UNIT – I: NUMBERS, QUANTIFICATION AND NUMERICAL APPLICATIONS

CHAPTER-1

NUMBER & QUANTIFICATION

TOPIC-1

Binary Numbers

Revision Notes

➢ **Decimal Number System**

The numbers system that we are familiar with, that we use everyday is commonly referred to as decimal number system. It is also known as Base-10 system, since 10 digits (0 to 9) are used. A number is presented by its two values— symbol value (any digit from 0 to 9) and positional value (in terms of base value).

For example : Consider a decimal number 7238. This number can be represented as

7 2 3 8

10^3 position (*i, e,* thousound position)

10^2 position (*i, e,* hundreds position)

10^1 position (*i, e,* tens position)

10° position (*i, e,* ones position)

● **Binary Number System**

The binary number system, also referred to as base 2, makes use of only two digits *i.e.,* 1 and 0. '*Bi*' in binary is analogous to *bi* in bicycle (two wheels). Each digit of the binary system called a *bit* originating from *binary digit.* Some examples of binary numbers are 1001011, 1011.101, 1111.01 etc. A binary number can be mapped to an equivalent decimal number that can be easily understood by human.

Decimal	Binary
0	0
1	1
2	10
3	11
4	100
5	101
6	110
7	111
8	1000
9	1001

Table 1 : Binary value for (0-9) digits of decimal number system

➢ **Conversion of Decimal to Binary**

To convert a decimal number into binary number, we divide decimal number repeatedly by 2 because the bases value of binary system is 2. Record the remainder after each division and finally write the remainders in reverse order in which they computed.

For example : Convert decimal number 25 into binary number.

Sol. Step 1 : Divide the decimal number 25 dy 2.

2	25		
2	12	1	1^{st} remainder
2	6	0	2^{nd} remainder
2	3	0	3^{rd} remainder
2	1	1	4^{th} remainder
	0	1	5^{th} remainder

Step 2 : Collect the remainder from the bottom to top to get binary equivalent.

$$(25)_{10} = (11001)_2$$

- **Conversion of Binary to Decimal**

Since, binary number system has base 2, the positional values are computed in terms of powers of 2.

For example : Convert $(11001)_2$ into decimal number

Digit	1	1	0	0	1
Position Number	4	3	2	1	0
Positional Value	2^4	2^3	2^2	2^1	2^0

Decimal Number $= 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$

$= 16 + 8 + 0 + 0 + 1 = 25$

Thus, $(11001)_2 = (25)_{10}$

- **Additional of Binary Numbers**

The binary number system used only two digits 0 and 1 due to which their addition is simple. There are four basic operations for binary addition as mentioned below.

(i) $0 + 0 = 0$, **(ii)** $0 + 1 = 1$, **(iii)** $1 + 0 = 1$, **(iv)** $1 + 1 = 10$ (with carry 1)

For example : Consider two binary number 11101 and 11011, then

$$\begin{array}{cccccc} (1) & (1) & (1) & (1) & (1) & & \leftarrow \text{Carry} \\ & 1 & 1 & 1 & 0 & 1 \\ + & 1 & 1 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ \hline \end{array}$$

The above sum is carried out by following steps

$1 + 1 = 10 = 0$ with carry of 1

$1 + 0 + 1 = 10 = 0$ with carry of 1

$1 + 1 + 0 = 10 = 0$ with carry of 1

$1 + 1 + 1 = 10 + 1 = 11$ with carry of 1

$1 + 1 + 1 = 11$

Thus, $11101 + 11011 = 111000$

Note : $10 + 1 = 11$, which is equivalent to two + one = three (the next binary number after 10.)

- **Substraction of Binary Numbers**

The subtraction of the binary digit depends on the four basic operations

(i) $0 – 0 = 0$ **(ii)** $1 – 0 = 1$ **(iii)** $1 – 1 = 0$ **(iv)** $10 – 1 = 1$ OR ($0 – 1 = 1$ by borrowing 1)

The above first three operations are easy to understand as they are identical to decimal substraction the fourth operation can be understand with the logic two minus one is one (*i.e*, $2 – 1 = 1$).

For example : Consider to binary numbers 1100 and 1010, then

$$\begin{array}{ccccc} (1) & 0 & 10 & & \leftarrow \text{borrow} \\ 1 & 1 & 0 & 0 \\ -\,1 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline \end{array}$$

The above substraction is carried out through following steps —

For $0 – 0 = 0$

for $0 – 1 = 1$, taking borrow 1 and then $10 – 1 = 1$

for $1 – 0$, since 1 has already been given, it becomes $0 – 0 = 0$

for $1 – 1 = 0$

Thus, $1100 – 1010 = 0010$

- **Multiplication of Binary Numbers**

The multiplication process is the same for the binary numbers as it is for numerals.

For example : Multiple $(1101)_2$ by $(1010)_2$

Sol.

$$
\begin{array}{cccccccc}
 & & & & 1 & 1 & 0 & 1 \\
 & & & & 1 & 0 & 1 & 0 \\
\hline
 & & & & 0 & 0 & 0 & 0 \\
 & & & 1 & 1 & 0 & 1 & \times \\
 & & 0 & 0 & 0 & 0 & \times & \times \\
+ & 1 & 1 & 0 & 1 & \times & \times & \times \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
\end{array}
$$

➢ **Division of Binary Numbers**

The binary division is similar to the decimal number division method.

For example : Divide $(1010)_2$ = by $(10)_2$

Sol.

$$
\begin{array}{l}
10\overline{)1010}(101 \\
\underline{-10} \\
\quad 010 \\
\quad \underline{\;\;10} \\
\quad\;\;\; 0
\end{array}
$$

Prime and Complex Numbers

➢ **Prime Numbers**

- In earlier classes, we have studied about prime numbers. Prime numbers have only two divisors 1 and itself. They cannot be written as product of other numbers. Prime numbers are central to number theory.

  For example: 2, 3, 5, 7 are prime number as these numbers are divisible by 1 and itself.

  List of prime number less than 200 is

2	3	5	7	11	13	17	23	29	31	37	41
43	47	53	59	61	67	71	73	79	83	89	97
101	103	107	113	127	131	137	139	149	151	163	167
173	179	181	191	193	197	199					

- Two numbers a and b are **relatively prime (co-prime)** if have no common divisors apart from 1.

  For example: 8 and 15 are relatively prime since factors of 8 are 1, 2, 4, 8 and 15 are 1, 3, 5, 15 and 1 is the only common factor.

➢ **Encryption using Prime Numbers**

- Encryption is the method by which information is converted into secret codes that hides the information's true meaning. The science of encryption and decryption (conversion of secert code into original message or information) information is called cryptography.
- In computing, unencrypted message (data) is known as **plaintext** (original message) and encrypted message is called **ciphertext.** The formula used to encode and decode massages are called **encryption algorithms or ciphers.**

  The ciphers (cryptography techniques) widely fall into two categories –

  (i) Symmetric Ciphers (Symmetric Key Cryptography)

  (ii) Asymmetric Ciphers (Asymmetric Key Cryptography)

(i) Symmetric Ciphers is also known as **secret key encryption,** use a single key. This key is sometime referred to as a shared key because sender or computing system doing the encryption must share the secret key with all entities authorized to decrypt the message. The most widely used symmetric key cipher is the **Advance Encryption Standard** (AES), which was designed to protect government – classified information. Symmetric key encryption is usually much faster than asymmetric encryption.

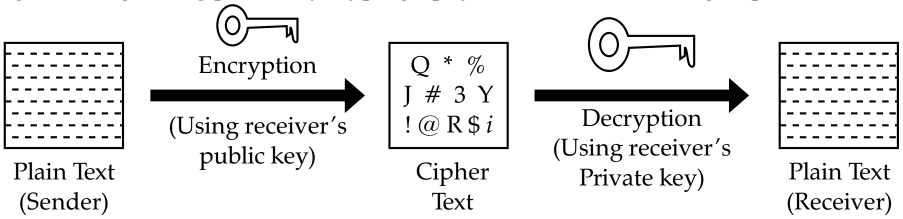
(ii) Asymmetric Ciphers

In this technique,

(i) Sender and receiver use different keys to encrypt and decrypt the message.
(ii) It is called so because sender and receiver use different keys.
(iii) It is also called as public key cryptography.
(iv) This type of cryptography often uses prime numbers to create keys.

