NCERT Ex. 5.1, Q. 32, Page 161]

Ans. Let $g(x) = |x|$ and $h(x) = \cos x$, then $goh(x) = g(h(x)) = f(x) = |\cos x|$.

We know that if $g(x)$ and $h(x)$ are continuous function then $goh(x) = g(h(x))$ is a continuous function. **[1]**

Since $g(x) = |x|$ and $h(x) = \cos x$ is continuous function for all real value of x, then $goh(x) = g(h(x)) = f(x) = |\cos x|$ is also a continuous function. **[1]**

Q. 20. Examine that $\sin |x|$ is a continuous function.

[NCERT Ex. 5.1, Q. 33, Page 161]

Ans. Let $g(x) = \sin x$ and $h(x) = |x|$, then $goh(x) = g(h(x)) = f(x) = \sin|x|$.

We know that if $g(x)$ and $h(x)$ are continuous function then $goh(x) = g(h(x))$ is a continuous function. **[1]**

Since $g(x) = \sin x$ and $h(x) = |x|$ is continuous function for all real value of x, then $goh(x) = g(h(x)) = f(x) = \sin|x|$ is also a continuous function. **[1]**

Q. 21. Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.

[NCERT Misc Ex. Q. 21, Page 192]

Ans. The function $f(x) = |x - 4| + |x - 5|$ is continuous for all real points of x but it is not differentiable at exactly two points $x = 4$ and $x = 5$. **[2]**

Q. 22. Find all points of discontinuity of the function $f(t) = \frac{1}{t^2 + t - 2}$, where $t = \frac{1}{x - 1}$.

[NCERT Exemp. Ex. 5.3, Q. 18, Page 109]

Ans. Given that,

$$f(t) = \frac{1}{t^2 + t - 2}$$

Put $t = \frac{1}{x-1}$, we have

$$f\left(\frac{1}{x-1}\right) = \frac{1}{\left(\frac{1}{x-1}\right)^2 + \left(\frac{1}{x-1}\right) - 2}$$
$$= \frac{(x-1)^2}{(2x-1)(2-x)}$$

[1]

Since it is not defined at $(2x-1)(2-x) = 0 \Rightarrow x = 2, \frac{1}{2}$, then

The given function is discontinuous function at $x = 2$ and $\frac{1}{2}$. **[1]**

Q. 23. Given the function $f(x) = \frac{1}{x+2}$. Find the points of discontinuity of the composite function $y = f(f(x))$.

[NCERT Exemp. Ex. 5.3, Q. 17, Page 108]

Ans. Since the given function is not defined at $x = -2$, the function is discontinuous function at $x = -2$.

$$\text{Now, } f(f(x)) = \frac{1}{f(x) + 2}$$
$$= \frac{1}{\left(\frac{1}{x+2}\right) + 2}$$
$$= \frac{x+2}{2x+5}$$

[1]

Thus, the function $f(f(x)) = \frac{x+2}{2x+5}$ is discontinuous at $x = -2$ and $-\frac{5}{2}$. **[1]**

Short Answer Type Questions (3 or 4 marks each)

Q. 1. Examine the following functions for continuity of $f(x)=\dfrac{x^2-25}{x+5}$.
[NCERT Ex. 5.1, Q. 3(c), Page 159]

Ans. Let $k \neq -5$ be any real number.

Given function is $f(x)=\dfrac{x^2-25}{x+5}$.

$$\text{LHL (at } x=k) = \lim_{x\to k^-} f(x)$$
$$= \lim_{h\to 0} f(k-h)$$
$$= \lim_{h\to 0}\left[\frac{(k-h)^2-25}{k-h+5}\right]$$
$$= \frac{k^2-25}{k-5}$$
$$= \frac{(k-5)(k+5)}{k-5}$$
$$= k+5$$
[1]

$$\text{RHL (at } x=k) = \lim_{x\to k^+} f(x)$$
$$= \lim_{h\to 0} f(k+h)$$
$$= \lim_{h\to 0}\left[\frac{(k+h)^2-25}{k+h+5}\right]$$
$$= \frac{k^2-25}{k-5}$$
$$= \frac{(k-5)(k+5)}{k-5}$$
$$= k+5$$
[1]

And, $f(k)=\dfrac{k^2-25}{k-5}$
$$= \frac{(k-5)(k+5)}{k-5}$$
$$= k+5$$

Since $\text{LHL}=\text{RHL}=f(k)=k+5$, then the given function is continuous at $x = k \neq -5$

Thus, the given function is continuous for all real numbers except –5. **[1]**

Q. 2. Is the function f defined by $f(x)=\begin{cases} x, & \text{if } x\le 1 \\ 5, & \text{if } x>1 \end{cases}$ continuous at $x = 0$? At $x = 1$? At $x = 2$?
[NCERT Ex. 5.1, Q. 5, Page 159]

Ans. Given function is $f(x)=\begin{cases} x, & \text{if } x\le 1 \\ 5, & \text{if } x>1 \end{cases}$.

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0} (-h)$$
$$= 0$$

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x)$$
$$= \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0} (h)$$
$$= 0$$

And, $(0) \quad 0$

Since $\text{LHL}=\text{RHL}=f(0)=0$, then the given function is continuous at $x = 0$. **[1]**

$$\text{LHL (at } x=1) = \lim_{x\to 1^-} f(x)$$
$$= \lim_{h\to 0} f(1-h)$$
$$= \lim_{h\to 0} (1-h)$$
$$= 1$$

$$\text{RHL (at } x=1) = \lim_{x\to 1^+} f(x)$$
$$= \lim_{h\to 0} f(1+h)$$
$$= \lim_{h\to 0} 5$$
$$= 5$$

And, $f(1)=1$

Since $\text{LHL}\neq\text{RHL}\neq f(1)=1$, then the given function is not continuous at $x = 1$. **[1]**

$$\text{LHL (at } x=2) = \lim_{x\to 2^-} f(x)$$
$$= \lim_{h\to 0} f(2-h)$$
$$= \lim_{h\to 0} 5$$
$$= 5$$

$$\text{RHL (at } x=2) = \lim_{x\to 2^+} f(x)$$
$$= \lim_{h\to 0} f(2+h)$$
$$= \lim_{h\to 0} 5$$
$$= 5$$

And, $f(2)=5$

Since $\text{LHL}=\text{RHL}=f(2)=5$, then the given function is continuous at $x = 2$. **[1]**

Q. 3. Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases} |x|+3, & \text{if } x\le -3 \\ -2x, & \text{if } -3<x<3 \\ 6x+2, & \text{if } x\ge 3 \end{cases}$$

[NCERT Ex. 5.1, Q. 7, Page 159]

Ans. Given function is $f(x)=\begin{cases} |x|+3, & \text{if } x\le -3 \\ -2x, & \text{if } -3<x<3 \\ 6x+2, & \text{if } x\ge 3 \end{cases}$.

From the function, $x = -3$ and 3

$$\text{LHL (at } x=-3) = \lim_{x\to -3^-} f(x)$$
$$= \lim_{h\to 0} f(-3-h)$$
$$= \lim_{h\to 0}\left[|-3-h|+3\right]$$
$$= |-3|+3$$
$$= 6$$

$$\text{RHL (at } x=-3) = \lim_{x\to -3^+} f(x)$$
$$= \lim_{h\to 0} f(-3+h)$$
$$= \lim_{h\to 0}\left[-2(-3+h)\right]$$
$$= -2(-3)$$
$$= 6$$ **[1]**

And, $f(-3) = |-3| + 3 = 6$

Since LHL = RHL = $f(-3) = 6$, then the given function is continuous at $x = -3$.

$$\text{LHL (at } x=3) = \lim_{x\to 3^-} f(x)$$
$$= \lim_{h\to 0} f(3-h)$$
$$= \lim_{h\to 0}\left[-2(3-h)\right]$$
$$= -6$$ **[1]**

$$\text{RHL (at } x=3) = \lim_{x\to 3^+} f(x)$$
$$= \lim_{h\to 0} f(3+h)$$
$$= \lim_{h\to 0}\left[6(3+h)+2\right]$$
$$= 6(3)+2$$
$$= 20$$

And, $f(3) = 6(2) + 2 = 14$

Since LHL ≠ RHL ≠ $f(3) = 14$, then the given function is discontinuous at $x = 3$. **[1]**

Q. 4. Discuss the continuity of the function *f*, where *f* is defined by

$$f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}.$$

[NCERT Ex. 5.1, Q. 14, Page 160]

Ans. Given function is $f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$.

From the function, $x = 1$ and 3

$$\text{LHL (at } x=1) = \lim_{x\to 1^-} f(x)$$
$$= \lim_{h\to 0} f(1-h)$$
$$= \lim_{h\to 0}[3]$$
$$= 3$$

$$\text{RHL (at } x=1) = \lim_{x\to 1^+} f(x)$$
$$= \lim_{h\to 0} f(1+h)$$
$$= \lim_{h\to 0}[4]$$
$$= 4$$ **[1]**

And, $f(1) = 3$

Since LHL ≠ RHL ≠ $f(1) = 3$, then the given function is discontinuous at $x = 1$.

Thus, the given function is not continuous at $x = 1$.

$$\text{LHL (at } x=3) = \lim_{x\to 3^-} f(x)$$
$$= \lim_{h\to 0} f(3-h)$$
$$= \lim_{h\to 0}[4]$$
$$= 4$$ **[1]**

$$\text{RHL (at } x=3) = \lim_{x\to 3^+} f(x)$$
$$= \lim_{h\to 0} f(3+h)$$
$$= \lim_{h\to 0}[5]$$
$$= 5$$

And, $f(3) = 5$

Since LHL ≠ RHL ≠ $f(1) = 3$, then the given function is discontinuous at $x = 3$.

Thus, the given function is not continuous at $x = 3$. **[1]**

Q. 5. Discuss the continuity of the function *f*, where *f* is defined by

$$f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$$

[NCERT Ex. 5.1, Q. 15, Page 160]

Ans. Given function is $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1. \\ 4x, & \text{if } x > 1 \end{cases}$

From the function, $x = 0$ and 1

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0}\left[2(-h)\right]$$
$$= 0$$ **[1]**

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x)$$
$$= \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0}[0]$$
$$= 0$$

And, $f(0) = 0$

Since LHL = RHL = $f(0) = 0$, then the given function is continuous at $x = 0$. **[1]**

$$\text{LHL (at } x=1) = \lim_{x\to 1^-} f(x)$$
$$= \lim_{h\to 0} f(1-h)$$
$$= \lim_{h\to 0}[0]$$
$$= 0$$

$$\text{RHL (at } x=1) = \lim_{x\to 1^+} f(x)$$
$$= \lim_{h\to 0} f(1+h)$$
$$= \lim_{h\to 0}\left[4(1+h)\right]$$
$$= 4$$

And, $f(1) = 4$

Since LHL ≠ RHL ≠ $f(1) = 4$, then the given function is discontinuous at $x = 1$. **[1]**

Q. 6. Discuss the continuity of the function *f*, where *f* is defined by

$$f(x)=\begin{cases}-2, & \text{if } x\le -1\\ 2x, & \text{if } -1<x\le 1\\ 2, & \text{if } x>1\end{cases}$$

[NCERT Ex. 5.1, Q. 16, Page 160]

Ans. Given function is $f(x)=\begin{cases}-2, & \text{if } x\le -1\\ 2x, & \text{if } -1<x\le 1.\\ 2, & \text{if } x>1\end{cases}$

From the function, $x = -1$ and 1

$$\text{LHL (at } x=-1) = \lim_{x\to -1^-} f(x) = \lim_{h\to 0} f(-1-h) = \lim_{h\to 0}[-2] = -2$$

$$\text{RHL (at } x=-1) = \lim_{x\to -1^+} f(x) = \lim_{h\to 0} f(-1+h) = \lim_{h\to 0}[2(-1+h)] = -2$$ **[1]**

And, $f(-1)=-2$

Since LHL = RHL = $f(-1)=-2$, then the given function is continuous at $x=-1$. **[1]**

$$\text{LHL (at } x=1) = \lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}[2(1-h)] = 2$$

$$\text{RHL (at } x=1) = \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}[2] = 2$$

And, $f(1)=2(1)=2$

Since LHL = RHL = $f(1)=2$, then the given function is continuous at $x = 1$. **[1]**

Q. 7. Find the relationship between *a* and *b* so that the function *f* defined by

$$f(x)=\begin{cases}ax+1, & \text{if } x\le 3\\ bx+3, & \text{if } x>3\end{cases}$$ **is continuous at $x = 3$.**

[NCERT Ex. 5.1, Q. 17, Page 160]

Ans. Given function is $f(x)=\begin{cases}ax+1, & \text{if } x\le 3\\ bx+3, & \text{if } x>3\end{cases}$.

$$\text{LHL (at } x=3) = \lim_{x\to 3^-} f(x) = \lim_{h\to 0} f(3-h) = \lim_{h\to 0}[a(3-h)+1] = 3a+1$$ **[1]**

$$\text{RHL (at } x=3) = \lim_{x\to 3^+} f(x) = \lim_{h\to 0} f(3+h) = \lim_{h\to 0}[b(3+h)+3] = 3b+3$$

And, $f(3)=3a+1$

Since the given function is continuous at x = 3, then LHL = RHL = $f(3)=3a+1$. **[1]**

LHL=RHL

$\Rightarrow 3a+1=3b+3$

$\Rightarrow 3a=3b+2$

$\Rightarrow a=b+\frac{2}{3}$ **[1]**

Q. 8. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x.

[NCERT Ex. 5.1, Q. 19, Page 160]

Ans. Let k be any integer.

Given function is $g(x) = x - [x]$.

$$\text{LHL (at } x=k) = \lim_{x\to k^-} f(x) = \lim_{h\to 0} f(k-h) = \lim_{h\to 0}\{(k-h)-[k-h]\} = \lim_{h\to 0}\{(k-h)-(k-1)\} = k-(k-1) = 1$$ **[1]**

$$\text{RHL (at } x=k) = \lim_{x\to k^+} f(x) = \lim_{h\to 0} f(k+h) = \lim_{h\to 0}\{(k+h)-[k+h]\} = \lim_{h\to 0}\{(k+h)-k\} = k-k = 0$$ **[1]**

And, $f(k)=k-[k]=k-k=0$

Since LHL $\neq$ RHL = $f(k)=0$, then the given function is discontinuous for all integers. **[1]**

Q. 9. Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

[NCERT Ex. 5.1, Q. 20, Page 160]

Ans. Given function is $f(x) = x^2 - \sin x + 5$.

$$\text{LHL (at } x=\pi) = \lim_{x\to \pi^-} f(x) = \lim_{h\to 0} f(\pi-h) = \lim_{h\to 0}\{(\pi-h)^2-\sin(\pi-h)+5\} = \lim_{h\to 0}\{(\pi-h)^2-\sin(h)+5\} = \pi^2-0+5 = \pi^2+5$$ **[1]**

$$\text{LHL (at } x=\pi) = \lim_{x\to \pi^+} f(x) = \lim_{h\to 0} f(\pi+h) = \lim_{h\to 0}\{(\pi+h)^2-\sin(\pi+h)+5\} = \lim_{h\to 0}\{(\pi+h)^2+\sin(h)+5\} = \pi^2+0+5 = \pi^2+5$$ **[1]**

And, $f(\pi)=(\pi)^2+\sin(\pi)+5=\pi^2+5$

Since $LHL = RHL = f(\pi) = \pi^2 + 5$, then the given function is continuous at $x = \pi$. **[1]**

Q. 10. Find all points of discontinuity of *f*, where

$$f(x)=\begin{cases}\dfrac{\sin x}{x}, & \text{if } x<0\\ x+1, & \text{if } x\ge 0\end{cases}.$$

[NCERT Ex. 5.1, Q. 23, Page 160]

Ans. Given function is $f(x)=\begin{cases}\dfrac{\sin x}{x}, & \text{if } x<0\\ x+1, & \text{if } x\ge 0\end{cases}$.

From the function, $x = 0$

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0}\left[\frac{\sin(-h)}{-h}\right] = \lim_{h\to 0}\left[\frac{-\sinh}{-h}\right] = \lim_{h\to 0}\left[\frac{\sinh}{h}\right] = 1$$

[1]

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0}[h+1] = 1$$

[1]

And, $f(0)=1$

Since $LHL = RHL = f(0) = 1$, then the given function is continuous at $x = 0$.

Thus, there is no point of discontinuity. **[1]**

Q. 11. Determine if *f* defined by

$$f(x)=\begin{cases}x^2\sin\dfrac{1}{x}, & \text{if } x\ne 0\\ 0, & \text{if } x=0\end{cases}$$ **is a continuous function?**

[NCERT Ex. 5.1, Q. 24, Page 160]

Ans. Given function is $f(x)=\begin{cases}x^2\sin\dfrac{1}{x}, & \text{if } x\ne 0\\ 0, & \text{if } x=0\end{cases}$.

From the function, $x = 0$

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0}\left[(-h)^2\sin\frac{1}{(-h)}\right] = 0\times\begin{bmatrix}\text{Any value}\\ \text{between } -1 \text{ and } 1\end{bmatrix} = 0$$

[1]

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0}\left[(h)^2\sin\frac{1}{(h)}\right] = 0\times\begin{bmatrix}\text{Any value}\\ \text{between } -1 \text{ and } 1\end{bmatrix} = 0$$

[1]

And, $f(0)=0$

Since $LHL = RHL = f(0) = 1$, then the given function is continuous at $x = 0$.

Thus, the given function is continuous for all real value of x. **[1]**

Q. 12. Examine the continuity of *f*, where *f* is defined by

$$f(x)=\begin{cases}\sin x-\cos x, & \text{if } x\ne 0\\ -1, & \text{if } x=0\end{cases}.$$

[NCERT Ex. 5.1, Q. 25, Page 161]

Ans. Given function is $f(x)=\begin{cases}\sin x-\cos x, & \text{if } x\ne 0\\ -1, & \text{if } x=0\end{cases}$.

From the function, $x = 0$

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0}[\sin(-h)-\cos(-h)] = \lim_{h\to 0}[-\sin h-\cos h] = -1$$

[1]

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0}[\sin(h)-\cos(h)] = -1$$

[1]

And, $f(0)=-1$

Since $LHL = RHL = f(0) = -1$, then the given function is continuous at $x = 0$.

Thus, the given function is continuous for all real value of x. **[1]**

Q. 13. Find the values of *k* so that the function *f* is continuous at the indicated point in

$$f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\ne\dfrac{\pi}{2}\\ 3, & \text{if } x=\dfrac{\pi}{2}\end{cases} \text{ at } x=\frac{\pi}{2}.$$

[NCERT Ex. 5.1, Q. 26, Page 161]

Ans. Given function is $f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\ne\dfrac{\pi}{2}\\ 3, & \text{if } x=\dfrac{\pi}{2}\end{cases}$.

From the function, $x = \frac{\pi}{2}$

$$\text{LHL}\left(\text{at } x=\frac{\pi}{2}\right) = \lim_{x\to\frac{\pi}{2}^-} f(x)$$
$$= \lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)$$
$$= \lim_{h\to 0}\left[\frac{k\cos\left(\frac{\pi}{2}-h\right)}{\pi-2\left(\frac{\pi}{2}-h\right)}\right]$$
$$= \lim_{h\to 0}\left[\frac{k\sin h}{\pi-\pi+2h}\right]$$
$$= \lim_{h\to 0}\left[\frac{k}{2}\left(\frac{\sin h}{h}\right)\right]$$
$$= \frac{k}{2}$$ **[1]**

$$\text{RHL}\left(\text{at } x=\frac{\pi}{2}\right) = \lim_{x\to\frac{\pi}{2}^+} f(x)$$
$$= \lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)$$
$$= \lim_{h\to 0}\left[\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}\right]$$
$$= \lim_{h\to 0}\left[\frac{-k\sin h}{\pi-\pi-2h}\right]$$
$$= \lim_{h\to 0}\left[\frac{k}{2}\left(\frac{\sin h}{h}\right)\right]$$
$$= \frac{k}{2}$$ **[1]**

And, $f\left(\frac{\pi}{2}\right) = 3$

Since the given function is continuous function at $x = \frac{\pi}{2}$, then $\text{LHL} = \text{RHL} = f\left(\frac{\pi}{2}\right) = 3$.

Thus, $\frac{k}{2} = 3 \Rightarrow k = 6$. **[1]**

Q. 14. Find the values of *k* so that the function *f* is continuous at the indicated point in

$$f(x) = \begin{cases} kx+1, & \text{if } x \le 5 \\ 3x-5, & \text{if } x > 5 \end{cases} \quad \text{at } x = 5$$

[NCERT Ex. 5.1, Q. 29, Page 161]

Ans. Given function is $f(x) = \begin{cases} kx+1, & \text{if } x \le 5 \\ 3x-5, & \text{if } x > 5 \end{cases}$.

From the function, $x = 5$

$$\text{LHL}(\text{at } x=5) = \lim_{x\to 5^-} f(x)$$
$$= \lim_{h\to 0} f(5-h)$$
$$= \lim_{h\to 0}\left[k(5-h)+1\right]$$
$$= 5k+1$$ **[1]**

$$\text{RHL}(\text{at } x=5) = \lim_{x\to 5^+} f(x)$$
$$= \lim_{h\to 0} f(5+h)$$
$$= \lim_{h\to 0}\left[3(5+h)-5\right]$$
$$= 10$$ **[1]**

And, $f(5) = 5k+1$

Since the given function is continuous function at $x = 5$, then $\text{LHL} = \text{RHL} = f(5) = 5k+1$. Thus,

$5k+1 = 10 \Rightarrow k = \frac{9}{5}$ **[1]**

Q. 15. Prove that the function f given by $f(x) = |x-1|$, $x \in R$ is not differentiable at $x = 1$.

[NCERT Ex. 5.2, Q. 9, Page 166]

Ans. Given function is $f(x) = |x-1|$

$$\text{LHD}(\text{at } x=1) = \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}$$
$$= \lim_{h\to 0}\frac{f(1-h)-f(1)}{1-h-1}$$
$$= \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$
$$= \lim_{h\to 0}\frac{|1-h-1|-|1-1|}{-h}$$
$$= \lim_{h\to 0}\frac{|-h|-0}{-h}$$
$$= \lim_{h\to 0}\left(\frac{h}{-h}\right)$$
$$= \lim_{h\to 0}(-1)$$
$$= -1$$ **[1½]**

$$\text{RHD}(\text{at } x=1) = \lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$$
$$= \lim_{h\to 0}\frac{f(1+h)-f(1)}{1+h-1}$$
$$= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to 0}\frac{|1+h-1|-|1-1|}{h}$$
$$= \lim_{h\to 0}\frac{|h|-0}{h}$$
$$= \lim_{h\to 0}\left(\frac{h}{h}\right)$$
$$= \lim_{h\to 0}(1)$$
$$= 1$$ **[1½]**

Since LHD ≠ RHD, then the given function is not differentiable at $x = 1$.

Q. 16. Examine the continuity of the function $f(x) = x^3 + 2x^2 - 1$ at $x = 1$.

[NCERT Exemp. Ex. 5.3, Q. 1, Page 107]

Ans. Given function is $f(x) = x^3 + 2x^2 - 1$.

$$\text{LHL}(\text{at } x=1) = \lim_{x\to 1^-} f(x)$$
$$= \lim_{h\to 0} f(1-h)$$
$$= \lim_{h\to 0}(1-h)^3 + 2(1-h)^2 - 1$$
$$= 1+2-1$$
$$= 2$$ **[1]**

$$\text{RHL (at } x=1) = \lim_{x\to 1^+} f(x)$$
$$= \lim_{h\to 0} f(1+h)$$
$$= \lim_{h\to 0} (1+h)^3 + 2(1+h)^2 - 1$$
$$= 1+2-1$$
$$= 2$$ **[1]**

And, $f(1) = 1+2-1 = 2$

Since LHL = RHL = $f(1) = 2$, then the given function is continuous at $x = 1$. **[1]**

Q. 17. Is the function $f(x) = \begin{cases} 3x+5, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases}$ continuous or discontinuous at $x = 2$?

[NCERT Exemp. Ex. 5.3, Q. 2, Page 107]

Ans. Given function is $f(x) = \begin{cases} 3x+5, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases}$

$$\text{LHL (at } x=2) = \lim_{x\to 2^-} f(x)$$
$$= \lim_{h\to 0} f(2-h)$$
$$= \lim_{h\to 0} (2-h)^2$$
$$= 4$$ **[1½]**

$$\text{RHL (at } x=2) = \lim_{x\to 2^+} f(x)$$
$$= \lim_{h\to 0} f(2+h)$$
$$= \lim_{h\to 0} 3(2+h)+5$$
$$= 11$$ **[1½]**

Since LHL ≠ RHL, then the given function is discontinuous at $x = 2$.

Q. 18. Is the function $f(x) = \begin{cases} \frac{1-\cos 2x}{x^2}, & \text{if } x \neq 0 \\ 5, & \text{if } x = 0 \end{cases}$ continuous or discontinuous at $x = 0$?

[NCERT Exemp. Ex. 5.3, Q. 3, Page 107]

Ans. Given function is $f(x) = \begin{cases} \frac{1-\cos 2x}{x^2}, & \text{if } x \neq 0 \\ 5, & \text{if } x = 0 \end{cases}$.

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0} \left[\frac{1-\cos 2(-h)}{(-h)^2}\right]$$
$$= \lim_{h\to 0} \left[\frac{1-\cos 2h}{h^2}\right]$$
$$= \lim_{h\to 0} \left[\frac{2\sin^2 h}{h^2}\right]$$
$$= \lim_{h\to 0} \left[2\left(\frac{\sin h}{h}\right)^2\right]$$
$$= 2$$ **[1½]**

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x)$$
$$= \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0} \left[\frac{1-\cos 2(h)}{(h)^2}\right]$$
$$= \lim_{h\to 0} \left[\frac{1-\cos 2h}{h^2}\right]$$
$$= \lim_{h\to 0} \left[\frac{2\sin^2 h}{h^2}\right]$$
$$= \lim_{h\to 0} \left[2\left(\frac{\sin h}{h}\right)^2\right]$$
$$= 2$$ **[1½]**

And, $f(0) = 5$

Since LHL = RHL ≠ $f(0) = 5$, then the given function is discontinuous at $x = 0$.

Q. 19. Is the function $f(x) = \begin{cases} \frac{|x-4|}{2(x-4)}, & \text{if } x \neq 4 \\ 0, & \text{if } x = 4 \end{cases}$ continuous or discontinuous at $x = 4$?

[NCERT Exemp. Ex. 5.3, Q. 5, Page 107]

Ans. Given function is $f(x) = \begin{cases} \frac{|x-4|}{2(x-4)}, & \text{if } x \neq 4 \\ 0, & \text{if } x = 4 \end{cases}$.

$$\text{LHL (at } x=4) = \lim_{x\to 4^-} f(x)$$
$$= \lim_{h\to 0} f(4-h)$$
$$= \lim_{h\to 0} \left[\frac{|4-h-4|}{2(4-h-4)}\right]$$
$$= \lim_{h\to 0} \left[\frac{|-h|}{-2h}\right]$$
$$= \lim_{h\to 0} \left[\frac{h}{-2h}\right]$$
$$= \lim_{h\to 0} \left[-\frac{1}{2}\right]$$
$$= -\frac{1}{2}$$ **[1]**

$$\text{RHL (at } x=4) = \lim_{x\to 4^+} f(x)$$
$$= \lim_{h\to 0} f(4+h)$$
$$= \lim_{h\to 0} \left[\frac{|4+h-4|}{2(4+h-4)}\right]$$
$$= \lim_{h\to 0} \left[\frac{|h|}{2h}\right]$$
$$= \lim_{h\to 0} \left[\frac{h}{2h}\right]$$
$$= \lim_{h\to 0} \left[\frac{1}{2}\right]$$
$$= \frac{1}{2}$$ **[1]**

Since LHL ≠ RHL, then the given function is discontinuous at $x = 4$. **[1]**

Q. 20. Is the function $f(x)=\begin{cases}|x|\cos\frac{1}{x}, \text{if } x\neq 0\\ 0, \quad \text{if } x=0\end{cases}$ **continuous or discontinuous at** $x = 0$?

[NCERT Exemp. Ex. 5.3, Q. 6, Page 107]

Ans. Given function is $f(x)=\begin{cases}|x|\cos\frac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$.

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0}\left[|-h|\cos\frac{1}{(-h)}\right]$$
$$= \lim_{h\to 0}\left[h\cos\frac{1}{h}\right]$$
$$= 0\times\left[\begin{array}{l}\text{An oscillating number}\\ \text{between } -1 \text{ and } 1\end{array}\right]$$
$$= 0$$ **[1]**

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x)$$
$$= \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0}\left[|h|\cos\frac{1}{(h)}\right]$$
$$= \lim_{h\to 0}\left[h\cos\frac{1}{h}\right]$$
$$= 0\times\left[\begin{array}{l}\text{An oscillating number}\\ \text{between } -1 \text{ and } 1\end{array}\right]$$
$$= 0$$ **[1]**

And, $f(0)=0$

Since LHL=RHL=$f(0)$=0 , then the given function is continuous at $x = 0$. **[1]**

Q. 21. Is the function $f(x)=\begin{cases}|x-a|\sin\frac{1}{x-a}, & \text{if } x\neq a\\ 0, & \text{if } x=a\end{cases}$ **continuous or discontinuous at** $x = a$?

[NCERT Exemp. Ex. 5.3, Q. 7, Page 107]

Ans. Given function is $f(x)=\begin{cases}|x-a|\sin\frac{1}{x-a}, & \text{if } x\neq a\\ 0, & \text{if } x=a\end{cases}$.

$$\text{LHL (at } x=a) = \lim_{x\to a^-} f(x)$$
$$= \lim_{h\to 0} f(a-h)$$
$$= \lim_{h\to 0}\left[|a-h-a|\sin\frac{1}{(a-h-a)}\right]$$
$$= \lim_{h\to 0}\left[|-h|\sin\frac{1}{(-h)}\right]$$
$$= \lim_{h\to 0}\left[-h\sin\frac{1}{h}\right]$$
$$= 0\times\left[\begin{array}{l}\text{An oscillating number}\\ \text{between } -1 \text{ and } 1\end{array}\right]$$
$$= 0$$ **[1]**

$$\text{RHL (at } x=a) = \lim_{x\to a^+} f(x)$$
$$= \lim_{h\to 0} f(a+h)$$
$$= \lim_{h\to 0}\left[|a+h-a|\sin\frac{1}{(a+h-a)}\right]$$
$$= \lim_{h\to 0}\left[|h|\sin\frac{1}{(h)}\right]$$
$$= \lim_{h\to 0}\left[h\sin\frac{1}{h}\right]$$
$$= 0\times\left[\begin{array}{l}\text{An oscillating number}\\ \text{between } -1 \text{ and } 1\end{array}\right]$$
$$= 0$$ **[1]**

And, $f(a)=0$

Since LHL = RHL = $f(0)=0$, then the given function is continuous at $x = a$. **[1]**

Q. 22. Is the function $f(x)=\begin{cases}\dfrac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$ **continuous or discontinuous at** $x = 0$?

[NCERT Exemp. Ex. 5.3, Q. 8, Page 107]

Ans. Given function is $f(x)=\begin{cases}\dfrac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$.

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0}\left[\frac{e^{\frac{1}{-h}}}{1+e^{\frac{1}{-h}}}\right]$$
$$= \left[\frac{e^{-\infty}}{1+e^{-\infty}}\right]$$
$$= \frac{0}{1+0}$$
$$= 0$$ **[1]**

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x)$$
$$= \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0}\left[\frac{e^{\frac{1}{h}}}{1+e^{\frac{1}{h}}}\right]$$
$$= \lim_{h\to 0}\left[\frac{1}{1+e^{-\frac{1}{h}}}\right]$$
$$= \frac{1}{1+e^{-\infty}}$$
$$= \frac{1}{1+0}$$
$$= 1$$ **[1]**

And, $f(0)=0$

Since LHL $\neq$ RHL $= f(0)=0$, then the given function is discontinuous at $x = 0$. **[1]**

Q. 23. Is the function $f(x)=\begin{cases} \frac{x^2}{2}, & \text{if } 0 \le x \le 1 \\ 2x^2-3x+\frac{3}{2}, & \text{if } 1 < x \le 2 \end{cases}$ continuous or discontinuous at x = 1?

[NCERT Exemp. Ex. 5.3, Q. 9, Page 107]

Ans. Given function is $f(x)=\begin{cases} \frac{x^2}{2}, & \text{if } 0 \le x \le 1 \\ 2x^2-3x+\frac{3}{2}, & \text{if } 1 < x \le 2 \end{cases}$.

$$\text{LHL (at } x=1) = \lim_{x\to1^-} f(x) = \lim_{h\to0} f(1-h) = \lim_{h\to0}\left[\frac{(1-h)^2}{2}\right] = \frac{1}{2}$$ **[1]**

$$\text{RHL (at } x=1) = \lim_{x\to1^+} f(x) = \lim_{h\to0} f(1+h) = \lim_{h\to0}\left[2(1+h)^2-3(1+h)+\frac{3}{2}\right] = 2(1)^2-3(1)+\frac{3}{2} = \frac{1}{2}$$ **[1]**

And, $f(1)=\frac{1}{2}$

Since LHL = RHL $= f(1)=\frac{1}{2}$, then the given function is continuous at $x = 1$. **[1]**

Q. 24. Is the function $f(x)=|x|+|x-1|$ continuous or discontinuous at $x = 1$?

[NCERT Exemp. Ex. 5.3, Q. 10, Page 107]

Ans. Given function is $f(x)=|x|+|x-1|$.

$$\text{LHL (at } x=1) = \lim_{x\to1^-} f(x) = \lim_{h\to0} f(1-h) = \lim_{h\to0}\left[|1-h|+|1-h-1|\right] = \lim_{h\to0}\left[|1-h|+|-h|\right] = 1+0 = 1$$ **[1]**

$$\text{RHL (at } x=1) = \lim_{x\to1^+} f(x) = \lim_{h\to0} f(1+h) = \lim_{h\to0}\left[|1+h|+|1+h-1|\right] = \lim_{h\to0}\left[|1+h|+|h|\right] = 1+0 = 1$$ **[1]**

And, $f(1)=1$

Since LHL = RHL $= f(1)=1$, then the given function is continuous at $x = 1$. **[1]**

Q. 25. Find the value of k in $f(x)=\begin{cases} 3x-8, & \text{if } x \le 5 \\ 2k, & \text{if } x > 5 \end{cases}$ so that the function f is continuous at $x = 5$?

[NCERT Exemp. Ex. 5.3, Q. 11, Page 108]

Ans. Given function is $f(x)=\begin{cases} 3x-8, & \text{if } x \le 5 \\ 2k, & \text{if } x > 5 \end{cases}$.

$$\text{LHL (at } x=5) = \lim_{x\to5^-} f(x) = \lim_{h\to0} f(5-h) = \lim_{h\to0}\left[3(5-h)-8\right] = 3(5)-8 = 7$$ **[1]**

$$\text{RHL (at } x=5) = \lim_{x\to5^+} f(x) = \lim_{h\to0} f(5+h) = \lim_{h\to0}[2k] = 2k$$ **[1]**

And, $f(5)=7$

Since the given function is continuous at $x = 5$, then LHL = RHL $= f(5)=7$.

LHL=RHL

$\Rightarrow 7=2k$

$\Rightarrow k=\frac{7}{2}$ **[1]**

Q. 26. Find the value of k in $f(x)=\begin{cases} \frac{2^{x+2}-16}{4^x-16}, & \text{if } x \ne 2 \\ k, & \text{if } x = 2 \end{cases}$ so that the function f is continuous at $x = 2$?

[NCERT Exemp. Ex. 5.3, Q. 12, Page 108]

Ans. Given function is

$$f(x)=\begin{cases} \frac{2^{x+2}-16}{4^x-16}, & \text{if } x \ne 2 \\ k, & \text{if } x = 2 \end{cases}$$

$$\Rightarrow f(x)=\begin{cases} \frac{4\times2^x-16}{4^x-16}, & \text{if } x \ne 2 \\ k, & \text{if } x = 2 \end{cases}.$$

Since the given function is continuous at $x = 2$, then LHL = RHL $= f(2)=k$. **[1]**

$$k=\lim_{x\to2} f(x) = \lim_{x\to2}\left(\frac{2^{x+2}-16}{4^x-16}\right) = \lim_{x\to2}\frac{4(2^x-4)}{(2^x-4)(2^x+4)} = \lim_{x\to2}\frac{4}{(2^x+4)} = \frac{4}{4+4} = \frac{1}{2}$$ **[2]**

Q. 27. Prove that the function f defined by

$$f(x)=\begin{cases}\dfrac{x}{|x|+2x^2}, & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}$$

remains discontinuous at $x = 0$, regardless the choice of k.