- **Working of Asymmetric Cipher**

  The message exchange using public key cryptography involves the following steps –



Asymmetric Cipher

Step 1:

At sender side,

* Sender encrypt the message using receivers public key
* The public key of receiver is publicly available and know to everyone.
* Encryption converts the message into a cipher text.
* This cipher text can be decrypted only using the receiver's private key.

Step 2:

* The cipher text is sent to the receiver over the communication channel.

Step 3:

At receiver side,

* Receiver decrypts the cipher text using his private key.
* The private key of the receiver is known only to the receiver.
* Using public key, it is not possible for anyone to determine the receiver's private key.
* After decryption, cipher text converts back into a readable format.

TOPIC-2
Indices

Revision Notes

We have learnt in earlier classes that result of a repeated addition can be held by multiplication.

e.g., : $3 + 3 + 3 + 3 + 3 = 5(3) = 15$

$a + a + a + a + a = 5(a) = 5a$

Now, $3 \times 3 \times 3 \times 3 \times 3 = 3^5$

$a \times a \times a \times a \times a = a^5$

It may be noticed that in the first case 3 in multiplied 5 times and in the second case 'a' is multiplied 5 times. In all such cases, a factor which multiplies is called the '**base**' and the number of times it is multiplied is called the '**power**' or the '**index**' or the '**exponent**'. Therefore, '3' and 'a' are the bases and '5' is the index for both.

Thus, if n is a positive integer, and 'a' is a real number, *i.e.*, $n \in N$ and $a \in R$ (where N is the set of positive integers and R is the set of real numbers), 'a' is used to denote the continued product of n factors each equal to 'a' as shown below :

$$a^n = a \times a \times a \times \text{............ to } n \text{ factors}$$

Here, a^n is the power of 'a' whose base is 'a' and the index is 'n'.

e.g., : In $5 \times 5 \times 5 \times 5 = 5^4$, 5 is base and 4 is index or power.

- **Zero Index**

  Any base raised to the power zero is defined to be 1.

  i.e., $a^0 = 1$

  e.g., : $2^0 = 1, 3^0 = 1, x^0 = 1$ etc.

- **Negative Index**

  For any non-zero number and positive integer, we define $a^{-1} = \frac{1}{a}$ and $a^{-n} = \frac{1}{a^n}$

  e.g., : $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$, $3^{-3} = \frac{1}{3^3} = \frac{1}{27}$

- **Fractional Index**

  For any positive integer n and positive number a, we define

$$a^{1/n} = \sqrt[n]{a}$$

e.g., : $27^{1/3} = \sqrt[3]{27} = 3$

Note : We also define $\sqrt[r]{a} = a^{1/r}$

➢ Index Laws

- **Law 1 : Product of Index**

$$a^m \times a^n = a^{m+n}, \text{ where } m \text{ and } n \text{ are positive integers,}$$

by definition, $a^m = a \times a \times$ .... to m factors and $a^n = a \times a \times$ .... to n factors.

$\therefore$
$$\begin{aligned} a^m \times a^n &= (a \times a \times \text{.......... to } m \text{ factors}) \times (a \times a \times \text{........ to } n \text{ factors}) \\ &= a \times a \times \text{........ to } (m+n) \text{ factors} \\ &= a^{m+n} \end{aligned}$$

Now, we extend this logic to negative integers and fractions. Consider a negative integer *i.e.*, 'm' will be replaced by $-n$.

By the definition of $\quad a^m \times a^n = a^{m+n}$

we get $\quad a^{-n} \times a^n = a^{-n+n} = a^0 = 1$

e.g., : $\quad 5^3 \times 5^4 = (5 \times 5 \times 5) \times (5 \times 5 \times 5 \times 5)$
$$= 5^{3+4} = 5^7$$

- **Law 2 : Quotient of Index**

$$\frac{a^m}{a^n} = a^{m-n}, \text{ where } m \text{ and } n \text{ are positive integers and } m > n.$$

By definition, $\quad a^m = a \times a \times$ .... to m factors

Therefore,
$$\begin{aligned} a^m \div a^n &= \frac{a^m}{a^n} = \frac{a \times a \times \text{..........to } m \text{ factors}}{a \times a \times \text{..........to } n \text{ factors}} \\ &= a \times a \times \text{....... to } (m-n) \text{ factors} \\ &= a^{m-n} \end{aligned}$$

For example : $\quad 2^7 \div 2^4 = \dfrac{2^7}{2^4} = 2^{7-4} = 2^3 = 2 \times 2 \times 2 = 8.$

- **Law 3 : Index of Index**

$$(a^m)^n = a^{mn}, \text{ where } m \text{ and } n \text{ are positive integers}$$

By definition,
$$\begin{aligned} (a^m)^n &= a^m \times a^m \times \text{....... to } n \text{ factors} \\ &= (a \times a \times \text{.... } m \text{ factors}) \ a \times a \times \text{.... to } n \text{ factors .... to } n \text{ times.} \\ &= a \times a \times \text{........ to } mn \text{ factors} \\ &= a^{mn} \end{aligned}$$

Following above, $\quad (a^m)^n = (a^m)^{p/q}$

Here, we will keep m as it is and replace n by $^p/_q$, where p and q are positive integers.

Now, $\quad$ the q^{th} power of $(a^m)^{p/q} = \left\{\left(a^m\right)^{p/q}\right\}^q$
$$\begin{aligned} &= (a^m)^{(p/q) \times q} \\ &= a^{mp} \end{aligned}$$

e.g., : $\quad (2^4)^3 = 2^{12} = 4096.$

If we take q^{th} root of the above, we obtain

$$(a^{mp})^{1/q} = \sqrt[q]{a^{mp}}$$

e.g., : $\quad (2^{12})^{1/6} = 2^2 = 4$

- **Law 4 : Index of Product**

$$(ab)^n = a^n b^n, \text{ where } n \text{ can take all the values.}$$

$$\begin{aligned} (ab)^n &= ab \times ab \times \text{..... to } n \text{ factors} \\ &= (a \times a \times \text{.... to } n \text{ factors}) \times (b \times b \times \text{.... to } n \text{ factors}) \\ &= a^n \times b^n \end{aligned}$$

When n is a positive fraction, we will replace n by p/q.

Then we will have,

$$(ab)^n = (ab)^{p/q}$$

Also, the q^{th} power of $(ab)^{p/q} = \left\{(ab)^{p/q}\right\}^q = (ab)^p$

e.g., : $\quad (2 \times 3)^4 = 2^4 \times 3^4 = 16 \times 81 = 1296.$

- **Law 5 : Index of Quotient**

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \ (\text{provided } b \neq 0), \text{ where } a, b, n \text{ are real.}$$

$$\left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \text{...... to } n \text{ factors}$$

$$= \frac{a \times a \times \text{.......to } n \text{ factors}}{b \times b \times \text{........to } n \text{ factors}} = \frac{a^n}{b^n}$$

e.g., : $$\left(\frac{5}{2}\right)^3 = \frac{5^3}{2^3} = \frac{125}{8}.$$

TOPIC-3
Logarithms and Antilogarithms

Revision Notes

If a is a positive real number, other than 1 and $a^x = n$, then we write :

$$x = \log_a n$$

and we say that the value of log n to the base 'a' is 'x'.

In other words, logarithm of a number to a given base is the index or power to which the base must be raised to produce the number *i.e.*, to make it equal to the given number. If there are three quantities indicated by a, x, and n, they are related as follows :

If $a^x = n$, where $n > 0$, $a > 0$ and $a \neq 1$, then x is said to be the logarithm of number n to the base 'a' symbolically it can be expressed as follows :

$$\log_a n = x$$

i.e., the logarithm of n to the base 'a' is x.