[NCERT Exemp. Ex. 5.3, Q. 15, Page 108]

Ans. Given function is $f(x)=\begin{cases}\dfrac{x}{|x|+2x^2}, & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}$.

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0}\left[\frac{-h}{|-h|+2(-h)^2}\right]$$
$$= \lim_{h\to 0}\left[\frac{-h}{h+2h^2}\right]$$
$$= \lim_{h\to 0}\left[\frac{-1}{1+2h}\right]$$
$$= -1$$ [1½]

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x)$$
$$= \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0}\left[\frac{h}{|h|+2(h)^2}\right]$$
$$= \lim_{h\to 0}\left[\frac{h}{h+2h^2}\right]$$
$$= \lim_{h\to 0}\left[\frac{1}{1+2h}\right]$$
$$= 1$$ [1½]

Since LHL $\neq$ RHL, then the given function is discontinuous at $x = 0$ regardless the choice of k.

Q. 28. Find the values of a and b such that the function f defined by $f(x)=\begin{cases}\dfrac{x-4}{|x-4|}+a, & \text{if } x<4\\ a+b, & \text{if } x=4\\ \dfrac{x-4}{|x-4|}+b, & \text{if } x>4\end{cases}$ **is a continuous function at $x = 4$.**

[NCERT Exemp. Ex. 5.3, Q. 16, Page 108]

Ans. Given that $f(x)=\begin{cases}\dfrac{x-4}{|x-4|}+a, & \text{if } x<4\\ a+b, & \text{if } x=4\\ \dfrac{x-4}{|x-4|}+b, & \text{if } x>4\end{cases}$

$$\text{LHL (at } x=4) = \lim_{x\to 4^{-1}} f(x)$$
$$= \lim_{h\to 0} f(4-h)$$
$$= \lim_{h\to 0}\left[\frac{4-h-4}{|4-h-4|}+a\right]$$
$$= \lim_{h\to 0}\left[\frac{-h}{h}+a\right]$$
$$= [a-1]$$ [1½]

$$\text{RHL (at } x=4) = \lim_{x\to 4^+} f(x)$$
$$= \lim_{h\to 0} f(4+h)$$
$$= \lim_{h\to 0}\left[\frac{4+h-4}{|4+h-4|}+b\right]$$
$$= \lim_{h\to 0}\left[\frac{h}{h}+b\right]$$
$$= [b+1]$$

And, $f(4)=a+b$

Since the given function is continuous at $x = 4$, then LHL = RHL = $f(4)=a+b$.

Thus, $a-1=b+1=a+b \Rightarrow a=1$ and $b=-1$. [1½]

Q. 29. Show that the function $f(x) = |\sin x + \cos x|$ is continuous at $x = \pi$.

[NCERT Exemp. Ex. 5.3, Q. 19, Page 109]

Ans. Given function is $f(x)=|\sin x+\cos x|$.

$$\text{LHL (at } x=\pi) = \lim_{x\to \pi^-} f(x)$$
$$= \lim_{h\to 0} f(\pi-h)$$
$$= \lim_{h\to 0}\left[|\sin(\pi-h)+\cos(\pi-h)|\right]$$
$$= \lim_{h\to 0}\left[|\sin h-\cos h|\right]$$
$$= |0-1|$$
$$= 1$$ [1½]

$$\text{RHL (at } x=\pi) = \lim_{x\to \pi^+} f(x)$$
$$= \lim_{h\to 0} f(\pi+h)$$
$$= \lim_{h\to 0}\left[|\sin(\pi+h)+\cos(\pi+h)|\right]$$
$$= \lim_{h\to 0}\left[|-\sin h-\cos h|\right]$$
$$= |-0-1|$$
$$= 1$$

And, $f(\pi)=|-1|=1$

Since LHL=RHL=$f(\pi)$=1, then the given function is continuous at $x = \pi$. [1½]

Q. 30. A function $f : R \to R$ satisfies the equation $f(x + y) = f(x)\, f(y)$ for all $x, y \in R$, $f(x) \neq 0$. Suppose that the function is differentiable at $x = 0$ and $f'(0) = 2$. Prove that $f'(x) = 2f(x)$.

[NCERT Exemp. Ex. 5.3, Q. 24, Page 109]

Ans. Since the given function is differentiable at $x = 0$, then

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{x+h-x}$$
$$= \lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h}$$
$$= f(x)\left[\lim_{h\to 0}\frac{f(h)-1}{h}\right]$$ [1½]

Since

$$f(0+0) = f(0)f(0)$$
$$\Rightarrow \quad [f(0)]^2 - f(0) = 0$$
$$\Rightarrow \quad f(0)[f(0)-1] = 0$$
$$\Rightarrow \quad f(0) = 1$$

and $f'(0) = 2$, then

$$f'(x) = f(x)\left[\lim_{h\to 0}\frac{f(h)-f(0)}{h}\right]$$
$$f'(x) = f(x)f'(0)$$
$$\Rightarrow \quad f'(x) = 2f(x)$$

[1½]

Q. 31. Find the values of p and q so that

$$f(x) = \begin{cases} x^2+3x+p, & \text{if } x \le 1 \\ qx+2, & \text{if } x > 1 \end{cases}$$ **is differentiable at $x = 1$.**

[NCERT Exemp. Ex. 5.3, Q. 79, Page 112]

Ans. Given that $f(x) = \begin{cases} x^2+3x+p, & \text{if } x \le 1 \\ qx+2, & \text{if } x > 1 \end{cases}$ is differentiable at $x = 1$.

Thus, it is also a continuous function at $x = 1$.

$$\Rightarrow \quad \text{LHL} = \text{RHL} = f(1)$$
$$\Rightarrow \quad 1+3+p = q+2 = 1+3+p$$
$$\Rightarrow \quad p - q = -2 \qquad \ldots\text{(i)}$$

[1½]

Since $f'(x) = \begin{cases} 2x+3, & \text{if } x < 1 \\ q, & \text{if } x > 1 \end{cases}$ is differentiable at $x = 1$.

Thus,

$$\Rightarrow \text{LHL} = \text{RHL}$$
$$\Rightarrow 2+3 = q$$
$$\Rightarrow \quad q = 5$$

From equation (i), $p = 3$

Thus, $p = 3$ and $q = 5$. [1½]

Q. 32. Find the values of a and b, if the function f defined by

$$f(x) = \begin{cases} x^2+3x+a, & x \le 1 \\ bx+2, & x > 1 \end{cases}$$ **is differentiable at $x = 1$.**

[CBSE Board, Delhi Region, 2016]

Ans. Given that $f(x) = \begin{cases} x^2+3x+a, & x \le 1 \\ bx+2, & x > 1 \end{cases}$ is differentiable at $x = 1$.

Thus, it is also a continuous function at $x = 1$.

$$\Rightarrow \text{LHL} = \text{RHL} = f(1)$$
$$\Rightarrow 1+3+a = b+2 = 1+3+p$$
$$\Rightarrow a - b = -2 \qquad \ldots\text{(i)}$$

[2]

Since $f'(x) = \begin{cases} 2x+3, & \text{if } x < 1 \\ b, & \text{if } x > 1 \end{cases}$ is differentiable at $x = 1$.

Thus,

$$\Rightarrow \text{LHD} = \text{RHD}$$
$$\Rightarrow 2+3 = b$$
$$\Rightarrow \quad b = 5$$

From equation (i), $a = 3$

Thus, $a = 3$ and $b = 5$. [2]

Long Answer Type Questions

(5 or 6 marks each)

Q. 1. Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.

[NCERT Ex. 5.1, Q. 1, Page 159]

Ans. Given function is $f(x) = 5x - 3$.

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0} (5(-h)-3)$$
$$= -3$$

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x)$$
$$= \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0} (5(h)-3)$$
$$= -3$$

And, $f(0) = 5(0) - 3 = -3$

Since LHL = RHL = $f(0) = -3$, then the given function is continuous at $x = 0$. [1½]

$$\text{LHL (at } x=-3) = \lim_{x\to -3^-} f(x)$$
$$= \lim_{h\to 0} f(-3-h)$$
$$= \lim_{h\to 0} (5(-3-h)-3)$$
$$= -18$$

$$\text{RHL (at } x=-3) = \lim_{x\to -3^+} f(x)$$
$$= \lim_{h\to 0} f(-3+h)$$
$$= \lim_{h\to 0} (5(-3+h)-3)$$
$$= -18$$

And, $f(-3) = 5(-3) - 3 = -18$

Since LHL = RHL = $f(-3) = -18$, then the given function is continuous at x = –3. [1½]

$$\text{LHL (at } x=5) = \lim_{x\to 5^-} f(x)$$
$$= \lim_{h\to 0} f(5-h)$$
$$= \lim_{h\to 0} (5(5-h)-3)$$
$$= 22$$

$$\text{RHL (at } x=5) = \lim_{x\to 5^+} f(x)$$
$$= \lim_{h\to 0} f(5+h)$$
$$= \lim_{h\to 0} (5(5+h)-3)$$
$$= 22$$

And, $f(5) = 5(5) - 3 = 22$

Since LHL = RHL = $f(5) = 22$, then the given function is continuous at $x = 5$. [2]

Q. 2. For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2-2x), & \text{if } x \le 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$$ **continuous at $x = 0$?**

What about continuity at $x = 1$?

[NCERT Ex. 5.1, Q. 18, Page 160]

Ans. Given function is $f(x)=\begin{cases}\lambda(x^2-2x), & \text{if } x\le 0\\ 4x+1, & \text{if } x>0\end{cases}$.

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0}\left[\lambda\left((-h)^2-2(-h)\right)\right] = 0$$ **[1]**

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0}[4h+1] = 1$$

And, $f(0)=\lambda\left((0)^2-2(0)\right)=0$ **[1]**

Since LHL $\neq$ RHL $\neq f(0)=0$, then there is no value of λ which makes the given function is continuous at $x = 0$.

$$\text{LHL (at } x=1) = \lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}\left[4(1-h)+1\right] = 5$$ **[1]**

$$\text{RHL (at } x=1) = \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}\left[4(1+h)+1\right] = 5$$

And, $f(1)=4(1)+1=5$

Since LHL = RHL = $f(1)=5$, then the given function is continuous at $x = 1$ for any value of λ. **[2]**

Q. 3. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

[NCERT Ex. 5.1, Q. 22, Page 160]

Ans. Let k be any real number and let $f(x)=\cos x$. Then,

$$\text{LHL (at } x=k) = \lim_{x\to k^-} f(x) = \lim_{h\to 0} f(k-h) = \lim_{h\to 0}\left[\cos(k-h)\right] = \lim_{h\to 0}[\cos k\cosh+\sin k\sinh] = \cos k(1)+\sin k(0) = \cos k$$ **[1]**

$$\text{RHL (at } x=k) = \lim_{x\to k^+} f(x) = \lim_{h\to 0} f(k+h) = \lim_{h\to 0}\left[\cos(k+h)\right] = \lim_{h\to 0}[\cos k\cosh-\sin k\sinh] = \cos k(1)-\sin k(0) = \cos k$$ **[1]**

And, $f(k)=\cos k$

Since LHL = RHL = $f(k)=\cos k$, then the given function is continuous.

Thus, $f(x)=\cos x$ is a continuous function for any real number.

Let k be any real number and let $g(x)=\sin x$. Then,

$$\text{LHL (at } x=k) = \lim_{x\to k^-} g(x) = \lim_{h\to 0} g(k-h) = \lim_{h\to 0}\left[\sin(k-h)\right] = \lim_{h\to 0}[\sin k\cosh-\cos k\sinh] = \sin k$$ **[1]**

$$\text{RHL (at } x=k) = \lim_{x\to k^+} g(x) = \lim_{h\to 0} g(k+h) = \lim_{h\to 0}\left[\sin(k+h)\right] = \lim_{h\to 0}[\sin k\cosh+\cos k\sinh] = \sin k$$

And, $g(k)=\sin k$

Since LHL = RHL = $g(k)=\sin k$, then the given function is continuous. **[1]**

Thus, $g(x)=\sin x$ is a continuous function for any real number.

Let $h(x)=\text{cosec}\,x=\dfrac{1}{\sin x}$

Now, we know that if $g(x)$ is continuous functions, then $\dfrac{1}{g(x)}$ is also a continuous function but $g(x)\neq 0 \Rightarrow \sin x\neq 0 \Rightarrow x\neq n\pi$.

Thus, $h(x)=\text{cosec}\,x$ is also a continuous function except $x=n\pi$.

Let $h(x)=\sec x=\dfrac{1}{\cos x}$ but

$f(x)\neq 0$

$\Rightarrow \cos x\neq 0$

$\Rightarrow x\neq \dfrac{(2n+1)\pi}{2}$

Thus, $h(x)=\sec x$ is also a continuous function except $x=\dfrac{(2n+1)\pi}{2}$.

Let $h(x)=\cot x=\dfrac{\cos x}{\sin x}$ but

$g(x)\neq 0$

$\Rightarrow \sin x\neq 0$

$\Rightarrow x\neq n\pi$

Thus, $h(x)=\cot x$ is also a continuous function except $x=n\pi$. **[1]**

Q. 4. Find the values of a and b such that the function defined by

$$f(x)=\begin{cases}5, & \text{if } x\le 2\\ ax+b, & \text{if } 2<x<10\\ 21, & \text{if } x\ge 10\end{cases}$$ **is a continuous function.**

[NCERT Ex. 5.1, Q. 30, Page 161]

Ans. Given function is $f(x)=\begin{cases}5, & \text{if } x\le 2\\ ax+b, & \text{if } 2<x<10\\ 21, & \text{if } x\ge 10\end{cases}$.

From the function, $x = 2$ and 10

$$\text{LHL (at } x=2) = \lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0}[5] = 5$$ **[1½]**

$$\text{RHL (at } x=2) = \lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0}[a(2+h)+b] = 2a+b$$

And, $f(2)=5$

Since the given function is continuous function at $x=2$, then LHL = RHL = $f(2)=5$.

Thus, $2a+b=5$ ...(i)

[1½]

$$\text{LHL (at } x=10) = \lim_{x\to 10^-} f(x) = \lim_{h\to 0} f(10-h) = \lim_{h\to 0}[a(10-h)+b] = 10a+b$$

$$\text{RHL (at } x=10) = \lim_{x\to 10^+} f(x) = \lim_{h\to 0} f(10+h) = \lim_{h\to 0}[21] = 21$$

And, $f(10)=21$

Since the given function is continuous function at $x=10$, then LHL = RHL = $f(10)=21$.

Thus, $10a+b=21$ ...(ii)

Solving equations (i) and (ii), we have

$a = 2$ and $b = 1$ **[2]**

Q. 5. Find all the points of discontinuity of *f* defined by *f* (*x*) = |*x*| – |*x* + 1|.

[NCERT Ex. 5.1, Q. 34, Page 161]

Ans.

Case 1 : For $x<-1$, $f(x)=-x+(x+1)=1$

Case 2 : For $-1\le x<0$, $f(x)=-x-(x+1)=-2x-1$

Case 3 : For $x\ge 0$, $f(x)=x-(x+1)=-1$ **[1]**

Thus, the given function is

$$f(x)=\begin{cases}1, & \text{for } x<-1\\ -2x-1, & \text{for} -1\le x<0\\ -1, & \text{for } x\ge 0\end{cases}$$

From the function, $x = -1$ and 0

$$\text{LHL (at } x=-1) = \lim_{x\to -1^-} f(x) = \lim_{h\to 0} f(-1-h) = \lim_{h\to 0}[1] = 1$$ **[1]**

$$\text{RHL (at } x=-1) = \lim_{x\to -1^+} f(x) = \lim_{h\to 0} f(-1+h) = \lim_{h\to 0}[-2(-1+h)-1] = 2-1 = 1$$

And, $f(-1)=-2(-1)-1=1$ **[1]**

Since LHL = RHL = $f(-1)=1$, then the given function is continuous at $x = -1$.

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0}[-2(-h)-1] = -1$$ **[1]**

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0}[-1] = -1$$

And, $f(0)=-1$

Since LHL = RHL = $f(0)=-1$, then the given function is continuous at $x = 0$.

Thus, there is no point of discontinuity of the given function. **[1]**

Q. 6. Prove that the greatest integer function defined by *f*(*x*) = [*x*], 0 < *x* < 3 is not differentiable at *x* = 1 and *x* = 2. [NCERT Ex. 5.2, Q. 10, Page 166]

Ans. Given function is $f(x)=[x]$

For $x = 1$

$$\text{LHD (at } x=1) = \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{h\to 0}\frac{f(1-h)-f(1)}{1-h-1} = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{[1-h]-[1]}{-h} = \lim_{h\to 0}\frac{0-1}{-h} = \lim_{h\to 0}\left(\frac{1}{h}\right) = \infty \text{ (not defined)}$$ **[1½]**

$$\text{RHD (at } x=1) = \lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1} = \lim_{h\to 0}\frac{f(1+h)-f(1)}{1+h-1} = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{[1+h]-[1]}{h}$$
$$= \lim_{h\to 0}\frac{1-1}{h}$$
$$= \lim_{h\to 0}\left(\frac{0}{h}\right)$$
$$= 0$$

Since $\text{LHD} \neq \text{RHD}$, then the given function is not differentiable at $x = 1$. **[1½]**

For $x = 2$

$$\text{LHD (at } x = 2) = \lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}$$
$$= \lim_{h\to 0}\frac{f(2-h)-f(2)}{2-h-2}$$
$$= \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$$
$$= \lim_{h\to 0}\frac{[2-h]-[2]}{-h}$$
$$= \lim_{h\to 0}\frac{1-2}{-h}$$
$$= \lim_{h\to 0}\left(\frac{-1}{-h}\right)$$
$$= \infty \text{ (not defined)}$$ **[1]**

$$\text{RHD (at } x = 2) = \lim_{x\to 1^+}\frac{f(x)-f(2)}{x-2}$$
$$= \lim_{h\to 0}\frac{f(2+h)-f(2)}{2+h-2}$$
$$= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0}\frac{[2+h]-[2]}{h}$$
$$= \lim_{h\to 0}\frac{2-2}{h}$$
$$= \lim_{h\to 0}\left(\frac{0}{h}\right)$$
$$= 0$$

Since $\text{LHD} \neq \text{RHD}$, then the given function is not differentiable at $x = 2$. **[1]**

Q. 7. Is the function $f(x) = \begin{cases} \dfrac{2x^2-3x-2}{x-2}, & \text{if } x \neq 2 \\ 5, & \text{if } x = 2 \end{cases}$ **continuous or discontinuous at $x = 2$?**

[NCERT Exemp. Ex. 5.3, Q. 4, Page 107]

Ans. Given function is $f(x) = \begin{cases} \dfrac{2x^2-3x-2}{x-2}, & \text{if } x \neq 2 \\ 5, & \text{if } x = 2 \end{cases}$.

$$\text{LHL (at } x = 2) = \lim_{x\to 2^-} f(x)$$
$$= \lim_{h\to 0} f(2-h)$$
$$= \lim_{h\to 0}\left[\frac{2(2-h)^2-3(2-h)-2}{(2-h)-2}\right]$$
$$= \lim_{h\to 0}\left[\frac{8+2h^2-8h-6+3h-2}{-h}\right]$$
$$= \lim_{h\to 0}\left[\frac{2h^2-5h}{-h}\right]$$
$$= \lim_{h\to 0}\left[\frac{h(2h-5)}{-h}\right]$$
$$= \lim_{h\to 0}[5-2h]$$
$$= 5$$ **[2]**

$$\text{RHL (at } x = 2) = \lim_{x\to 2^+} f(x)$$
$$= \lim_{h\to 0} f(2+h)$$
$$= \lim_{h\to 0}\left[\frac{2(2+h)^2-3(2+h)-2}{(2+h)-2}\right]$$
$$= \lim_{h\to 0}\left[\frac{8+2h^2+8h-6-3h-2}{h}\right]$$
$$= \lim_{h\to 0}\left[\frac{2h^2+5h}{h}\right]$$
$$= \lim_{h\to 0}\left[\frac{h(2h+5)}{h}\right]$$
$$= \lim_{h\to 0}[5+2h]$$
$$= 5$$ **[2]**

And, $f(2) = 5$

Since $\text{LHL} = \text{RHL} = f(2) = 5$, then the given function is continuous at $x = 2$. **[1]**

Q. 8. Find the value of k in

$$f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \le x \le 1 \end{cases}$$

so that the function f is continuous at $x = 0$?

[NCERT Exemp. Ex. 5.3, Q. 13, Page 108]

Ans. Given function is

$$f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \le x \le 1 \end{cases}$$. **[1]**

$$\text{LHL(at } x = 0) = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0}\left[\frac{\sqrt{1-kh}-\sqrt{1+kh}}{-h}\right]$$
$$= \lim_{h\to 0}\left[\frac{\sqrt{1+kh}-\sqrt{1-kh}}{h}\times\frac{\sqrt{1+kh}+\sqrt{1-kh}}{\sqrt{1+kh}+\sqrt{1-kh}}\right]$$
$$= \lim_{h\to 0}\left[\frac{1+kh-1+kh}{h\left(\sqrt{1+kh}+\sqrt{1-kh}\right)}\right]$$
$$= \lim_{h\to 0}\left[\frac{2kh}{h\left(\sqrt{1+kh}+\sqrt{1-kh}\right)}\right]$$
$$= \lim_{h\to 0}\left[\frac{2k}{\left(\sqrt{1+kh}+\sqrt{1-kh}\right)}\right]$$
$$= \frac{2k}{2}$$
$$= k$$ **[2]**

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x)$$
$$= \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0}\left[\frac{2h+1}{h-1}\right]$$
$$= -1$$

And, $f(0) = -1$

Since the given function is continuous at $x = 0$, then $\text{LHL} = \text{RHL} = f(0) = -1$.

$\text{LHL} = \text{RHL}$

$\Rightarrow \quad k = -1$ **[1]**

Q. 9. Find the value of k in

$$f(x) = \begin{cases} \dfrac{1-\cos kx}{x\sin x}, & \text{if } x \neq 0 \\ \dfrac{1}{2}, & \text{if } x = 0 \end{cases}$$

so that the function f is continuous at $x = 0$?

[NCERT Exemp. Ex. 5.3, Q. 14, Page 108]

Ans. Given function is $f(x) = \begin{cases} \dfrac{1-\cos kx}{x\sin x}, & \text{if } x \neq 0 \\ \dfrac{1}{2}, & \text{if } x = 0 \end{cases}$.

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0}\left[\frac{1-\cos k(-h)}{(-h)\sin(-h)}\right]$$
$$= \lim_{h\to 0}\left[\frac{1-\cos kh}{h\sin h}\right]$$
$$= \lim_{h\to 0}\left[\frac{2\sin^2\left(\frac{kh}{2}\right)}{h\sin h}\right]$$
$$= \lim_{h\to 0}\left[\frac{2\left(\frac{\sin\frac{kh}{2}}{\frac{kh}{2}}\right)^2 \times \frac{(kh)^2}{4}}{h\sin h}\right]$$
$$= \lim_{h\to 0}\left[\frac{\frac{k^2}{2}\left(\frac{\sin\frac{kh}{2}}{\frac{kh}{2}}\right)^2}{\left(\frac{\sin h}{h}\right)}\right]$$
$$= \frac{k^2}{2}$$ **[3]**

And, $f(0) = \frac{1}{2}$

Since the given function is continuous at $x = 0$, then $\text{LHL} = \text{RHL} = f(0) = \frac{1}{2}$

$\text{LHL} = f(0)$

$\Rightarrow \quad \frac{k^2}{2} = \frac{1}{2}$

$\Rightarrow \quad k = \pm 1$ **[2]**

Q. 10. Examine the differentiability of f, where f is defined by

$$f(x) = \begin{cases} x[x], & \text{if } 0 \le x < 2 \\ (x-1)x, & \text{if } 2 \le x < 3 \end{cases} \text{ at x = 2.}$$

[NCERT Exemp. Ex. 5.3, Q. 20, Page 109]

Ans. Given function is $f(x) = \begin{cases} x[x], & \text{if } 0 \le x < 2 \\ (x-1)x, & \text{if } 2 \le x < 3 \end{cases}$

$$\text{LHD (at } x=2) = \lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}$$
$$= \lim_{h\to 0}\frac{f(2-h)-f(2)}{2-h-2}$$
$$= \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$$
$$= \lim_{h\to 0}\frac{(2-h)[2-h]-2}{-h}$$
$$= \lim_{h\to 0}\frac{(2-h)-2}{-h}$$
$$= \lim_{h\to 0}\left(\frac{-h}{-h}\right)$$
$$= 1$$ **[2]**

$$\text{RHD (at } x=2) = \lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}$$
$$= \lim_{h\to 0}\frac{f(2+h)-f(2)}{2+h-2}$$
$$= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0}\frac{(2+h-1)(2+h)-2}{h}$$
$$= \lim_{h\to 0}\frac{h^2+3h}{h}$$
$$= 3$$ **[2]**

Since $\text{LHD} \neq \text{RHD}$, then the given function is not differentiable at $x = 2$. **[1]**

Q. 11. Examine the differentiability of f, where f is defined by

$$f(x) = \begin{cases} x^2\sin\dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

at $x = 0$. [NCERT Exemp. Ex. 5.3, Q. 21, Page 109]

Ans. Given function is $f(x) = \begin{cases} x^2\sin\dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

$$\text{LHD (at } x=0) = \lim_{x\to 2^-} \frac{f(x)-f(0)}{x-0}$$
$$= \lim_{h\to 0} \frac{f(0-h)-f(0)}{0-h}$$
$$= \lim_{h\to 0} \frac{f(-h)-f(0)}{-h}$$
$$= \lim_{h\to 0} \frac{(-h)^2 \sin\frac{1}{(-h)}}{(-h)}$$
$$= \lim_{h\to 0}\left(h\sin\frac{1}{h}\right)$$
$$= 0\times\begin{bmatrix}\text{An oscillating number}\\ \text{between } -1 \text{ and } 1\end{bmatrix}$$
$$= 0$$ **[2]**

$$\text{RHD (at } x=0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$$
$$= \lim_{h\to 0} \frac{f(0+h)-f(0)}{0+h}$$
$$= \lim_{h\to 0} \frac{f(h)-f(0)}{h}$$
$$= \lim_{h\to 0} \frac{(-h)^2 \sin\frac{1}{(-h)}}{(-h)}$$
$$= \lim_{h\to 0}\left(h\sin\frac{1}{h}\right)$$
$$= 0\times\begin{bmatrix}\text{An oscillating number}\\ \text{between } -1 \text{ and } 1\end{bmatrix}$$
$$= 0$$ **[2]**

Since $\text{LHD} = \text{RHD}$, then the given function is differentiable at $x = 0$. **[1]**

Q. 12. Examine the differentiability of f, where f is defined by

$$f(x) = \begin{cases} 1+x, & \text{if } x \le 2 \\ 5-x, & \text{if } x > 2 \end{cases}$$

at x = 2. [NCERT Exemp. Ex. 5.3, Q. 22, Page 109]

Ans. Given function is $f(x) = \begin{cases} 1+x, & \text{if } x \le 2 \\ 5-x, & \text{if } x > 2 \end{cases}$

$$\text{LHD (at } x=2) = \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2}$$
$$= \lim_{h\to 0} \frac{f(2-h)-f(2)}{2-h-2}$$
$$= \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h}$$
$$= \lim_{h\to 0} \frac{1+(2-h)-1-2}{(-h)}$$
$$= \lim_{h\to 0}\left(\frac{-h}{(-h)}\right)$$
$$= 1$$ **[2]**

$$\text{RHD (at } x=2) = \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2}$$
$$= \lim_{h\to 0} \frac{f(2+h)-f(2)}{2+h-2}$$
$$= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0} \frac{5-(2+h)-3}{h}$$
$$= \lim_{h\to 0}\left(\frac{-h}{(h)}\right)$$
$$= -1$$ **[2]**

Since $\text{LHD} \ne \text{RHD}$, then the given function is differentiable at $x = 2$. **[1]**

Q. 13. Show that $f(x) = |x - 5|$ is continuous but not differentiable at $x = 5$.

[NCERT Exemp. Ex. 5.3, Q. 23, Page 109]

Ans. We know that the modulus function is continuous function for all real values. Thus, the given function $f(x) = |x - 5|$ is continuous at $x = 5$. **[1]**

Now, $f(x) = |x-5| = \begin{cases} x-5, & \text{if } x \ge 5 \\ 5-x, & \text{if } x < 5 \end{cases}$

$$\text{LHD (at } x=5) = \lim_{x\to 5^-} \frac{f(x)-f(5)}{x-5}$$
$$= \lim_{h\to 0} \frac{f(5-h)-f(5)}{5-h-5}$$
$$= \lim_{h\to 0} \frac{f(5-h)-f(5)}{-h}$$
$$= \lim_{h\to 0} \frac{5-(5-h)}{(-h)}$$
$$= \lim_{h\to 0}\left(\frac{h}{(-h)}\right)$$
$$= -1$$ **[2]**

$$\text{RHD (at } x=5) = \lim_{x\to 5^+} \frac{f(x)-f(5)}{x-5}$$
$$= \lim_{h\to 0} \frac{f(5+h)-f(5)}{5+h-5}$$
$$= \lim_{h\to 0} \frac{f(5+h)-f(5)}{h}$$
$$= \lim_{h\to 0} \frac{5+h-5}{(h)}$$
$$= \lim_{h\to 0}\left(\frac{h}{(h)}\right)$$
$$= 1$$ **[2]**

Since $\text{LHD} \ne \text{RHD}$, then the given function is differentiable at $x = 5$.

Q. 14. Find the values of p and q for which

$$f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x < \dfrac{\pi}{2} \\ p, & \text{if } x = \dfrac{\pi}{2} \\ \dfrac{q(1-\sin x)}{(\pi - 2x)^2}, & \text{if } x > \dfrac{\pi}{2} \end{cases}$$

is continuous at $x = \frac{\pi}{2}$.

[CBSE Board, Delhi Region, 2017, CBSE Board, Delhi Region, 2016]

Ans. Given function is $f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x<\dfrac{\pi}{2}\\ p, & \text{if } x=\dfrac{\pi}{2}\\ \dfrac{q(1-\sin x)}{(\pi-2x)^2}, & \text{if } x>\dfrac{\pi}{2}\end{cases}$.

$$\text{LHL}\left(\text{at } x=\frac{\pi}{2}\right)=\lim_{x\to\frac{\pi}{2}^-} f(x)$$
$$=\lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)$$
$$=\lim_{h\to 0}\left[\frac{1-\sin^3\left(\frac{\pi}{2}-h\right)}{3\cos^2\left(\frac{\pi}{2}-h\right)}\right]$$
$$=\lim_{h\to 0}\left[\frac{1-\cos^3(h)}{3\sin^2(h)}\right]$$
$$=\lim_{h\to 0}\left[\frac{(1-\cos(h))(1+\cos h+\cos^2 h)}{3(1-\cos^2(h))}\right]$$
$$=\lim_{h\to 0}\left[\frac{(1+\cos h+\cos^2 h)}{3(1+\cos(h))}\right]$$
$$=\frac{1+1+1}{3(1+1)}$$
$$=\frac{1}{2}$$
[2]

$$\text{RHL}\left(\text{at } x=\frac{\pi}{2}\right)=\lim_{x\to\frac{\pi}{2}^+} f(x)$$
$$=\lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)$$
$$=\lim_{h\to 0}\left[\frac{q\left(1-\sin\left(\frac{\pi}{2}+h\right)\right)}{\left(\pi-2\left(\frac{\pi}{2}+h\right)\right)^2}\right]$$
$$=\lim_{h\to 0}\left[\frac{q(1-\cos h)}{h^2}\right]$$
$$=\lim_{h\to 0}\left[\frac{2q\sin^2\left(\frac{h}{2}\right)}{h^2}\right]$$
$$=\lim_{h\to 0}\left[\frac{q}{2}\left(\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right)^2\right]$$
$$=\frac{q}{2}$$
[2]

And, $f\left(\dfrac{\pi}{2}\right)=p$

Since the given function is continuous at $x=\dfrac{\pi}{2}$, then

$$\text{LHL}=\text{RHL}=f\left(\frac{\pi}{2}\right)$$
$$\frac{1}{2}=\frac{q}{2}=p$$
$$\Rightarrow\quad p=\frac{1}{2} \text{ and } q=1$$
[1]

Q. 15. Determine the value of the constant 'k' so that the function

$$f(x)=\begin{cases}\dfrac{kx}{|x|}, & \text{if } x<0\\ 3, & \text{if } x\ge 0\end{cases}\quad \textbf{is continuous at } x=0.$$

[CBSE Board, Delhi Region, 2017]

Ans. Given function is $f(x)=\begin{cases}\dfrac{kx}{|x|}, & \text{if } x<0\\ 3, & \text{if } x\ge 0\end{cases}$.

$$\text{LHL}(\text{at } x=0)=\lim_{x\to 0^-} f(x)$$
$$=\lim_{h\to 0} f(0-h)$$
$$=\lim_{h\to 0} f(-h)$$
$$=\lim_{h\to 0}\left[\frac{k(-h)}{|-h|}\right]$$
$$=\lim_{h\to 0}\left[\frac{kh}{h}\right]$$
$$=\lim_{h\to 0}[k]$$
$$=k$$
[2]

$$\text{RHL}(\text{at } x=0)=\lim_{x\to 0^+} f(x)$$
$$=\lim_{h\to 0} f(0+h)$$
$$=\lim_{h\to 0} f(h)$$
$$=\lim_{h\to 0}[3]$$
$$=3$$
[2]

And, $f(0)=3$

Since the given function is continuous at $x = 0$, then

$$\text{LHL}=\text{RHL}=f(0)$$
$$k=3=3$$
$$\Rightarrow\quad k=3$$
[1]

Q. 16. Determine the value of 'k' for which the following function is continuous at $x = 3$

$$f(x)=\begin{cases}\dfrac{(x+3)^2-36}{x-3}, & x\ne 3\\ k, & x=3\end{cases}.$$

[CBSE Board, All India Region, 2017]

Ans. Given function is $f(x)=\begin{cases}\dfrac{(x+3)^2-36}{x-3}, & x\ne 3\\ k, & x=3\end{cases}$.

$$\text{LHL}(\text{at } x=3)=\lim_{x\to 3^-} f(x)$$
$$=\lim_{h\to 0} f(3-h)$$
$$=\lim_{h\to 0}\left[\frac{(3-h+3)^2-36}{3-h-3}\right]$$
$$=\lim_{h\to 0}\left[\frac{36-12h+h^2-36}{-h}\right]$$
$$=\lim_{h\to 0}[12-h]$$
$$=12$$
[2]

$$\text{RHL (at } x=3) = \lim_{x\to 3^+} f(x)$$
$$= \lim_{h\to 0} f(3+h)$$
$$= \lim_{h\to 0}\left[\frac{(3+h+3)^2-36}{3+h-3}\right]$$
$$= \lim_{h\to 0}\left[\frac{36+12h+h^2-36}{h}\right]$$
$$= \lim_{h\to 0}[12+h]$$
$$= 12 \quad \textbf{[2]}$$

And, $f(3)=k$

Since the given function is continuous at $x = 3$, then

$$\text{LHL} = \text{RHL} = f(3)$$
$$12 = 12 = k$$
$$\Rightarrow \quad k = 12 \quad \textbf{[1]}$$

Q. 17. For what value of 'k' is the function

$$f(x)=\begin{cases}\dfrac{\sin 5x}{3x}+\cos x, & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}$$ **continuous at $x = 0$?**

[CBSE Board, Foreign Scheme, 2017]

Ans. Given function is $f(x)=\begin{cases}\dfrac{\sin 5x}{3x}+\cos x, & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}$.

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0}\left[\frac{\sin 5(-h)}{3(-h)}+\cos(-h)\right]$$
$$= \lim_{h\to 0}\left[\frac{\sin 5h}{3h}+\cosh\right]$$
$$= \lim_{h\to 0}\left[\frac{\left(\frac{\sin 5h}{5h}\right)\times 5h}{3h}+\cosh\right]$$
$$= \frac{5}{3}+1$$
$$= \frac{8}{3} \quad \textbf{[2]}$$

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x)$$
$$= \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0}\left[\frac{\sin 5(h)}{3(h)}+\cos(h)\right]$$
$$= \lim_{h\to 0}\left[\frac{\sin 5h}{3h}+\cosh\right]$$
$$= \lim_{h\to 0}\left[\frac{\left(\frac{\sin 5h}{5h}\right)\times 5h}{3h}+\cosh\right]$$
$$= \frac{5}{3}+1$$
$$= \frac{8}{3} \quad \textbf{[2]}$$

And, $f(0) = k$

Since the given function is continuous at $x = 0$, then

$$\text{LHL}=\text{RHL}=f(3)$$
$$\frac{8}{3}=\frac{8}{3}=k$$
$$\Rightarrow k=\frac{8}{3} \quad \textbf{[1]}$$

Q. 18. Show that the function

$$f(x)=\begin{cases}\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, & \text{if } x\neq 0\\ -1, & \text{if } x=0\end{cases}$$

is discontinuous function at $x = 0$.

[CBSE Board, All India Region, 2016]

Ans. Given function is $f(x)=\begin{cases}\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, & \text{if } x\neq 0\\ -1, & \text{if } x=0\end{cases}$.

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0}\left[\frac{e^{\frac{1}{-h}}-1}{1+e^{\frac{1}{-h}}}\right]$$
$$= \left[\frac{e^{-\infty}-1}{1+e^{-\infty}}\right]$$
$$= \frac{0-1}{1+0}$$
$$= -1 \quad \textbf{[2]}$$

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x)$$
$$= \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0}\left[\frac{e^{\frac{1}{h}}-1}{1+e^{\frac{1}{h}}}\right]$$
$$= \lim_{h\to 0}\left[\frac{1-e^{-\frac{1}{h}}}{1+e^{-\frac{1}{h}}}\right]$$
$$= \frac{1-e^{-\infty}}{1+e^{-\infty}}$$
$$= \frac{1-0}{1+0}$$
$$= 1 \quad \textbf{[2]}$$

And, $f(0)=-1$

Since LHL $\neq$ RHL $\neq f(0)=-1$, then the given function is discontinuous at $x = 0$. **[1]**

Q. 19. If $f(x)=\begin{cases}\dfrac{\sin(a+1)x+2\sin x}{x}, & x<0\\ 2, & x=0\\ \dfrac{\sqrt{1+bx}-1}{x}, & x>0\end{cases}$ **is continuous at $x = 0$ then find the values of a and b.**

[CBSE Board, All India Region, 2016]

Ans. Given function is

$$f(x)=\begin{cases}\dfrac{\sin(a+1)x+2\sin x}{x}, & x<0\\ 2, & x=0\\ \dfrac{\sqrt{1+bx}-1}{x}, & x>0\end{cases}.$$

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0}\left[\frac{\sin(a+1)(-h)+2\sin(-h)}{-h}\right]$$
$$= \lim_{h\to 0}\left[\frac{-\sin(a+1)h-2\sinh}{-h}\right]$$
$$= \lim_{h\to 0}\left[(a+1)\left\{\frac{\sin(a+1)h}{(a+1)h}\right\}+2\left(\frac{\sinh}{h}\right)\right]$$
$$= a+3$$

[2]

$$\text{RHL (at } x=0) = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0}\left[\frac{\sqrt{1+bh}-1}{h}\times\frac{\sqrt{1+bh}+1}{\sqrt{1+bh}+1}\right]$$
$$= \lim_{h\to 0}\left[\frac{1+bh-1}{h\left(\sqrt{1+bh}+1\right)}\right]$$
$$= \lim_{h\to 0}\left[\frac{b}{\left(\sqrt{1+bh}+1\right)}\right]$$
$$= \frac{b}{2}$$

[2]

And, $f(0)=2$

Since the given function is continuous at $x = 0$, then

$$\text{LHL} = \text{RHL} = f(0)$$
$$a+3=\frac{b}{2}=2$$
$$\Rightarrow \quad a=-1 \text{ and } b=4$$

[1]

Q. 20. Find k, if $f(x)=\begin{cases}k\sin\frac{\pi}{2}(x+1), & x\le 0\\ \dfrac{\tan x-\sin x}{x^3}, & x>0\end{cases}$ is continuous function at $x = 0$.

[CBSE Board, All India Region, 2016]

Ans. Given function is $f(x)=\begin{cases}k\sin\frac{\pi}{2}(x+1), & x\le 0\\ \dfrac{\tan x-\sin x}{x^3}, & x>0\end{cases}.$

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0}\left[k\sin\left\{\frac{\pi}{2}(-h+1)\right\}\right]$$
$$= \lim_{h\to 0}\left[k\sin\left\{\frac{\pi}{2}-\frac{\pi h}{2}\right\}\right]$$
$$= \lim_{h\to 0}\left[k\cos\left\{\frac{\pi h}{2}\right\}\right]$$
$$= k$$

[2]

$$\text{RHL (at } x=0) = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0}\left[\frac{\tan x-\sin x}{x^3}\right]$$
$$= \lim_{h\to 0}\left[\frac{\tan x(1-\cos x)}{x^3}\right]$$
$$= \lim_{h\to 0}\left[2\left(\frac{\tan x}{x}\right)\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\times\frac{1}{4}\right]$$
$$= \left[2(1)(1)^2\times\frac{1}{4}\right]$$
$$= \frac{1}{2}$$

[2]

And, $f(0)=k$

Since the given function is continuous at $x = 0$, then

$$\text{LHL} = \text{RHL} = f(0)$$
$$k=\frac{1}{2}=k$$
$$\Rightarrow \quad k=\frac{1}{2}$$

[1]

TOPIC-2
Derivatives

Quick Review

- Algebra of derivatives : If u and v are the function of x, then

$$\frac{d}{dx}(u \pm v) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v)$$

$$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) \pm v\frac{d}{dx}(u)$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

- Derivatives of composite function : Let f be a real valued function which is a composite of two functions u and v; i.e., $f = v \text{ o } u$. Suppose $t = u(x)$ and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist, then $\frac{df}{dx} = \frac{dv}{dt}\frac{dt}{dx}$.

TRICKS...

- *The differentiation of a function f(x) = x^n with respect to x is* $\frac{d}{dx}(x^n) = nx^{n-1}$.
- *The differentiation of a constant function f(x) = c with respect to x is* $\frac{d}{dx}(c) = 0$.

- Derivatives of implicit function : When a relationship between x and y is expressed in a way that it is easy to solve for y and write $y = f(x)$, we say that y is given as an explicit function of x. When a relationship between x and y is expressed in a way that it is not necessary that functions are always expressed in this form. It does not seem that there is an easy way to solve for y**.** There is no doubt about the dependence of y on x in either of the cases. In that case, differentiate the given function of x and y with respect to x and find the value of $\frac{dy}{dx}$. Hence, we get the derivative of implicit function.
- Derivatives of trigonometric function : Following are some of the standard derivatives (in appropriate domains) :

$\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$, $\frac{d}{dx}(\tan x) = \sec^2 x$,

$\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$, $\frac{d}{dx}(\sec x) = \sec x \tan x$, $\frac{d}{dx}(\text{cosec} x) = -\text{cosec} x \cot x$

- Derivatives of inverse trigonometric function : Following are some of the standard derivatives (in appropriate domains) :

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$,

$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$, $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$, $\frac{d}{dx}(\text{cos}ec^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$

- Exponential function : The exponential function with positive base $b > 1$ is the function $y = f(x) = b^x$. Its domain is R, the set of all real numbers and range is the set of all positive real numbers. Exponential function with base 10 is called the common exponential function and with base e is called the natural exponential function.
- Derivatives of exponential function : The derivative of $y = f(x) = e^x$ with respect to x is $\frac{dy}{dx} = e^x$.
- Logarithmic function : Let $b > 1$ be a real number. Then we say logarithm of a to base b is x if $b^x = a$, Logarithm of a to the base b is denoted by $\log_b a$. If the base $b = 10$, we say it is common logarithm and if $b = e$, then we say it is natural logarithms. $f(x) = \log x$ denotes the logarithm function to base e. The domain of logarithm function is R^+, the set of all positive real numbers and the range is the set of all real numbers.
- Logarithmic rules : The properties of logarithmic function to any base $b > 1$ are as below.

$\log_b(xy) = \log_b(x) + \log_b(y)$, $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$, $\log_b(x^n) = n\log_b(x)$,

$\log_b(x) = \dfrac{\log_c(x)}{\log_c(b)}$, $\log_b(x) = \dfrac{1}{\log_x(b)}$, $\log_b(b) = 1$ and $\log_b(1) = 0$,

❖ Derivatives of logarithmic function : The derivative of $y = f(x) = \log x$ with respect to x is $\dfrac{dy}{dx} = \dfrac{1}{x}$.

❖ Derivative of function in parametric forms : If a relation expressed between two variables x and y in the form $x = f(t)$, $y = g(t)$ is said to be parametric form with t as a parameter. In order to find derivative of function in such form, we use $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \left(\dfrac{dy}{dt}\right) \times \left(\dfrac{dt}{dx}\right)$.

❖ Second order of derivative : If a function $y = f(x)$is differentiate with respect to x, then $\dfrac{dy}{dx} = f'(x)$ and if it is again differentiate with respect to x, then $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}[f'(x)] \Rightarrow \dfrac{d^2y}{dx^2} = f''(x)$. It is known as second order of derivative of $f(x)$.

Know the Links

☞ *http ://www.intuitive-calculus.com/solving-derivatives.html*
☞ *http ://www.sosmath.com/calculus/diff/der02/der02.html*
☞ *https ://www.simplylearnt.com/tips-tricks/Differentiation-of-Functions*

Multiple Choice Questions **(1 mark each)**

Q. 1. If $y = \log\left(\dfrac{1-x^2}{1+x^2}\right)$, then $\dfrac{dy}{dx}$ is equal to

(a) $\dfrac{4x^3}{1-x^4}$ (b) $\dfrac{-4x}{1-x^4}$

(c) $\dfrac{1}{4-x^4}$ (d) $\dfrac{-4x^3}{1-x^4}$

[NCERT Exemp. Ex. 5.3, Q. 91, Page 114-115]

Ans. Correct option : (b)

Explanation : Given that,

$$y = \log\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow y = \log(1-x^2) - \log(1+x^2).$$

Differentiate with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(1-x^2)\right] - \frac{d}{dx}\left[\log(1+x^2)\right]$$

$$= \frac{-2x}{1-x^2} - \frac{2x}{1+x^2}$$

$$= -2x\left(\frac{2}{(1-x^2)(1+x^2)}\right)$$

$$= -\frac{4x}{1-x^4}$$

Q. 2. If $y = \sqrt{\sin x + y}$, then $\dfrac{dy}{dx}$is equal to

(a) $\dfrac{\cos x}{2y-1}$ (b) $\dfrac{\cos x}{1-2y}$

(c) $\dfrac{\sin x}{1-2y}$ (d) $\dfrac{\sin x}{2y-1}$

[NCERT Exemp. Ex. 5.3, Q. 92, Page 115]

Ans. Correct option : (a)

Explanation : Given that,

$$y = \sqrt{\sin x + y}$$

$$\Rightarrow y^2 = \sin x + y.$$

Differentiate with respect to x, we have

$$\Rightarrow \quad 2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$\Rightarrow (2y-1)\frac{dy}{dx} = \cos x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos x}{2y-1}$$

Q. 3. The derivative of $\cos^{-1}(2x^2-1)$ w.r.t. $\cos^{-1}x$ is

(a) 2 (b) $\dfrac{-1}{2\sqrt{1-x^2}}$

(c) $\dfrac{2}{x}$ (d) $1-x^2$

[NCERT Exemp. Ex. 5.3, Q. 93, Page 115]

Ans. Correct option : (a)

Explanation : Let

$$u = \cos^{-1}(2x^2-1)$$

$$\Rightarrow \frac{du}{dx} = -\frac{4x}{\sqrt{1-(2x^2-1)^2}}$$

$$\Rightarrow \frac{du}{dx} = -\frac{4x}{\sqrt{1-4x^4+4x^2-1}}$$

$$\Rightarrow \frac{du}{dx} = -\frac{4x}{\sqrt{-4x^4+4x^2}}$$

$$\Rightarrow \frac{du}{dx} = -\frac{2}{\sqrt{1-x^2}}$$

And, $v = \cos^{-1} x$

$\Rightarrow \quad \frac{dv}{dx} = -\frac{1}{\sqrt{1-x^2}}$

Thus, $\frac{du}{dv}$

Q. 4. If $x = t^2$ and $y = t^3$ then $\frac{d^2y}{dx^2}$ is

(a) $\frac{3}{2}$ (b) $\frac{3}{4t}$

(c) $\frac{3}{2t}$ (d) $\frac{3}{4}$

[NCERT Exemp. Ex. 5.3, Q. 94, Page 115]

Ans. Correct option : (b)

Explanation : Given that,

$x = t^2$ and $y = t^3$

Then, $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2$

Thus,

$\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{3}{2}$

Q. 5. Fill in the blanks :

Derivative of x^2 with respect to x^3 is __________

[NCERT Exemp. Ex. 5.3, Q. 98, Page 116]

Ans. $\frac{2}{3x}$

Since $u = x^2 \Rightarrow \frac{du}{dx} = 2x$ and $v = x^3 \Rightarrow \frac{dv}{dx} = 3x^2$, then

$\frac{du}{dv} = \frac{2x}{3x^2} = \frac{2}{3x}.$

Q. 6. Fill in the blanks :

If $f(x) = |\cos x|$ then $f'\left(\frac{\pi}{4}\right) =$ ________

[NCERT Exemp. Ex. 5.3, Q. 99, Page 116]

Ans. $-\frac{1}{\sqrt{2}}$

$f(x) = |\cos x| = \cos x$ for $0 < x < \frac{\pi}{2}$

Thus,

$f'(x) = \frac{d}{dx}(\cos x) = -\sin x$

$\Rightarrow f'\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

Q. 7. Fill in the blanks :

If $f(x) = |\cos x - \sin x|$ then $f'\left(\frac{\pi}{3}\right) =$ _________

[NCERT Exemp. Ex. 5.3, Q. 100, Page 116]

Ans. $\frac{\sqrt{3}+1}{2}$

$f(x) = |\cos x - \sin x| = -(\cos x - \sin x)$ for $\frac{\pi}{4} < x < \frac{\pi}{2}$

Thus,

$f'(x) = \cos x + \sin x$

$f'\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} + \sin\frac{\pi}{3}$

$= \frac{\sqrt{3}+1}{2}$

Q. 8. Fill in the blanks :

For the curve $\sqrt{x} + \sqrt{y} = 1$, $\frac{dy}{dx}$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$ is _______.

[NCERT Exemp. Ex. 5.3, Q. 101, Page 116]

Ans. –1

Given that,

$\sqrt{x} + \sqrt{y} = 1$

Differentiable with respect to x, we have

$\sqrt{x} + \sqrt{y} = 1$

$\Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = -\sqrt{\frac{y}{x}}$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{\left(\frac{1}{4}, \frac{1}{4}\right)} = -\sqrt{\frac{\frac{1}{4}}{\frac{1}{4}}} = -1$

Very Short Answer Type Questions (1 or 2 marks each)

Q. 1. Differentiate the functions $\sin(x^2+5)$ with respect to x. **[NCERT Ex. 5.2, Q. 1, Page 166]**

Ans. Given that,

$y = \sin(x^2+5)$

Differentiating with respect to x, we have

$\frac{dy}{dx} = \frac{d}{dx}\left[\sin(x^2+5)\right]$

$= \cos(x^2+5)\frac{d}{dx}(x^2+5)$

$= \cos(x^2+5)(2x)$

$= 2x\cos(x^2+5)$

[2]

Q. 2. Differentiate the functions $\cos(\sin x)$ with respect to x. **[NCERT Ex. 5.2, Q. 2, Page 166]**

Ans. Given that,

$y = \cos(\sin x)$

Differentiating with respect to x, we have

$\frac{dy}{dx} = \frac{d}{dx}\left[\cos(\sin x)\right]$

$= -\sin(\sin x)\frac{d}{dx}(\sin x)$

$= -\sin(\sin x)(\cos x)$

$= -\cos x \sin(\sin x)$

[2]

Q. 3. Differentiate the functions $\sin(ax+b)$ with respect to x. **[NCERT Ex. 5.2, Q. 3, Page 166]**

Ans. Given that,

$y=\sin(ax+b)$

Differentiating with respect to x, we have

$$\frac{dy}{dx}=\frac{d}{dx}[\sin(ax+b)]$$
$$=\cos(ax+b)\frac{d}{dx}(ax+b)$$
$$=\cos(ax+b)(a)$$
$$=a\cos(ax+b)$$

[2]

Q. 4. Differentiate the functions $\sec(\tan(\sqrt{x}))$ with respect to x. **[NCERT Ex. 5.2, Q. 4, Page 166]**

Ans. Given that,

$y=\sec(\tan(\sqrt{x}))$

Differentiating with respect to x, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sec(\tan(\sqrt{x}))\right]$$
$$=\sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\frac{d}{dx}(\tan(\sqrt{x}))$$
$$=\sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\sec^2(\sqrt{x})\frac{d}{dx}(\sqrt{x})$$
$$=\sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\sec^2(\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)$$
$$=\frac{1}{2\sqrt{x}}\sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\sec^2(\sqrt{x})$$

[2]

Q. 5. Differentiate the functions $\cos x^3\sin^2(x^5)$ with respect to x. **[NCERT Ex. 5.2, Q. 6, Page 166]**

Ans. Given that,

$y=\cos x^3\sin^2(x^5)$

Differentiating with respect to x, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\cos x^3\sin^2(x^5)\right]$$
$$=\cos x^3\frac{d}{dx}\left[\sin^2(x^5)\right]+\sin^2(x^5)\frac{d}{dx}\left[\cos x^3\right]$$
$$=\cos x^3\left[2\sin(x^5)\cos(x^5)\right]\frac{d}{dx}\left[x^5\right]+\sin^2(x^5)\left[-\sin x^3\right]\frac{d}{dx}\left[x^3\right]$$
$$=\cos x^3\left[2\sin(x^5)\cos(x^5)\right]\left[5x^4\right]+\sin^2(x^5)\left[-\sin x^3\right]\left[3x^2\right]$$
$$=10x^4\cos x^3\sin(x^5)\cos(x^5)-3x^2\sin x^3\sin^2(x^5)$$

[2]

Q. 6. Differentiate the functions $\cos(\sqrt{x})$ with respect to x. **[NCERT Ex. 5.2, Q. 8, Page 166]**

Ans. Given that,

$y=\cos(\sqrt{x})$

Differentiating with respect to x, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\cos(\sqrt{x})\right]$$
$$=-\sin(\sqrt{x})\frac{d}{dx}\left[\sqrt{x}\right]$$
$$=-\sin(\sqrt{x})\left[\frac{1}{2\sqrt{x}}\right]$$
$$=-\frac{\sin(\sqrt{x})}{2\sqrt{x}}$$

[2]

Q. 7. Find $\frac{dy}{dx}$ of $2x+3y=\sin x$. **[NCERT Ex. 5.3, Q. 1, Page 169]**

Ans. Given that,

$2x+3y=\sin x$

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}[2x+3y]=\frac{d}{dx}[\sin x]$$
$$\Rightarrow\quad 2+3\frac{dy}{dx}=\cos x$$
$$\Rightarrow\quad \frac{dy}{dx}=\frac{\cos x-2}{3}$$

[2]

Q. 8. Find $\frac{dy}{dx}$ of $2x+3y=\sin y$. **[NCERT Ex. 5.3, Q. 2, Page 169]**

Ans. Given that,

$2x+3y=\sin y$

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}[2x+3y]=\frac{d}{dx}[\sin y]$$
$$\Rightarrow\quad 2+3\frac{dy}{dx}=\cos y\frac{dy}{dx}$$
$$\Rightarrow(\cos y-3)\frac{dy}{dx}=2$$
$$\Rightarrow\quad \frac{dy}{dx}=\frac{2}{\cos y-3}$$

[2]

Q. 9. Find $\frac{dy}{dx}$ of $ax+by^2=\cos y$. **[NCERT Ex. 5.3, Q. 3, Page 169]**

Ans. Given that,

$ax+by^2=\cos y$

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}\left[ax+by^2\right]=\frac{d}{dx}[\cos y]$$
$$\Rightarrow\quad a+2by\frac{dy}{dx}=-\sin y\frac{dy}{dx}$$
$$\Rightarrow(2by+\sin y)\frac{dy}{dx}=-a$$
$$\Rightarrow\quad \frac{dy}{dx}=\frac{-a}{2by+\sin y}$$

[2]

Q. 10. Find $\frac{dy}{dx}$ of $xy+y^2=\tan x+y$. **[NCERT Ex. 5.3, Q. 4, Page 169]**

Ans. Given that,

$xy+y^2=\tan x+y$

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}\left[xy+y^2\right]=\frac{d}{dx}\left[\tan x+y\right.$$

$$\Rightarrow x\frac{dy}{dx}+y+2y\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(x+2y-1)=\sec^2 x-y$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sec^2 x-y}{x+2y-1}$$

[2]

Q. 11. Find $\frac{dy}{dx}$ of $x^2+xy+y^2=100$.

[NCERT Ex. 5.3, Q. 5, Page 169]

Ans. Given that,

$x^2+xy+y^2=100$

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}\left[x^2+xy+y^2\right]=\frac{d}{dx}[100]$$

$$\Rightarrow 2x+x\frac{dy}{dx}+y+2y\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}(x+2y)=-(2x+y)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{2x+y}{x+2y}$$

[2]

Q. 12. Find $\frac{dy}{dx}$ of $x^3+x^2y+xy^2+y^3=81$.

[NCERT Ex. 5.3, Q. 6, Page 169]

Ans. Given that,

$x^3+x^2y+xy^2+y^3=81$

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}\left[x^3+x^2y+xy^2+y^3\right]=\frac{d}{dx}[81]$$

$$\Rightarrow 3x^2+2xy+x^2\frac{dy}{dx}+y^2+2xy\frac{dy}{dx}+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}\left(x^2+2xy+3y^2\right)=-\left(3x^2+2xy+y^2\right)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{3x^2+2xy+y^2}{x^2+2xy+3y^2}$$

[2]

Q. 13. Find $\frac{dy}{dx}$ of $\sin^2 y+\cos xy=\pi$.

[NCERT Ex. 5.3, Q. 7, Page 169]

Ans. Given that,

$\sin^2 y+\cos xy=\pi$

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}\left[\sin^2 y+\cos xy\right]=\frac{d}{dx}[\pi]$$

$$\Rightarrow 2\sin y\cos y\frac{dy}{dx}-\sin xy\left(x\frac{dy}{dx}+y\right)=0$$

$$\Rightarrow 2\sin y\cos y\frac{dy}{dx}-x\sin xy\frac{dy}{dx}-y\sin xy=0$$

$$\Rightarrow \frac{dy}{dx}(2\sin y\cos y-x\sin xy)=y\sin xy$$

$$\Rightarrow \frac{dy}{dx}=\frac{y\sin xy}{2\sin y\cos y-x\sin xy}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y\sin xy}{\sin 2y-x\sin xy}$$

[2]

Q. 14. Find $\frac{dy}{dx}$ of $\sin^2 x+\cos^2 y=1$.

[NCERT Ex. 5.3, Q. 8, Page 169]

Ans. Given that,

$\sin^2 x+\cos^2 y=1$

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}\left[\sin^2 x+\cos^2 y\right]=\frac{d}{dx}[1]$$

$$2\sin x\cos x+2\cos y(-\sin y)\frac{dy}{dx}=0$$

$$2\sin y\cos y\frac{dy}{dx}=2\sin x\cos x$$

$$\frac{dy}{dx}=\frac{2\sin x\cos x}{2\sin y\cos y}=\frac{\sin 2x}{\sin 2y}$$

[2]

Q. 15. Find $\frac{dy}{dx}$ of $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

[NCERT Ex. 5.3, Q. 9, Page 169]

Ans. Given that,

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Put $x=\tan\theta$, we have

$$y=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$\Rightarrow y=\sin^{-1}(\sin 2\theta)$$

$$\Rightarrow y=2\theta$$

[1]

Since $x=\tan\theta\Rightarrow\theta=\tan^{-1}x$, then

$\Rightarrow y=2\tan^{-1}x$

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left(2\tan^{-1}x\right)=\frac{2}{1+x^2}$$

[1]

Q. 16. Find $\frac{dy}{dx}$ of $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$, $-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$.

[NCERT Ex. 5.3, Q. 10, Page 169]

Ans. Given that,

$$y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),\ -\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$$

Put $x=\tan\theta$, we have

$$y=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$$\Rightarrow y=\tan^{-1}(\tan 3\theta)$$

$$\Rightarrow y=3\theta$$

[1]

Since $x=\tan\theta\Rightarrow\theta=\tan^{-1}x$, then

$\Rightarrow y=3\tan^{-1}x$

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(3\tan^{-1}x\right)$$
$$= \frac{3}{1+x^2}$$ **[1]**

Q. 17. Find $\frac{dy}{dx}$ of $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $0 < x < 1$.

[NCERT Ex. 5.3, Q. 11, Page 169]

Ans. Given that,

$$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ 0<x<1$$

Put $x = \tan\theta$, we have

$$y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$
$$= \cos^{-1}(\cos 2\theta)$$
$$= 2\theta$$ **[1]**

Since $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$, then

$$\Rightarrow y = 2\tan^{-1}x$$

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(2\tan^{-1}x\right)$$
$$= \frac{2}{1+x^2}$$ **[1]**

Q. 18. Differentiate $\frac{e^x}{\sin x}$ with respect to x. **[NCERT Ex. 5.4, Q. 1, Page 174]**

Ans. Given that,

$$y = \frac{e^x}{\sin x}$$

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x}{\sin x}\right)$$
$$= \frac{\sin x\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(\sin x)}{\sin^2 x}$$
$$= \frac{\sin x(e^x) - e^x\cos x}{\sin^2 x}$$
$$= \frac{e^x(\sin x - \cos x)}{\sin^2 x},\ x \ne n\pi$$ **[2]**

Q. 19. Differentiate $e^{\sin^{-1}x}$ with respect to x.

[NCERT Ex. 5.4, Q. 2, Page 174]

Ans. Given that $y = e^{\sin^{-1}x}$

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin^{-1}x}\right)$$
$$= e^{\sin^{-1}x}\frac{d}{dx}\left(\sin^{-1}x\right)$$
$$= e^{\sin^{-1}x}\left(\frac{1}{\sqrt{1-x^2}}\right)$$
$$= \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}},\ -1<x<1$$ **[2]**

Q. 20. Differentiate e^{x^3} with respect to x.

[NCERT Ex. 5.4, Q. 3, Page 174]

Ans. Given that $y = e^{x^3}$

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x^3}\right)$$
$$= e^{x^3}\frac{d}{dx}\left(x^3\right)$$
$$= e^{x^3}\left(3x^2\right)$$
$$= 3x^2e^{x^3}$$ **[2]**

Q. 21. Differentiate $\sin\left(\tan^{-1}e^{-x}\right)$ with respect to x.

[NCERT Ex. 5.4, Q. 4, Page 174]

Ans. Given that $y = \sin\left(\tan^{-1}e^{-x}\right)$

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin\left(\tan^{-1}e^{-x}\right)\right]$$
$$= \cos\left(\tan^{-1}e^{-x}\right)\frac{d}{dx}\left(\tan^{-1}e^{-x}\right)$$
$$= \cos\left(\tan^{-1}e^{-x}\right)\left(\frac{1}{1+\left(e^{-x}\right)^2}\right)\frac{d}{dx}\left(e^{-x}\right)$$
$$= \frac{\cos\left(\tan^{-1}e^{-x}\right)}{1+e^{-2x}}\left(-e^{-x}\right)$$
$$= -\frac{e^{-x}\cos\left(\tan^{-1}e^{-x}\right)}{1+e^{-2x}}$$ **[2]**

Q. 22. Differentiate $\log\left(\cos e^x\right)$ with respect to x.

[NCERT Ex. 5.4, Q. 5, Page 174]

Ans. Given that $y = \log\left(\cos e^x\right)$

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\cos e^x\right)\right]$$
$$= \frac{1}{\cos e^x}\frac{d}{dx}\left(\cos e^x\right)$$
$$= \frac{1}{\cos e^x}\times\left(-\sin e^x\right)\frac{d}{dx}\left(e^x\right)$$
$$= \frac{1}{\cos e^x}\times\left(-\sin e^x\right)\times\left(e^x\right)$$
$$= \frac{-e^x\sin e^x}{\cos e^x}$$
$$= -e^x\tan e^x$$ **[2]**

Q. 23. Differentiate $e^x + e^{x^2} + \ldots + e^{x^5}$ with respect to x.

[NCERT Ex. 5.4, Q. 6, Page 174]

Ans. Given that $y = e^x + e^{x^2} + \ldots + e^{x^5}$

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x + e^{x^2} + \ldots + e^{x^5}\right)$$
$$= \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(e^{x^2}\right) + \ldots + \frac{d}{dx}\left(e^{x^5}\right)$$
$$= e^x + e^{x^2}(2x) + \ldots + e^{x^5}\left(5x^4\right)$$
$$= e^x + 2xe^{x^2} + \ldots + 5x^4e^{x^5}$$ **[2]**

Q. 24. Differentiate $\sqrt{e^{\sqrt{x}}}$, $x > 0$ with respect to x.
[NCERT Ex. 5.4, Q. 7, Page 174]

Ans. Given that,

$y = \sqrt{e^{\sqrt{x}}}$, $x > 0$

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{e^{\sqrt{x}}}\right)$$
$$= \frac{1}{2\sqrt{e^{\sqrt{x}}}} \times \frac{d}{dx}\left(e^{\sqrt{x}}\right)$$
$$= \frac{1}{2\sqrt{e^{\sqrt{x}}}} \times e^{\sqrt{x}} \frac{d}{dx}\left(\sqrt{x}\right)$$
$$= \frac{1}{2\sqrt{e^{\sqrt{x}}}} \times e^{\sqrt{x}} \times \frac{1}{2\sqrt{x}}$$
$$= \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}, \; x > 0$$

[2]

Q. 25. Differentiate $\log(\log x)$, $x > 1$ with respect to x.
[NCERT Ex. 5.4, Q. 8, Page 174]

Ans. Given that $y = \log(\log x)$, $x > 1$

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(\log x)\right]$$
$$= \frac{1}{\log x} \times \frac{d}{dx}(\log x)$$
$$= \frac{1}{\log x} \times \frac{1}{x}$$
$$= \frac{1}{x \log x}, \; x > 1$$

[2]

Q. 26. Differentiate $\dfrac{\cos x}{\log x}$, $x > 0$ with respect to x.
[NCERT Ex. 5.4, Q. 9, Page 174]

Ans. Given that $y = \dfrac{\cos x}{\log x}$, $x > 0$

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\cos x}{\log x}\right)$$
$$= \frac{\log x \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(\log x)}{(\log x)^2}$$
$$= \frac{\log x(-\sin x) - \cos x\left(\frac{1}{x}\right)}{(\log x)^2}$$
$$= \frac{-x \sin x \log x - \cos x}{x(\log x)^2}$$
$$= -\frac{x \sin x \log x + \cos x}{x(\log x)^2}, \; x > 0$$

[2]

Q. 27. Differentiate $\cos(\log x + e^x)$, $x > 0$ with respect to x.
[NCERT Ex. 5.4, Q. 10, Page 174]

Ans. Given that $y = \cos(\log x + e^x)$, $x > 0$

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cos(\log x + e^x)\right]$$
$$= -\sin(\log x + e^x)\frac{d}{dx}(\log x + e^x)$$
$$= -\sin(\log x + e^x)\left(\frac{1}{x} + e^x\right)$$
$$= -\left(\frac{1}{x} + e^x\right)\sin(\log x + e^x), \; x > 0$$

[2]

Q. 28. Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ in three ways by using product rule.
[NCERT Ex. 5.5, Q. 17(i), Page 178]

Ans. $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = (x^2 - 5x + 8)\frac{d}{dx}(x^3 + 7x + 9) + (x^3 + 7x + 9)\frac{d}{dx}(x^2 - 5x + 8)$$
$$= (x^2 - 5x + 8)(3x^2 + 7) + (x^3 + 7x + 9)(2x - 5)$$
$$= 3x^4 + 7x^2 - 15x^3 - 35x + 24x^2 + 56 + 2x^4 - 5x^3 + 14x^2 - 35x + 18x - 45$$
$$= 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

[2]

Q. 29. Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ in three ways by expanding the product to obtain a single polynomial. **[NCERT Ex. 5.5, Q. 17(ii), Page 178]**

Ans. $y = x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72$

$= x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$$
$$= 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

[2]

Q. 30. Find $\dfrac{dy}{dx}$ of $x = 2at^2$, $y = at^4$.
[NCERT Ex. 5.6, Q. 1, Page 181]

Ans. Given that $x = 2at^2$, $y = at^4$

Then,

$$\frac{dx}{dt} = \frac{d}{dt}(2at^2)$$
$$= 4at$$

And,

$$\frac{dy}{dt} = \frac{d}{dt}(at^4)$$
$$= 4at^3$$

[1]

Thus,

$$\frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$
$$= \frac{4at^3}{4at}$$
$$= t^2$$

[1]

Q. 31. Find $\dfrac{dy}{dx}$ of $x = a\cos\theta$, $y = b\cos\theta$.
[NCERT Ex. 5.6, Q. 2, Page 181]

Ans. Given that $x = a\cos\theta$, $y = b\cos\theta$

Then,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos\theta)$$
$$= -a\sin\theta$$

[1]

And,

$$\frac{dy}{d\theta}=\frac{d}{d\theta}(b\cos\theta)$$
$$=-b\sin\theta$$

Thus,

$$\frac{dy}{dx}=\left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)$$
$$=\frac{-b\sin\theta}{-a\sin\theta}$$
$$=\frac{b}{a}$$ **[1]**

Q. 32. Find $\frac{dy}{dx}$ of $x=\sin t,\ y=\cos 2t$.
[NCERT Ex. 5.6, Q. 3, Page 181]

Ans. Given that $x=\sin t,\ y=\cos 2t$

Then,

$$\frac{dx}{dt}=\frac{d}{dt}(\sin t)$$
$$=\cos t$$ **[1]**

And,

$$\frac{dy}{dt}=\frac{d}{dt}(\cos 2t)$$
$$=-2\sin 2t$$

Thus,

$$\frac{dy}{dx}=\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$
$$=\frac{-2\sin 2t}{\cos t}$$
$$=-\frac{2\times 2\sin t\cos t}{\cos t}$$
$$=-4\sin t$$ **[1]**

Q. 33. Find $\frac{dy}{dx}$ of $x=4t,\ y=\frac{4}{t}$.
[NCERT Ex. 5.6, Q. 4, Page 181]

Ans. Given that $x=4t,\ y=\frac{4}{t}$

Then,

$$\frac{dx}{dt}=\frac{d}{dt}(4t)$$
$$=4$$ **[1]**

And,

$$\frac{dy}{dt}=\frac{d}{dt}\left(\frac{4}{t}\right)$$
$$=-\frac{4}{t^2}$$

Thus,

$$\frac{dy}{dx}=\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$
$$=\frac{-\frac{4}{t^2}}{4}$$
$$=-\frac{1}{t^2}$$ **[1]**

Q. 34. Find $\frac{dy}{dx}$ of $x=\cos\theta-\cos 2\theta,\ y=\sin\theta-\sin 2\theta$.
[NCERT Ex. 5.6, Q. 5, Page 181]

Ans. Given that $x=\cos\theta-\cos 2\theta,\ y=\sin\theta-\sin 2\theta$

Then,

$$\frac{dx}{d\theta}=\frac{d}{d\theta}(\cos\theta-\cos 2\theta)$$
$$=(-\sin\theta)-(-2\sin 2\theta)$$
$$=2\sin 2\theta-\sin\theta$$ **[1]**

And,

$$\frac{dy}{d\theta}=\frac{d}{d\theta}(\sin\theta-\sin 2\theta)$$
$$=\cos\theta-2\cos 2\theta$$

Thus,

$$\frac{dy}{dx}=\left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)$$
$$=\frac{\cos\theta-2\cos 2\theta}{2\sin 2\theta-\sin\theta}$$ **[1]**