➢ **Common Logarithm :** Logarithm to the base 10 are known as common logarithm. When base is not mentioned, it is taken as 10.

e.g., :

(i) $2^4 = 16 \Rightarrow \log_2 16 = 4$ *i.e.*, the logarithm of 16 to the base 2 is equal to 4.

(ii) $10^3 = 1000 \Rightarrow \log_{10}(1000) = 3$ *i.e.*, the logarithm of 1000 to the base 10 is 3.

(iii) $5^{-3} = \frac{1}{125} \Rightarrow \log_5\left(\frac{1}{125}\right) = -3$ *i.e.*, the logarithm of $\frac{1}{125}$ to the base 5 is – 3.

Remarks :

- The two equations $a^x = n$ and $x = \log_a n$ are only transformations of each other and should be remembered to change one form of the relation into the other.

  For example :

  (i) Express $2^{-3} = \frac{1}{8}$ in log

  Sol. $2^{-3} = \frac{1}{8} \Rightarrow \log_2 \frac{1}{8} = -3$

  (ii) Express $\log_3 81 = 4$ in the index form.

  Sol. $\log_3 81 = 4 \Rightarrow 3^4 = 81$.

- The logarithm of 1 to any base is zero. This is because any number raised to the power zero is one.

  Since, $a^0 = 1$, $\boxed{\log_a 1 = 0}$

  e.g., : $\log_7 1 = 0$, $\log_{2\sqrt{3}} 1 = 0$

- The logarithm of any quantity to the same base is unity. This is because any quantity raised to the power 1 is that quantity only.

  Since, $a^1 = 1$, $\boxed{\log_a a = 1}$

e.g., : $\log_{34} 34 = 1$, $\log_{21} 21 = 1$

➢ **Fundamental Laws of Logarithms**

Law 1 : Logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers to the same base, *i.e.*,

$$\boxed{\log_a mn = \log_a m + \log_a n}$$

e.g., $$\log_{10} 12 = \log_{10}(2\times2\times3) = \log_{10} 2 + \log_{10} 2 + \log_{10} 3$$
$$= 2\log_{10} 2 + \log_{10} 3$$

Law 2 : The logarithm of the quotient of two numbers is equal to the difference of their logarithms to the same base, *i.e.*,

$$\boxed{\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n}$$

e.g., : **(i)**

$$\log_2\left(\frac{9}{5}\right) = \log_2 9 - \log_2 5$$
$$= \log_2 (3\times3) - \log_2 5 = \log_2 3 + \log_2 3 - \log_2 5$$
$$= 2\log_2 3 - \log_2 5$$

(ii)

$$\log_{10}\left(\frac{1}{2}\right) = \log_{10} 1 - \log_{10} 2$$
$$= 0 - \log_{10} 2 = -\log_{10} 2. \qquad [\because \log_a 1 = 0]$$

Law 3 : Logarithm of the number raised to the power is equal to the index of the power multiplied by the logarithm of the number to the same base *i.e.*,

$$\boxed{\log_a m^n = n \log_a m}$$

e.g., : $\log_{10} (32) = \log_{10}(2^5) = 5 \log_{10} 2$

- **Change of Base**

If the logarithm of a number to any base is given, then the logarithm of the same number to any other base can be determined from the following relation :

$$\boxed{\log_a m = \log_b m \times \log_a b \Rightarrow \log_b m = \frac{\log_a m}{\log_a b}}$$

For example : Change the base of $\log_5 31$ into the common logarithmic base.

Sol. Since,

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$\therefore \quad \log_5 31 = \frac{\log_{10} 31}{\log_{10} 5}$$

- **Relation between Indices and Logarithms**

- Let $x = \log_a m$ and $y = \log_a n$

$\therefore$ $a^x = m$ and $a^y = n$

So, $a^x . a^y = mn$

or, $a^{x+y} = mn$

or, $x+y = \log_a mn$

or, $\log_a m + \log_a n = \log_a mn$

- Also, $\frac{m}{n} = \frac{a^x}{a^y}$

or, $\frac{m}{n} = a^{x-y}$

or, $\log_a\left(\frac{m}{n}\right) = x - y$

or $\boxed{\log_a\left(\frac{m}{n}\right) = \log_a m - \log_b n}$

- Again, $m^n = m.m.m.........$ to n times

So, $\log_a m^n = \log (m.m.m..........$ to n times$)$

or, $\log_a m^n = \log_a m + \log_a m + \log_a m .... + \log_a m$

or, $\log_a m^n = n\log_a m$

Now, $a^\circ = 1 \Rightarrow 0 = \log_a 1$

- Let $\log_b a = x$ and $\log_a b = y$

$\therefore$ $a = b^x$ and $b = a^y$

So, $a = (a^y)^x$

or, $a^{xy} = a$

or, $xy = 1$ (Same base has equal power)

or, $\boxed{\log_b a \times \log_a b = 1}$

- **Logarithm Table**

It is not always necessary to find the logarithm of a number by mere calculation. We can also use logarithm table to find the logarithm of a number. The logarithm of a number comprises of two parts. The characteristics part and the mantissa part.

- **Characteristic Part :** The whole or the integral part of a number is called the characteristic part. The characteristic of any number greater than one is positive, and it is one less than the number of digits to the left of the decimal point in a given number. If the number is less than one, the characteristic is negative and is one more than the number of zeros to the right of the decimal point.

e.g., :

Number	Characteristic
4	0 [one less than the number of digits to the left of the decimal point]
21	1
111	2
0.1	–1 or $\bar{1}$ [one more than the number of zero on the right immediately after the decimal point]
0.025	– 2 or $\bar{2}$
0.000010	– 5 or $\bar{5}$

- **Mantissa Part :** The decimal part of the logarithm number is said to be the mantissa part and it should always be a positive value. If the mantissa part is in a negative value, then convert into the positive value.

e.g.,: – 5.2592 = – 6 + (1– 0.2592) = – 6 + 0.7428

For example :

Find the value of $\log_{10}2.872$.

Sol.

Step 1 : Characteristic Part = 2 and mantissa part = 872

Step 2 : Check the row number 28 and column number 7. So the value obtained is 4579.

Step 3 : Check the mean difference value for row number 28 and mean difference column 2. The value corresponding to the row and column is 3.

Step 4 : Add the values obtained in step 2 and 3, we get 4582. This is the mantissa part.

Step 5 : Since the number of digits to the left side of the decimal part is 1, the characteristic part is less than 1. So the characteristic part is 0.

Step 6 : Finally combine the characteristic part and the mantissa part. So it becomes 0.4582.

Therefore the value of log 2.872 is 0.4582.

- **Antilogarithms**

If x is the logarithm of a given number n with a given base, then n is called the antilogarithm (antilog) of x to that base.

i.e., If $\log_a n = x$, then n = antilog x

***e.g.,* :** If log 61720 = 4.7904, then

61720 = antilog (4.7904)

Number	Logarithm	Antilogarithm
206	2.3139	206.0
20.6	1.3139	20.60
2.06	0.3139	2.060
0.206	–1.3139	0.2060
0.0206	–2.3139	0.02060

Remarks : Logarithm using base e is called Natural logarithm.

{e = 2.33 (approx) called exponential number and $\log_e e = 1$}

***e.g.,* :** Find the number whose logarithm is –2.4678.

Sol. – 2.4678 = – 3 + 2 – 2.4678 = – 3 + 0.5322 = $\bar{3}.5322$

For mantissa 0.532, the number = 3404

For mean difference 2, the number = 2

∴ For mantissa 0.5322, the number = 3404 + 2 = 3406

The characteristic is – 3, therefore, the number is less than one and there must be two zeroes just after the decimal point.

Thus, Antilog (–2.4678) = 0.003406

➢ **Applications of Logarithm and Antilogarithm**

The logarithm is used in the following of real life applications.

(i) Earth quake Intensity Measurement
(ii) Acidic Measurement of Solutions (pH Value)
(iii) Sound Intensity Measurement
(iv) Express larger values
(v) Balance on financial investment
(vi) Population growth
(vii) Radioactive Decay
(viii) Computation of Time

- At the Richter scale, a logarithm function is used to measure the magnitude of earthquakes by the following formula :

$$R = \log\left(\frac{A}{A_0}\right)$$

where, A — the measure of the amplitude of the earthquake wave.
A — the amplitude of the smallest detectable wave (or standard wave)

For example : An earthquake is measured with a wave amplitude 392 times as great as A_0. What is the magnitude of this earthquake using the Richter scale, to nearest tenth ?

Sol. We know that, $R = \log\left(\frac{A}{A_0}\right)$

Given , $A = 392\,A_0$

Therefore, $R = \log\left(\frac{392\,A_0}{A_0}\right)$

$\Rightarrow$ $R = \log 392$

Now, according to log table, $\log 392 = 2.5932$

Hence, $R = 2.5932 \cong 2.6$

Thus, the magnitude of this earthquake is 2.6 on the Richter scale.

- The measure of acidity of a liquid is called the pH of the liquid. The formula for pH is

$$\text{pH} = -\log [H^+]$$

where, $[H^+]$ is the concentration of hydrogen, given in a unit called mol/L (moles per liter), one mole is 6.022×10^{23} molecules or atoms.

For example : If lime juice has a pH of 1.7, what is the concentration of hydrogen ions (in mol/L) in lime juice, to the nearest hundredth ?

Sol. We know that,

$$\text{pH} = -\log [H^+]$$

Since, given pH $= 1.7$

Therefore, $1.7 = -\log [H^+]$

$\Rightarrow$ $1.7 = -\log x$ (let H^+ represents with variable x)

$\Rightarrow$ $\log x = -1.7$

$\Rightarrow$ $\log x = -2 + 2 - 1.7 = -2 + 0.3 = \bar{2}.30$

$\Rightarrow$ $x = \text{antilog}\,(\bar{2}.30) = 0.01995 \cong 0.02$

Hence, the concentration of hydrogen ions in lime juice is 0.02.

- Exponential function including population and bacterial growth, radioactive decay, compound interest, cooling if objects and growth of phenomena such as virus infection, internet usage and popularity of foods etc. are solved with the help of logarithm.

Important Formulae

- **Compound Interest is calculated by the formula**

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where,
$A(t)$ = amount after t years
P = Principal amount
r = Interest rate
n = number of times the interest is compounded per year
t = number of years

For example :

(1) How long will it take to the nearest whole year, for money to double if it is invested at 10% compounded monthly ? [Use $\log_e 2 = 0.6931$, $\log_e(1.0083) = 0.008298$]

Sol. Let principal amount or initial investment be P.

Given, $A = 2P$ (Let money be doubled in t year)

$$r = 10\% = \frac{10}{100} = 0.1$$

$$n = 1 \text{ year} = 12 \text{ months}$$

We know that, $A = P\left(1+\frac{r}{n}\right)^{nt}$

$$\therefore \quad 2P = P\left(1+\frac{0.1}{12}\right)^{12t}$$

$$\Rightarrow \quad 2 = \left(1+\frac{0.1}{12}\right)^{12t}$$

Taking log on both sides, we get

$$\log_e 2 = \log_e\left[\left(1+\frac{0.1}{12}\right)^{12t}\right]$$

$$\Rightarrow \quad \log_e 2 = 12t\,\log_e\left(1+\frac{0.1}{12}\right)$$

$$\Rightarrow \quad t = \frac{\log_e 2}{12\log_e\left(1+\frac{0.1}{12}\right)}$$

$$t = \frac{0.6931}{12\log_e(1.008333)}$$

$$\Rightarrow \quad t = \frac{0.6931}{12(0.008298)} \quad \text{[Given]}$$

$$\Rightarrow \quad t = \frac{0.6931}{0.099581}$$

$$t \cong 6.96$$

Hence, money will double in about 7 years.

- **Compounded Continuously Interest** is calculated by the formula

$$A(t) = Pe^{rt}$$

Where, $A(t)$ = amount after t years
P = Principal amount
r = interest rate
t = number of years

- **Exponential Growth** of a population increases according to the formula

$$P(t) = P_0\, e^{rt}$$

Where, $P(t)$ = Population after time t
P = initial population
r = growth rate
t = time

- **Exponential Decay** of a substance (radioactive substance) is given by the following formula

$$m(t) = m_0\, e^{-rt}$$

where, $m(t)$ = mass remaining after time t
m_0 = initial mass
r = decay rate
t = time

Its **half-life** is given by $h = \dfrac{\log 2}{r}$

For example :
If a 325 mg sample of a radioactive material decays to 195 mg in 72 hours, find the half-life of the element.
[Use $\log_e (0.6) = -0.51081$ and $\log_e 2 = 0.6931$]

Sol. Since, half life of a radioactive substance, $h = \dfrac{\log 2}{r}$

Therefore, we need to find the decay rate.
We know that

$$m(t) = m_0 e^{-rt}$$

Given, $m_0 = 325$, $m(t) = 195$ and $t = 72$

$\therefore \quad 195 = 325e^{-72^{r}}$

$\Rightarrow \quad 0.6 = e^{-72r}$

Taking log on both sides, we get

$$\log_e (0.6) = \log_e (e^{-72r})$$

$\Rightarrow \quad \log_e (0.6) = -72r \log_e e$

$\Rightarrow \quad \log_e (0.6) = -72r \qquad [\because \log_e e = 1]$

$\Rightarrow \quad r = \dfrac{\log_e(0.6)}{-72}$

$\Rightarrow \quad r = \dfrac{-0.51082}{-72}$

$\Rightarrow \quad r = 0.007094$

Therefore, substituting this into the formula for half-life,

we get $\quad h = \dfrac{\log_e 2}{r} = \dfrac{0.6931}{0.007094} = 97.6979$

Hence, half-life is approximately 97.7 hours.

CHAPTER-2

NUMERICAL APPLICATIONS

TOPIC-1
Average

Revision Notes

➢ **Averages**

An average (or mean or arithmetic mean) of a list of data is the expression of the central value of a set of data. Mathematically, average of a number of quantities of the same kind is equal to their sum divided by the number of those quantities.

For given set of data, if $x_1, x_2, x_3 \ldots\ldots\, x_n$, are n observations, then

$$\text{Average} = \frac{\text{Sum of observations}}{\text{Number of observations}}$$

or $\quad \bar{x} = \dfrac{x_1 + x_2 + x_3 + \ldots\ldots + x_n}{n}$

The average is basically mean of the values which are represented by $\bar{x}$. It is also denoted by the symbol 'μ'. Arithmetic average is used for all averages like : average income, average profit, average age, average marks etc.

For example : Find the average of all prime numbers between 30 and 50.

Sol. There are five all prime numbers between 30 and 50.

They are 31, 37, 41, 43, and 47

$\therefore$ Required average $= \dfrac{31+37+41+43+47}{5} = \dfrac{199}{5} = 39.8$

- **Weighted Average**

  Let Group A contain $\boldsymbol{m}$ quantities and their average is '$\boldsymbol{a}$' and Group B contains $\boldsymbol{n}$ quantities and their average is 'b', the **average of group C** containing '$a + b$' quantities $= \dfrac{ma+nb}{m+n}$.

If we have two or more groups of members whose individual averages are known, then combined average of all the members of all the group is known as **weighted average.**

If there are k groups having member of number $n_i, n_1, n_2, n_3, ..., n_k$, with average $A_1, A_2, A_3, ..., A_k$, respectively.

Then, weighted average $= \dfrac{n_1A_1 + n_2A_2 + ... + n_kA_k}{n_1 + n_2 + ... + n_k}$

For example : The average monthly expenditure of a family was ₹ 2200 during the first 3 months; ₹ 2250 during the next 4 months and ₹ 3120 during the last 5 months of a year. If the total saving during the year were ₹ 1260, then calculate average monthly income of family.