Q. 35. Find $\frac{dy}{dx}$ of $x=a(\theta-\sin\theta),\ y=a(1+\cos\theta)$.
[NCERT Ex. 5.6, Q. 6, Page 181]

Ans. Given that $x=a(\theta-\sin\theta),\ y=a(1+\cos\theta)$

Then,

$$\frac{dx}{d\theta}=\frac{d}{d\theta}[a(\theta-\sin\theta)]$$
$$=a(1-\cos\theta)$$ **[1]**

And,

$$\frac{dy}{d\theta}=\frac{d}{d\theta}[a(1+\cos\theta)]$$
$$=a(0-\sin\theta)$$
$$=-a\sin\theta$$

Thus,

$$\frac{dy}{dx}=\left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)$$
$$=\frac{-a\sin\theta}{a(1-\cos\theta)}$$
$$=-\frac{\sin\theta}{1-\cos\theta}$$
$$=-\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$
$$=-\cot\frac{\theta}{2}$$ **[1]**

Q. 36. Find $\frac{dy}{dx}$ of $x=a\sec\theta,\ y=b\tan\theta$.
[NCERT Ex. 5.6, Q. 9, Page 181]

Ans. Given that $x=a\sec\theta,\ y=b\tan\theta$

Then,

$$\frac{dx}{d\theta}=\frac{d}{d\theta}(a\sec\theta)$$
$$=a\sec\theta\tan\theta$$ **[1]**

And,

$$\frac{dy}{d\theta}=\frac{d}{d\theta}(b\tan\theta)$$
$$=b\sec^2\theta$$

Thus,

$$\frac{dy}{dx}=\left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)$$
$$=\frac{b\sec^2\theta}{a\sec\theta\tan\theta}$$
$$=\frac{b}{a}\text{cosec}\theta$$

[1]

Q. 37. Find $\frac{dy}{dx}$ of $x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\theta\cos\theta)$.

[NCERT Ex. 5.6, Q. 10, Page 181]

Ans. Given that

$x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\theta\cos\theta)$

Then,

$$\frac{dx}{d\theta}=\frac{d}{d\theta}[a(\cos\theta+\theta\sin\theta)]$$
$$=a(-\sin\theta+\theta\cos\theta+\sin\theta)$$
$$=a\theta\cos\theta$$

[1]

And,

$$\frac{dy}{d\theta}=\frac{d}{d\theta}[\ (\sin\theta-\theta\cos\theta)]$$
$$=\ (\cos\theta+\theta\sin\theta-\cos\theta)$$
$$\theta\sin\theta$$

Thus,

$$\frac{dy}{dx}=\left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)$$
$$=\frac{a\theta\sin\theta}{a\theta\cos\theta}$$
$$=\tan\theta$$

[1]

Q. 38. Find the second-order derivatives of the function x^2+3x+2. **[NCERT Ex. 5.7, Q. 1, Page 183]**

Ans. Given that $y=x^2+3x+2$

Differentiating with respect to x, we have

$$\frac{dy}{dx}=\frac{d}{dx}(x^2+3x+2)$$
$$=2x+3$$

[1]

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(2x+3)$$
$$=2$$

[1]

Q. 39. Find the second-order derivatives of the function x^{20}. **[NCERT Ex. 5.7, Q. 2, Page 183]**

Ans. Given that $y=x^{20}$

Differentiating with respect to x, we have

$$\frac{dy}{dx}=\frac{d}{dx}(x^{20})$$
$$=20x^{19}$$

[1]

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(20x^{19})$$
$$=20\times19x^{18}$$
$$=380x^{18}$$

[1]

Q. 40. Find the second-order derivatives of the function $x\cos x$. **[NCERT Ex. 5.7, Q. 3, Page 183]**

Ans. Given that $y=x\cos x$

Differentiating with respect to x, we have

$$\frac{dy}{dx}=\frac{d}{dx}(x\cos x)$$
$$=x(-\sin x)+\cos x$$
$$=-x\sin x+\cos x$$

[1]

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(-x\sin x+\cos x)$$
$$=\frac{d}{dx}(-x\sin x)+\frac{d}{dx}(\cos x)$$
$$=-x\cos x-\sin x-\sin x$$
$$=-x\cos x-2\sin x$$
$$=-(x\cos x+2\sin x)$$

[1]

Q. 41. Find the second-order derivatives of the function $\log x$. **[NCERT Ex. 5.7, Q. 4, Page 183]**

Ans. Given that $y=\log x$

Differentiating with respect to x, we have

$$\frac{dy}{dx}=\frac{d}{dx}(\log x)$$
$$=\frac{1}{x}$$

[1]

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\right)$$
$$=-\frac{1}{x^2}$$

[1]

Q. 42. Find the second-order derivatives of the function $x^3\log x$. **[NCERT Ex. 5.7, Q. 5, Page 183]**

Ans. Given that $y=x^3\log x$

Differentiating with respect to x, we have

$$\frac{dy}{dx}=\frac{d}{dx}(x^3\log x)$$
$$=x^3\times\frac{1}{x}+3x^2\log x$$
$$=x^2+3x^2\log x$$

[1]

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(x^2+3x^2\log x)$$
$$=2x+6x\log x+3x$$
$$=6x\log x+5x$$

[1]

Q. 43. Differentiate w.r.t. x the function $(3x^2-9x+5)^9$.

[NCERT Misc Ex. Q. 1, Page 191]

Ans. Given that $y=(3x^2-9x+5)^9$

Differentiating with respect to x, we have

$$\frac{dy}{dx} = 9\left(3x^2 - 9x + 5\right)^8 (6x - 9)$$
$$= 27\left(3x^2 - 9x + 5\right)^8 (2x - 3)$$
[2]

Q. 44. Differentiate w.r.t. x the function $\sin^3 x + \cos^6 x$.
[NCERT Misc Ex. Q. 2, Page 191]

Ans. Given that $y = \sin^3 x + \cos^6 x$

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^3 x + \cos^6 x\right)$$
$$= \frac{d}{dx}\left(\sin^3 x\right) + \frac{d}{dx}\left(\cos^6 x\right)$$
$$= 3\sin^2 x \cos x + 6\cos^5 x(-\sin x)$$
$$= 3\sin^2 x \cos x - 6\sin x \cos^5 x$$
$$= 3\sin x \cos x\left(\sin x - 2\cos^4 x\right)$$
[2]

Q. 45. Differentiate w.r.t. x the function $(5x)^{3\cos 2x}$.
[NCERT Misc Ex. Q. 3, Page 191]

Ans. Given that $y = (5x)^{3\cos 2x}$

Taking log both sides, we have

$$\log y = \log\left[(5x)^{3\cos 2x}\right]$$
$$= 3\cos 2x \log(5x)$$

Differentiating with respect to x, we have

$$\frac{d}{dx}(\log y) = 3\frac{d}{dx}\left[\cos 2x \log(5x)\right]$$
$$\frac{1}{y}.\frac{dy}{dx} = 3\left[\cos 2x \frac{d}{dx}\{\log(5x)\} + \log(5x)\frac{d}{dx}(\cos 2x)\right]$$
$$\frac{dy}{dx} = 3y\left[\cos 2x \times \left(\frac{5}{5x}\right) + \log(5x)(-2\sin 2x)\right]$$
$$= 3(5x)^{3\cos 2x}\left[\frac{\cos 2x}{x} - 2\sin 2x \log(5x)\right]$$
[2]

Q. 46. Differentiate w.r.t. x the function $\sin^{-1}\left(x\sqrt{x}\right)$, $0 \le x \le 1$.
[NCERT Misc Ex. Q. 4, Page 191]

Ans. Given that $y = \sin^{-1}\left(x\sqrt{x}\right)$

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin^{-1}\left(x\sqrt{x}\right)\right]$$
$$= \frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^2}}\frac{d}{dx}\left(x\sqrt{x}\right)$$
$$= \frac{1}{\sqrt{1-x^3}}\frac{d}{dx}\left(x^{\frac{3}{2}}\right)$$
$$= \frac{1}{\sqrt{1-x^3}}\left(\frac{3}{2}x^{\frac{1}{2}}\right)$$
$$= \frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$
[2]

Q. 47. Differentiate w.r.t. x the function cos(a cos x + b sin x) for some constant a and b.
[NCERT Misc Ex. Q. 8, Page 191]

Ans. Given that $y = \cos(a\cos x + b\sin x)$

Differentiating with respect to x, we have

$$\frac{d}{dx}(y) = \frac{d}{dx}\left[\cos(a\cos x + b\sin x)\right]$$
$$\frac{dy}{dx} = -\sin(a\cos x + b\sin x)\frac{d}{dx}[a\cos x + b\sin x]$$
$$= -\sin(a\cos x + b\sin x)[-a\sin x + b\cos x]$$
$$= \sin(a\cos x + b\sin x)[a\sin x - b\cos x]$$
[2]

Q. 48. Find $\frac{dy}{dx}$, if $y = 12(1 - \cos t)$ and $x = 10(t - \sin t)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$. **[NCERT Misc Ex. Q. 12, Page 191]**

Ans. Given that $y = 12(1 - \cos t)$ and $x = 10(1 - \sin t)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

Then,

$$\frac{dx}{dt} = \frac{d}{dt}\left[10(t - \sin t)\right]$$
$$= 10(1 - \cos t)$$
[1]

And,

$$\frac{dy}{dt} = \frac{d}{dt}\left[12(1 - \cos t)\right]$$
$$= 12\sin t$$

Thus,

$$\frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$
$$= \frac{12\sin t}{10(1 - \cos t)}$$
$$= \frac{6 \times 2\sin\frac{t}{2}\cos\frac{t}{2}}{5 \times 2\sin^2\frac{t}{2}}$$
$$= \frac{6}{5}\cot\frac{t}{2}$$
[1]

Q. 49. Find $\frac{dy}{dx}$, if $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$, $-1 \le x \le 1$.
[NCERT Misc Ex. Q. 13, Page 191]

Ans. Given that $y = \sin^{-1} x + \sin^{-1}\sqrt{1-x^2}$

Put $x = \cos\theta$, we have

$$y = \sin^{-1}(\cos\theta) + \sin^{-1}\sqrt{1-\cos^2\theta}$$
$$= \sin^{-1}\left[\sin\left(\frac{\pi}{2} - \theta\right)\right] + \sin^{-1}(\sin\theta)$$
$$= \frac{\pi}{2} - \theta + \theta$$
$$= \frac{\pi}{2}$$
[2]

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = 0$$

Q. 50. If $x = a$ (cos t + t sin t) and $y = a$ (sin t – t cos t), find $\frac{d^2y}{dx^2}$. **[NCERT Misc Ex. Q. 17, Page 192]**

Ans. Given that $y = a(\sin t - t\cos t)$ and $x = a(\cos t + t\sin t)$

Then,

$$\frac{dx}{dt} = \frac{d}{dt}\left[a(\cos t + t\sin t)\right]$$
$$= a(-\sin t + t\cos t + \sin t)$$
$$= at\cos t$$

And,

$$\frac{dy}{dt} = \frac{d}{dt}\left[a(\sin t - t\cos t)\right]$$
$$= a(\cos t + t\sin t - \cos t)$$
$$= at\sin t \quad \textbf{[1]}$$

Thus,

$$\frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$
$$= \frac{at\sin t}{at\cos t}$$
$$= \tan t$$
$$\frac{d^2y}{dx^2} = \frac{d}{dt}(\tan t) \times \frac{dt}{dx}$$
$$= \sec^2 t \times \frac{1}{at\cos t}$$
$$= \frac{\sec^3 t}{at} \quad \textbf{[1]}$$

Q. 51. If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it. [NCERT Misc Ex. Q. 18, Page 192]

Ans. By the definition of modulus function, we have

$$f(x) = |x|^3 = \begin{cases} x^3, & x \geq 0 \\ -x^3, & x < 0 \end{cases} \quad \textbf{[1]}$$

If $x \geq 0$,

$$f(x) = x^3$$
$$f'(x) = 3x^2$$
$$f''(x) = 6x$$

If $x < 0$,

$$f(x) = -x^3$$
$$f'(x) = -3x^2$$
$$f''(x) = -6x$$

Thus, $f''(x)$ exists for all real x. **[1]**

Q. 52. Differentiate the function $2^{\cos^2 x}$ w.r.t. x. [NCERT Exemp. Ex. 5.3, Q. 25, Page 109]

Ans. Given that $y = 2^{\cos^2 x}$

Taking log both sides, we have

$$\log y = \cos^2 x \log 2$$
$$= \log 2 \cos^2 x \quad \textbf{[1]}$$

Differentiating both sides, we have

$$\frac{d}{dx}(\log y) = \log 2 \frac{d}{dx}(\cos^2 x)$$
$$\frac{1}{y}\frac{dy}{dx} = \log 2(-2\cos x \sin x)$$
$$\frac{dy}{dx} = -2^{\cos^2 x}(\sin 2x)\log 2 \quad \textbf{[1]}$$

Q. 53. Differentiate the function $\frac{8^x}{x^8}$ w.r.t. x. [NCERT Exemp. Ex. 5.3, Q. 26, Page 109]

Ans. Given that $y = \frac{8^x}{x^8}$

Taking log both sides, we have

$$\log y = \log 8^x - \log x^8$$
$$= x\log 8 - 8\log x \quad \textbf{[1]}$$

Differentiating both sides, we have

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(x\log 8 - 8\log x)$$
$$\frac{1}{y}\frac{dy}{dx} = \log 8 - \frac{8}{x}$$
$$\frac{dy}{dx} = y\left(\log 8 - \frac{8}{x}\right)$$
$$= \frac{8^x}{x^8}\left(\log 8 - \frac{8}{x}\right) \quad \textbf{[1]}$$

Q. 54. Differentiate the function $\log\left(x + \sqrt{x^2 + a}\right)$ w.r.t. x. [NCERT Exemp. Ex. 5.3, Q. 27, Page 109]

Ans. Given that $y = \log\left(x + \sqrt{x^2 + a}\right)$

Differentiating both sides, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(x + \sqrt{x^2 + a}\right)\right]$$
$$= \frac{1}{\left(x + \sqrt{x^2 + a}\right)} \times \left(1 + \frac{2x}{2\sqrt{x^2 + a}}\right)$$
$$= \frac{1}{\left(x + \sqrt{x^2 + a}\right)} \times \left(\frac{x + \sqrt{x^2 + a}}{\sqrt{x^2 + a}}\right)$$
$$= \frac{1}{\sqrt{x^2 + a}} \quad \textbf{[2]}$$

Q. 55. Differentiate the function $\log\left[\log(\log x^5)\right]$ w.r.t. x. [NCERT Exemp. Ex. 5.3, Q. 28, Page 109]

Ans. Given that,

$$y = \log [\log (\log x^5)\}]$$

Differentiating both sides, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left\{\log\left(\log x^5\right)\right\}\right]$$
$$= \frac{1}{\log\left(\log x^5\right)} \times \frac{1}{\log x^5} \times \frac{1}{x^5} \times 5x^4$$
$$= \frac{5}{x\left(\log x^5\right)\log\left(\log x^5\right)} \quad \textbf{[2]}$$

Q. 56. Differentiate the function $\sin\sqrt{x} + \cos^2\sqrt{x}$ w.r.t. x. [NCERT Exemp. Ex. 5.3, Q. 29, Page 109]

Ans. Given that,

$$y = \sin\sqrt{x} + \cos^2\sqrt{x}$$
$$= \sin\sqrt{x} + \left(\cos\sqrt{x}\right)^2$$

Differentiating both sides, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin\sqrt{x}+\left(\cos\sqrt{x}\right)^2\right]$$
$$=\frac{d}{dx}\left(\sin\sqrt{x}\right)+\frac{d}{dx}\left[\left(\cos\sqrt{x}\right)^2\right]$$
$$=\frac{\cos\sqrt{x}}{2\sqrt{x}}+\left[2\cos\sqrt{x}\left(-\frac{\sin\sqrt{x}}{2\sqrt{x}}\right)\right]$$
$$=\frac{1}{2\sqrt{x}}\left(\cos\sqrt{x}-\sin 2\sqrt{x}\right)$$

[2]

Q. 57. Differentiate the function $\sin^n\left(ax^2+bx+c\right)$ w.r.t. x.
[NCERT Exemp. Ex. 5.3, Q. 30, Page 109]

Ans. Given that,

$$y=\sin^n\left(ax^2+bx+c\right)$$

Differentiating both sides, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin\left(ax^2+bx+c\right)\right]^n$$
$$=n\left[\sin\left(ax^2+bx+c\right)\right]^{n-1}\cos\left(ax^2+bx+c\right)\left(2ax+b\right)$$
$$=n\left(2ax+b\right)\cos\left(ax^2+bx+c\right)\left[\sin\left(ax^2+bx+c\right)\right]^{n-1}$$

[2]

Q. 58. Differentiate the function $\cos\left(\tan\sqrt{x+1}\right)$ w.r.t. x.
[NCERT Exemp. Ex. 5.3, Q. 31, Page 109]

Ans. Given that $y=\cos\left(\tan\sqrt{x+1}\right)$

Differentiating both sides, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\cos\left(\tan\sqrt{x+1}\right)\right]$$
$$=\left[-\sin\left(\tan\sqrt{x+1}\right)\right]\left(\sec^2\sqrt{x+1}\right)\left(\frac{1}{2\sqrt{x+1}}\right)$$
$$=-\frac{1}{2\sqrt{x+1}}\sin\left(\tan\sqrt{x+1}\right)\sec^2\sqrt{x+1}$$

[2]

Q. 59. Differentiate the function $\sin x^2+\sin^2 x+\sin^2\left(x^2\right)$ w.r.t. x. **[NCERT Exemp. Ex. 5.3, Q. 32, Page 109]**

Ans. Given that $y=\sin x^2+\sin^2 x+\sin^2\left(x^2\right)$

Differentiating both sides, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin x^2+\sin^2 x+\sin^2\left(x^2\right)\right]$$
$$=\frac{d}{dx}\left(\sin x^2\right)+\frac{d}{dx}\left(\sin^2 x\right)+\frac{d}{dx}\left[\sin^2\left(x^2\right)\right]$$
$$=2x\cos x^2+2\sin x\cos x+2\sin\left(x^2\right)\cos\left(x^2\right)\times\left(2x\right)$$
$$=2x\cos x^2+\sin 2x+2x\sin\left(2x^2\right)$$

[2]

Q. 60. Differentiate the function $\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$ w.r.t. x.
[NCERT Exemp. Ex. 5.3, Q. 33, Page 109]

Ans. Given that $y=\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$ Differentiating both sides, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)\right]$$
$$=\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{x+1}}\right)^2}}\times\frac{d}{dx}\left[\left(x+1\right)^{-\frac{1}{2}}\right]$$
$$=\frac{1}{\sqrt{\frac{x+1-1}{x+1}}}\times\left[-\frac{1}{2}\left(x+1\right)^{-\frac{3}{2}}\right]$$
$$=\sqrt{\frac{x+1}{x}}\left[-\frac{1}{2\left(x+1\right)\sqrt{x+1}}\right]$$
$$=-\frac{1}{2\sqrt{x}\left(x+1\right)}$$

[2]

Q. 61. Differentiate the function $\sin^m x\cos^n x$ w.r.t. x.
[NCERT Exemp. Ex. 5.3, Q. 35, Page 109]

Ans. Given that $y=\sin^m x\cos^n x$

Differentiating both sides, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sin^m x\cos^n x\right)$$
$$=\sin^m x\frac{d}{dx}\left(\cos^n x\right)+\cos^n x\frac{d}{dx}\left(\sin^m x\right)$$
$$=\sin^m x\left(-n\cos^{n-1}x\sin x\right)+\cos^n x\left(m\sin^{m-1}x\cos x\right)$$
$$=\sin^m x\cos^n x\left(m\cot x-n\tan x\right)$$

[2]

Q. 62. Differentiate the function $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$, $-\frac{\pi}{4}<x<\frac{\pi}{4}$ w.r.t. x.
[NCERT Exemp. Ex. 5.3, Q. 38, Page 110]

Ans. Given that,

$$y=\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$$
$$=\tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$$
$$=\tan^{-1}\left(\tan\frac{x}{2}\right)$$
$$=\frac{x}{2}$$

[1]

Differentiating both sides, we have

$$\frac{dy}{dx}=\frac{1}{2}$$

[1]

Q. 63. Find $\frac{dy}{dx}$ of the function expressed in parametric form $x=t+\frac{1}{t}$, $y=t-\frac{1}{t}$.
[NCERT Exemp. Ex. 5.3, Q. 44, Page 110]

Ans. Given that $x=t+\frac{1}{t}$, $y=t-\frac{1}{t}$

$$x=t+\frac{1}{t}$$
$$\frac{dx}{dt}=\frac{d}{dt}\left(t+\frac{1}{t}\right)=1-\frac{1}{t^2}$$
$$y=t-\frac{1}{t}$$
$$\frac{dy}{dt}=\frac{d}{dt}\left(t-\frac{1}{t}\right)=1+\frac{1}{t^2}$$

[1]

Thus,

$$\frac{dy}{dx}=\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$

$$=\frac{1+\frac{1}{t^2}}{1-\frac{1}{t^2}}$$

$$=\frac{t^2+1}{t^2-1}$$ **[1]**

Q. 64. Find $\frac{dy}{dx}$ of the function expressed in parametric form $\sin x=\frac{2t}{1+t^2}$, $\tan y=\frac{2t}{1-t^2}$.

[NCERT Exemp. Ex. 5.3, Q. 47, Page 110]

Ans. Given that $\sin x=\frac{2t}{1+t^2}$, $\tan y=\frac{2t}{1-t^2}$

Put $t=\tan\theta$ in $\sin x=\frac{2t}{1+t^2}$, we have

$$\sin x=\frac{2\tan\theta}{1+\tan^2\theta}$$

$\Rightarrow \sin x=\sin 2\theta$

$\Rightarrow \quad x=2\theta$

$\Rightarrow \quad \frac{dx}{d\theta}=2$ **[1]**

Put $t=\tan\theta$ in $\tan y=\frac{2t}{1-t^2}$, we have

$$\tan y=\frac{2\tan\theta}{1-\tan^2\theta}$$

$\Rightarrow \tan y=\tan 2\theta$

$\Rightarrow \quad y=2\theta$

$\Rightarrow \quad \frac{dy}{d\theta}=2$ **[1]**

Thus, $\frac{dy}{dx}=1$

Q. 65. Find $\frac{dy}{dx}$ when x and y are connected by the relation $\tan^{-1}(x^2+y^2)=a$.

[NCERT Exemp. Ex. 5.3, Q. 56, Page 111]

Ans. Given that,

$\tan^{-1}(x^2+y^2)=a$

$\Rightarrow \quad x^2+y^2=\tan a$

Differentiating with respect to x, we have

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(\tan a)$$

$\Rightarrow 2x+2y\frac{dy}{dx}=0$

$\Rightarrow \quad \frac{dy}{dx}=-\frac{x}{y}$ **[2]**

Q. 66. Find $\frac{dy}{dx}$ when x and y are connected by the relation $(x^2+y^2)^2=xy$.

[NCERT Exemp. Ex. 5.3, Q. 57, Page 111]

Ans. Given that $(x^2+y^2)^2=xy$

Differentiating with respect to x, we have

$$\frac{d}{dx}(x^2+y^2)^2=\frac{d}{dx}(xy)$$

$$\Rightarrow 2(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right)=x\frac{dy}{dx}+y$$

$$\Rightarrow 4x(x^2+y^2)+4y(x^2+y^2)\frac{dy}{dx}=x\frac{dy}{dx}+y$$

$$\Rightarrow \left[4y(x^2+y^2)-x\right]\frac{dy}{dx}=y-4x(x^2+y^2)$$

$$\Rightarrow (4x^2y+4y^3-x)\frac{dy}{dx}=y-4x^3-4xy^2$$

$$\therefore \quad \frac{dy}{dx}=\frac{y-4x^3-4xy^2}{4x^2y+4y^3-x}$$ **[2]**

Short Answer Type Questions (3 or 4 marks each)

Q. 1. Differentiate the functions $\frac{\sin(ax+b)}{\cos(cx+d)}$ with respect to x.

[NCERT Ex. 5.2, Q. 5, Page 166]

Ans. Given that,

$$y=\frac{\sin(ax+b)}{\cos(cx+d)}$$

Differentiating with respect to x, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{\sin(ax+b)}{\cos(cx+d)}\right]$$

$$=\frac{\left[\cos(cx+d)\frac{d}{dx}[\sin(ax+b)]-\sin(ax+b)\frac{d}{dx}[\cos(cx+d)]\right]}{\cos^2(cx+d)}$$

$$=\frac{\left[\cos(cx+d)\cos(ax+b)\frac{d}{dx}(ax+b)-\sin(ax+b)[-\sin(cx+d)]\frac{d}{dx}(cx+d)\right]}{\cos^2(cx+d)}$$

$$=\frac{\left[\cos(cx+d)\cos(ax+b)(a)+\sin(ax+b)\sin(cx+d)(c)\right]}{\cos^2(cx+d)}$$

$$=\frac{\left[a\cos(cx+d)\cos(ax+b)+c\sin(ax+b)\sin(cx+d)\right]}{\cos^2(cx+d)}$$

$$=\left[\frac{a\cos(cx+d)\cos(ax+b)}{\cos^2(cx+d)}+\frac{c\sin(ax+b)\sin(cx+d)}{\cos^2(cx+d)}\right]$$ **[3]**

Q. 2. Differentiate the functions $2\sqrt{\cot(x^2)}$ with respect to x. **[NCERT Ex. 5.2, Q. 7, Page 166]**

Ans. Given that $y = 2\sqrt{\cot(x^2)}$

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[2\sqrt{\cot(x^2)}\right]$$

$$= 2\frac{d}{dx}\left[\sqrt{\cot(x^2)}\right]$$

$$= 2\times\frac{1}{2\sqrt{\cot(x^2)}}\times\frac{d}{dx}\left[\cot(x^2)\right]$$

$$= \frac{1}{\sqrt{\cot(x^2)}}\times\left[-\operatorname{cosec}^2(x^2)\right]\frac{d}{dx}(x^2)$$

$$= -\frac{\operatorname{cosec}^2(x^2)}{\sqrt{\cot(x^2)}}(2x)$$

$$= \frac{-2x}{\sin^2(x^2)\sqrt{\frac{\cos(x^2)}{\sin(x^2)}}}$$

$$= \frac{-2x}{\sin(x^2)\sqrt{\sin^2(x^2)\times\frac{\cos(x^2)}{\sin(x^2)}}}$$

$$= \frac{-2\sqrt{2}x}{\sin(x^2)\sqrt{2\cos(x^2)\sin(x^2)}}$$

$$= \frac{-2\sqrt{2}x}{\sin(x^2)\sqrt{\sin 2(x^2)}}$$

[3]

Q. 3. Find $\frac{dy}{dx}$ of $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $0 < x < 1$. **[NCERT Ex. 5.3, Q. 12, Page 169]**

Ans. Given that,

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ 0<x<1$$

Put $x = \tan\theta$, we have

$$y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\cos 2\theta)$$

$$= \sin^{-1}\left[\sin\left(\frac{\pi}{2}-2\theta\right)\right]$$

$$= \frac{\pi}{2}-2\theta$$

[1]

Since $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$, then

$$\Rightarrow y = \frac{\pi}{2}-2\tan^{-1}x$$

[1]

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}-2\tan^{-1}x\right)$$

$$= 0-\frac{2}{1+x^2}$$

$$= -\frac{2}{1+x^2}$$

[1]

Q. 4. Find $\frac{dy}{dx}$ of $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$, $-1 < x < 1$. **[NCERT Ex. 5.3, Q. 13, Page 169]**

Ans. Given that,

$$y = \cos^{-1}\left(\frac{2x}{1+x^2}\right),\ -1<x<1$$

Put $x = \tan\theta$, we have

$$y = \cos^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \cos^{-1}(\sin 2\theta)$$

$$= \cos^{-1}\left[\cos\left(\frac{\pi}{2}-2\theta\right)\right]$$

$$= \frac{\pi}{2}-2\theta$$

[1]

Since $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$, then

$$\Rightarrow y = \frac{\pi}{2}-2\tan^{-1}x$$

[1]

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}-2\tan^{-1}x\right)$$

$$= 0-\frac{2}{1+x^2}$$

$$= -\frac{2}{1+x^2}$$

[1]

Q. 5. Find $\frac{dy}{dx}$ of $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. **[NCERT Ex. 5.3, Q. 14, Page 169]**

Ans. Given that,

$$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right),\ -\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$$

Put $x = \sin\theta$, we have

$$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$$

$$= \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$$= \sin^{-1}(2\sin\theta\cos\theta)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta$$

[1]

Since $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$, then

$$\Rightarrow y = 2\sin^{-1}x$$

[1]

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(2\sin^{-1}x\right)$$

$$= \frac{2}{\sqrt{1-x^2}}$$

[1]

Q. 6. Find $\frac{dy}{dx}$ of $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$, $0 < x < \frac{1}{\sqrt{2}}$. **[NCERT Ex. 5.3, Q. 15, Page 169]**

Ans. Given that,

$$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right),\ 0<x<\frac{1}{\sqrt{2}}$$

Put $x = \cos\theta$, we have

$$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$$
$$= \sec^{-1}\left(\frac{1}{2\cos^2\theta-1}\right)$$
$$= \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$$
$$= \sec^{-1}(\sec 2\theta)$$
$$= 2\theta$$ **[1]**

Since $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$, then

$\Rightarrow y = 2\cos^{-1}x$ **[1]**

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(2\cos^{-1}x\right)$$
$$= -\frac{2}{\sqrt{1-x^2}}$$ **[1]**

Q. 7. Differentiate the function $(\log x)^{\cos x}$ with respect to x. [NCERT Ex. 5.5, Q. 3, Page 178]

Ans. Given that $y = (\log x)^{\cos x}$

Taking log both sides, we have

$$\log y = \log\left[(\log x)^{\cos x}\right]$$
$$= \cos x \log(\log x)$$ **[1]**

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(\log y) = \frac{d}{dx}\left[\cos x \log(\log x)\right]$$
$$\frac{1}{y}\frac{dy}{dx} = \cos x\frac{d}{dx}\left[\log(\log x)\right] + \log(\log x)\frac{d}{dx}(\cos x)$$
$$\frac{dy}{dx} = y\left[\cos x \times \frac{1}{\log x} \times \frac{1}{x} + \log(\log x)(-\sin x)\right]$$
$$= (\log x)^{\cos x}\left[\frac{\cos x}{x\log x} - \sin x \log(\log x)\right]$$ **[2]**

Q. 8. Differentiate the function $\cos x \cos 2x \cos 3x$ with respect to x. [NCERT Ex. 5.5, Q. 1, Page 178]

Ans. Given that $y = \cos x \cos 2x \cos 3x$

Taking log both sides, we have

$$\log y = \log(\cos x \cos 2x \cos 3x)$$
$$= \log(\cos x) + \log(\cos 2x) + \log(\cos 3x)$$ **[1]**

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(\log y) = \begin{bmatrix} \frac{d}{dx}(\log\cos x) + \frac{d}{dx}(\log\cos 2x) \\ + \frac{d}{dx}(\log\cos 3x) \end{bmatrix}$$
$$\frac{1}{y}\frac{dy}{dx} = \begin{bmatrix} \frac{1}{\cos x}\frac{d}{dx}(\cos x) + \frac{1}{\cos 2x}\frac{d}{dx}(\cos 2x) \\ + \frac{1}{\cos 3x}\frac{d}{dx}(\cos 3x) \end{bmatrix}$$
$$\frac{dy}{dx} = y\begin{bmatrix} \frac{1}{\cos x}\times(-\sin x) + \frac{1}{\cos 2x}\times(-2\sin 2x) \\ + \frac{1}{\cos 3x}\times(-3\sin 3x) \end{bmatrix}$$
$$= \cos x\cos 2x\cos 3x\left(-\frac{\sin x}{\cos x} - \frac{2\sin 2x}{\cos 2x} - \frac{3\sin 3x}{\cos 3x}\right)$$
$$= \cos x\cos 2x\cos 3x(-\tan x - 2\tan 2x - 3\tan 3x)$$
$$= -\cos x\cos 2x\cos 3x(\tan x + 2\tan 2x + 3\tan 3x)$$ **[2]**

Q. 9. Differentiate the function $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$ with respect to x.

[NCERT Ex. 5.5, Q. 2, Page 178]

Ans. Given that,

$$y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Taking log both sides, we have

$$\log y = \log\left(\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\right)$$
$$= \frac{1}{2}\begin{bmatrix} \log(x-1) + \log(x-2) - \log(x-3) \\ -\log(x-4) - \log(x-5) \end{bmatrix}$$ **[1]**

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(\log y) = \frac{1}{2}\frac{d}{dx}\begin{bmatrix} \log(x-1) + \log(x-2) - \log(x-3) \\ -\log(x-4) - \log(x-5) \end{bmatrix}$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\begin{bmatrix} \frac{d}{dx}\{\log(x-1)\} + \frac{d}{dx}\{\log(x-2)\} \\ -\frac{d}{dx}\{\log(x-3)\} - \frac{d}{dx}\{\log(x-4)\} \\ -\frac{d}{dx}\{\log(x-5)\} \end{bmatrix}$$
$$\frac{dy}{dx} = \frac{y}{2}\left(\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right)$$
$$= \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\begin{pmatrix} \frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} \\ -\frac{1}{x-4} - \frac{1}{x-5} \end{pmatrix}$$ **[2]**

Q. 10. Find the derivative of the function given by $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$. [NCERT Ex. 5.5, Q. 16, Page 178]

Ans. Given that,

$$y = (1+x)(1+x^2)(1+x^4)(1+x^8)$$

Taking log both sides, we have

$$\log y = \log\left[(1+x)(1+x^2)(1+x^4)(1+x^8)\right]$$
$$= \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^8)$$ **[1]**

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(\log y) = \frac{d}{dx}\begin{bmatrix} \log(1+x) + \log(1+x^2) \\ +\log(1+x^4) + \log(1+x^8) \end{bmatrix}$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}$$
$$\frac{dy}{dx} = y\left(\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right)$$
$$= \begin{bmatrix} (1+x)(1+x^2)(1+x^4)(1+x^8) \\ \left(\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right) \end{bmatrix}$$ **[1]**

Put $x = 1$, we have

$$f'(1) = \left(\frac{dy}{dx}\right)_{x=1}$$

$$= (2\times2\times2\times2)\left(\frac{1}{2}+\frac{2}{2}+\frac{4}{2}+\frac{8}{2}\right)$$

$$= 16\times\frac{15}{2}$$

$$= 120$$ **[1]**

Q. 11. Find $\frac{dy}{dx}$ of $xy = e^{(x-y)}$.
[NCERT Ex. 5.5, Q. 15, Page 178]

Ans. Given that,

$$xy = e^{(x-y)}$$

Taking log both sides, we have

$$\log(xy) = \log\left[e^{(x-y)}\right]$$

$$\log x + \log y = (x-y)\log e$$

$$= (x-y)$$ **[1]**

Differentiating both sides with respect to x, we have

$$\Rightarrow \frac{d}{dx}(\log x + \log y) = \frac{d}{dx}(x-y)$$

$$\Rightarrow \frac{1}{x}+\frac{1}{y}\frac{dy}{dx} = 1-\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}+\frac{dy}{dx} = 1-\frac{1}{x}$$

$$\Rightarrow \left(\frac{1+y}{y}\right)\frac{dy}{dx} = \frac{x-1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\left(\frac{x-1}{y+1}\right)$$ **[2]**

Q. 12. Find $\frac{dy}{dx}$ of $(\cos x)^y = (\cos y)^x$.
[NCERT Ex. 5.5, Q. 14, Page 178]

Ans. Given that,

$$(\cos x)^y = (\cos y)^x$$

Taking log both sides, we have

$$\log\left[(\cos x)^y\right] = \log\left[(\cos y)^x\right]$$

$$\Rightarrow y\log\cos x = x\log\cos y$$ **[1]**

Differentiating both sides with respect to x, we have

$$\Rightarrow \frac{d}{dx}(y\log\cos x) = \frac{d}{dx}(x\log\cos y)$$

$$\Rightarrow y\left(-\frac{\sin x}{\cos x}\right)+\log\cos x\frac{dy}{dx} = x\left(-\frac{\sin y}{\cos y}\frac{dy}{dx}\right)+\log\cos y$$

$$\Rightarrow -\frac{y\sin x}{\cos x}+\log\cos x\frac{dy}{dx} = -\frac{x\sin y}{\cos y}\frac{dy}{dx}+\log\cos y$$

$$\Rightarrow \log\cos x\frac{dy}{dx}+\frac{x\sin y}{\cos y}\frac{dy}{dx} = \log\cos y+\frac{y\sin x}{\cos x}$$

$$\Rightarrow \log\cos x\frac{dy}{dx}+x\tan y\frac{dy}{dx} = \log\cos y + y\tan x$$

$$\Rightarrow (\log\cos x + x\tan y)\frac{dy}{dx} = \log\cos y + y\tan x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\log\cos y + y\tan x}{\log\cos y + y\tan x}$$ **[2]**