Sol. Total annual income

$$= 3 \times 2200 + 4 \times 2250 + 5 \times 3120 + 1260 \text{ (saving).}$$
$$= 6600 + 9000 + 15600 + 1260$$
$$= 32460$$

$\therefore$ Average monthly income $= \dfrac{32460}{12} = ₹\ 2705$

Hence, average monthly income of family is ₹ 2705.

- **Geometric average (mean)**

Geometric mean of $x_1, x_2, ... x_n$, is denoted by G.M. $= \sqrt[n]{x_1 \times x_2 \times .... x_n}$

For example : The production of a company for three successive years has increased by 10%, 20%, and 40%, respectively. What is the average increase of production.

Sol. G.M. $= (10 \times 20 \times 40)^{1/3} = (8000)^{1/3} = (8 \times 1000)^{1/3}$

$$= (2^3 \times 10^3)^{1/3}$$
$$= \left[(2 \times 10)^3\right]^{1/3}$$
$$= 20\%.$$

- **Some facts about average**

(i) If X is the average of $x_1 + x_2 + x_3 + ... + x_n$, then

(a) The average of $x_1 + a, x_2 + a, x_3 + a, ... , x_n + a$ is $X + a$.

(b) The average of $x_1 - a, x_2 - a, x_3 - a, ... , x_n - a$ is $X - a$.

(c) The average of $ax_1, ax_2, ..., ax_n$ is aX, provided $a \neq 0$.

(d) The average of $\dfrac{x_1}{a}, \dfrac{x_2}{a}, \dfrac{x_3}{a}, ..., \dfrac{x_n}{a}$ is $\dfrac{X}{a}$, provided $a \neq 0$.

(ii) If, in a group, one or more new quantities are added or excluded, then the new quantity or sum of added or excluded quantities = [change in number of quantities × original average]
± [change in average × final number of quantities]

Take +ve sign, if quantities are added and
Take – ve sign, if quantities are removed.

For example : The average weight of 40 students in a class is 15 years, When 10 new students are added, the average is increased by 0.2 year. Find the average age of the new students.

Sol. Here, 10 new students are added.

Therefore, change in average is +ve

$\therefore$ Sum of the quantities added = (change in no. of quantities × original average) + (change in average × final no. of quantities)

$\Rightarrow$ Sum of the weight of 10 new students added $= (10 \times 15) + (0.2 \times 50)$

$$= 150 + 10$$
$$= 160 \text{ kg}$$

$\therefore$ Average age of 10 new students $= \dfrac{160}{10} = 16$

Hence, average age of 10 new students is 16 years.

TOPIC-2
Calendar and Clock

Revision Notes

➢ **Calendar**

We know that, a **Calendar** is a chart or series of pages showing the days, weeks and months of a particular year. Solar year consists of 365 days 5 hrs 48 minutes, 48 seconds. In 47 BC, Julius caesar arranged a calendar known as the Julius Calendar, in which a year was taken as 365 days and in order to get rid of the odd quarter of a day, an extra day was added once in every 4^{th} year and this was called as **leap year** or Bissextile. Now-a-days, the calendar which is mostly used, is arranged by Pope Gregory II and known as **Gregorian calendar**

Basic structure of a Calendar

- **Century year**

  A year is century year if it is divisible by 100.
- **Non-century year**

  A year is non-century year if it is not a century year.
- **Ordinary year**

  1 ordinary year = 365 days = 52 weeks + 1 day.

  A year which is not divisible by 4, is an **Ordinary year.** 1879, 2009, 2019 etc. are few examples of ordinary years.
- **Leap year**

  1 leap year = 366 days = 52 weeks + 2 days

  (i) Every year divisible by 4 is a **leap year**, if it is not a century.

  (ii) In case of the century year, if the number of year is exactly divisible by 400, then it is a **leap year.**

  (iii) Every 4^{th} century is a leap year and no other century is a leap year.

  2012, 2016, 2020 etc. are few examples of leap year.
- **Odd Days**

  In a given period, the number of days more than the complete week are called odd days *i.e.*, the total number of days for a specific period of time is when divided by 7, the remainder obtain in such a case is term as odd days (s).

  Counting of Odd Days

  (i) Since, 1 ordinary year = 365 days = 52 weeks + 1 day

  Therefore, 1 ordinary year has 1 odd day.

  (ii) Since, 1 leap year = 366 day = 52 weeks + 2 days.

  Therefore, 1 leap year has 2 odd days

  (iii) 100 years = 76 ordinary years + 24 leap years

  = (76 × 1 + 24 × 2) odd days = 124 odd days

  = 17 weeks + 5 days (On dividing 124 by 7)

  = 5 odd days

  Thus,

  Number of odd days in 100 years = 5

  Number of odd days in 200 years = (5 × 2) = 3 odd days

  Number of odd days in 300 years = (5 × 3) = 1 odd day

  Number of odd days in 400 years = (5 × 4 + 1) = 0 odd day

  Similarly, each one of 800 years, 1200 years, 1600 years, 2000 years etc. has 0 (zero) odd day.

  Day of the week Related to Odd Days

No. of days	0	1	2	3	4	5	6
Day	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday

  Number of odd days in Months

  The month with 31 days contain (4 × 7 + 3) *i.e.*, 3 odd days and the month with 30 days contains (4 × 7 + 2) *i.e.*, 2 odd days.

For example : (1) What day of the week was 15^{th} August 1949 ?

15^{th} August 1949 means :

1948 complete year + first 7 months of the year 1949 + 15 days of August 1949

Counting of the odd days

Number of odd days in 1600 years = 0

Number of odd days in 300 years = 1

48 years = 12 leap years + 36 ordinary years [$\because$ 48 ÷ 4 = 12 leap years and remaining 48 – 12 = 36 ordinary years]

$= (12 \times 2 + 36 \times 1)$ odd days

= 60 odd days

= (8 weeks + 4 days) [on dividing 60 by 7]

$\equiv$ 4 odd days

$\therefore$ 1948 years have = (0 + 1 + 4) odd days = 5 odd days

Jan.	Feb.	March	April	May	June	July	August
(31 +	28 +	31 +	30 +	31 +	30 +	31 +	15) = 227 day

227 days = (32 weeks + 4 days)

$\equiv$ 3 odd days

$\therefore$ Total number of odd days = 5 + 3 = 8 days

$\equiv$ 1 odd day

This means that 15^{th} August fell on 1^{st} day. Therefore, the required day was Monday.

(2) Prove that the calender for 1990 will same for 2001 also.

Sol. For the calendar for 1990 will serve for 2001, it must have same day on 01.01.1990 (1st January 1990) and 01.01.2001 (1st January 2001). For that the number of odd days between 31st December 1989 and 31st December 2000 must be zero.

This period from 1990 to 2001 has 3 leap years and 8 ordinary years.

Number of odd days = $(3 \times 2 + 8 \times 1)$ = 14 days = 2 weeks + 0 odd day

$\therefore$ Calendar for the year 1990 will serve for the year 2001.

- **Clock**

The face or dial of a watch is a circle whose circumference is divided in 60 equal parts, called minute spaces.

A clock has two hands : the smaller one is called Hour hand or Short hand while the larger one is called the Minute hand or Long hand.

(i) Angle traced by minute hand in 60 min. = 360°

(ii) Angle traced by hour hand in 12 hours = 360°

(iii) In 60 minutes, the minute hand gains 55 minutes on the hour hand.

(iv) One minute, the 60 minute hand moves 60° *i.e.*, one minute division = $\frac{360°}{60°}$ = 6° apart

(v) In one hour, the hour hand moves 30° apart *i.e.*, one hour division = 6° × 5 = 30° apart (each hour number is separated by five minute divisions)

(vi) In one minute, the hour hand moves = $\frac{30°}{60°} = \frac{1°}{2}$ apart.

(vii) In one minute, the minute hand gains $5\frac{1°}{2}$ more than hour hand *i.e.*, one minute hand moves 6° and one hour hand moves $\frac{1°}{2}$.

- **Relative Position of the hands**

Any relative position of the hands of a clock is repeated 11 times in every 12 hours.

(i) When both hands are at right angles, they are 15 minute apart spaces.

(ii) The hands are in the same straight line when they are coincident or opposite to each other.

(iii) In every hour, both the hands coincide once.

(iv) In a day, the hands are coinciding 22 times.

(v) In every 12 hours, the hands of clock coincide 11 times.

(vi) In every 12 hours, the hands of clock are in opposite direction 11 times.

(vii) In every 12 hours, the hands of clock are at right angles 22 times.

(viii) In every hour, the two hands are at right angles 2 times.

(ix) In every hour, the two hands are in opposite direction once.

(x) In a day, the two hands are at right angles 44 times.

(xi) If both the hands coincide, then they will again coincide after $65\frac{5}{11}$ minutes *i.e.*, in correct clock both the hands coincide at an interval of $65\frac{5}{11}$ minutes.

For example : (1) At what time between 4 and 5 o'clock will the hands of clock be at right angles?

Sol. At 4 o'clock the minute hand will be 20 min. spaces behind the hour hand. Now, when the two hands are at right angles, they are 15 min. spaces apart.

So, they are at right angles in the following two cases :

Case 1 : When minute hand is 15 min. spaces behind the hour hand :

In this case min. hand will have to gain (20 – 15) = 5 minute spaces

We know that,

$\because$ 55 min. spaces are gained by hour hand in 60 min.

$\therefore$ 5 min. spaces will be gained by it in $\left(\frac{60}{55}\times 5\right)$ min $= 5\frac{5}{11}$ min.

$\therefore$ They are at right angles at $5\frac{5}{11}$ min. past 4.

Case 2 : When minute hand is 15 min spaces ahead of the hour hand :

To be in this position, the minute hand will be have to gain (20 + 15) = 35 min. spaces.

$\because$ 55 min. spaces are gained in 60 min.

$\therefore$ 35 min space are gained in $\left(\frac{60}{55}\times 35\right)$ min. $= 38\frac{2}{11}$ min.

$\therefore$ They are at right angles at $38\frac{2}{11}$ min past 4.

(2) At what time between 2 and 3 o'clock will the hands of a clock be together ?

Sol. At 2 o'clock, the hour hand is at 2 and the minute hand is at 12, *i.e.*, they are 10 min spaces apart. To be together, the minute hand must gain 10 minutes over the hour hand.

$\because$ 55 minutes are gained by it in 60 min

$\therefore$ 10 minutes will be gained in $\left(\frac{60}{55}\times 10\right)$ min. $= 10\frac{10}{11}$ min.

$\therefore$ The hands will coincide at $10\frac{10}{11}$ min. past 2.

- **Incorrect Clock**

If a clock or watch indicates 8.15, when the correct time is 8, it is said to be 15 minutes **too fast.**

On the other hand,. if it indicates 7.45, when the correct time is 8, it is said to be 15 minutes **too slow.**

Also,

(i) If both hands coincides at an interval of x minutes and $x < 65\frac{5}{11}$, then total gained time = $\left(\frac{65\frac{5}{11} - x}{x}\right)$ minutes and clock is said to 'fast'.

(ii) If both hands coincide at an interval of x minutes and $x > 65\frac{5}{11}$, then total time lost = $\left(\frac{x - 65\frac{5}{11}}{x}\right)$ minutes and clock is said to be 'slow'.

For example : The minute hand of a clock overtakes the hour hand at intervals of 65 minutes of the correct time. How much a day does the clock gain or lose ?

Sol. In a correct clock, the minute hand gain 55 min. over the hour hand in 60 minutes.

To be together again, the minute hand must gain 60 minutes over the hour hand.

60 min are gained in $\left(\frac{60}{55}\times 60\right)$ min $= 65\frac{5}{11}$ min

But, they are together after 65 min

$$\therefore \quad \text{Gain in 65 min} = \left(65\frac{5}{11} - 65\right) = \frac{5}{11} \text{ min.}$$

$$\text{Gain in 24 hours} = \left(\frac{5}{11}\times\frac{60\times 24}{65}\right) \text{min.} = 10\frac{10}{43} \text{ min}$$

Therefore, the clock gains $10\frac{10}{43}$ minutes in 24 hours.

TOPIC-3
Time, Work and Distance

Revision Notes

➢ **Time and Work**

Work is defined as something which has an effect or outcome; often the one desired or expected. The basic concept of time and work is similar to that across all arithmetic topics *i.e.*, the concept of proportionality. Efficiency is inversely proportional to the time taken when the amount of work done is constant.

➢ **General rules for time and Work**

Rule 1 : If M_1 persons can do W_1 work in D_1 days and M_2 persons can do W_2 work in D_2 days, then we can say :

$$M_1 \times D_1 \times W_1 = M_2 \times D_2 \times W_2$$

- If the persons work T_1 and T_2 hours per day, respective, then the equation gets modified to

$$M_1 \times D_1 \times W_1 \times T_1 = M_2 \times D_2 \times W_2 \times T_2$$

- It the persons efficiency are E_1 and E_2, respectively, then

$$M_1 \times D_1 \times W_1 \times T_1 \times E_1 = M_2 \times D_2 \times W_2 \times T_2 \times E_2$$

For example : If 15 men, working 9 hours a day, can reap a field in 16 days, in how many days will 18 men reap the field, working 8 hours a day ?

Sol. Here, $M_1 = 15, D_1 = 16, T_1 = 9$ hours, $M_2 = 18$ and $T_2 = 8$ hours

Using formula, $M_1 \times D_1 \times T_1 = M_2 \times D_2 \times T_2$

$$D_2 = \frac{M_1\times D_1\times T_1}{M_2\times T_2}$$

$$\Rightarrow \quad D_2 = \frac{15\times 16\times 9}{18\times 8} = 15 \text{ days}$$

Hence, required number of days are 15.

Rule 2 : If A can do a piece of work in n days, then the work done by A in one day $= \frac{1}{n}$ th part of whole work.

Rule 3 : If A's one day's work $= \frac{1}{n}$ th part of whole work, then A can finish the work in n days.

Rule 4 : If A can do a work in D_1 days and B can do the same work in D_2 days, then A and B together can do the same work in $\left(\frac{D_1\times D_2}{D_1+D_2}\right)$ days.

For example : A can do a piece of work in 5 days and B can do it in 6 days. How long will they take if both work together ?

Sol. Suppose, A can do a piece of work in D_1 days and B can do the same work in D_2 days.

Here, given $D_1 = 5$ days and $D_2 = 6$ days

Thus, A and B can do this work in $\dfrac{D_1 \times D_2}{D_1 + D_2}$ days

i.e., $$\frac{5\times 6}{5+6} = \frac{30}{11} = 2\frac{8}{11} \text{ days.}$$

Alternate Solution :

Sol. A's 1 day's work = $\dfrac{1}{5}$ th part whole work

and B's 1 day's work = $\dfrac{1}{6}$ th part of whole work

$\therefore$ $(A + B)$'s 1 day's work = $\left(\dfrac{1}{5}+\dfrac{1}{6}\right) = \dfrac{11}{30}$ th part of whole work

So, both together will finish the work in $\dfrac{30}{11}$ days = $2\dfrac{8}{11}$ days.

Rule 5 : If A, B and C can do a work in D_1, D_2 and D_3 days respectively, then all of them working together can finish the work in $\dfrac{D_1 \times D_2 \times D_3}{D_1 \times D_2 + D_2 \times D_3 + D_1 \times D_3}$ days.

Rule 6 : If A and B can do a piece of work in D_1 days and A alone can do it in D_2 days, then B alone can do the work in $\dfrac{D_1 \times D_2}{D_2 - D_1}$ days.