Q. 13. Differentiate $(x^2-5x+8)(x^3+7x+9)$ in three ways by logarithmic differentiation.
[NCERT Ex. 5.5, Q. 17(iii), Page 178]

Ans. Given that,

$$y = (x^2-5x+8)(x^3+7x+9)$$

Taking log both sides, we have

$$\log y = \log(x^2-5x+8)+\log(x^3+7x+9)$$ **[1]**

Differentiating both sides with respect to x, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}$$

$$\frac{dy}{dx} = y\left[\frac{(2x-5)(x^3+7x+9)+(3x^2+7)(x^2-5x+8)}{(x^2-5x+8)(x^3+7x+9)}\right]$$

$$= (x^2-5x+8)(x^3+7x+9)\left[\frac{2x^4+14x^2+18x-5x^3-35x-45+3x^4-15x^3+24x^2+7x^2-35x+56}{(x^2-5x+8)(x^3+7x+9)}\right]$$

$$= 5x^4-20x^3+45x^2-52x+11$$ **[2]**

Q. 14. If u, v, w are the function of x, then show that $\frac{d}{dx}(uvw) = \frac{du}{dx}(vw)+\frac{dv}{dx}(uw)+\frac{dw}{dx}(uv)$ in two ways, first repeating by product rule and second by logarithmic differentiation.
[NCERT Ex. 5.5, Q. 18, Page 179]

Ans. Differentiating by repeating by product rule,

$$y = uvw$$

$$y = u(vw)$$

$$\frac{dy}{dx} = u\frac{d}{dx}(vw)+(vw)\frac{du}{dx}$$

$$= u\left(v\frac{dw}{dx}+w\frac{d}{dx}\right)+(vw)\frac{du}{dx}$$

$$= (uv)\frac{dw}{dx}+(uw)\frac{dv}{dx}+(vw)\frac{du}{dx}$$ **[1½]**

Differentiating by repeating by logarithmic differentiation, we have

$$y = uvw$$

$$\log y = \log u + \log v + \log w$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}$$

$$\frac{dy}{dx} = y\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}\right)$$

$$= (uvw)\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}\right)$$

$$= (vw)\frac{du}{dx}+(uw)\frac{dv}{dx}+(uv)\frac{dw}{dx}$$ **[1½]**

Q. 15. Find $\frac{dy}{dx}$ of $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$, $y = a \sin t$.

[NCERT Ex. 5.6, Q. 8, Page 181]

Ans. Given that,

$$x = a\left(\cos t + \log \tan \frac{t}{2}\right),\ y = a \sin t$$

Then,

$$\frac{dx}{dt} = \frac{d}{dt}\left[a\left(\cos t + \log \tan \frac{t}{2}\right)\right]$$

$$= a\left(-\sin t + \frac{\frac{1}{2}\sec^2 \frac{t}{2}}{\tan \frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{2\sin \frac{t}{2}\cos \frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(\frac{1-\sin^2 t}{\sin t}\right)$$

$$= a\left(\frac{\cos^2 t}{\sin t}\right)$$

[2]

And,

$$\frac{dy}{dt} = \frac{d}{dt}(a \sin t)$$

$$= a \cos t$$

Thus,

$$\frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$

$$= \frac{a\cos t}{a\left(\frac{\cos^2 t}{\sin t}\right)}$$

$$= \tan t$$

[1]

Q. 16. If $x = \sqrt{a^{\sin^{-1}t}}$, $y = \sqrt{a^{\cos^{-1}t}}$, then show that $\frac{dy}{dx} = -\frac{y}{x}$.

[NCERT Ex. 5.6, Q. 11, Page 181]

Ans. Given that,

$$x = \sqrt{a^{\sin^{-1}t}},\ y = \sqrt{a^{\cos^{-1}t}}$$

Then,

$$\frac{dx}{dt} = \frac{d}{dt}\left(\sqrt{a^{\sin^{-1}t}}\right)$$

$$= \frac{1}{2\sqrt{a^{\sin^{-1}t}}} \times a^{\sin^{-1}t} \times \log a \times \frac{1}{\sqrt{1-t^2}}$$

$$= \frac{\log a\sqrt{a^{\sin^{-1}t}}}{2\sqrt{1-t^2}}$$

[1]

And,

$$\frac{dy}{dt} = \frac{d}{dt}\left(\sqrt{a^{\cos^{-1}t}}\right)$$

$$= \frac{1}{2\sqrt{a^{\cos^{-1}t}}} \times a^{\cos^{-1}t} \times \log a \times \frac{-1}{\sqrt{1-t^2}}$$

$$= -\frac{\log a\sqrt{a^{\cos^{-1}t}}}{2\sqrt{1-t^2}}$$

[1]

Thus,

$$\frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$

$$= \frac{\frac{\log a\sqrt{a^{\cos^{-1}t}}}{2\sqrt{1-t^2}}}{-\frac{\log a\sqrt{a^{\sin^{-1}t}}}{2\sqrt{1-t^2}}}$$

$$= -\frac{\sqrt{a^{\cos^{-1}t}}}{\sqrt{a^{\sin^{-1}t}}}$$

$$= -\frac{y}{x}$$

[1]

Q. 17. Find the second-order derivatives of the function $e^x \sin 5x$. **[NCERT Ex. 5.7, Q. 6, Page 183]**

Ans. Given that,

$$y = e^x \sin 5x$$

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x \sin 5x\right)$$

$$= 5e^x \cos 5x + e^x \sin 5x$$

[1]

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(5e^x \cos 5x + e^x \sin 5x\right)$$

$$= \frac{d}{dx}\left(5e^x \cos 5x\right) + \frac{d}{dx}\left(e^x \sin 5x\right)$$

$$= -25e^x \sin 5x + 5e^x \cos 5x + 5e^x \cos 5x + e^x \sin 5x$$

$$= -24e^x \sin 5x + 10e^x \cos 5x$$

$$= 10e^x \cos 5x - 24e^x \sin 5x$$

$$= 2e^x\left(5\cos 5x - 12\sin 5x\right)$$

[2]

Q. 18. Find the second-order derivatives of the function $e^{6x}\cos 3x$. **[NCERT Ex. 5.7, Q. 7, Page 183]**

Ans. Given that,

$$y = e^{6x}\cos 3x$$

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{6x}\cos 3x\right)$$

$$= e^{6x}\left(-3\sin 3x\right) + e^{6x}\cos 3x$$

$$= -3e^{6x}\sin 3x + 6e^{6x}\cos 3x$$

[1]

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-3e^{6x}\sin 3x + 6e^{6x}\cos 3x\right)$$

$$= \frac{d}{dx}\left(-3e^{6x}\sin 3x\right) + 6\frac{d}{dx}\left(e^{6x}\cos 3x\right)$$

$$= -3\left(3e^{6x}\cos 3x + 6e^{6x}\sin 3x\right) + 6\left(-3e^{6x}\sin 3x + 6e^{6x}\cos 3x\right)$$

$$= -9e^{6x}\cos 3x - 18e^{6x}\sin 3x - 18e^{6x}\sin 3x + 36e^{6x}\cos 3x$$

$$= 27e^{6x}\cos 3x - 36e^{6x}\sin 3x$$

$$= 9e^{6x}\left(3\cos 3x - 4\sin 3x\right)$$

[2]

Q. 19. Find the second-order derivatives of the function $\tan^{-1}x$. **[NCERT Ex. 5.7, Q. 8, Page 183]**

Ans. Given that,

$y = \tan^{-1} x$

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1} x\right)$$

$$= \frac{1}{1+x^2}$$ **[1]**

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{1+x^2}\right)$$

$$= \frac{d}{dx}\left[\left(1+x^2\right)^{-1}\right]$$

$$= -\left(1+x^2\right)^{-2} \times (2x)$$

$$= -\frac{2x}{\left(1+x^2\right)^2}$$

[2]

Q. 20. Find the second-order derivatives of the function $\log(\log x)$. **[NCERT Ex. 5.7, Q. 9, Page 183]**

Ans. Given that,

$y = \log(\log x)$

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(\log x)\right]$$

$$= \frac{1}{x \log x}$$ **[1]**

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x \log x}\right)$$

$$= \frac{d}{dx}\left[(x \log x)^{-1}\right]$$

$$= -(x \log x)^{-2} \times \left(x \times \frac{1}{x} + \log x\right)$$

$$= -\frac{1+\log x}{(x \log x)^2}$$ **[2]**

Q. 21. Find the second-order derivatives of the function $\sin(\log x)$. **[NCERT Ex. 5.7, Q. 10, Page 183]**

Ans. Given that,

$y = \sin(\log x)$

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin(\log x)\right]$$

$$= \cos(\log x) \times \frac{1}{x}$$

$$= \frac{\cos(\log x)}{x}$$ **[1]**

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$

$$= \frac{x\frac{d}{dx}\left[\cos(\log x)\right] - \cos(\log x)\frac{d}{dx}(x)}{x^2}$$

$$= \frac{x\left[\frac{-\sin(\log x)}{x}\right] - \cos(\log x)}{x^2}$$

$$= \frac{-\sin(\log x) - \cos(\log x)}{x^2}$$

$$= -\frac{\sin(\log x) + \cos(\log x)}{x^2}$$

[2]

Q. 22. If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$. **[NCERT Ex. 5.7, Q. 11, Page 183]**

Ans. Given that,

$y = 5\cos x - 3\sin x$

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}(5\cos x - 3\sin x)$$

$$= -5\sin x - 3\cos x$$

$$= -(5\sin x + 3\cos x)$$

[1]

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-(5\sin x + 3\cos x)\right]$$

$$\frac{d^2y}{dx^2} = -(5\cos x - 3\sin x)$$

$$\frac{d^2y}{dx^2} = -y$$

$$\frac{d^2y}{dx^2} + y = 0$$

[2]

Q. 23. If $y = \cos^{-1}x$, find $\frac{d^2y}{dx^2}$ in terms of y alone. **[NCERT Ex. 5.7, Q. 12, Page 184]**

Ans. Given that,

$$y = \cos^{-1} x$$

$$\Rightarrow \cos y = x$$

Differentiating with respect to x, we have

$$-\sin y \frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = -\text{cosec } y$$

[1]

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-\text{cosec } y)$$

$$= -\text{cosec } y \cot y \frac{dy}{dx}$$

$$= (-\text{cosec } y \cot y)(-\text{cosec } y)$$

$$= -\text{cosec}^2 y \cot y$$

[2]

Q. 24. Differentiate w.r.t. x the function

$$\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}, -2 < x < 2.$$

[NCERT Misc Ex. Q. 5, Page 191]

Ans. Given that,

$$y = \frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}, -2 < x < 2$$

Differentiate with respect to x, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}\right)$$

$$=\frac{\sqrt{2x+7}\frac{d}{dx}\left(\cos^{-1}\frac{x}{2}\right)-\cos^{-1}\frac{x}{2}\frac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^2}$$

$$=\frac{\sqrt{2x+7}\left(-\frac{\frac{1}{2}}{\sqrt{1-\frac{x^2}{4}}}\right)-\left(\frac{2}{2\sqrt{2x+7}}\right)\cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)^2}$$

$$=-\left[\frac{1}{\sqrt{4-x^2}\sqrt{2x+7}}+\frac{\cos^{-1}\frac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$ **[3]**

Q. 25. Differentiate w.r.t. x the function

$$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right],\ 0<x<\frac{\pi}{2}.$$

[NCERT Misc Ex. Q. 6, Page 191]

Ans. Given that,

$$y=\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right],\ 0<x<\frac{\pi}{2}$$

By applying ($\sin^2 x+\cos^2 x=1$) and ($\sin 2x = 2\sin x.\cos x$), we have

$$y=\cot^{-1}\left[\frac{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}+\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}-\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}\right]$$

$$=\cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right]$$

$$=\cot^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right)$$

$$=\cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right)$$

$$=\cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$=\frac{x}{2}$$ **[2]**

Differentiate with respect to x, we have

$$\frac{dy}{dx}=\frac{1}{2}$$ **[1]**

Q. 26. Differentiate w.r.t. x the function $(\log x)^{\log x}, x>1$.

[NCERT Misc Ex. Q. 7, Page 191]

Ans. Given that $y=(\log x)^{\log x}, x>1$

Taking log both sides, we have

$$\log y=\log\left[(\log x)^{\log x}\right]$$

$$=\log x\log(\log x)$$ **[1]**

Differentiate with respect to x, we have

$$\frac{d}{dx}(\log y)=\frac{d}{dx}\left[\log x\log(\log x)\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\log x\frac{d}{dx}\left[\log(\log x)\right]+\log(\log x)$$

$$\frac{d}{dx}(\log x)$$

$$\frac{dy}{dx}=y\left[\frac{\log x}{x\log x}+\frac{\log(\log x)}{x}\right]$$

$$=(\log x)^{\log x}\left[\frac{1+\log(\log x)}{x}\right]$$ **[2]**

Q. 27. Differentiate w.r.t. x the function

$$(\sin x-\cos x)^{(\sin x-\cos x)},\ \frac{\pi}{4}<x<\frac{3\pi}{4}.$$

[NCERT Misc Ex. Q. 9, Page 191]

Ans. Given that $y=(\sin x-\cos x)^{(\sin x-\cos x)}$

Taking log both sides, we have

$$\log y=\log\left[(\sin x-\cos x)^{(\sin x-\cos x)}\right]$$

$$=(\sin x-\cos x)\log(\sin x-\cos x)$$ **[1]**

Differentiating with respect to x, we have

$$\frac{d}{dx}(\log y)=\frac{d}{dx}\left[(\sin x-\cos x)\log(\sin x-\cos x)\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\left[\begin{array}{l}(\sin x-\cos x)\frac{d}{dx}\left[\log(\sin x-\cos x)\right]\\+\log(\sin x-\cos x)\frac{d}{dx}(\sin x-\cos x)\end{array}\right]$$

$$\frac{dy}{dx}=y\left[\begin{array}{l}(\sin x-\cos x)\left(\frac{\cos x+\sin x}{\sin x-\cos x}\right)\\+(\cos x+\sin x)\log(\sin x-\cos x)\end{array}\right]$$

$$=(\sin x-\cos x)^{(\sin x-\cos x)}\left[\begin{array}{l}\cos x+\sin x+(\cos x\\+\sin x)\log(\sin x-\cos x)\end{array}\right]$$ **[2]**

Q. 28. If $\cos y = x\cos(a+y)$, with $\cos a\neq \pm 1$, prove that $\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$.

[NCERT Misc Ex. Q. 16, Page 192]

Ans. Given that $\cos y = x\cos(a+y)$

$$x=\frac{\cos y}{\cos(a+y)}$$

Differentiating with respect to x, we have

$$\frac{dx}{dy}=\frac{\left[\cos(a+y)\right]\frac{d}{dy}(\cos y)-(\cos y)\frac{d}{dy}\left[\cos(a+y)\right]}{\cos^2(a+y)}$$

$$=\frac{-\sin y\cos(a+y)-(\cos y)\left[-\sin(a+y)\right]}{\cos^2(a+y)}$$

$$= \frac{-\sin y \cos(a+y) + \sin(a+y)\cos y}{\cos^2(a+y)}$$

$$= \frac{\sin(a+y)\cos y - \sin y \cos(a+y)}{\cos^2(a+y)}$$

$$= \frac{\sin(a+y-y)}{\cos^2(a+y)}$$

$$\frac{dx}{dy} = \frac{\sin a}{\cos^2(a+y)}$$

$$\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

[3]

Q. 29. If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, then prove that

$$\frac{dy}{dx} = \begin{vmatrix} f(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}.$$

[NCERT Misc Ex. Q. 22, Page 192]

Ans. Given that $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Differentiating with respect to x, we have **[1]**

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ \frac{dl}{dx} & \frac{dm}{dx} & \frac{dm}{dx} \\ a & b & c \end{vmatrix}$$

$$+ \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ \frac{da}{dx} & \frac{dm}{dx} & \frac{dn}{dx} \end{vmatrix}$$

$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix}$$

$$+ \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ 0 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix} + 0 + 0$$

$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

[2]

Q. 30. If $y = e^{a\cos^{-1}x}$, $-1 \le x \le 1$ show that

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0.$$

[NCERT Misc Ex. Q. 23, Page 192]

Ans. Given that $y = e^{a\cos^{-1}x}$, $-1 \le x \le 1$.

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{a\cos^{-1}x}\right)$$

$$= e^{a\cos^{-1}x} \times a\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \frac{-ay}{\sqrt{1-x^2}}$$

[1]

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{-ay}{\sqrt{1-x^2}}\right)$$

$$= (-a)\frac{\sqrt{1-x^2}\frac{dy}{dx} - y\frac{d}{dx}\left(\sqrt{1-x^2}\right)}{\left(\sqrt{1-x^2}\right)^2}$$

$$= (-a)\frac{\sqrt{1-x^2}\left(\frac{-ay}{\sqrt{1-x^2}}\right) - y\left(\frac{-2x}{2\sqrt{1-x^2}}\right)}{1-x^2}$$

$$= (-a)\frac{-ay + \frac{xy}{\sqrt{1-x^2}}}{1-x^2}$$

$$\left(1-x^2\right)\frac{d^2y}{dx^2} = a^2y + x\left(\frac{-ay}{\sqrt{1-x^2}}\right)$$

$$\left(1-x^2\right)\frac{d^2y}{dx^2} = a^2y + x\frac{dy}{dx}$$

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

[2]

Q. 31. Differentiate the function $(\sin x)^{\cos x}$ w.r.t. x.

[NCERT Exemp. Ex. 5.3, Q. 34, Page 109]

Ans. Given that $y = (\sin x)^{\cos x}$

Taking log both sides, we have

$$\log y = \log(\sin x)^{\cos x}$$

$$= \cos x(\log \sin x)$$

[1]

Differentiating both sides, we have

$$\frac{d}{dx}(\log y) = \frac{d}{dx}\left[\cos x(\log \sin x)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \cos x\frac{d}{dx}\left[(\log \sin x)\right] + \log \sin x\frac{d}{dx}(\cos x)$$

$$\frac{dy}{dx} = y\left[\cos x\frac{d}{dx}\{(\log \sin x)\} + \log \sin x\frac{d}{dx}(\cos x)\right]$$

$$= (\sin x)^{\cos x}\left(\cos x \times \frac{\cos x}{\sin x} - \sin x \log \sin x\right)$$

$$= (\sin x)^{\cos x}(\cos x \cot x - \sin x \log \sin x)$$

[2]

Q. 32. Differentiate the function $(x+1)^2(x+2)^3(x+3)^4$ w.r.t. x.

[NCERT Exemp. Ex. 5.3, Q. 36, Page 109]

Ans. Given that $y = (x+1)^2(x+2)^3(x+3)^4$

Taking log both sides, we have

$$\log y = \log\left[(x+1)^2(x+2)^3(x+3)^4\right]$$

$$= \log\left[(x+1)^2\right] + \log\left[(x+2)^3\right] + \log\left[(x+3)^4\right]$$

$$= 2\log(x+1) + 3\log(x+2) + 4\log(x+3)$$

[1]

Differentiating both sides, we have

$$\frac{d}{dx}(\log y)=\frac{d}{dx}\left[\begin{array}{l}2\log(x+1)+3\log(x+2)\\+4\log(x+3)\end{array}\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{2}{x+1}+\frac{3}{x+2}+\frac{4}{x+3}$$

$$\frac{dy}{dx}=y\left(\frac{2}{x+1}+\frac{3}{x+2}+\frac{4}{x+3}\right)$$

$$=\left[\begin{array}{l}(x+1)^2(x+2)^3(x+3)^4\\\left(\frac{2}{x+1}+\frac{3}{x+2}+\frac{4}{x+3}\right)\end{array}\right]$$

$$=(x+1)(x+2)^2(x+3)^3(9x^2+34x+29)$$ **[2]**

Q. 33. Differentiate the function

$\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right), -\frac{\pi}{4}<x<\frac{\pi}{4}$ **w.r.t. x.**

[NCERT Exemp. Ex. 5.3, Q. 37, Page 110]

Ans. Given that,

$$y=\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$$

$$=\cos^{-1}\left(\sin x\left(\frac{1}{\sqrt{2}}\right)+\cos x\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$=\cos^{-1}\left(\sin x\sin\frac{\pi}{4}+\cos x\cos\frac{\pi}{4}\right)$$

$$=\cos^{-1}\left(\cos\left(\frac{\pi}{4}-x\right)\right)$$

$$=\frac{\pi}{4}-x$$ **[2]**

Differentiating both sides, we have

$$\frac{dy}{dx}=-1$$ **[1]**

Q. 34. Differentiate the function

$\tan^{-1}(\sec x+\tan x), -\frac{\pi}{2}<x<\frac{\pi}{2}$ **w.r.t. x.**

[NCERT Exemp. Ex. 5.3, Q. 39, Page 110]

Ans. Given that,

$$y=\tan^{-1}(\sec x+\tan x)$$

Differentiating both sides, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\tan^{-1}(\sec x+\tan x)\right]$$

$$=\frac{1}{1+(\sec x+\tan x)^2}\times(\sec x\tan x+\sec^2 x)$$

$$=\frac{1}{1+\sec^2 x+\tan^2 x+2\sec x\tan x}\times\sec x(\tan x+\sec x)$$

$$=\frac{1}{\sec^2 x+\sec^2 x+2\sec x\tan x}\times\sec x(\tan x+\sec x)$$

$$=\frac{1}{2\sec^2 x+2\sec x\tan x}\times\sec x(\tan x+\sec x)$$

$$=\frac{1}{2\sec x(\sec x+\tan x)}\times\sec x(\tan x+\sec x)$$

$$=\frac{1}{2}$$ **[3]**

Q. 35. Differentiate the function

$\tan^{-1}\left(\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right), -\frac{\pi}{2}<x<\frac{\pi}{2}$ **and** $\frac{a}{b}\tan x>-1$

w.r.t. x. **[NCERT Exemp. Ex. 5.3, Q. 40, Page 110]**

Ans. Given that,

$$y=\tan^{-1}\left(\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right)$$

$$=\tan^{-1}\left(\frac{\frac{a\cos x}{b\cos x}-\frac{b\sin x}{b\cos x}}{\frac{b\cos x}{b\cos x}+\frac{a\sin x}{b\cos x}}\right)$$

$$=\tan^{-1}\left(\frac{\frac{a}{b}-\tan x}{1+\frac{a}{b}\tan x}\right)$$

$$=\tan^{-1}\frac{a}{b}-\tan^{-1}(\tan x)$$

$$=\tan^{-1}\frac{a}{b}-x$$ **[2]**

Differentiating both sides, we have

$$\frac{dy}{dx}=-1$$ **[1]**

Q. 36. Differentiate the function

$\sec^{-1}\left(\frac{1}{4x^3-3x}\right), 0<x<\frac{1}{\sqrt{2}}$ **w.r.t. x.**

[NCERT Exemp. Ex. 5.3, Q. 41, Page 110]

Ans. Given that,

$$y=\sec^{-1}\left(\frac{1}{4x^3-3x}\right),\ 0<x<\frac{1}{\sqrt{2}}$$

Put $x=\cos\theta\Rightarrow\theta=\cos^{-1}x$, then

$$y=\sec^{-1}\left(\frac{1}{4\cos^3\theta-3\cos\theta}\right)$$

$$=\sec^{-1}\left(\frac{1}{\cos 3\theta}\right)$$

$$=\sec^{-1}(\sec 3\theta)$$

$$=3\theta$$

$$=3\cos^{-1}x$$ **[2]**

Differentiating both sides, we have

$$\frac{dy}{dx}=3\frac{d}{dx}(\cos^{-1}x)$$

$$=-\frac{3}{\sqrt{1-x^2}}$$ **[1]**

Q. 37. Differentiate the function

$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right), -\frac{1}{\sqrt{3}}<\frac{x}{a}<\frac{1}{\sqrt{3}}$ **w.r.t. x.**

[NCERT Exemp. Ex. 5.3, Q. 42, Page 110]

Ans. Given that,

$$y=\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right), -\frac{1}{\sqrt{3}}<\frac{x}{a}<\frac{1}{\sqrt{3}}$$

$$y=\tan^{-1}\left(\frac{3\left(\frac{x}{a}\right)-\left(\frac{x}{a}\right)^3}{1-3\left(\frac{x}{a}\right)^2}\right)$$ **[1]**

Put $\frac{x}{a}=\tan\theta \Rightarrow \theta=\tan^{-1}\frac{x}{a}$, then

$$y=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$
$$=\tan^{-1}(\tan 3\theta)$$
$$=3\theta$$
$$=3\tan^{-1}\left(\frac{x}{a}\right)$$

[1]

Differentiating both sides, we have

$$\frac{dy}{dx}=3\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{a}\right)\right]$$
$$=3\times\frac{1}{1+\left(\frac{x}{a}\right)^2}$$
$$=\frac{3a}{a^2+x^2}$$

[1]

Q. 38. Differentiate the function

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), -1<x<1, x\neq 0 \text{ w.r.t. } x.$$

[NCERT Exemp. Ex. 5.3, Q. 43, Page 110]

Ans. Given that,

$$y=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), -1<x<1, x\neq 0$$

Put $x^2=\cos 2\theta \Rightarrow \theta=\frac{1}{2}\cos^{-1}x^2$, then

$$y=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$$
$$=\tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$
$$=\tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right)$$
$$=\tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$$
$$=\tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$
$$=\tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$
$$=\frac{\pi}{4}+\theta$$
$$=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

[2]

Differentiating both sides, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2\right)$$
$$=0+\frac{1}{2}\left(-\frac{2x}{\sqrt{1-x^4}}\right)$$
$$=-\frac{x}{\sqrt{1-x^4}}$$

[1]

Q. 39. Find $\frac{dy}{dx}$ of the function expressed in parametric form $x=3\cos\theta-2\cos^3\theta$, $y=3\sin\theta-2\sin^3\theta$.

[NCERT Exemp. Ex. 5.3, Q. 46, Page 110]

Ans. Given that,

$$x=3\cos\theta-2\cos^3\theta,\ y=3\sin\theta-2\sin^3\theta.$$
$$x=3\cos\theta-2\cos^3\theta$$

Differentiating both sides, we have

$$\frac{dx}{d\theta}=\frac{d}{d\theta}\left(3\cos\theta-2\cos^3\theta\right)$$
$$=-3\sin\theta+6\cos^2\theta\sin\theta$$

[1]

And,

$$y=3\sin\theta-2\sin^3\theta$$

Differentiating both sides, we have

$$\frac{dy}{d\theta}=\frac{d}{d\theta}\left(3\sin\theta-2\sin^3\theta\right)$$
$$=3\cos\theta-6\sin^2\theta\cos\theta$$

[1]

Thus,

$$\frac{dy}{dx}=\left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)$$
$$=\frac{3\cos\theta-6\sin^2\theta\cos\theta}{-3\sin\theta+6\cos^2\theta\sin\theta}$$
$$=\frac{3\cos\theta\left(1-2\sin^2\theta\right)}{3\sin\theta\left(-1+2\cos^2\theta\right)}$$
$$=\cot\theta\times\frac{\cos 2\theta}{\cos 2\theta}$$
$$=\cot\theta$$

[1]

Q. 40. If $x=e^{\cos 2t}$ and $y=e^{\sin 2t}$, prove that $\frac{dy}{dx}=-\frac{y\log x}{x\log y}$.

[NCERT Exemp. Ex. 5.3, Q. 49, Page 110]

Ans. Given that $x=e^{\cos 2t}$ and $y=e^{\sin 2t}$

Since

$$x=e^{\cos 2t}$$
$$\log x=\cos 2t$$
$$(\log x)^2=\cos^2 2t$$
$$y=e^{\sin 2t}$$
$$\log y=\sin 2t$$
$$(\log y)^2=\sin^2 2t$$

[1]

Then

$$(\log x)^2+(\log y)^2=\sin^2 2t+\cos^2 2t$$
$$=1$$

[1]

Thus,

$$\Rightarrow \frac{2(\log x)}{x}+\frac{2(\log y)}{y}\frac{dy}{dx}=0$$
$$\Rightarrow \frac{dy}{dx}=-\frac{y\log x}{x\log y}$$

[1]

Q. 41. If $x=3\sin t-\sin 3t$ and $y=3\cos t-\cos 3t$, find $\frac{dy}{dx}$ at $t=\frac{\pi}{3}$.

[NCERT Exemp. Ex. 5.3, Q. 51, Page 110]

Ans. Given that,

$x = 3\sin t - \sin 3t$ and $y = 3\cos t - \cos 3t$.

$x = 3\sin t - \sin 3t$

Differentiating both sides, we have

$$\frac{dx}{dt} = 3\cos t - 3\cos 3t$$

And,

$y = 3\cos t - \cos 3t$

Differentiating both sides, we have

$$\frac{dy}{dt} = -3\sin t + 3\sin 3t$$ **[1]**

Thus,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-3\sin t + 3\sin 3t}{3\cos t - 3\cos 3t} = \frac{-\sin t + \sin 3t}{\cos t - \cos 3t}$$ **[1]**

Therefore, $\frac{dy}{dx}$ at $t = \frac{\pi}{3}$ is

$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{3}} = \frac{-\sin\frac{\pi}{3} + \sin\pi}{\cos\frac{\pi}{3} - \cos\pi} = \frac{-\frac{\sqrt{3}}{2} + 0}{\frac{1}{2} + 1} = -\frac{1}{\sqrt{3}}$$ **[1]**

Q. 42. Differentiate $\frac{x}{\sin x}$ with respect to $\sin x$.

[NCERT Exemp. Ex. 5.3, Q. 52, Page 111]

Ans. Let $u = \frac{x}{\sin x}$ and $v = \sin x$.

Then,

$u = \frac{x}{\sin x}$

Differentiating both sides, we have

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{x}{\sin x}\right) = \frac{\sin x \frac{d}{dx}(x) - x\frac{d}{dx}(\sin x)}{\sin^2 x} = \frac{\sin x - x\cos x}{\sin^2 x}$$ **[1]**

And,

$v = \sin x$

Differentiating both sides, we have

$$\frac{dv}{dx} = \frac{d}{dx}(\sin x) = \cos x$$ **[1]**

Thus,

$$\frac{du}{dv} = \frac{\frac{\sin x - x\cos x}{\sin^2 x}}{\cos x} = \frac{\sin x - x\cos x}{\sin^2 x\cos x} = \frac{\tan x - x}{\sin^2 x}$$ **[1]**

Q. 43. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}x$ when $x \neq 0$.

[NCERT Exemp. Ex. 5.3, Q. 53, Page 111]

Ans. Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \tan^{-1}x$.

Put $x = \tan\theta$ in $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, we have

$$u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$ **[2]**

Thus, $\frac{du}{dx} = \frac{1}{2(1+x^2)}$

And,

$v = \tan^{-1}x \Rightarrow \frac{dv}{dx} = \frac{1}{1+x^2}$

Thus,

$$\frac{du}{dv} = \frac{\frac{1}{2(1+x^2)}}{\frac{1}{(1+x^2)}} = \frac{1}{2}$$ **[1]**

Q. 44. Find $\frac{dy}{dx}$ when x and y are connected by the relation $\sin(xy) + \frac{x}{y} = x^2 - y$.

[NCERT Exemp. Ex. 5.3, Q. 54, Page 111]

Ans. Given that $\sin(xy) + \frac{x}{y} = x^2 - y$.

Differentiating with respect to x, we have

$$\frac{d}{dx}\left[\sin(xy) + \frac{x}{y}\right] = \frac{d}{dx}(x^2 - y)$$

$$\Rightarrow \frac{d}{dx}[\sin(xy)] + \frac{d}{dx}\left(\frac{x}{y}\right) = \frac{d}{dx}(x^2) - \frac{d}{dx}(y)$$

$$\Rightarrow \cos(xy)\left(x\frac{dy}{dx}+y\right)+\left(\frac{y-x\frac{dy}{dx}}{y^2}\right)=2x-\frac{dy}{dx}$$

$$\Rightarrow \cos(xy)\left(xy^2\frac{dy}{dx}+y^3\right)+y-x\frac{dy}{dx}=2xy^2-\frac{dy}{dx}y^2$$

$$\Rightarrow xy^2\cos(xy)\frac{dy}{dx}+y^3\cos(xy)+y-x\frac{dy}{dx}=2xy^2-y^2\frac{dy}{dx}$$

$$\Rightarrow xy^2\cos(xy)\frac{dy}{dx}-x\frac{dy}{dx}+y^2\frac{dy}{dx}=2xy^2-y^3\cos(xy)-y$$

$$\Rightarrow \frac{dy}{dx}\left[xy^2\cos(xy)-x+y^2\right]=2xy^2-y^3\cos(xy)-y$$

$$\Rightarrow \frac{dy}{dx}=\frac{2xy^2-y^3\cos(xy)-y}{xy^2\cos(xy)-x+y^2}$$ **[3]**

Q. 45. Find $\frac{dy}{dx}$ when x and y are connected by the relation $\sec(x+y)=xy$.

[NCERT Exemp. Ex. 5.3, Q. 55, Page 111]

Ans. Given that, $\sec(x+y)=xy$.

Differentiating with respect to x, we have

$$\frac{d}{dx}\left[\sec(x+y)\right]=\frac{d}{dx}(xy)$$

$$\Rightarrow \sec(x+y)\tan(x+y)\left(1+\frac{dy}{dx}\right)=x\frac{dy}{dx}+y$$

$$\Rightarrow \sec(x+y)\tan(x+y)+\sec(x+y)\tan(x+y)\frac{dy}{dx}=x\frac{dy}{dx}+y$$

$$\Rightarrow \sec(x+y)\tan(x+y)\frac{dy}{dx}-x\frac{dy}{dx}=y-\sec(x+y)\tan(x+y)$$

$$\Rightarrow \left[\sec(x+y)\tan(x+y)-x\right]\frac{dy}{dx}=y-\sec(x+y)\tan(x+y)$$

$$\therefore \quad \frac{dy}{dx}=\frac{y-\sec(x+y)\tan(x+y)}{\sec(x+y)\tan(x+y)-x}$$ **[3]**

Q. 46. If $ax^2+2hxy+by^2+2gx+2fy+c=0$, then show that $\frac{dy}{dx}\times\frac{dx}{dy}=1$.

[NCERT Exemp. Ex. 5.3, Q. 58, Page 111]

Ans. Given that, $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Differentiating with respect to x, we have

$$2ax+2h\left(x\frac{dy}{dx}+y\right)+2by\frac{dy}{dx}+2g+2f\frac{dy}{dx}+0=0$$

$$\Rightarrow 2ax+2hx\frac{dy}{dx}+2hy+2by\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0$$

$$\Rightarrow (2hx+2by+2f)\frac{dy}{dx}=-(2ax+2hy+2g)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{2ax+2hy+2g}{2hx+2by+2f}=-\frac{ax+hy+g}{hx+by+f}$$

$$\Rightarrow \frac{dx}{dy}=-\frac{hx+by+f}{ax+hy+g}$$ **[2]**

Thus,

$$\Rightarrow \frac{dy}{dx}\times\frac{dx}{dy}=\left(-\frac{ax+hy+g}{hx+by+f}\right)\left(-\frac{hx+by+f}{ax+hy+g}\right)=1$$ **[1]**

Q. 47. If $x=e^{\frac{x}{y}}$, then prove that $\frac{dy}{dx}=\frac{x-y}{x\log x}$.

[NCERT Exemp. Ex. 5.3, Q. 59, Page 111]

Ans. Given that, $x=e^{\frac{x}{y}}$.