For example : A and B together can complete a piece of work in 4 days. If A alone can complete the same work in 12 days, in how many days can B alone complete that work ?

Sol. Suppose A and B can do a piece of work in D_1 days and A alone can do the same piece of work in D_2 days.

Here, given $D_1 = 4$ days and $D_2 = 12$ days

Thus, B alone can complete the work in $\dfrac{D_1 \times D_2}{D_2 - D_1}$ days.

i.e., $$\frac{4\times 12}{12-4} = \frac{48}{8} = 6 \text{ days}$$

Alternate Solution :

$$(A + B)\text{'s 1 day's work} = \frac{1}{4}$$

$$A\text{'s 1 day's work} = \frac{1}{12}$$

$$\therefore \quad B\text{'s 1 day's work} = \left(\frac{1}{4}-\frac{1}{12}\right) = \frac{1}{6}$$

Hence, B alone can complete the work in 6 days.

Rule 7. If A and B together can do a piece of work in x days, B and C together can do in y days and C and A together can do in z days, the same work can be done

By A alone in $\left(\dfrac{2xyz}{xy+yz-zx}\right)$ days

By B alone in $\left(\dfrac{2xyz}{yz+zx-xy}\right)$ days

By C alone in $\left(\dfrac{2xyz}{zx+xy-yz}\right)$ days

By A, B and C together in $\left(\dfrac{2xyz}{xy+yz+zx}\right)$ days.

For example : A and B can do a certain piece of work in 18 days, B and C can do it in 12 days and C and A can do it in 24 days. How long would each take separately to do it ?

Sol. Given$A$ and B together can do the piece of work in 18 days.

B and C together can do the piece of work in 12 days.

and C and A together can do the piece of work in 24 days.

Then, A alone can do the work in $= \left(\dfrac{2xyz}{xy+yz-zx}\right)$ days

$$= \frac{2\times18\times12\times24}{(18\times12)+(12\times24)-(24\times18)} = 144 \text{ days}$$

B alone can do the work in $= \left(\dfrac{2xyz}{yz+zx-xy}\right)$ days

$$= \frac{2\times18\times12\times24}{(12\times24)+(24\times18)-(18\times12)} = \frac{144}{7} \text{ days}$$

C alone can do the work in $= \left(\dfrac{2xyz}{zx+xy-yz}\right)$ days

$$= \frac{2\times18\times12\times24}{(18\times24)+(12\times18)-(12\times24)} = \frac{144}{5} \text{ days}$$

Alternate Solution :

$$(A + B)\text{'s 1 day's work} = \frac{1}{18}$$

$$(A + C)\text{'s 1 day's work} = \frac{1}{24}$$

$$(B + C)\text{'s 1 day's work} = \frac{1}{12}$$

Now, add up all three equations :

$$2\,(A + B + C)\text{'s 1 days' work} = \frac{1}{18}+\frac{1}{24}+\frac{1}{12}=\frac{13}{72}$$

$$(A + B + C)\text{'s 1 day's work} = \frac{13}{144}$$

$$\text{A's 1 day's work} = (A + B + C)\text{'s 1 day's work} - (B + C)\text{'s 1 day's work}$$

$$= \frac{13}{144}-\frac{1}{12}=\frac{1}{144}$$

Since, A completes the work in 1 day, therefore he will complete 1 work in $\dfrac{144}{1}$ = 144 days

Similarly, using the same logic, we can find that B needs $\dfrac{144}{7}$ days and C needs $\dfrac{144}{5}$ days, respectively to complete the same work in 1 day.

Rule 8 : If A and B can do a work in 'X' days and 'Y' days respectively. They started the work together but A left 'a' days before completion of the work. Then, time taken to finish the work is $\dfrac{Y(X+a)}{X+Y}$.

For example : A and B can do alone a job in 6 days and 12 days. They begin the work together but 3 days before the completion of job, A leaves off. In how many days will the work be completed ?

Sol. Using the direct formula *i.e.*,

$$\text{Time taken to finish the work} = \frac{Y(X+a)}{X+Y}$$

Here, given $X = 6$, $Y = 12$ and $a = 3$

$$\text{Therefore, time taken to finish the work} = \frac{12(6+3)}{6+12} = 6 \text{ days}$$

Alternate solution :

Let work will be completed in x days.

Then, work done by A in $(x - 3)$ days + work done by B in x days = 1 *i.e.,* $$\frac{x-3}{6}+\frac{x}{12} = 1$$

$$\Rightarrow \quad \frac{3x-6}{12} = 1$$

$$\Rightarrow \quad x = 6 \text{ days.}$$

Rule 9 : If 'A' is 'a' times efficient (good) than B and A can finish a work in X days, then working together, they can finish the work in $\frac{aX}{a+1}$ days.

Rule 10 : If 'A' is 'a' times efficient (good) than B and working together they finish a work in Z days, then time taken by $A = \frac{Z(a+1)}{a}$ days and time taken by $B = Z(a + 1)$ days.

For example : A is twice good as workman as B and together they finish a piece of work in 18 days. In how many days will A alone finish the work ?

Sol. Here we apply formula

$$\text{Time taken by A} = \frac{Z(a+1)}{a}$$

Given, $a = 2$ and Z = 18

Therefore, $$\text{time taken by A} = \frac{18(2+1)}{2} = 18\times\frac{3}{2} = 27 \text{ days.}$$

Alternate solution :

$$\text{(A's 1 day's work) : (B's 1 days work)} = 2:1$$

$$\text{(A + B)'s 1 day's work} = \frac{1}{18}$$

Divide $\frac{1}{18}$ in the ratio 2 : 1

$$\therefore \quad \text{A's 1 day's work} = \frac{1}{18}\times\frac{2}{3} = \frac{1}{27}$$

Hence, A alone can finish the work in 27 days.

Remark :

- If A is twice as good as workman as B, then A will take half time taken by B to complete a piece of work.
- If A is thrice as good as workman as B, then A will take one-third of the time taken by B to complete a piece of work.

Rule 11 : If A working alone takes 'X' days more than A and B together, and B working alone takes 'Y' days more than A and B together ; then the number of days taken by A and B working together is given by $\left(\sqrt{XY}\right)$ days.

➢ **Time and Distance**

The concept of time and distance is based on the formula

$$\text{Speed} = \frac{\text{Distance}}{\text{time}}$$

i.e., The rate at which any moving body (object) covers a particular distance in a particular time is called its speed. The speed is measured in terms of km/h. The SI unit of speed is metre per second (m/sec). It is also measured in miles per hour (mph).

➢ **Conversion of units**

1 hour = 60 mins = 60 × 60 sec

1 km = 1000 m

1 km = 0.625 mile

1 mile = 1.60 km *i.e*, 8 km = 5 miles

1 yard = 3 feet

1 km/h = $\frac{5}{18}$ m/sec

$$1 \text{ m/sec} = \frac{18}{5} \text{ km/hr}$$

$$1 \text{ miles/hr} = \frac{22}{15} \text{ ft/sec}$$

- While travelling a certain distance d, if a man changes his speed in the ratio $m : n$, then the ratio of time taken becomes $m : n$.
- **Average speed**

$$\text{Average speed} = \frac{\text{Total distance covered}}{\text{Total time taken}}$$

(i) When the distance is constant :

$$\text{Average speed} = \frac{2xy}{x+y}$$

Where, x and y are the two speeds at which the same distance has been covered.

(ii) When time taken is constant :

$$\text{Average time} = \frac{x+y}{2}$$

where, x and y are the two speeds at which we travelled for the same time.

For example : A person goes from A to B at the speed of 40 kmph and comes back at the speed of 60 kmph, what is his average speed for the whole journey.

Sol. Since, the distance travelled on both sides is the same, we can use the (i) formula, written above.

i.e., $$\text{Average speed} = \frac{2xy}{x+y}$$

Here, $x = 40$ kmph and $y = 60$ kmph

$\therefore$ $$\text{Average speed} = \frac{2\times40\times60}{40+60} = \frac{4800}{100} = 48 \text{ kmph}$$

Hence, the average speed of whole journey is 48 kmph.