Differentiating with respect to x, we have

$$\frac{d}{dx}(x)=\frac{d}{dx}\left(e^{\frac{x}{y}}\right)$$

$$1=e^{\frac{x}{y}}\left[\frac{y\frac{d}{dx}(x)-x\frac{dy}{dx}}{y^2}\right]$$

$$y^2=e^{\frac{x}{y}}\left(y-x\frac{dy}{dx}\right)$$

$$=ye^{\frac{x}{y}}-xe^{\frac{x}{y}}\frac{dy}{dx}$$

$$xe^{\frac{x}{y}}\frac{dy}{dx}=ye^{\frac{x}{y}}-y^2$$

$$\frac{dy}{dx}=\frac{y\left(e^{\frac{x}{y}}-y\right)}{xe^{\frac{x}{y}}}$$

$$=\frac{\left(e^{\frac{x}{y}}-y\right)}{\frac{x}{y}e^{\frac{x}{y}}}$$

$$=\frac{x-y}{x\log x}$$ **[3]**

Q. 48. If $y^x=e^{y-x}$, then prove that $\frac{dy}{dx}=\frac{(1+\log y)^2}{\log y}$.

[NCERT Exemp. Ex. 5.3, Q. 60, Page 111]

Ans. Given that, $y^x=e^{y-x}$

Taking log both sides, we have

$$x\log y=y-x$$

$$x(1+\log y)=y$$

$$x=\frac{y}{1+\log y}$$

Differentiating w.r.t. y both sides, we have

$$\frac{dy}{dx}=\frac{(1+\log y)\frac{d}{dy}(y)-y\frac{d}{dy}(1+\log y)}{(1+\log y)^2}$$

$$=\frac{(1+\log y)-y.\frac{1}{y}}{(1+\log y)^2}$$

$$\frac{dy}{dx}=\frac{\log y}{(1+\log y)^2}$$

$$\therefore \quad \frac{dy}{dx}=\frac{(1+\log y)^2}{\log y}$$ **[2]**

Q. 49. If $y=(\cos x)^{(\cos x)^{(\cos x)\ldots\infty}}$, then show that $\frac{dy}{dx}=\frac{y^2\tan x}{y\log\cos x-1}$.

[NCERT Exemp. Ex. 5.3, Q. 61, Page 111]

Ans. Given that,

$$y = (\cos x)^{(\cos x)^{(\cos x)\ldots\infty}}$$

$$\Rightarrow y = (\cos x)^y$$

Taking log both sides, we have

$\log y = y \log \cos x$ **[1]**

Differentiating with respect to x, we have

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(y \log \cos x)$$

$$\frac{1}{y}\frac{dy}{dx} = y\frac{d}{dx}(\log \cos x) + \log \cos x \frac{dy}{dx}$$

$$\left(\frac{1}{y} - \log \cos x\right)\frac{dy}{dx} = y\left(\frac{-\sin x}{\cos x}\right)$$

$$\left(\frac{1 - y\log \cos x}{y}\right)\frac{dy}{dx} = -y\tan x$$

$$\frac{dy}{dx} = -\frac{y^2 \tan x}{1 - y\log \cos x}$$

$$= \frac{y^2 \tan x}{y \log \cos x - 1}$$ **[2]**

Q. 50. If $x\sin(a+y) + \sin a\cos(a+y) = 0$, then prove that $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$.

[NCERT Exemp. Ex. 5.3, Q. 62, Page 111]

Ans. Given that,

$$x\sin(a+y) + \sin a\cos(a+y) = 0$$

$$x = \frac{-\sin a\cos(a+y)}{\sin(a+y)}$$

$$= -\sin a\cot(a+y)$$ **[1]**

Differentiating with respect to y, we have

$$\frac{dx}{dy} = -\sin a\frac{d}{dy}[\cot(a+y)]$$

$$= -\sin a[-\operatorname{cosec}^2(a+y)]$$

$$\frac{dx}{dy} = \frac{\sin a}{\sin^2(a+y)}$$

$$\therefore \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$ **[2]**

Q. 51. If $y = \tan^{-1}x$, then $\dfrac{d^2y}{dx^2}$ in terms of y alone.

[NCERT Exemp. Ex. 5.3, Q. 64, Page 111]

Ans. Given that, $y = \tan^{-1}x \Rightarrow x = \tan y$

Differentiating with respect to x, we have

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan y)$$

$$\Rightarrow \quad 1 = \sec^2 y\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \cos^2 y$$ **[1½]**

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\cos^2 y)$$

$$= 2\cos y(-\sin y)\frac{dy}{dx}$$

$$= 2\cos y(-\sin y)\cos^2 y$$

$$= -\sin 2y\cos^2 y$$ **[1½]**

Q. 52. If $x^m y^n = (x+y)^{m+n}$, then prove that $\dfrac{d^2y}{dx^2} = 0$.

[NCERT Exemp. Ex. 5.3, Q. 80(ii), Page 113, CBSE Board, Delhi Region, 2017]

Ans. Given that, $x^m y^n = (x+y)^{m+n}$

Taking log both sides, we have

$$\log(x^m y^n) = \log(x+y)^{m+n}$$

$$\Rightarrow m\log x + n\log y = (m+n)\log(x+y)$$ **[1]**

Differentiating with respect to x, we have

$$\Rightarrow \frac{d}{dx}(m\log x + n\log y) = \frac{d}{dx}[(m+n)\log(x+y)]$$

$$\Rightarrow \quad m\times\frac{1}{x} + n\times\frac{1}{y}\frac{dy}{dx} = (m+n)\times\frac{1}{x+y}\times\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{my-nx}{x(x+y)} = \frac{my-nx}{y(x+y)}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$ **[1]**

Again, differentiating with respect to x, we have

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{x\times\frac{y}{x} - y}{x^2} = 0$$ **[1]**

Q. 53. If $x^m y^n = (x+y)^{m+n}$, then prove that $\dfrac{dy}{dx} = \dfrac{y}{x}$.

[NCERT Exemp. Ex. 5.3, Q. 80(i), Page 113]

Ans. Given that, $x^m y^n = (x+y)^{m+n}$

Taking log both sides, we have

$$\log(x^m y^n) = \log(x+y)^{m+n}$$

$$\Rightarrow m\log x + n\log y = (m+n)\log(x+y)$$ **[1]**

Differentiating with respect to x, we have

$$\Rightarrow \frac{d}{dx}(m\log x + n\log y) = \frac{d}{dx}[(m+n)\log(x+y)]$$

$$\Rightarrow \quad m\times\frac{1}{x} + n\times\frac{1}{y}\frac{dy}{dx} = (m+n)\times\frac{1}{x+y}\times\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{my-nx}{x(x+y)} = \frac{my-nx}{y(x+y)}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$ **[2]**

Q. 54. If $y = x^x$, then prove that $\dfrac{d^2y}{dx^2} - \dfrac{1}{y}\left(\dfrac{dy}{dx}\right)^2 - \dfrac{y}{x} = 0$.

[CBSE Board, Delhi Region, 2017]

Ans. Given that, $y = x^x$

Taking log both sides, we have

$\log y = x\log x$

Differentiating with respect to x, we have

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(x\log x)$$

$$\frac{1}{y}\frac{dy}{dx} = x\times\frac{1}{x} + \log x$$

$$\frac{dy}{dx}=y(1+\log x)$$
$$=x^x(1+\log x)$$ **[1½]**

Again, differentiating with respect to x, we have

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left[x^x(1+\log x)\right]$$
$$\Rightarrow \frac{d^2y}{dx^2}=\left(x^x\right)\frac{d}{dx}(1+\log x)+(1+\log x)\frac{d}{dx}\left(x^x\right)$$
$$\Rightarrow \frac{d^2y}{dx^2}=y\left(\frac{1}{x}\right)+(1+\log x)\frac{dy}{dx}$$
$$\Rightarrow \frac{d^2y}{dx^2}=\frac{y}{x}+\frac{1}{y}\left(\frac{dy}{dx}\right)^2$$
$$\Rightarrow \frac{d^2y}{dx^2}-\frac{1}{y}\left(\frac{dy}{dx}\right)^2-\frac{y}{x}=0$$ **[1½]**

Q. 55. Find $\frac{dy}{dx}$ at $x = 1$, $y = \frac{\pi}{4}$ if $\sin^2 y + \cos xy = k$.

[CBSE Board, Delhi Region, 2017]

Ans. Given that, $\sin^2 y+\cos xy=k$.

Differentiating with respect to x, we have

$$\frac{d}{dx}\left(\sin^2 y+\cos xy\right)=\frac{d}{dx}(k)$$
$$\Rightarrow 2\sin y\cos y\frac{dy}{dx}-\sin xy\left(x\frac{dy}{dx}+y\right)=0$$
$$\Rightarrow \sin 2y\frac{dy}{dx}-x\sin xy\frac{dy}{dx}-y\sin xy=0$$
$$\Rightarrow \frac{dy}{dx}=\frac{y\sin xy}{\sin 2y-x\sin xy}$$ **[2]**

Put $x = 1$, $y = \frac{\pi}{4}$, we have

$$\left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{4}\right)}=\frac{\frac{\pi}{4}\sin\frac{\pi}{4}}{\sin\frac{\pi}{2}-\sin\frac{\pi}{4}}$$
$$=\frac{\frac{\pi}{4\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}$$
$$=\frac{\pi}{4\left(\sqrt{2}-1\right)}$$ **[1]**

Q. 56. If $y=\sin^{-1}\left(6x\sqrt{1-9x^2}\right)$, $-\frac{1}{3\sqrt{2}}<x<\frac{1}{3\sqrt{2}}$, then find $\frac{dy}{dx}$.

[CBSE Board, Delhi Region, 2017]

Ans. Given that,

$$y=\sin^{-1}\left(6x\sqrt{1-9x^2}\right),\ -\frac{1}{3\sqrt{2}}<x<\frac{1}{3\sqrt{2}}$$

Put $x=\frac{1}{3}\sin\theta \Rightarrow \theta=\sin^{-1}3x$, we have

$$y=\sin^{-1}\left[6\times\frac{1}{3}\sin\theta\sqrt{1-9\left(\frac{1}{3}\sin\theta\right)^2}\right]$$
$$=\sin^{-1}(2\sin\theta\cos\theta)$$
$$=\sin^{-1}(\sin 2\theta)$$
$$=2\theta$$
$$=2\sin^{-1}(3x)$$ **[1½]**

Differentiating with respect to x, we have

$$\frac{dy}{dx}=2\frac{d}{dx}\left[\sin^{-1}(3x)\right]$$
$$=2\left(\frac{3}{\sqrt{1-(3x)^2}}\right)$$
$$=\frac{6}{\sqrt{1-9x^2}}$$ **[1½]**

Q. 57. If $e^y(x+1)=1$, then show that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$.

[CBSE Board, All India Region, 2017, NCERT Ex. 5-7 Q. 16, Page 184]

Ans. Given that, $e^y(x+1)=1$

Differentiating with respect to x, we have

$$\frac{d}{dx}\left[e^y(x+1)\right]=\frac{d}{dx}(1)$$
$$\Rightarrow e^y(1)+(x+1)e^y\frac{dy}{dx}=0$$
$$\Rightarrow e^y+(x+1)e^y\frac{dy}{dx}=0$$
$$\therefore \frac{dy}{dx}=-\frac{1}{1+x}$$ **[1½]**

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(-\frac{1}{1+x}\right)$$
$$=-\frac{d}{dx}\left[(1+x)^{-1}\right]$$
$$=-\frac{d}{dx}\left[-(1+x)^{-2}\right]$$
$$=\left[-\frac{1}{(1+x)}\right]^2$$
$$=\left(\frac{dy}{dx}\right)^2$$ **[1½]**

Q. 58. Find $\frac{dy}{dx}$ at $t=\frac{2\pi}{3}$ when $x=10(t-\sin t)$ and $y=12(1-\cos t)$.

[CBSE Board, Foreign Scheme, 2017]

Ans. Given that $x=10(t-\sin t)$ and $y=12(1-\cos t)$.

Then,

$$\frac{dx}{dt}=10\frac{d}{dt}(t-\sin t)$$
$$=10(1-\cos t)$$ **[1]**

And,

$$\frac{dy}{dt}=12\frac{d}{dt}(1-\cos t)$$
$$=12\sin t$$ **[1]**

Thus,

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$\left(\frac{dy}{dx}\right)_{t=\frac{2\pi}{3}} = \frac{6\sin\frac{2\pi}{3}}{5\left(1-\sin\frac{2\pi}{3}\right)}$$

$$= \frac{6\times\frac{\sqrt{3}}{2}}{5\left(1-\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{3\sqrt{3}}{5\left(2-\sqrt{3}\right)}$$ **[1]**

Q. 59. If $xy = e^{(x-y)}$, then show that $\frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$.

[CBSE Board, Foreign Scheme, 2017]

Ans. Given that,

$xy = e^{(x-y)}$

Taking log both sides, we have

$$\log(xy) = \log\left[e^{(x-y)}\right]$$

$$\log x + \log y = (x-y)\log e$$
$$= (x-y)$$ **[1½]**

Differentiating both sides with respect to x, we have

$$\Rightarrow \frac{d}{dx}(\log x + \log y) = \frac{d}{dx}(x-y)$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} + \frac{dy}{dx} = 1 - \frac{1}{x}$$

$$\Rightarrow \left(\frac{1+y}{y}\right)\frac{dy}{dx} = \frac{x-1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\left(\frac{x-1}{y+1}\right)$$ **[1½]**

Q. 60. If $\log y = \tan^{-1}x$, then show that

$$\mathbf{(1+x^2)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0.}$$

[CBSE Board, Foreign Scheme, 2017]

Ans. Given that, $\log y = \tan^{-1}x$

Differentiating with respect to x, we have

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(\tan^{-1}x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\therefore \frac{dy}{dx} = \frac{y}{1+x^2}$$ **[1]**

Again, differentiating with respect to x, we have

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y}{1+x^2}\right)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{(1+x^2)\frac{dy}{dx} - y(2x)}{(1+x^2)^2}$$

$$\Rightarrow (1+x^2)^2\frac{d^2y}{dx^2} = (1+x^2)\frac{dy}{dx} - 2x(1+x^2)\frac{dy}{dx}$$

$$\Rightarrow (1+x^2)^2\frac{d^2y}{dx^2} = (1+x^2)\frac{dy}{dx} - 2x(1+x^2)\frac{dy}{dx}$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} = \frac{dy}{dx} - 2x\frac{dy}{dx}$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$$ **[2]**

Q. 61. Find $\frac{dy}{dx}$ if $y = \sin^{-1}\left(\frac{6x-4\sqrt{1-4x^2}}{5}\right)$.

[CBSE Board, All India Region, 2016]

Ans. Given that, $y = \sin^{-1}\left(\frac{6x-4\sqrt{1-4x^2}}{5}\right)$

Put $2x = \sin\theta$, we have

$$y = \sin^{-1}\left(\frac{3\sin\theta - 4\sqrt{1-\sin^2\theta}}{5}\right)$$

$$= \sin^{-1}\left(\frac{3\sin\theta - 4\cos\theta}{5}\right)$$

$$= \sin^{-1}\left(\frac{3}{5}\sin\theta - \frac{4}{5}\cos\theta\right)$$

$$= \sin^{-1}(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$$

$$= \sin^{-1}\sin(\theta-\alpha)$$

$$= \theta - \alpha$$

$$= \sin^{-1}(2x) - \alpha$$ **[1½]**

Differentiate with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin^{-1}(2x)\right] - 0$$

$$= \frac{2}{\sqrt{1-4x^2}}$$ **[1½]**

Q. 62. Differentiate $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to x.

[CBSE Board, Delhi Region, 2018]

Ans. Given that,

$$y = \tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$= \tan^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\cot\frac{x}{2}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{2} - \frac{x}{2}\right)\right]$$

$$= \frac{\pi}{2} - \frac{x}{2}$$ **[2]**

Differentiating with respect to x, we have

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}$$ **[1]**

Q. 63. If $y = \sin(\sin x)$, then prove that

$$\mathbf{\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x = 0}.$$

[CBSE Board, Delhi Region, 2018]

Ans. Given that, $y = \sin(\sin x)$

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(\sin x)]$$

$$= \cos x \cos(\sin x)$$ **[1]**

Again, differentiating with respect to x, we have

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[\cos x \cos(\sin x)]$$

$$\Rightarrow \frac{d^2y}{dx^2} = \cos x \frac{d}{dx}[\cos(\sin x)] + \cos(\sin x)\frac{d}{dx}(\cos x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \cos x[-\cos x \sin(\sin x)] + \cos(\sin x)(-\sin x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\cos^2 x \sin(\sin x) - \sin x \cos(\sin x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -y\cos^2 x - \tan x \cos x \cos(\sin x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -y\cos^2 x - \tan x \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} + y\cos^2 x + \tan x \frac{dy}{dx} = 0$$ **[2]**

Q. 64. If $(x^2 + y^2)^2 = xy$, then find $\frac{dy}{dx}$.

[CBSE Board, Delhi Region, 2018]

Ans. Given that, $(x^2 + y^2)^2 = xy$

Differentiating with respect to x, we have

$$\frac{d}{dx}(x^2 + y^2)^2 = \frac{d}{dx}(xy)$$

$$\Rightarrow 2(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow 4x(x^2 + y^2) + 4y(x^2 + y^2)\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$\Rightarrow (4y(x^2 + y^2) - x)\frac{dy}{dx} = y - 4x(x^2 + y^2)$$

$$\Rightarrow (4x^2y + 4y^3 - x)\frac{dy}{dx} = y - 4x^3 - 4xy^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{y - 4x^3 - 4xy^2}{4x^2y + 4y^3 - x}$$ **[3]**

Q. 65. If $x = a(2\theta - \sin 2\theta)$ and $y = a(1 - \cos 2\theta)$, then find $\frac{dy}{dx}$ when $\theta = \frac{\pi}{3}$.

[CBSE Board, Delhi Region, 2018]

Ans. Given that, $x = a(2\theta - \sin 2\theta),\ y = a(1 - \cos 2\theta)$

Then,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}[a(2\theta - \sin 2\theta)]$$

$$= a(2 - 2\cos 2\theta)$$

$$= 2a(1 - \cos 2\theta)$$ **[1]**

And,

$$\frac{dy}{d\theta} = \frac{d}{d\theta}[a(1 - \cos 2\theta)]$$

$$= a(0 + 2\sin 2\theta)$$

$$= 2a\sin\theta$$ **[1]**

Thus,

$$\frac{dy}{dx} = \left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)$$

$$= \frac{2a\sin\theta}{2a(1 - \cos 2\theta)}$$

$$= \frac{\sin\theta}{1 - \cos 2\theta}$$

$$= \frac{\sin\theta}{2\sin^2\theta}$$

$$= \frac{1}{2\sin\theta}$$

Put $\theta = \frac{\pi}{3}$, we get

$$\left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{3}} = \frac{1}{2\sin\frac{\pi}{3}}$$

$$= \frac{1}{2 \times \frac{\sqrt{3}}{2}}$$

$$= \frac{1}{\sqrt{3}}$$ **[1]**

Long Answer Type Questions

(5 or 6 marks each)

Q. 1. Differentiate the function $x^x - 2^{\sin x}$ with respect to x.

[NCERT Ex. 5.5, Q. 4, Page 178]

Ans. Given that, $y = x^x - 2^{\sin x}$

Let $u = x^x$ and $v = 2^{\sin x}$. Then

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \quad \ldots \text{(i)}$$

Now, $u = x^x$

Taking log both sides, we have

$$\log u = \log[x^x]$$

$$= x\log(x)$$ **[1]**

Differentiating both sides with respect to x, we have

$$\Rightarrow \frac{d}{dx}(\log u) = \frac{d}{dx}[x\log(x)]$$

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}[\log(x)] + \log(x)\frac{d}{dx}(x)$$

$\Rightarrow \quad \dfrac{du}{dx} = u\left(x \times \dfrac{1}{x} + \log x \times 1\right)$

$\therefore \quad \dfrac{du}{dx} = x^x(1+\log x)$ **[1½]**

For, $v = 2^{\sin x}$

Taking log both sides, we have

$\log v = \log\left(2^{\sin x}\right)$

$= \sin x \log(2)$ **[1]**

Differentiating both sides with respect to x, we have

$\Rightarrow \dfrac{d}{dx}(\log v) = \dfrac{d}{dx}\left[\sin x \log(2)\right]$

$\Rightarrow \quad \dfrac{1}{v}\dfrac{dv}{dx} = \log 2 \dfrac{d}{dx}(\sin x)$

$\Rightarrow \quad \dfrac{dv}{dx} = v\left[\log 2 \dfrac{d}{dx}(\sin x)\right]$

$\therefore \quad \dfrac{dv}{dx} = 2^{\sin x}\log 2 \cos x$

From equation (i), we have

$\dfrac{dy}{dx} = x^x(1+\log x) - 2^{\sin x}\log 2 \cos x$ **[1½]**

Q. 2. Differentiate the function $(x+3)^2(x+4)^3(x+5)^4$ with respect to x.

[NCERT Ex. 5.5, Q. 5, Page 178]

Ans. Given that, $y = (x+3)^2(x+4)^3(x+5)^4$

Taking log both sides, we have

$\log y = \log\left[(x+3)^2(x+4)^3(x+5)^4\right]$

$= \log(x+3)^2 + \log(x+4)^3 + \log(x+5)^4$

$= 2\log(x+3) + 3\log(x+4) + 4\log(x+5)$ **[2]**

Differentiating both sides with respect to x, we have

$\Rightarrow \dfrac{d}{dx}(\log y) = \dfrac{d}{dx}\begin{bmatrix} 2\log(x+3)+3\log(x+4) \\ +4\log(x+5) \end{bmatrix}$

$\Rightarrow \quad \dfrac{1}{y}\dfrac{dy}{dx} = 2 \times \dfrac{1}{x+3} + 3 \times \dfrac{1}{x+4} + 4 \times \dfrac{1}{x+5}$

$\Rightarrow \dfrac{dy}{dx} = y\left[\dfrac{2}{x+3} + \dfrac{3}{x+4} + \dfrac{4}{x+5}\right]$

$\Rightarrow \dfrac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4 \left[\dfrac{2}{x+3} + \dfrac{3}{x+4} + \dfrac{4}{x+5}\right]$

$\Rightarrow \dfrac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4 \dfrac{\begin{bmatrix} 2(x+4)(x+5)+3(x+3) \\ (x+5)+4(x+3)(x+4) \end{bmatrix}}{(x+3)(x+4)(x+5)}$

$\Rightarrow \dfrac{dy}{dx} = (x+3)(x+4)^2(x+5)^3 \begin{bmatrix} 2(x^2+9x+20) \\ +3(x^2+8x+15) \\ +4(x^2+7x+12) \end{bmatrix}$

$\Rightarrow \dfrac{dy}{dx} = (x+3)(x+4)^2(x+5)^3(9x^2+70x+133)$ **[3]**

Q. 3. Differentiate the function $\left(x+\dfrac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$ with respect to x.

[NCERT Ex. 5.5, Q. 6, Page 178]

Ans. Given that,

$y = \left(x+\dfrac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Let $u = \left(x+\dfrac{1}{x}\right)^x$ and $v = x^{\left(1+\frac{1}{x}\right)}$. Then

$\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ … (i)

Now, $u = \left(x+\dfrac{1}{x}\right)^x$ **[1]**

Taking log both sides, we have

$\log u = \log\left[\left(x+\dfrac{1}{x}\right)^x\right]$

$= x\log\left(x+\dfrac{1}{x}\right)$ **[1]**

Differentiating both sides with respect to x, we have

$\Rightarrow \dfrac{d}{dx}(\log u) = \dfrac{d}{dx}\left[x\log\left(x+\dfrac{1}{x}\right)\right]$

$\Rightarrow \dfrac{1}{u}\dfrac{du}{dx} = x\dfrac{d}{dx}\left[\log\left(x+\dfrac{1}{x}\right)\right] + \log\left(x+\dfrac{1}{x}\right)\dfrac{d}{dx}[x]$

$\Rightarrow \quad \dfrac{du}{dx} = u\left[x \times \dfrac{1}{\left(x+\frac{1}{x}\right)} \times \left(1-\dfrac{1}{x^2}\right) + \log\left(x+\dfrac{1}{x}\right)\right]$

$\Rightarrow \quad \dfrac{du}{dx} = \left(x+\dfrac{1}{x}\right)^x\left[\dfrac{x^2-1}{x^2+1} + \log\left(x+\dfrac{1}{x}\right)\right]$ **[1½]**

For, $v = x^{\left(1+\frac{1}{x}\right)}$

Taking log both sides, we have

$\Rightarrow \log v = \log\left[x^{\left(1+\frac{1}{x}\right)}\right]$

$\Rightarrow \log v = \left(1+\dfrac{1}{x}\right)\log(x)$

Differentiating both sides with respect to x, we have

$\Rightarrow \dfrac{d}{dx}(\log v) = \dfrac{d}{dx}\left[\left(1+\dfrac{1}{x}\right)\log(x)\right]$

$\Rightarrow \quad \dfrac{1}{v}\dfrac{dv}{dx} = \left(1+\dfrac{1}{x}\right)\dfrac{d}{dx}[\log x] + \log x \dfrac{d}{dx}\left(1+\dfrac{1}{x}\right)$

$\Rightarrow \quad \dfrac{dv}{dx} = v\left[\left(1+\dfrac{1}{x}\right)\dfrac{1}{x} + \log x\left(-\dfrac{1}{x^2}\right)\right]$

$\Rightarrow \quad \dfrac{dv}{dx} = x^{\left(1+\frac{1}{x}\right)}\left[\dfrac{1}{x} + \dfrac{1}{x^2} - \dfrac{\log x}{x^2}\right]$

$\Rightarrow \quad \dfrac{dv}{dx} = x^{\left(1+\frac{1}{x}\right)}\left[\dfrac{x+1-\log x}{x^2}\right]$ **[1½]**

From equation (i), we have

$\dfrac{dy}{dx} = \left(x+\dfrac{1}{x}\right)^x\left[\dfrac{x^2-1}{x^2+1} + \log\left(x+\dfrac{1}{x}\right)\right] + x^{\left(1+\frac{1}{x}\right)}\left[\dfrac{x+1-\log x}{x^2}\right]$

Q. 4. Differentiate the function $(\log x)^x + x^{\log x}$ with respect to x.

[NCERT Ex. 5.5, Q. 7, Page 178]

Ans. Given that, $y = (\log x)^x + x^{\log x}$

Let $u = (\log x)^x$ and $v = x^{\log x}$. Then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \ldots \text{(i)}$$

Now, $u = (\log x)^x$ **[1]**

Taking log both sides, we have

$$\log u = \log\left[(\log x)^x\right]$$
$$= x\log(\log x)$$

[1]

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(\log u) = \frac{d}{dx}\left[x\log(\log x)\right]$$
$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\left[\log(\log x)\right] + \log(\log x)\frac{d}{dx}[x]$$
$$\frac{du}{dx} = u\left[x \times \frac{1}{\log x} \times \frac{1}{x} + \log(\log x) \times 1\right]$$
$$= (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right]$$
$$= (\log x)^x\left[\frac{1 + \log x\log(\log x)}{\log x}\right]$$

[1½]

For, $v = x^{\log x}$

Taking log both sides, we have

$$\log v = \log\left(x^{\log x}\right)$$
$$= \log x \log x$$
$$= (\log x)^2$$

Differentiating both sides with respect to x, we have

$$\Rightarrow \frac{d}{dx}(\log v) = \frac{d}{dx}\left[(\log x)^2\right]$$
$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = 2\log x \times \left(\frac{1}{x}\right)$$
$$\Rightarrow \frac{dv}{dx} = v\left[\frac{2}{x}\log x\right]$$
$$\Rightarrow \frac{dv}{dx} = x^{\log x}\left(\frac{2}{x}\log x\right)$$
$$\Rightarrow \frac{dv}{dx} = 2x^{\log x - 1}(\log x)$$

[1½]

From equation (i), we have

$$\frac{dy}{dx} = (\log x)^x\left[\frac{1 + \log x\log(\log x)}{\log x}\right] + 2x^{\log x - 1}(\log x)$$

Q. 5. Differentiate the function $(\sin x)^x + \sin^{-1}\sqrt{x}$ with respect to x.

[NCERT Ex. 5.5, Q. 8, Page 178]

Ans. Given that, $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Let $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$. Then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \ldots \text{(i)}$$

Now, $u = (\sin x)^x$ **[1]**

Taking log both sides, we have

$$\log u = \log\left[(\sin x)^x\right]$$
$$= x\log(\sin x)$$

[1]

Differentiating both sides with respect to x, we have

$$\Rightarrow \frac{d}{dx}(\log u) = \frac{d}{dx}\left[x\log(\sin x)\right]$$
$$\Rightarrow \frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\left[\log(\sin x)\right] + \log(\sin x)\frac{d}{dx}[x]$$
$$\Rightarrow \frac{du}{dx} = u\left[x \times \frac{1}{\sin x} \times \cos x + \log(\sin x) \times 1\right]$$
$$\Rightarrow \frac{du}{dx} = (\sin x)^x\left[\frac{x\cos x}{\sin x} + \log(\sin x)\right]$$

[1½]

For, $v = \sin^{-1}\sqrt{x}$

Differentiating both sides with respect to x, we have

$$\Rightarrow \frac{dv}{dx} = \frac{d}{dx}\left[\sin^{-1}\sqrt{x}\right]$$
$$\Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$$
$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{x - x^2}}$$

From equation (i), we have

$$\frac{dy}{dx} = (\sin x)^x\left[\frac{x\cos x}{\sin x} + \log(\sin x)\right] + \frac{1}{2\sqrt{x - x^2}}$$

[1½]

Q. 6. Differentiate the function $y = x^{\sin x} + (\sin x)^{\cos x}$ with respect to x.

[NCERT Ex. 5.5, Q. 9, Page 178]

Ans. Given that, $y = x^{\sin x} + (\sin x)^{\cos x}$

Let $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$. Then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \ldots \text{(i)}$$

Now, $u = x^{\sin x}$ **[1]**

Taking log both sides, we have

$$\log u = \log\left(x^{\sin x}\right)$$
$$= \sin x \log x$$

[1]

Differentiating both sides with respect to x, we have

$$\Rightarrow \frac{d}{dx}(\log u) = \frac{d}{dx}[\sin x \log x]$$
$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \sin x\frac{d}{dx}[\log x] + \log x\frac{d}{dx}[\sin x]$$
$$\Rightarrow \frac{du}{dx} = u\left[\sin x \times \frac{1}{x} + \log x \times \cos x\right]$$
$$\Rightarrow \frac{du}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \log x\cos x\right]$$

[1½]

For, $v = (\sin x)^{\cos x}$

Taking log both sides, we have

$$\Rightarrow \log v = \log\left[(\sin x)^{\cos x}\right]$$
$$\Rightarrow \log v = \cos x\log(\sin x)$$

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(\log v)=\frac{d}{dx}\left[\cos x\log(\sin x)\right]$$

$$\frac{1}{v}\frac{dv}{dx}=\cos x\frac{d}{dx}\left[\log(\sin x)\right]$$

$$+\log(\sin x)\frac{d}{dx}(\cos x)$$

$$\frac{dv}{dx}=v\begin{bmatrix}\cos x\times\frac{1}{\sin x}\times\cos x\\ +\log(\sin x)\times(-\sin x)\end{bmatrix}$$

$$=(\sin x)^{\cos x}\left[\cos x\cot x-\sin x\log(\sin x)\right]$$ **[1½]**

From equation (i), we have

$$\frac{dy}{dx}=x^{\sin x}\left(\frac{\sin x}{x}+\log x\cos x\right)$$

$$+(\sin x)^{\cos x}\left[\cos x\cot x-\sin x\log(\sin x)\right]$$

Q. 7. Differentiate the function $x^{x\cos x}+\frac{x^2+1}{x^2-1}$ with respect to x.

[NCERT Ex. 5.5, Q. 10, Page 178]

Ans. Given that, $y=x^{x\cos x}+\frac{x^2+1}{x^2-1}$

Let $u=x^{x\cos x}$ and $v=\frac{x^2+1}{x^2-1}$. Then

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \quad \dots \text{(i)}$$

Now, $u=x^{x\cos x}$ **[1]**

Taking log both sides, we have

$$\log u=\log\left(x^{x\cos x}\right)$$

$$=x\cos x\log x$$ **[1]**

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(\log u)=\frac{d}{dx}(x\cos x\log x)$$

$$\frac{1}{u}\frac{du}{dx}=x\cos x\frac{d}{dx}(\log x)+x\log x\frac{d}{dx}(\cos x)$$

$$+\cos x\log x\frac{d}{dx}(x)$$

$$\frac{du}{dx}=u\begin{bmatrix}x\cos x\times\frac{1}{x}+x\log x\times(-\sin x)\\ +\cos x\log x\end{bmatrix}$$

$$=x^{x\cos x}(\cos x-x\log x\sin x+\cos x\log x)$$

$$=x^{x\cos x}\left[\cos x(1+\log x)-x\log x\sin x\right]$$

For, $v=\frac{x^2+1}{x^2-1}$ **[1½]**

Taking log both sides, we have

$$\log v=\log\left(\frac{x^2+1}{x^2-1}\right)$$

$$=\log(x^2+1)-\log(x^2-1)$$

Differentiating both sides with respect to x, we have

$$\Rightarrow \frac{d}{dx}(\log v)=\frac{d}{dx}\left[\log(x^2+1)-\log(x^2-1)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\frac{d}{dx}\left[\log(x^2+1)\right]-\frac{d}{dx}\left[\log(x^2-1)\right]$$

$$\Rightarrow \frac{dv}{dx}=v\left[\frac{2x}{x^2+1}-\frac{2x}{x^2-1}\right]$$

$$\Rightarrow \frac{dv}{dx}=\frac{x^2+1}{x^2-1}\left[\frac{2x}{x^2+1}-\frac{2x}{x^2-1}\right]$$

$$\Rightarrow \frac{dv}{dx}=\frac{x^2+1}{x^2-1}\left[\frac{2x^3-2x-2x^3-2x}{(x^2-1)(x^2+1)}\right]$$

$$\Rightarrow \frac{dv}{dx}=-\left[\frac{4x}{(x^2-1)^2}\right]$$

From equation (i), we have

$$\frac{dy}{dx}=x^{x\cos x}\left[\cos x(1+\log x)-x\log x\sin x\right]$$

$$-\left[\frac{4x}{(x^2-1)^2}\right]$$ **[1½]**

Q. 8. Differentiate the function $(x\cos x)^x+(x\sin x)^{\frac{1}{x}}$ with respect to x.

[NCERT Ex. 5.5, Q. 11, Page 178]

Ans. Given that, $y=(x\cos x)^x+(x\sin x)^{\frac{1}{x}}$

Let $u=(x\cos x)^x$ and $v=(x\sin x)^{\frac{1}{x}}$. Then

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \quad \dots \text{(i)}$$

Now, $u=(x\cos x)^x$ **[1]**

Taking log both sides, we have

$$\log u=\log\left[(x\cos x)^x\right]$$

$$=x\log(x\cos x)$$

$$=x\left[\log(x)+\log(\cos x)\right]$$ **[1]**

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(\log u)=\frac{d}{dx}\left[x\{\log(x)+\log(\cos x)\}\right]$$

$$\frac{1}{u}\frac{du}{dx}=x\frac{d}{dx}\left[\log(x)+\log(\cos x)\right]$$

$$+\left[\log(x)+\log(\cos x)\right]\frac{d}{dx}(x)$$

$$\frac{du}{dx}=u\begin{bmatrix}x\left\{\frac{1}{x}+\frac{1}{\cos x}\times(-\sin x)\right\}\\ +\log(x)+\log(\cos x)\end{bmatrix}$$

$$=(x\cos x)^x(1-x\tan x+\log x\cos x)$$

For, $v=(x\sin x)^{\frac{1}{x}}$ **[1½]**

Taking log both sides, we have

$$\log v = \log\left[(x\sin x)^{\frac{1}{x}}\right]$$
$$= \frac{1}{x}\log(x\sin x)$$
$$= \frac{1}{x}(\log x + \log\sin x)$$

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(\log v) = \frac{d}{dx}\left[\frac{1}{x}(\log x + \log\sin x)\right]$$
$$\frac{1}{v}\frac{dv}{dx} = \frac{1}{x}\frac{d}{dx}\left[(\log x + \log\sin x)\right]$$
$$+ (\log x + \log\sin x)\frac{d}{dx}\left(\frac{1}{x}\right)$$
$$\frac{dv}{dx} = v\left[\frac{1}{x}\left(\frac{1}{x} + \frac{\cos x}{\sin x}\right) + \left(\begin{matrix}\log x\\ +\log\sin x\end{matrix}\right)\left(\frac{-1}{x^2}\right)\right]$$
$$= (x\sin x)^{\frac{1}{x}}\left[\frac{1}{x^2} + \frac{\cot x}{x} - \frac{1}{x^2}\{\log(x\sin x)\}\right]$$
$$= (x\sin x)^{\frac{1}{x}}\left(\frac{1 + x\cot x - \log x\sin x}{x^2}\right)$$

From equation (i), we have

$$\frac{dy}{dx} = (x\cos x)^x(1 - x\tan x + \log x\cos x)$$
$$+ (x\sin x)^{\frac{1}{x}}\left(\frac{1 + x\cot x - \log x\sin x}{x^2}\right)$$

[1½]

Q. 9. Find $\frac{dy}{dx}$ of $y^x = x^y$.

[NCERT Ex. 5.5, Q. 13, Page 178]

Ans. Given that, $y^x = x^y$

Let $v = y^x$ and $u = x^y$. Then

$$\frac{dv}{dx} = \frac{du}{dx} \quad \dots \text{(i)}$$

For, $v = y^x$

Taking log both sides, we have

$$\log v = \log(y^x)$$
$$= x\log(y)$$

[1]

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(\log v) = \frac{d}{dx}[x\log(y)]$$
$$\frac{1}{v}\frac{dv}{dx} = x\frac{d}{dx}(\log y) + \log y\frac{d}{dx}(x)$$
$$\frac{dv}{dx} = v\left(\frac{x}{y}\frac{dy}{dx} + \log y\right)$$
$$= y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right)$$

[1]

Now, $u = x^y$

Taking log both sides, we have

$$\log u = \log(x^y)$$
$$= y\log(x)$$

[1]

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(\log u) = \frac{d}{dx}[y\log(x)]$$
$$\frac{1}{u}\frac{du}{dx} = y\frac{d}{dx}[\log(x)] + \log x\frac{dy}{dx}$$
$$\frac{du}{dx} = u\left(\frac{y}{x} + \log x\frac{dy}{dx}\right)$$
$$= x^y\left(\frac{y}{x} + \log x\frac{dy}{dx}\right)$$

[1]

From equation (i), we have

$$y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = x^y\left(\frac{y}{x} + \log x\frac{dy}{dx}\right)$$
$$xy^{x-1}\frac{dy}{dx} + y^x\log y = x^{y-1}y + x^y\log x\frac{dy}{dx}$$
$$\left(xy^{x-1} - x^y\log x\right)\frac{dy}{dx} = x^{y-1}y - y^x\log y$$
$$\frac{dy}{dx} = \frac{x^{y-1}y - y^x\log y}{xy^{x-1} - x^y\log x}$$
$$= \frac{\frac{y}{x} - \log y}{\frac{x}{y} - \log x} \quad [\because y^x = x^y]$$
$$= \frac{y}{x}\left(\frac{y - x\log y}{x - y\log x}\right)$$

[1]

Q. 10. Find $\frac{dy}{dx}$ of $x^y + y^x = 1$.

[NCERT Ex. 5.5, Q. 12, Page 178]

Ans. Given that $x^y + y^x = 1$

Let $u = x^y$ and $v = y^x$. Then

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \quad \dots \text{(i)}$$

Now, $u = x^y$

Taking log both sides, we have

$$\log u = \log(x^y)$$
$$= y\log(x)$$

[1]

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(\log u) = \frac{d}{dx}[y\log(x)]$$
$$\frac{1}{u}\frac{du}{dx} = y\frac{d}{dx}[\log(x)] + \log x\frac{dy}{dx}$$
$$\frac{du}{dx} = u\left(\frac{y}{x} + \log x\frac{dy}{dx}\right)$$
$$= x^y\left(\frac{y}{x} + \log x\frac{dy}{dx}\right)$$

[1]

For, $v = y^x$

Taking log both sides, we have

$$\log v = \log(y^x)$$
$$= x\log(y)$$

[1]

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(\log v) = \frac{d}{dx}[x\log(y)]$$
$$\frac{1}{v}\frac{dv}{dx} = x\frac{d}{dx}(\log y) + \log y\frac{d}{dx}(x)$$

$$\frac{dv}{dx} = v\left(\frac{x}{y}\frac{dy}{dx} + \log y\right)$$
$$= y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right)$$ **[1]**

From equation (i), we have

$$x^y\left(\frac{y}{x} + \log x\frac{dy}{dx}\right) + y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = 0$$
$$\Rightarrow x^{y-1}y + x^y\log x\frac{dy}{dx} + xy^{x-1}\frac{dy}{dx} + y^x\log y = 0$$
$$\Rightarrow \left(x^y\log x + xy^{x-1}\right)\frac{dy}{dx} = -\left(x^{y-1}y + y^x\log y\right)$$
$$\therefore \frac{dy}{dx} = -\frac{x^{y-1}y + y^x\log y}{x^y\log x + xy^{x-1}}$$ **[1]**

Q. 11. Find $\frac{dy}{dx}$ of $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$, $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$.

[NCERT Ex. 5.6, Q. 7, Page 181]

Ans. Given that,

$$x = \frac{\sin^3 t}{\sqrt{\cos 2t}},\ y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$$

Then,

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{\sin^3 t}{\sqrt{\cos 2t}}\right)$$
$$= \frac{\sqrt{\cos 2t}\,\frac{d}{dt}\left(\sin^3 t\right) - \sin^3 t\,\frac{d}{dt}\left(\sqrt{\cos 2t}\right)}{\left(\sqrt{\cos 2t}\right)^2}$$
$$= \frac{\sqrt{\cos 2t}\left(3\sin^2 t\cos t\right) - \sin^3 t\left(\frac{-2\sin 2t}{2\sqrt{\cos 2t}}\right)}{\cos 2t}$$
$$= \frac{3\sin^2 t\cos t\cos 2t + \sin^3 t\sin 2t}{\cos 2t.\sqrt{\cos 2t}}$$ **[1½]**

And,

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{\cos^3 t}{\sqrt{\cos 2t}}\right)$$
$$= \frac{\sqrt{\cos 2t}\,\frac{d}{dt}\left(\cos^3 t\right) - \cos^3 t\,\frac{d}{dt}\left(\sqrt{\cos 2t}\right)}{\left(\sqrt{\cos 2t}\right)^2}$$
$$= \frac{\sqrt{\cos 2t}\left(-3\cos^2 t\sin t\right) - \cos^3 t\left(\frac{-2\sin 2t}{2\sqrt{\cos 2t}}\right)}{\cos 2t}$$
$$= \frac{-3\cos^2 t\sin t\cos 2t + \cos^3 t\sin 2t}{\cos 2t\sqrt{\cos 2t}}$$ **[1½]**

Thus,

$$\frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{\frac{-3\cos^2 t\sin t\cos 2t + \cos^3 t\sin 2t}{\cos 2t\sqrt{\cos 2t}}}{\frac{3\sin^2 t\cos t\cos 2t + \sin^3 t\sin 2t}{\cos 2t\sqrt{\cos 2t}}}$$
$$= -\frac{3\cos^2 t\sin t\cos 2t - \cos^3 t\sin 2t}{3\sin^2 t\cos t\cos 2t + \sin^3 t\sin 2t}$$
$$= -\frac{3\cos^2 t\sin t\cos 2t - 2\cos^3 t\sin t\cos t}{3\sin^2 t\cos t\cos 2t + 2\sin^3 t\sin t\cos t}$$
$$= -\frac{\cos^2 t\sin t\left(3\cos 2t - 2\cos^2 t\right)}{\sin^2 t\cos t\left(3\cos 2t + 2\sin^2 t\right)}$$
$$= -\frac{\cos t\left(3\left(2\cos^2 t - 1\right) - 2\cos^2 t\right)}{\sin t\left(3\left(1 - 2\sin^2 t\right) + 2\sin^2 t\right)}$$
$$= -\frac{\cos t\left(6\cos^2 t - 3 - 2\cos^2 t\right)}{\sin t\left(3 - 6\sin^2 t + 2\sin^2 t\right)}$$
$$= -\frac{\cos t\left(4\cos^2 t - 3\right)}{\sin t\left(3 - 4\sin^2 t\right)}$$
$$= -\frac{4\cos^3 t - 3\cos t}{3\sin t - 4\sin^3 t}$$
$$= -\frac{\cos 3t}{\sin 3t} = -\cos 3t$$ **[2]**

Q. 12. If $y = 3\cos(\log x) + 4\sin(\log x)$, then $x^2y_2 + xy_1 + y = 0$.

[NCERT Ex. 5.7, Q. 13, Page 184]

Ans. Given that, $y = 3\cos(\log x) + 4\sin(\log x)$

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[3\cos(\log x) + 4\sin(\log x)\right]$$
$$\Rightarrow \frac{dy}{dx} = 3\frac{d}{dx}\left[\cos(\log x)\right] + 4\frac{d}{dx}\left[\sin(\log x)\right]$$
$$\Rightarrow \frac{dy}{dx} = 3\left[-\frac{\sin(\log x)}{x}\right] + 4\left(\frac{\cos(\log x)}{x}\right)$$
$$\Rightarrow\ y_1 = \frac{1}{x}\left[-3\sin(\log x) + 4\cos(\log x)\right]$$
$$\Rightarrow xy_1 = -3\sin(\log x) + 4\cos(\log x)$$ **[2]**

Again, differentiating with respect to x, we have

$$\frac{d}{dx}(xy_1) = \frac{d}{dx}\left[-3\sin(\log x) + 4\cos(\log x)\right]$$
$$\Rightarrow xy_2 + y_1 = -3\frac{d}{dx}\left[\sin(\log x)\right] + 4\frac{d}{dx}\left[\cos(\log x)\right]$$
$$\Rightarrow xy_2 + y_1 = -3\left[\frac{\cos(\log x)}{x}\right] + 4\left[\frac{-\sin(\log x)}{x}\right]$$
$$\Rightarrow xy_2 + y_1 = \left[\frac{-3\cos(\log x) - 4\sin(\log x)}{x}\right]$$
$$\Rightarrow x^2y_2 + xy_1 = -\left[3\cos(\log x) + 4\sin(\log x)\right]$$
$$\Rightarrow x^2y_2 + xy_1 = -y$$
$$\Rightarrow x^2y_2 + xy_1 + y = 0$$ **[3]**

Q. 13. If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$.

[NCERT Ex. 5.7, Q. 14, Page 184]

Ans. Given that,

$$y = Ae^{mx} + Be^{nx} \quad \ldots(i)$$

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(Ae^{mx} + Be^{nx}\right)$$
$$= \frac{d}{dx}\left(Ae^{mx}\right) + \frac{d}{dx}\left(Be^{nx}\right)$$
$$= A\frac{d}{dx}\left(e^{mx}\right) + B\frac{d}{dx}\left(e^{nx}\right)$$
$$= A\left(me^{mx}\right) + B\left(ne^{nx}\right) \quad \ldots(ii)$$ **[1½]**

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[A\left(me^{mx}\right) + B\left(ne^{nx}\right)\right]$$
$$= Am\frac{d}{dx}\left(e^{mx}\right) + Bn\frac{d}{dx}\left(e^{nx}\right)$$
$$= Am\left(me^{mx}\right) + Bn\left(ne^{nx}\right) \quad \ldots\text{(iii)}$$

[1½]

From equations (i), (ii) and (iii), we get

$$\text{LHS} = \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$
$$= Am^2e^{mx} + Bn^2e^{nx} - (m+n)$$
$$\left(Ame^{mx} + Bne^{nx}\right) + mn\left(Ae^{mx} + Be^{nx}\right)$$
$$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx}$$
$$- Amne^{mx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx}$$
$$= 0$$
$$= \text{RHS}$$

[2]

Q. 14. If $y = 500e^{7x} + 600e^{-7x}$, then $\frac{d^2y}{dx^2} = 49y$.

[NCERT Ex. 5.7, Q. 15, Page 184]

Ans. Given that, $y = 500e^{7x} + 600e^{-7x}$

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(500e^{7x} + 600e^{-7x}\right)$$
$$= \frac{d}{dx}\left(500e^{7x}\right) + \frac{d}{dx}\left(600e^{-7x}\right)$$
$$= 500\frac{d}{dx}\left(e^{7x}\right) + 600\frac{d}{dx}\left(e^{-7x}\right)$$
$$= 500\left(7e^{7x}\right) + 600\left(-7e^{-7x}\right)$$
$$= 3500e^{7x} - 4200e^{-7x}$$

[2]

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(3500e^{7x} - 4200e^{-7x}\right)$$
$$= \frac{d}{dx}\left(3500e^{7x}\right) - \frac{d}{dx}\left(4200e^{-7x}\right)$$
$$= 3500\left(7e^{7x}\right) - 4200\left(-7e^{-7x}\right)$$
$$= 49\left(500e^{7x} + 600e^{-7x}\right)$$
$$= 49y$$

[3]

Q. 15. If $y = \left(\tan^{-1}x\right)^2$, show that $(x^2 + 1)^2 y_2 + 2x(x^2 + 1) y_1 = 2$. **[NCERT Ex. 5.7, Q. 17, Page 184]**

Ans. Given that, $y = \left(\tan^{-1}x\right)^2$

Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\left(\tan^{-1}x\right)^2\right]$$
$$= \frac{2\left(\tan^{-1}x\right)}{1+x^2} \quad \ldots\text{(i)}$$

[1]

Again, differentiating with respect to x, we have

$$\Rightarrow \frac{d^2y}{dx^2} = 2\frac{d}{dx}\left(\frac{\tan^{-1}x}{1+x^2}\right)$$
$$\Rightarrow \frac{d^2y}{dx^2} = 2\left[\frac{\left(1+x^2\right)\times\frac{1}{1+x^2} - \left(\tan^{-1}x\right)\times(2x)}{\left(1+x^2\right)^2}\right]$$
$$\Rightarrow \left(1+x^2\right)^2\frac{d^2y}{dx^2} = 2\left(1 - 2x\tan^{-1}x\right)$$
$$\Rightarrow \left(1+x^2\right)^2\frac{d^2y}{dx^2} + 4x\tan^{-1}x = 2$$

[2]

From equation (i), we have

$$\frac{dy}{dx} = \frac{2\left(\tan^{-1}x\right)}{1+x^2}$$
$$\therefore 2\left(\tan^{-1}x\right) = \left(1+x^2\right)\frac{dy}{dx}$$

Thus,

$$\Rightarrow \left(1+x^2\right)^2\frac{d^2y}{dx^2} + 2x\left(1+x^2\right)\frac{dy}{dx} = 2$$
$$\Rightarrow \quad \left(1+x^2\right)^2 y_2 + 2x\left(1+x^2\right)y_1 = 2$$

[2]

Q. 16. Differentiate w.r.t. x the function $x^x + x^a + a^x + a^a$, for some values of $a > 0$ and $x > 0$.

[NCERT Misc Ex. Q. 10, Page 191]

Ans. Given that, $y = x^x + x^a + a^x + a^a$

Differentiate with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^x + x^a + a^x + a^a\right)$$
$$= \frac{d}{dx}\left(x^x\right) + \frac{d}{dx}\left(x^a\right) + \frac{d}{dx}\left(a^x\right) + \frac{d}{dx}\left(a^a\right)$$
$$= \frac{d}{dx}\left(x^x\right) + ax^{a-1} + a^x\log a + 0$$
$$= \frac{d}{dx}\left(x^x\right) + ax^{a-1} + a^x\log a$$

[2]

Let $u = x^x$

Taking log both sides, we have

$$\log u = \log\left(x^x\right)$$
$$= x\log x$$

[1]

Differentiate with respect to x, we have

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x\log x)$$
$$\frac{du}{dx} = u\left(x\times\frac{1}{x} + \log x\right)$$
$$= x^x(1+\log x)$$

Thus,

$$\frac{dy}{dx} = x^x(1+\log x) + ax^{a-1} + a^x\log a$$

[2]

Q. 17. Differentiate w.r.t. x the function $x^{x^2-3} + (x-3)^{x^2}$ for x > 3. **[NCERT Misc Ex. Q. 11, Page 191]**

Ans. Given that, $y = x^{x^2-3} + (x-3)^{x^2}$

Let $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$. Then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \ldots\text{(i)}$$

Now, $u = x^{x^2-3}$

Taking log both sides, we have

$$\log u = \log\left(x^{x^2-3}\right)$$
$$= \left(x^2-3\right)\log x$$

[1]

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(\log u) = \frac{d}{dx}\left[\left(x^2-3\right)\log x\right]$$

$$\frac{1}{u}\frac{du}{dx} = \left(x^2-3\right)\frac{d}{dx}\left[\log(x)\right] + \log(x)\frac{d}{dx}\left(x^2-3\right)$$

$$\frac{du}{dx} = u\left[\frac{x^2-3}{x} + 2x\log(x)\right]$$

$$= x^{x^2-3}\left[\frac{x^2-3}{x} + 2x\log(x)\right]$$

[2]

For, $v = (x-3)^{x^2}$

Taking log both sides, we have

$$\log v = \log\left[(x-3)^{x^2}\right]$$

$$= x^2\log(x-3)$$

[1]

Differentiating both sides with respect to x, we have

$$\Rightarrow \frac{d}{dx}(\log v) = \frac{d}{dx}\left[x^2\log(x-3)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = x^2\frac{d}{dx}\left[\log(x-3)\right] + \log(x-3)\frac{d}{dx}\left[x^2\right]$$

$$\Rightarrow \frac{dv}{dx} = v\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$$

$$\Rightarrow \frac{dv}{dx} = (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$$

From equation (i), we have

$$\frac{dy}{dx} = x^{x^2-3}\left[\frac{x^2-3}{x} + 2x\log(x)\right] + (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$$

[1]

Q. 18. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ for –1 < x < 1, then prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$.

[NCERT Misc Ex. Q. 14, Page 191]

Ans. Given that, $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$$

$$\Rightarrow \left(x\sqrt{1+y}\right)^2 = \left(-y\sqrt{1+x}\right)^2$$

$$\Rightarrow x^2(1+y) = y^2(1+x)$$

$$\Rightarrow x^2 + x^2y = y^2 + xy^2$$

$$\Rightarrow x^2 + x^2y - y^2 - xy^2 = 0$$

$$\Rightarrow \left(x^2-y^2\right) + \left(x^2y - xy^2\right) = 0$$

$$\Rightarrow (x+y)(x-y) + xy(x-y) = 0$$

$$\Rightarrow (x-y)(x+y-xy) = 0$$

$$\Rightarrow x+y-xy = 0 \quad [\because x \neq y]$$

$$\Rightarrow y = -\frac{x}{1+x}$$

[3]

Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = -\frac{d}{dx}\left(\frac{x}{1+x}\right)$$

$$= -\left[\frac{(1+x)\frac{d}{dx}(x) - (x)\frac{d}{dx}(1+x)}{(1+x)^2}\right]$$

$$= -\left[\frac{(1+x)-(x)}{(1+x)^2}\right]$$

$$= -\frac{1}{(1+x)^2}$$

[2]

Q. 19. If $(x-a)^2 + (y-b)^2 = c^2$ for some $c > 0$, then prove that $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ is a constant independent of a and b.

[NCERT Misc Ex. Q. 15, Page 191]

Ans. Given that $(x-a)^2 + (y-b)^2 = c^2$

Differentiating both sides with respect to x, we have

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x-a}{y-b}$$

[1]

Again, differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = -\frac{(y-b)\frac{d}{dx}(x-a) - (x-a)\frac{d}{dx}(y-b)}{(y-b)^2}$$

$$\frac{d^2y}{dx^2} = -\frac{(y-b) - (x-a)\frac{dy}{dx}}{(y-b)^2}$$

$$= -\frac{(y-b) - (x-a)\left(-\frac{x-a}{y-b}\right)}{(y-b)^2}$$

$$= -\frac{(y-b)^2 + (x-a)^2}{(y-b)^3}$$

$$= -\frac{c^2}{(y-b)^3}$$

[2]

Thus,

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\left(-\frac{x-a}{y-b}\right)^2\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}$$

$$= \frac{\frac{\left[(x-a)^2+(y-b)^2\right]^{\frac{3}{2}}}{(y-b)^{2\times\frac{3}{2}}}}{-\frac{c^2}{(y-b)^3}}$$

$$= \frac{\frac{\left[c^2\right]^{\frac{3}{2}}}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}} = -c$$

[2]

is a constant independent of a and b.

Q. 20. Using mathematical induction prove that $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$ for all positive integers n.

[NCERT Misc Ex. Q. 19, Page 192]

Ans. Given that, $P(n):\frac{d}{dx}\left(x^n\right)=nx^{n-1}$

Put $n = 1$ in LHS and RHS.

$$\text{LHS}=\frac{d}{dx}\left(x^1\right)=1$$

$$\text{RHS}=1x^{1-1}=1$$ **[1]**

Thus, P(n) is true for $n = 1$.

Let $P(k):\frac{d}{dx}\left(x^k\right)=kx^{k-1}$ is true for k. **[1]**

Now, we need to prove

$$P(k+1):\frac{d}{dx}\left(x^{k+1}\right)=(k+1)x^k.$$

$$\begin{aligned}\text{LHS}&=\frac{d}{dx}\left(x^{k+1}\right)\\&=\frac{d}{dx}\left(x^k\times x\right)\\&=\left(x^k\right)\frac{d}{dx}(x)+(x)\frac{d}{dx}\left(x^k\right)\\&=x^k+x\times kx^{k-1}\\&=(k+1)x^k\\&=\text{RHS}\end{aligned}$$ **[2]**

Thus, $P(k+1):\frac{d}{dx}\left(x^{k+1}\right)=(k+1)x^k$ is true for $k+1$.

Therefore, $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$ for all positive integers n. **[1]**

Q. 21. Using the fact that sin (A + B) = sin A cos B + cos A sin B and the differentiation, obtain the sum formula for cosines.

[NCERT Misc Ex. Q. 20, Page 192]

Ans. Given that,

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

Differentiating with respect to x, we have **[1]**

$$\frac{d}{dx}\left[\sin(A+B)\right]=\frac{d}{dx}(\sin A\cos B+\cos A\sin B)$$

$$\cos(A+B)\left(\frac{dA}{dx}+\frac{dB}{dx}\right)=\frac{d}{dx}(\sin A\cos B)+\frac{d}{dx}(\cos A\sin B)$$

$$=\sin A\frac{d}{dx}(\cos B)+\cos B\frac{d}{dx}(\sin A)+\cos A\frac{d}{dx}(\sin B)+\sin B\frac{d}{dx}(\cos A)$$

$$=\sin A(-\sin B)\frac{dB}{dx}+\cos B\cos A\frac{dA}{dx}+\cos A\cos B\frac{dB}{dx}+\sin B(-\sin A)\frac{dA}{dx}$$

$$=-\sin A\sin B\left(\frac{dA}{dx}+\frac{dB}{dx}\right)+\cos A\cos B\left(\frac{dA}{dx}+\frac{dB}{dx}\right)$$

$$=(\cos A\cos B-\sin A\sin B)\left(\frac{dA}{dx}+\frac{dB}{dx}\right)$$

$$\cos(A+B)=(\cos A\cos B-\sin A\sin B)$$ **[4]**

Q. 22. Find $\frac{dy}{dx}$ of the function expressed in parametric form $x=e^\theta\left(\theta+\frac{1}{\theta}\right)$, $y=e^{-\theta}\left(\theta-\frac{1}{\theta}\right)$.

[NCERT Exemp. Ex. 5.3, Q. 45, Page 110]

Ans. Given that,

$$x=e^\theta\left(\theta+\frac{1}{\theta}\right),\ y=e^{-\theta}\left(\theta-\frac{1}{\theta}\right)$$

$$x=e^\theta\left(\theta+\frac{1}{\theta}\right)$$

$$\begin{aligned}\Rightarrow\frac{dx}{d\theta}&=\frac{d}{d\theta}\left[e^\theta\left(\theta+\frac{1}{\theta}\right)\right]\\&=\left[e^\theta\frac{d}{d\theta}\left(\theta+\frac{1}{\theta}\right)+\left(\theta+\frac{1}{\theta}\right)\frac{d}{d\theta}\left(e^\theta\right)\right]\\&=e^\theta\left(1-\frac{1}{\theta^2}\right)+\left(\theta+\frac{1}{\theta}\right)\left(e^\theta\right)\\&=e^\theta\left(1-\frac{1}{\theta^2}+\theta+\frac{1}{\theta}\right)\\&=e^\theta\left(\frac{\theta^3+\theta^2+\theta-1}{\theta^2}\right)\end{aligned}$$ **[2]**

And,

$$y=e^{-\theta}\left(\theta-\frac{1}{\theta}\right)$$

$$\begin{aligned}\Rightarrow\frac{dy}{d\theta}&=\frac{d}{d\theta}\left[e^{-\theta}\left(\theta-\frac{1}{\theta}\right)\right]\\&=\left[e^{-\theta}\frac{d}{d\theta}\left(\theta-\frac{1}{\theta}\right)+\left(\theta-\frac{1}{\theta}\right)\frac{d}{d\theta}\left(e^{-\theta}\right)\right]\\&=e^{-\theta}\left(1+\frac{1}{\theta^2}\right)-\left(\theta-\frac{1}{\theta}\right)\left(e^{-\theta}\right)\\&=e^{-\theta}\left(1+\frac{1}{\theta^2}-\theta+\frac{1}{\theta}\right)\\&=e^{-\theta}\left(\frac{-\theta^3+\theta^2+\theta+1}{\theta^2}\right)\end{aligned}$$ **[2]**

Thus,

$$\frac{dy}{dx}=e^{-2\theta}\left(\frac{-\theta^3+\theta^2+\theta+1}{\theta^3+\theta^2+\theta+1}\right)$$ **[1]**

Q. 23. Find $\frac{dy}{dx}$ of the function expressed in parametric form $x=\frac{1+\log t}{t^2}$, $y=\frac{3+2\log t}{t}$.

[NCERT Exemp. Ex. 5.3, Q. 48, Page 110]

Ans. Given that,

$$x=\frac{1+\log t}{t^2},\ y=\frac{3+2\log t}{t}$$

$$x=\frac{1+\log t}{t^2}$$

Differentiating both sides with respect to t, we have

$$\frac{dx}{dt}=\frac{t^2\frac{d}{dt}(1+\log t)-(1+\log t)\frac{d}{dt}\left(t^2\right)}{t^4}$$

$$= \frac{t^2 \times \frac{1}{t} - 2t(1+\log t)}{t^4}$$
$$= \frac{t - 2t(1+\log t)}{t^4}$$
$$= \frac{1 - 2(1+\log t)}{t^3}$$
$$= \frac{-1 - 2\log t}{t^3}$$ **[2]**

And,
$$y = \frac{3 + 2\log t}{t}$$

Differentiating both sides with respect to t, we have

$$\frac{dy}{dt} = \frac{t\frac{d}{dt}(3+2\log t) - (3+2\log t)\frac{d}{dt}(t)}{t^2}$$
$$= \frac{t \times \frac{2}{t} - (3+2\log t)}{t^2}$$
$$= \frac{2 - 3 - 2\log t}{t^2}$$
$$= \frac{-1 - 2\log t}{t^2}$$ **[2]**

Thus,
$$\frac{dy}{dx} = \frac{\frac{-1-2\log t}{t^2}}{\frac{-1-2\log t}{t^3}} = t$$ **[1]**

Q. 24. If $x = a\sin 2t(1+\cos 2t)$ and $y = b\cos 2t(1-\cos 2t)$, show that $\left(\frac{dy}{dx}\right)_{\text{at } t=\frac{\pi}{4}} = \frac{b}{a}$.

[NCERT Exemp. Ex. 5.3, Q. 50, Page 110]

Ans. Given that,
$x = a\sin 2t(1+\cos 2t)$ and $y = b\cos 2t(1-\cos 2t)$.

$x = a\sin 2t(1+\cos 2t)$

Differentiating both sides with respect to t, we have

$$\frac{dx}{dt} = a\left[\sin 2t\frac{d}{dt}(1+\cos 2t) + (1+\cos 2t)\frac{d}{dt}(\sin 2t)\right]$$
$$= a\left[\sin 2t(-2\sin 2t) + (1+\cos 2t)(2\cos 2t)\right]$$
$$= a\left[-2\sin^2 2t + 2\cos 2t + 2\cos^2 2t\right]$$
$$= -2a\sin^2 2t + 2a\cos 2t + 2a\cos^2 2t$$ **[2]**

And,
$$\frac{dy}{dt} = b\frac{d}{dt}\left[\cos 2t(1-\cos 2t)\right]$$
$$= b\left[\cos 2t\frac{d}{dt}(1-\cos 2t) + (1-\cos 2t)\frac{d}{dt}(\cos 2t)\right]$$
$$= b\left[2\sin 2t\cos 2t - 2\sin 2t(1-\cos 2t)\right]$$
$$= 2b\sin 2t\cos 2t - 2b\sin 2t(1-\cos 2t)$$ **[1]**

Thus,
$$\frac{dy}{dx} = \frac{2b\sin 2t\cos 2t - 2b\sin 2t(1-\cos 2t)}{-2a\sin^2 2t + 2a\cos 2t + 2a\cos^2 2t}$$
$$= \frac{b\left[\sin 4t - 2\sin 2t(1-\cos 2t)\right]}{-2a\left[\sin^2 2t - \cos 2t - \cos^2 2t\right]}$$ **[1]**

Therefore,
$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \frac{b\left[\sin(\pi) - 2\sin\left(\frac{\pi}{2}\right)\left(1-\cos\left(\frac{\pi}{2}\right)\right)\right]}{-2a\left[\sin^2\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right) - \cos^2\left(\frac{\pi}{2}\right)\right]}$$
$$= \frac{b\left[0 - 2(1-0)\right]}{-2a\left[1-0-0\right]}$$
$$= \frac{b}{a}$$ **[1]**

Q. 25. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, then prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

[NCERT Exemp. Ex. 5.3, Q. 63, Page 111]

Ans. Given that, $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$

Put $x = \sin\alpha$ and $y = \sin\beta$, we have

$$\sqrt{1-\sin^2\alpha} + \sqrt{1-\sin^2\beta} = a(\sin\alpha - \sin\beta)$$
$$\Rightarrow \sqrt{\cos^2\alpha} + \sqrt{\cos^2\beta} = a(\sin\alpha - \sin\beta)$$
$$\Rightarrow \cos\alpha + \cos\beta = a(\sin\alpha - \sin\beta)$$
$$\Rightarrow 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2a\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
$$\Rightarrow \cos\frac{\alpha-\beta}{2} = a\sin\frac{\alpha-\beta}{2}$$
$$\Rightarrow \cot\frac{\alpha-\beta}{2} = a$$
$$\Rightarrow \frac{\alpha-\beta}{2} = \cot^{-1}a$$
$$\Rightarrow \alpha - \beta = 2\cot^{-1}a$$
$$\Rightarrow \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a$$ **[3]**

Differentiating with respect to x, we have

$$\Rightarrow \frac{d}{dx}\left(\sin^{-1}x - \sin^{-1}y\right) = \frac{d}{dx}\left(2\cot^{-1}a\right)$$
$$\Rightarrow \frac{d}{dx}\left(\sin^{-1}x\right) - \frac{d}{dx}\left(\sin^{-1}y\right) = \frac{d}{dx}\left(2\cot^{-1}a\right)$$
$$\Rightarrow \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$
$$\therefore \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$ **[2]**

Q. 26. If $x = \sin t$ and $y = \sin pt$, prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$.

[NCERT Exemp. Ex. 5.3, Q. 81, Page 113]

Ans. Given that,
$x = \sin t$ and $y = \sin pt$.

Thus,
$x = \sin t$
$$\Rightarrow \frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t$$ **[1]**

And,
$y = \sin pt$
$$\Rightarrow \frac{dy}{dt} = \frac{d}{dt}(\sin pt) = p\cos pt$$ **[1]**

Therefore,

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{p\cos pt}{\cos t}$$

$$\Rightarrow \cos t\frac{dy}{dx}=p\cos pt$$

$$\Rightarrow \cos^2 t\left(\frac{dy}{dx}\right)^2=p^2\cos^2 pt$$

$$\Rightarrow \left(1-\sin^2 t\right)\left(\frac{dy}{dx}\right)^2=p^2\left(1-\sin^2 pt\right)$$

$$\Rightarrow \left(1-x^2\right)\left(\frac{dy}{dx}\right)^2=p^2\left(1-y^2\right)$$ **[1½]**

Differentiating with respect to x, we have

$$\Rightarrow \left(1-x^2\right)\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right]+\left(\frac{dy}{dx}\right)^2\frac{d}{dx}\left[\left(1-x^2\right)\right]$$

$$=\frac{d}{dx}\left[p^2\left(1-y^2\right)\right]$$

$$\Rightarrow \left(1-x^2\right)\times 2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)+\left(\frac{dy}{dx}\right)^2(-2x)$$

$$=p^2\left(-2y\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2}\left(1-x^2\right)-x\frac{dy}{dx}=-p^2y$$

$$\Rightarrow \frac{d^2y}{dx^2}\left(1-x^2\right)-x\frac{dy}{dx}+p^2y=0$$ **[1½]**

Q. 27. Find $\frac{dy}{dx}$, if $y=x^{\tan x}+\sqrt{\frac{x^2+1}{2}}$.

[NCERT Exemp. Ex. 5.3, Q. 82, Page 113]

Ans. Let $u=x^{\tan x}$ and $v=\sqrt{\frac{x^2+1}{2}}$

Now,

$$u=x^{\tan x}$$

$$\Rightarrow \log u=\tan x\log x$$ **[1]**

Differentiating both sides, we have

$$\frac{d}{dx}(\log u)=\frac{d}{dx}(\tan x\log x)$$

$$\frac{1}{u}\frac{du}{dx}=\frac{\tan x}{x}+\log x\sec^2 x$$

$$\frac{du}{dx}=u\left[\frac{\tan x}{x}+\log x\sec^2 x\right]$$

$$=x^{\tan x}\left[\frac{\tan x}{x}+\log x\sec^2 x\right]$$ **[2]**

Now,

$$v=\sqrt{\frac{x^2+1}{2}}$$

Differentiating both sides, we have

$$\frac{dv}{dx}=\frac{d}{dx}\left(\sqrt{\frac{x^2+1}{2}}\right)$$

$$=\frac{x}{\sqrt{2\left(x^2+1\right)}}$$ **[1]**

Thus,

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

$$=x^{\tan x}\left[\frac{\tan x}{x}+\log x\sec^2 x\right]+\frac{x}{\sqrt{2\left(x^2+1\right)}}$$ **[1]**

Q. 28. If $x=a\sin 2t\left(1+\cos 2t\right)$ and $y=b\cos 2t\left(1-\cos 2t\right)$, then find the value of $\frac{dy}{dx}$ at $t=\frac{\pi}{4}$ and $t=\frac{\pi}{3}$.

[CBSE Board, Delhi Region, 2017, CBSE Board, Delhi Region 2016]

Ans. Given that,

$x=a\sin 2t\left(1+\cos 2t\right)$ and $y=b\cos 2t\left(1-\cos 2t\right)$.

We have

$$x=a\sin 2t\left(1+\cos 2t\right)$$

Differentiating both sides with respect to t, we have

$$\frac{dx}{dt}=a\frac{d}{dt}\left[\sin 2t\left(1+\cos 2t\right)\right]$$

$$=a\left[\sin 2t\frac{d}{dt}\left(1+\cos 2t\right)+\left(1+\cos 2t\right)\frac{d}{dt}\left[\sin 2t\right]\right]$$

$$=a\left[\sin 2t\left(-2\sin 2t\right)+\left(1+\cos 2t\right)\left[2\cos 2t\right]\right]$$

$$=a\left[-2\sin^2 2t+2\cos 2t+2\cos^2 2t\right]$$

$$=a\left[2\cos 2t+2\left(\cos^2 2t-\sin^2 2t\right)\right]$$

$$=2a\left[\cos 2t+\cos 4t\right]$$ **[1½]**

And

$$y=b\cos 2t\left(1-\cos 2t\right)$$

Differentiating both sides with respect to t, we have

$$\frac{dy}{dt}=b\left[\cos 2t\frac{d}{dt}\left(1-\cos 2t\right)+\left(1-\cos 2t\right)\frac{d}{dt}\left(\cos 2t\right)\right]$$

$$=b\left[\cos 2t\left(2\sin 2t\right)+\left(1-\cos 2t\right)\left(-2\sin 2t\right)\right]$$

$$=b\left[2\sin 4t-2\sin 2t\right]$$

$$=2b\left[\sin 4t-\sin 2t\right]$$ **[1½]**

Thus,

$$\frac{dy}{dx}=\frac{2b\left[\sin 4t-\sin 2t\right]}{2a\left[\cos 2t+\cos 4t\right]}$$

$$=\frac{b\left[2\cos 3t\sin t\right]}{a\left[2\cos 3t\cos t\right]}$$

$$=\frac{b}{a}\tan t$$

Put $t=\frac{\pi}{4}\Rightarrow\frac{dy}{dx}=\frac{b}{a}\tan\frac{\pi}{4}=\frac{b}{a}(1)=\frac{b}{a}$

Put $t=\frac{\pi}{3}\Rightarrow\frac{dy}{dx}=\frac{b}{a}\tan\frac{\pi}{3}=\frac{b}{a}\left(\sqrt{3}\right)=\frac{\sqrt{3}b}{a}$ **[2]**

Q. 29. If $x^y+y^x=a^b$, then find $\frac{dy}{dx}$.

[CBSE Board, All India Region, 2017]

Ans. Given that, $x^y+y^x=a^b$

Let $u=x^y$ and $v=y^x$. Then

$$\frac{du}{dx}+\frac{dv}{dx}=0 \quad \ldots \text{(i)}$$ **[1]**

Now, $u=x^y$

Taking log both sides, we have

$\log u = \log\left[x^y\right]$

$= y\log(x)$ **[1]**

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(\log u) = \frac{d}{dx}\left[y\log(x)\right]$$

$$\frac{1}{u}\frac{du}{dx} = y\frac{d}{dx}\left[\log(x)\right] + \log x\frac{dy}{dx}$$

$$\frac{du}{dx} = u\left[\frac{y}{x} + \log x\frac{dy}{dx}\right]$$

$$= x^y\left[\frac{y}{x} + \log x\frac{dy}{dx}\right]$$ **[1]**

For, $v = y^x$

Taking log both sides, we have

$\log v = \log\left[y^x\right]$

$= x\log(y)$ **[1]**

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(\log v) = \frac{d}{dx}\left[x\log(y)\right]$$

$$\frac{1}{v}\frac{dv}{dx} = x\frac{d}{dx}(\log y) + \log y\frac{d}{dx}(x)$$

$$\frac{dv}{dx} = v\left[\frac{x}{y}\frac{dy}{dx} + \log y\right]$$

$$= y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right]$$

From equation (i), we have

$$x^y\left[\frac{y}{x} + \log x\frac{dy}{dx}\right] + y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] = 0$$

$$\Rightarrow x^{y-1}y + x^y\log x\frac{dy}{dx} + xy^{x-1}\frac{dy}{dx} + y^x\log y = 0$$

$$\Rightarrow \left(x^y\log x + xy^{x-1}\right)\frac{dy}{dx} = -\left(x^{y-1}y + y^x\log y\right)$$

$$\therefore \frac{dy}{dx} = -\frac{x^{y-1}y + y^x\log y}{x^y\log x + xy^{x-1}}$$ **[1]**

Q. 30. If $y = x^x$, then prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$.

[CBSE Board, Delhi Region, 2016]

Ans. Given that, $y = x^x$

Taking log both sides, we have

$\log y = x\log x$ **[1]**

Differentiate with respect to x, we have

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(x\log x)$$

$$\frac{1}{y}\frac{dy}{dx} = x\times\frac{1}{x} + \log x$$

$$\frac{dy}{dx} = y[1 + \log x]$$

$$= x^x[1 + \log x]$$ **[2]**

Again, differentiate with respect to x, we have

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(x^x[1 + \log x]\right)$$

$$\frac{d^2y}{dx^2} = \left(x^x\right)\frac{d}{dx}(1 + \log x) + (1 + \log x)\frac{d}{dx}\left(x^x\right)$$

$$\frac{d^2y}{dx^2} = y\left(\frac{1}{x}\right) + (1 + \log x)\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{y}{x} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2$$

$$\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right) - \frac{y}{x} = 0$$ **[2]**

Q. 31. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ if $x \in (-1, 1)$.

[CBSE Board, Delhi Region, 2016]

Ans. Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Put $x = \tan\theta$ in $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

$$u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{\theta}{2}$$

$$= \frac{1}{2}\tan^{-1}x$$ **[2]**

Thus, $\frac{du}{dx} = \frac{1}{2\left(1+x^2\right)}$ **[1]**

And,

$$v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$= \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta$$

$$= 2\tan^{-1}x$$ **[1]**

Thus, $\frac{dv}{dx} = \frac{2}{\left(1+x^2\right)}$

Therefore,

$$\frac{du}{dv} = \frac{\frac{1}{2\left(1+x^2\right)}}{\frac{2}{\left(1+x^2\right)}} = \frac{1}{4}$$ **[1]**

Q. 32. If $y = 2\cos(\log x) + 3\sin(\log x)$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$. [CBSE Board, All India Region, 2016]

Ans. Given that, $y = 2\cos(\log x) + 3\sin(\log x)$

Differentiate with respect to x, we have

$$\frac{dy}{dx}=2\frac{d}{dx}[\cos(\log x)]+3\frac{d}{dx}[\sin(\log x)]$$

$$\Rightarrow \frac{dy}{dx}=2\left[-\frac{\sin(\log x)}{x}\right]+3\left[\frac{\cos(\log x)}{x}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2\sin(\log x)+3\cos(\log x)}{x}$$

$$\Rightarrow x\frac{dy}{dx}=-2\sin(\log x)+3\cos(\log x)$$

[2]

Differentiate with respect to x, we have

$$\Rightarrow \frac{d}{dx}\left(x\frac{dy}{dx}\right)=-2\frac{d}{dx}[\sin(\log x)]+3\frac{d}{dx}[\cos(\log x)]$$

$$\Rightarrow x\frac{d^2y}{dx^2}+\frac{dy}{dx}=-2\left[\frac{\cos(\log x)}{x}\right]+3\left[\frac{-\sin(\log x)}{x}\right]$$

$$\Rightarrow x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{-2\cos(\log x)-3\sin(\log x)}{x}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-y$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

[3]

Q. 33. If $x\cos(a+y)=\cos y$ then prove that $\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$. Hence, show that $\sin a\frac{d^2y}{dx^2}+\sin 2(a+y)\frac{dy}{dx}=0$.

[CBSE Board, All India Region, 2016]

Ans. Given that, $\cos y = x\cos(a+y)$

$$x=\frac{\cos y}{\cos(a+y)}$$

$$\frac{dx}{dy}=\frac{(\cos(a+y))\frac{d}{dy}(\cos y)-(\cos y)\frac{d}{dy}(\cos(a+y))}{\cos^2(a+y)}$$

$$=\frac{-\sin y\cos(a+y)-(\cos y)(-\sin(a+y))}{\cos^2(a+y)}$$

$$=\frac{-\sin y\cos(a+y)+\sin(a+y)\cos y}{\cos^2(a+y)}$$

$$=\frac{\sin(a+y)\cos y-\sin y\cos(a+y)}{\cos^2(a+y)}$$

$$=\frac{\sin(a+y-y)}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy}=\frac{\sin a}{\cos^2(a+y)}$$

$$\therefore \frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$$

[3]

Thus,

$$\Rightarrow \frac{d}{dx}\left(\sin a\frac{dy}{dx}\right)=\cos^2(a+y)$$

$$\Rightarrow \sin a\frac{d^2y}{dx^2}=2\cos(a+y)(-\sin(a+y))\frac{dy}{dx}$$

$$\Rightarrow \sin a\frac{d^2y}{dx^2}+\sin 2(a+y)\frac{dy}{dx}=0$$

[2]

Q. 34. Differentiate $(\sin 2x)^x+\sin^{-1}\sqrt{3x}$ with respect to x.

[CBSE Board, All India Region, 2016]

Ans. Given that, $y=(\sin 2x)^x+\sin^{-1}\sqrt{3x}$

Let $u=(\sin 2x)^x$ and $v=\sin^{-1}\sqrt{3x}$. Then

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \quad \text{... (i)}$$

[1]

Now, $u=(\sin 2x)^x$

Taking log both sides, we have

$$\log u=\log\left[(\sin 2x)^x\right]$$

$$=x\log(\sin 2x)$$

[1]

Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(\log u)=\frac{d}{dx}[x\log(\sin 2x)]$$

$$\frac{1}{u}\frac{du}{dx}=x\frac{d}{dx}[\log(\sin 2x)]+\log(\sin 2x)\frac{d}{dx}[2x]$$

$$\frac{du}{dx}=u\left[x\times\frac{2}{\sin 2x}\times\cos 2x+\log(\sin 2x)\times 2\right]$$

$$=(\sin 2x)^x\left[\frac{2x\cos 2x}{\sin 2x}+2\log(\sin 2x)\right]$$

[2]

For, $v=\sin^{-1}\sqrt{3x}$

Differentiating both sides with respect to x, we have

$$\frac{dv}{dx}=\frac{d}{dx}\left[\sin^{-1}\sqrt{3x}\right]$$

$$=\frac{1}{\sqrt{1-3x}}\times\frac{3}{2\sqrt{3x}}$$

$$=\frac{3}{2\sqrt{3x-9x^2}}$$

From equation (i), we have

$$\frac{dy}{dx}=(\sin 2x)^x\left[\frac{2x\cos 2x}{\sin 2x}+\log(\sin 2x)\right]+\frac{3}{2\sqrt{3x-9x^2}}$$

[1]

Q. 35. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$ with respect to $\cos^{-1}x^2$.

[CBSE Board, All India Region, 2016]

Ans. Given that,

$$u=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right),\ v=\cos^{-1}x^2$$

Put $x^2=\cos 2\theta\Rightarrow\theta=\frac{1}{2}\cos^{-1}x^2$, then

$$u=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\theta\right)\right)$$

$$= \frac{\pi}{4}+\theta$$

$$= \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

[2]

Differentiating both sides, we have

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2\right)$$

$$= 0+\frac{1}{2}\left(-\frac{2x}{\sqrt{1-x^4}}\right)$$

$$= -\frac{x}{\sqrt{1-x^4}}$$

[1]

Differentiating both sides, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2\right)$$

$$= 0+\frac{1}{2}\left(-\frac{2x}{\sqrt{1-x^4}}\right)$$

$$= -\frac{x}{\sqrt{1-x^4}}$$

[1]

Now,

$v = \cos^{-1}x^2$

$$\Rightarrow \frac{dv}{dx} = \frac{d}{dx}\left(\cos^{-1}x^2\right) = -\frac{2x}{\sqrt{1-x^4}}$$

Thus, $$\frac{du}{dv} = \frac{-\frac{x}{\sqrt{1-x^4}}}{-\frac{2x}{\sqrt{1-x^4}}} = \frac{1}{2}$$ **[2]**

TOPIC-3
Rolle's Theorem and MVT

Quick Review

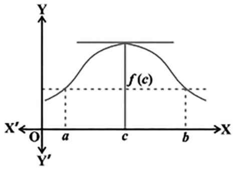
❖ Rolle's Theorem : For a function $f(x)$ to be applicable Rolle's Theorem, there are three conditions which is to be satisfied :

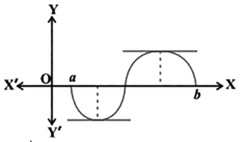
(xxxvi) $f(x)$ should be continuous on $[a, b]$

(xxxvii) $f(x)$ should be differentiable on (a, b)

(xxxviii) $f(a) = f(b)$, where a and b are some real numbers.

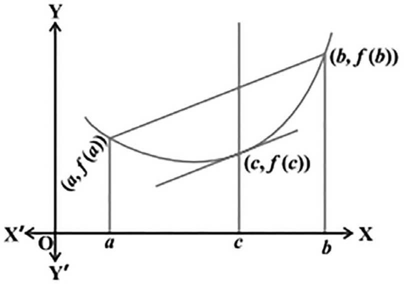
❖ If all the above conditions are satisfied, then there exists some c in (a, b) such that $f'(c) = 0$. Geometrically Rolle's theorem ensures that there is at least one point on the curve $y = f(x)$ at which tangent is parallel to x-axis [abscissa of the point lying in (a, b)].





❖ MVT (Mean Valued Theorem)

Let $f : [a, b]$ R be a continuous function on [a, b] and differentiable on (a, b). Then there exists at least one point c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ Geometrically, Mean Value Theorem states that there exists at least one point c in (a, b) such that the tangent at the point (c, f(c)) is parallel to the secant joining the points $(a, f(a))$ and $(b, f(b))$.



TIPS...

Rolle's theorem as the slope of the tangent at any point on the graph of y = f(x) is nothing but the derivative of f(x) at that point.

Know the Links

☞ *https ://www.cut-the-knot.org/Curriculum/Calculus/MVT.shtml*

☞ *https ://en.wikipedia.org/wiki/Rolle%27s_theorem*

☞ *http ://www.sosmath.com/calculus/diff/der11/der11.html*

Multiple Choice Questions

(1 mark each)

Q. 1. The value of c in the Rolle's Theorem for the function $f(x) = x^3 - 3x$ in the interval $[0, \sqrt{3}]$ is

(a) 1 **(b) –1**

(c) $\frac{3}{2}$ **(d) $\frac{1}{3}$**

[NCERT Exemp. Ex. 5.3, Q. 95, Page 115]

Ans. Correct option : (a)

Explanation : Given that $f(x) = x^3 - 3x$. It is continuous and differentiable. And, $f(0) = f(\sqrt{3}) = 0$.

Thus, by Rolle's Theorem, there exists c for which

$$f'(c) = 0$$

$$\Rightarrow 3c^2 - 3 = 0$$

$$\Rightarrow c^2 = 1$$

$$\Rightarrow c = \pm 1$$

$$\Rightarrow c = 1 \in [0, \sqrt{3}]$$

Q. 2. For the function $f(x) = x + \frac{1}{x}$, $x \in [1, 3]$, the value of c for the mean value theorem is

(a) 1 **(b) $\sqrt{3}$**

(c) 2 **(d) None of these**

[NCERT Exemp. Ex. 5.3, Q. 96, Page 116]

Ans. Correct option : (b)

Explanation : Given that $f(x) = x + \frac{1}{x}$, $x \in [1, 3]$. It is continuous and differentiable.

Thus, by mean value theorem, there exists c for which

$$f(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow 1 - \frac{1}{c^2} = \frac{\frac{10}{3} - 2}{3 - 1}$$

$$\Rightarrow c^2 = 3$$

$$\Rightarrow c = \sqrt{3} \in (1, 3)$$

Q. 3. State True or False for the statements :

Rolle's theorem is applicable for the function $f(x) = |x - 1|$ in [0, 2].

[NCERT Exemp. Ex. 5.3, Q. 102, Page 116]

Ans. False

Explanation : Since the given function $f(x) = |x - 1|$ is continuous function but not differentiable at x = 1 in [0, 2], then the Rolle's theorem is not applicable to the given function.

Very Short Answer Type Questions

(1 or 2 marks each)

Q. 1. Examine if Rolle's theorem is applicable to $f(x) = [x]$ for $x \in [5, 9]$. Can you say something about the converse of Rolle's theorem from these example? [NCERT Ex. 5.8, Q. 2(i), Page 186]

Ans. Given that $f(x) = [x]$

Since $f(x) = [x]$ is a greatest integer function which is neither continuous in $x \in [5, 9]$ nor differentiable in $x \in (5, 9)$ and also $f(5) \neq f(9)$, then the Rolle's theorem is not applicable to the given function. **[2]**

Q. 2. Examine if Rolle's theorem is applicable to $f(x) = [x]$ for $x \in [-2, 2]$. Can you say something about the converse of Rolle's theorem from these example? [NCERT Ex. 5.8, Q. 2(ii), Page 186]

Ans. Given that $f(x) = [x]$

Since $f(x) = [x]$ is a greatest integer function which is neither continuous in $x \in [-2, 2]$ nor differentiable in $x \in (-2, 2)$ and also $f(-2) \neq f(2)$, then the Rolle's theorem is not applicable to the given function. **[2]**

Q. 3. Examine if Rolle's theorem is applicable to $f(x) = x^2 - 1$ for $x \in [1, 2]$. Can you say something about the converse of Rolle's theorem from these example? [NCERT Ex. 5.8, Q. 2(iii), Page 186]

Ans. Given that $f(x) = x^2 - 1$

$f(x) = x^2 - 1$ is a polynomial function, then it is continuous in $x \in [1, 2]$

$f'(x) = 2x$ is a polynomial function, then it is differentiable in $x \in (1, 2)$ **[1]**

$f(1) = 1 - 1 = 0$ and $f(2) = 4 - 1 = 3 \Rightarrow f(1) \neq f(2)$

Since all the above conditions are not satisfied, then the Rolle's theorem is not verified for the given function. **[1]**

Q. 4. If $f : [-5, 5] \rightarrow R$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$. [NCERT Ex. 5.8, Q. 3, Page 186]

Ans. Since the function $f : [-5, 5] \rightarrow R$ is a differentiable function, then the given function is continuous in [–5, 5] and (–5, 5).

Thus, according to MVT, there exist a value $c \in (-5, 5)$ such that

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$ **[1]**

Since $f'(x)$ does not vanish anywhere, then

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)} \neq 0$$

$$\Rightarrow f(5) - f(-5) \neq 0$$

$$\Rightarrow f(5) \neq f(-5)$$ **[1]**

Short Answer Type Questions (3 or 4 marks each)

Q. 1. Verify Rolle's theorem for the function $f(x)=x^2+2x-8, x\in[-4, 2]$.

[NCERT Ex. 5.8, Q. 1, Page 186]

Ans. Given that $f(x)=x^2+2x-8$

$f(x)=x^2+2x-8$ is a polynomial function, then it is continuous in [–4, 2].

$f'(x)=2x+2$ is a polynomial function, then it is differentiable in (–4, 2).

$f(-4)=16-8-8=0$ and

$f(2)=4+4-8=0 \Rightarrow f(-4)=f(2)$ **[1½]**

Since all the above conditions are satisfied, then there exists some c in (–4, 2) such that $f'(c) = 0$

$f'(c)=0 \Rightarrow f'(c)=2c+2=0 \Rightarrow c=-1\in[-4, 2]$.

Thus, Rolle's theorem is verified for the given function. **[1½]**

Q. 2. Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval [a, b], where $a = 1$ and $b = 4$.

[NCERT Ex. 5.8, Q. 4, Page 186]

Ans. Given that $f(x)=x^2-4x-3$

$f(x)=x^2-4x-3$ is a polynomial function, then it is continuous in $x\in[1, 4]$.

$f'(x)=2x-4$ is a polynomial function, then it is differentiable in $x\in(1, 4)$. **[1½]**

Thus, according to MVT, there exists a value $c\in(1, 4)$ such that

$$f'(c)=\frac{f(4)-f(1)}{4-1}$$

$$\Rightarrow 2c-4=\frac{(16-16-3)-(1-4-3)}{3}$$

$$\Rightarrow 2c-4=\frac{-3-(-6)}{3}$$

$$\Rightarrow 2c-4=1$$

$$\Rightarrow c=\frac{5}{2}\in(1,4)$$

[1½]

Thus, the MVT theorem is verified for the given function.

Q. 3. Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval [a, b], where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$.

[NCERT Ex. 5.8, Q. 5, Page 186]

Ans. Given that $f(x)=x^3-5x^2-3x$

$f(x)=x^3-5x^2-3x$ is a polynomial function, then it is continuous in $x\in[1, 3]$

$f'(x)=3x^2-10x-3$ is a polynomial function, then it is differentiable in $x\in(1, 3)$ **[1]**

Thus, according to MVT, there exists a value $c\in(1, 3)$ such that

$$f'(c)=\frac{f(3)-f(1)}{3-1}$$

$$\Rightarrow 3c^2-10c-3=\frac{(27-45-9)-(1-5-3)}{2}$$

$$\Rightarrow 3c^2-10c-3=\frac{-27+7}{2}$$

$$\Rightarrow 3c^2-10c-3=-10$$

$$\Rightarrow 3c^2-10c+7=0$$

$$\Rightarrow (c-1)(3c-7)=0$$

$$\Rightarrow c=1, \frac{7}{3}$$

$$\Rightarrow c=\frac{7}{3}\in(1,3)$$

[2]

Q. 4. Verify the Rolle's theorem for the function $f(x)=x(x-1)^2$ in $[0, 1]$.

[NCERT Exemp. Ex. 5.3, Q. 65, Page 112]

Ans. Given that $f(x)=x(x-1)^2$ in $[0, 1]$

It is continuous and differentiable in [0, 1].

And, $f(0)=f(1)=0$ **[1]**

Thus, Rolle's theorem is applicable to the given function.

Then, there exists a value c such that

$$f'(c)=0$$

$$\Rightarrow 3c^2-4c+1=0$$

$$\Rightarrow (3c-1)(c-1)=0$$

$$\Rightarrow c=\frac{1}{3}\in(0,1)$$

[2]

Therefore, the Rolle's theorem is verified for the given function.

Q. 5. Verify the Rolle's theorem for the function $f(x)=\log(x^2+2)-\log 3$ in $[-1,1]$.

[NCERT Exemp. Ex. 5.3, Q. 67, Page 112]

Ans. Given that $f(x)=\log(x^2+2)-\log 3$ in $[-1,1]$

It is continuous in [–1, 1] and differentiable in (–1, 1). **[1]**

And, $f(-1)=f(1)=0$ **[1]**

Thus, Rolle's theorem is applicable to the given function.

Then, there exists a value c such that

$$f'(c)=0$$

$$\Rightarrow \frac{2c}{c^2+2}=0$$

$$\Rightarrow c=0\in(-1,1)$$

[1]

Therefore, the Rolle's theorem is verified for the given function.

Q. 6. Verify the Rolle's theorem for the function $f(x)=x(x+3)e^{-\frac{x}{2}}$ in $[-3, 0]$.

[NCERT Exemp. Ex. 5.3, Q. 68, Page 112]

Ans. Given that $f(x)=x(x+3)e^{-\frac{x}{2}}$ in $[-3, 0]$

It is continuous in [–3, 0] and differentiable in (–3, 0). [1]

And, $f(-3)=f(0)=0$ **[1]**

Thus, Rolle's theorem is applicable to the given function.

Then, there exists *a* value *c* such that

$$f'(c)=0$$

$$\Rightarrow (2c+3)e^{-\frac{c}{2}}-\frac{1}{2}c(c+3)e^{-\frac{c}{2}}=0$$

$$\Rightarrow \quad -\frac{1}{2}e^{-\frac{c}{2}}(c^2-c-6)=0$$

$$\Rightarrow \quad -\frac{1}{2}e^{-\frac{c}{2}}(c+2)(c-3)=0$$

$$\Rightarrow \quad c=-2\in(-3,0)$$ **[1]**

Therefore, the Rolle's theorem is verified for the given function.

Q. 7. Verify the Rolle's theorem for the function $f(x)=\sqrt{4-x^2}$ in $[-2,2]$.

[NCERT Exemp. Ex. 5.3, Q. 69, Page 112]

Ans. Given that $f(x)=\sqrt{4-x^2}$ in $[-2,2]$

It is continuous in [–2, 2] and differentiable in (–2, 2). **[1]**

And, $f(-2)=f(2)=0$ **[1]**

Thus, Rolle's theorem is applicable to the given function.

Then, there exists a value c such that

$$f'(c)=0$$

$$\Rightarrow \frac{-2c}{2\sqrt{4-c^2}}=0$$

$$\Rightarrow \quad c=0\in(-2,2)$$ **[1]**

Therefore, the Rolle's theorem is verified for the given function.

Q. 8. Find the points on the curve y = (cos x – 1) in [0, 2π], where the tangent is parallel to *x*-axis.

[NCERT Exemp. Ex. 5.3, Q. 71, Page 112]

Ans. Since the tangent of the given curve is parallel to *x*-axis, then

$$\frac{dy}{dx}=0$$

$$\Rightarrow -\sin x-0=0$$

$$\Rightarrow \quad \sin x=0$$ **[1]**

Thus,

$$\Rightarrow \quad x=\pi\in(0,2\pi)$$ **[1]**

Put $x=\pi$ in $y=\cos x-1$, we have

$$y=\cos\pi-1=-1-1=-2$$

Thus, the required point is $(\pi,-2)$. **[1]**

Q. 9. Using Rolle's theorem, find the point on the curve $y = x(x-4)$, $x\in[0, 4]$, where the tangent is parallel to *x*-axis. [NCERT Exemp. Ex. 5.3, Q. 72, Page 112]

Ans. Since the tangent of the given curve is parallel to *x*-axis, then

$$\frac{dy}{dx}=0$$

$$\Rightarrow \frac{d}{dx}\left[x^2-4x\right]=0$$

$$\Rightarrow \quad 2x-4=0$$

$$\Rightarrow \quad x=2$$ **[1]**

Thus,

$$\Rightarrow \quad x=2\in(0,4)$$ **[1]**

Put $x = 2$ in $y=x(x-4)$, we have

$$y=2(2-4)=-4$$

Thus, the required point is $(2,-4)$. **[1]**

Q. 10. Verify mean value theorem for the function $f(x)=\sqrt{25-x^2}$ in $[1, 5]$.

[NCERT Exemp. Ex. 5.3, Q. 76, Page 112]

Ans. Given that $f(x)=\sqrt{25-x^2}$ in $[1, 5]$

It is continuous in $[1, 5]$. **[1]**

Now, $f'(x)=\frac{-2x}{2\sqrt{25-x^2}}=-\frac{x}{\sqrt{25-x^2}}$, it is defined in $(1,5)$ and hence it is differentiable in $(1,5)$. **[1]**

Thus, the mean value theorem is applicable to the given function.

Then, there exists a value $c\in(1,5)$ such that

$$f'(c)=\frac{f(5)-f(1)}{5-1}$$

$$\Rightarrow -\frac{c}{\sqrt{25-c^2}}=\frac{0-\sqrt{24}}{4}$$

$$\Rightarrow \quad 16c^2=24(25-c^2)$$

$$\Rightarrow \quad 40c^2=600$$

$$\Rightarrow \quad c^2=15$$

$$\Rightarrow \quad c=\sqrt{15}\in(1,5)$$ **[1]**

Thus, the mean value verified for the given function.

Q. 11. Find a point on the curve $y = (x-3)^2$, where the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

[NCERT Exemp. Ex. 5.3, Q. 77, Page 112]

Ans. Since the tangent of the given curve is parallel to the chord joining the points (3, 0) and (4, 1), then

$$\frac{dy}{dx}=\frac{0-1}{3-4}$$

$$\Rightarrow \quad \frac{d}{dx}\left[(x-3)^2\right]=1$$

$$\Rightarrow \quad 2(x-3)=1$$

$$\Rightarrow \quad x=\frac{7}{2}$$ **[1]**

Thus,

$$\Rightarrow x=\frac{7}{2}\in(3,4)$$ **[1]**

Put x = $\frac{7}{2}$ in $y=(x-3)^2$, we have

$$y=\left(\frac{7}{2}-3\right)^2=\frac{1}{4}$$

Thus, the required point is $\left(\frac{7}{2},\frac{1}{4}\right)$. **[1]**

Long Answer Type Questions

(5 or 6 marks each)

Q. 1. Examine the applicability of Mean Value Theorem for all three functions :

(i) $f(x)=[x], x \in [5, 9]$

(ii) $f(x)=[x], x \in [-2, 2]$

(iii) $f(x)=x^2-1, x \in [1, 2]$

[NCERT Ex. 5.8, Q. 6, Page 186]

Ans. For a function $f(x)$ to be applicable MVT theorem, there are three conditions is to be satisfied :

$f(x)$ should be continuous on [a, b]

$f(x)$ should be differentiable on (a, b) **[1]**

(i) Given that $f(x)=[x]$

Since $f(x)=[x]$ is a greatest integer function which is neither continuous in $x \in [5, 9]$ nor differentiable in $x \in (5,9)$, then the MVT theorem is not applicable to the given function. **[1]**

(ii) Given that $f(x)=[x]$

Since $f(x)=[x]$ is a greatest integer function which is neither continuous in $x \in [-2, 2]$ nor differentiable in $x \in (-2,2)$, then the MVT theorem is not applicable to the given function. **[1½]**

(iii) Given that $f(x)=x^2-1$

$f(x)=x^2-1$ is a polynomial function, then it is continuous in $x \in [1, 2]$

$f'(x)=2x$ is a polynomial function, then it is differentiable in $x \in (1, 2)$

Hence mean value theorem is applicable for the given function. **[1½]**

Q. 2. Verify the Rolle's theorem for the function $f(x)=\sin^4 x+\cos^4 x$ in $\left[0, \frac{\pi}{2}\right]$.

[NCERT Exemp. Ex. 5.3, Q. 66, Page 112]

Ans. Given that,

$$f(x)=\sin^4 x+\cos^4 x \text{ in } \left[0, \frac{\pi}{2}\right]$$

It is continuous and differentiable in $\left[0, \frac{\pi}{2}\right]$ **[1]**

And, $f(0)=f\left(\frac{\pi}{2}\right)=1$ **[1]**

Thus, Rolle's theorem is applicable to the given function.

Then, there exists a value c such that

$$f'(c)=0$$

$$\Rightarrow 4\sin^3 c\cos c-4\cos^3 c\sin c=0$$

$$\Rightarrow 4\sin c\cos c\left(\sin^2 c-\cos^2 c\right)=0$$

$$\Rightarrow 4\sin c\cos c(-\cos 2c)=0$$

$$\Rightarrow -2(2\sin c\cos c)\cos 2c=0$$

$$\Rightarrow -2\sin 2c\cos 2c=0$$

$$\Rightarrow \sin 4c=0$$

$$\Rightarrow 4c=\pi$$

$$\Rightarrow c=\frac{\pi}{4}\in\left(0, \frac{\pi}{2}\right)$$

[3]

Therefore, the Rolle's theorem is verified for the given function.

Q. 3. Discuss the applicability of Rolle's theorem on the function given

$$f(x)=\begin{cases} x^2+1, \text{ if } 0 \le x \le 1 \\ 3-x, \text{ if } 1 \le x \le 2. \end{cases}$$

[NCERT Exemp. Ex. 5.3, Q. 70, Page 112]

Ans. Given that,

$$f(x)=\begin{cases} x^2+1, \text{ if } 0 \le x \le 1 \\ 3-x, \text{ if } 1 \le x \le 2 \end{cases}$$

At $x = 1$,

LHL $= \lim_{x\to 1^-}\left(x^2+1\right)=1+1=2$ **[1]**

RHL $= \lim_{x\to 1^+}(3-x)=3-1=2$ **[1]**

And, $f(1)=3-1=2$

Since LHL = RHL = $f(1)=2$, the function is continuous function at $x = 1$.

$$f(x)=\begin{cases} x^2+1, \text{ if } 0 \le x \le 1 \\ 3-x, \text{ if } 1 \le x \le 2 \end{cases} \Rightarrow f'(x)=\begin{cases} 2x, \text{ if } 0 < x < 1 \\ -1, \text{ if } 1 < x < 2 \end{cases}$$ **[2]**

Now,

LHD $= 2(1) = 2$

RHD $= -1$

Since LHD $\ne$ RHD, the given function is not differentiable at $x = 1$.

Thus, Rolle's theorem is not applicable to the given function. **[1]**

Q. 4. Verify mean value theorem for the function $f(x)=\frac{1}{4x-4}$ in $[1, 4]$.

[NCERT Exemp. Ex. 5.3, Q. 73, Page 112]

Ans. Given that $f(x)=\frac{1}{4x-4}$ in $[1, 4]$

It is continuous in [1, 4]. **[1]**

Now, $f'(x)=-\frac{4}{(4x-4)^2}$, it is defined in (1, 4) and hence it is differentiable in (1, 4).

Thus, the mean value theorem is applicable to the given function. **[1]**

Then, there exists a value $c \in (1,4)$ such that

$$f'(c)=\frac{f(4)-f(1)}{4-1}$$

$$\Rightarrow -\frac{4}{(4c-1)^2}=\frac{\frac{1}{16-1}-\frac{1}{4-1}}{3}$$

$$\Rightarrow -\frac{4}{(4c-1)^2}=\frac{\frac{1}{15}-\frac{1}{3}}{3}$$

$$\Rightarrow -\frac{4}{(4c-1)^2}=\frac{-4}{45}$$

$$\Rightarrow (4c-1)^2=45$$

$$\Rightarrow 4c-1=\pm 3\sqrt{5}$$

$$\Rightarrow c=\frac{3\sqrt{5}+1}{4}\in(1,4)$$

[3]

Thus, the mean value verified for the given function.

Q. 5. Verify mean value theorem for the function $f(x) = x^3 - 2x^2 - x + 3$ in $[0, 1]$.

[NCERT Exemp. Ex. 5.3, Q. 74, Page 112]

Ans. Given that $f(x) = x^3 - 2x^2 - x + 3$ in $[0, 1]$

It is continuous in [0, 1]. **[1]**

Now, $f'(x) = 3x^2 - 4x - 1$, it is defined in (0, 1) and hence it is differentiable in (0, 1).

Thus, the mean value theorem is applicable to the given function. **[1]**

Then, there exists a value $c \in (0,1)$ such that

$$f'(c) = \frac{f(0) - f(1)}{0 - 1}$$

$$\Rightarrow \quad 3c^2 - 4c - 1 = \frac{[0+3] - [1-2-1+3]}{0-1}$$

$$\Rightarrow \quad 3c^2 - 4c - 1 = -2$$

$$\Rightarrow \quad 3c^2 - 4c + 1 = 0$$

$$\Rightarrow (3c-1)(c-1) = 0$$

$$\Rightarrow \quad c = 1, \frac{1}{3}$$

$$\Rightarrow \quad c = \frac{1}{3} \in (0,1)$$

[3]

Thus, the mean value verified for the given function.

Q. 6. Verify mean value theorem for the function $f(x) = \sin x - \sin 2x$ in $[0, \pi]$.

[NCERT Exemp. Ex. 5.3, Q. 75, Page 112]

Ans. Given that $f(x) = \sin x - \sin 2x$ in $[0, \pi]$

It is continuous in $[0, \pi]$. **[1]**

Now, $f'(x) = \cos x - 2\cos 2x$, it is defined in $(0, \pi)$ and hence it is differentiable in $(0, \pi)$. **[1]**

Thus, the mean value theorem is applicable to the given function.

Then, there exists a value $c \in (0, \pi)$ such that

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}$$

$$\Rightarrow \quad \cos c - 2\cos 2c = \frac{\sin\pi - \sin 2\pi - \sin 0 + \sin 0}{\pi - 0}$$

$$\Rightarrow \quad 2\cos 2c - \cos c = 0$$

$$\Rightarrow 2(2\cos^2 c - 1) - \cos c = 0$$

$$\Rightarrow \quad 4\cos^2 c - \cos c - 2 = 0$$

$$\Rightarrow \quad \cos c = \frac{1 \pm \sqrt{1+32}}{8} = \frac{1 \pm \sqrt{33}}{8}$$

$$\Rightarrow \quad c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right) \in (0, \pi)$$

[3]

Thus, the mean value verified for the given function.

Q. 7. Using mean value theorem, prove that there is a point on the curve $y = 2x^2 - 5x + 3$ between the points A (1, 0) and B (2, 1), where tangent is parallel to the chord AB. Also, find that point.

[NCERT Exemp. Ex. 5.3, Q. 78, Page 112]

Ans. Given that $y = 2x^2 - 5x + 3$ in $[1, 2]$

It is continuous in [1, 2]. **[1]**

Now, $f'(x) = 4x - 5$, it is defined in $(1, 2)$ and hence it is differentiable in $(1, 2)$. **[1]**

Thus, the mean value theorem is applicable to the given function.

Then, there exists a value $c \in (1, 2)$ such that **[1]**

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

$$\Rightarrow 4c - 5 = \frac{1 - 0}{1}$$

$$\Rightarrow \quad c = \frac{3}{2} \in (1, 2)$$

[1]

Thus, the mean value verified for the given function.

Now, put $x = \frac{3}{2}$ in $y = 2x^2 - 5x + 3$, we get

$$y = 2\left(\frac{3}{2}\right)^2 - 5\left(\frac{3}{2}\right) + 3 = 0$$

Thus, the required point is $\left(\frac{3}{2}, 0\right)$. **[1]**

Some Commonly Made Errors

- Students do not care about the basic standard formula of other branches of mathematics like algebra, trigonometry and co-ordinate geometry.
- They must learn all the concepts and formulae of algebra, trigonometry and co-ordinate geometry.
- They should practice some questions of differentiation.

EXPERT ADVICE

- *Always focus on the definition of a function and its domain and range.*
- *It is particularly used in finding the left-hand limit (LHL), right-hand limit (RHL), LHD and RHD.*

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