- A man covers a certain distance D. If he moves S_1 speed faster, he would have taken t time less and if he moves S_2 speed slower, he would have taken t time more. Then his original speed S is given by

$$S = \frac{2(S_1 \times S_2)}{S_1 - S_2}$$

For example : A man covers a certain distance by scooter. If he had moved 3 km/hr faster, he would have taken 20 min less. If he had moved 2 km/hr slower, he would have taken 20 min more. Find the original speed.

Sol. Here, Faster speed, $S_1 = 3$ km/hr

Slower speed, $S_2 = 2$ km/hr

Then $$\text{Original speed, } S = \frac{2(S_1\times S_2)}{S_1 - S_2} = \frac{2(3\times2)}{3-2} = 12 \text{ km/hr}$$

Hence, the original speed of man is 12 km/hr.

- If a man with two different speeds U and V covers the same distance, then

$$\text{Required distance} = \frac{U\times V}{U-V} \times \text{Difference between arrival time}$$

Also, $$\text{Required distance} = \text{Total time taken} \times \frac{U\times V}{U-V}$$

For example : A boy walking at a speed of 10 km/hr reaches in school 12 min late. Next time at a speed of 15 km/hr he reaches his school 7 min late. Find the distance of his school from his house ?

Sol. Difference between time $= 12 - 7 = 5 \text{ min} = \frac{5}{60} = \frac{1}{12}$ hr

$$\text{Required distance} = \frac{U \times V}{U - V} \times \text{Difference between arrival times}$$

$$= \frac{15 \times 10}{15 - 10} \times \frac{1}{12} = 2.5 \text{ km} \qquad [\text{Here, } U = 15 \text{ and } V = 10]$$

➢ **Relative speed**

(i) When two bodies are moving in the same direction with speeds S_1 and S_2 respectively, their relative speed is the difference of their speeds.

i.e., Relative speed $= S_1 - S_2$

(ii) When two bodies are moving in the opposite directions with speeds S_1 and S_2, respectively, their relative speed is the sum of their speeds.

i.e., Relative speed $= S_1 + S_2$

For example : A person drives a car at the speed 68 km/hr locates a bus 40 metres ahead of him. After 10 seconds, the bus is 60 metres behind. Find the speed of the bus.

Sol. Let speed of the bus be S_B km/hr.

Now, in 10 sec, car covers the relative distance = (60 + 40) m = 100 m

$$\therefore \quad \text{Relative speed of car} = \frac{100}{10} = 10 \text{ m/sec}$$

$$= 10 \times \frac{18}{5} = 36 \text{ km/hr}$$

$$\therefore \quad 68 - S_B = 36$$

$$\Rightarrow \quad S_B = 32 \text{ km/hr}$$

Hence, speed of bus is 32 km/hr.

TOPIC-4
Mensuration and Seating Arrangement

Revision Notes

➢ **Mensuration**

Mensuration is the branch of Mathematics which deals with the skill of measuring the length of lines, areas of surfaces and volumes of solids from simple data of lines and angles. It is all about the process of measurement *i.e.,* Mensuration means measuring the various physical quantities such as perimeter, area, volume or length. There are two types of geometric shapes (i) 2*D* (2-dimension) and (ii) 3*D* (3 – dimension).

➢ **Perimeter**

A perimeter is a path that surrounds a two-dimensional shape. It is the sum of all the sides of 2D figure. It is measured in metre, centimetre, etc.

➢ **Area**

Area is a quantity that expresses the extent of a two dimensional surface or shape in a plane. The area of any figure is the amount of surface enclosed within its boundary lines. It is measured in square unit like m^2, cm^2, etc.

➢ **Volume**

It refers to the quantity of three-dimensional space enclosed by a closed surface. If an object is solid, then the space occupied by such an object is called its volume. It is measured in cubic unit like m^3, cm^3, etc.

➢ **Basic Conversions**

(I)

1 km = 10 hm (hm = hectometre & km = kilometre)
1 hm = 10 dam (dam = decametre)
1 dam = 10 m (m = metre)
1 m = 10 dm (dm = decimetre)
1 dm = 10 cm (cm = centimetre)
1 cm = 10 mm (mm = millimetre)
1 m = 100 cm = 1000 mm
1 km = 1000 m

(II)

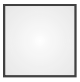
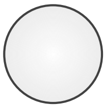
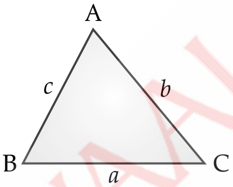
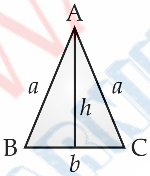
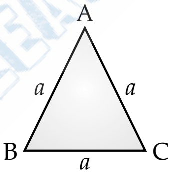
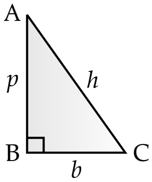
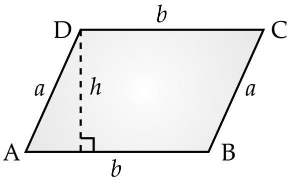
1 km = $\frac{5}{8}$ miles
1 mile = 1.6 km
1 inch = 2.54 cm

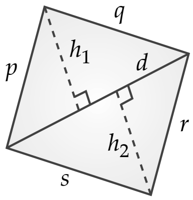
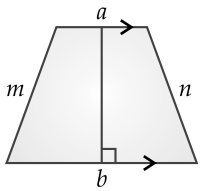
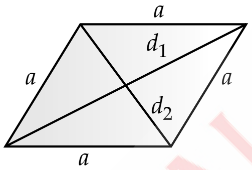
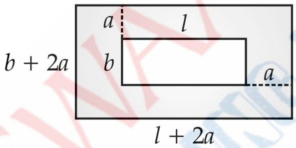
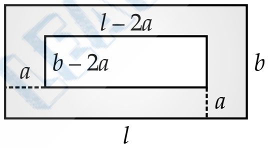
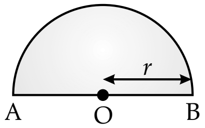
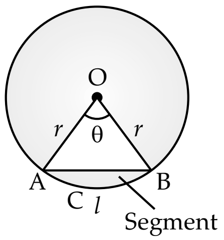
1 mile = 1760 yd = 5280 ft (yd = yards & ft = feets)
1 nautical mile (knot) = 6080 ft

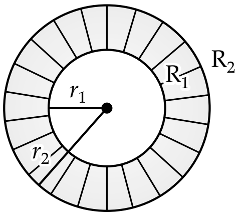
(III) 100 kg = 1 quintal
10 quintal = 1 tonne
1 kg = 2.2 pounds (approx.)

(IV) 1 litre = 1000 cc (cc = cubic centimetre)
1 acre = 100 m^2
1 hectare = 10000 m^2 (100 acre)

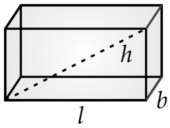
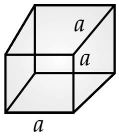
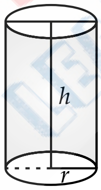
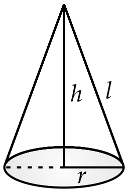
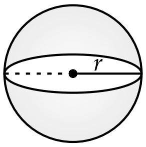
➤ **Plane Figures (2D-Figures)**

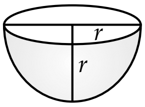
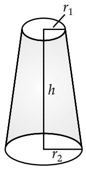
S. N.	Shape	Figure	Area (square units)	Perimeter (units)
1.	Square	 side of square = a	$A = a^2$,	$P = 4a$
2.	Rectangle	 length = l breadth = b	$A = l \times b$	$P = 2\,(l + b)$
3.	Circle	 radius = r	$A = \pi r^2$	$P = 2\pi r$ (circumference)
4.	Scalene Triangle		$A = \sqrt{s(s-a)(s-b)(s-c)}$ $s = \frac{a+b+c}{2}$	$P = a + b + c$
5.	Isosceles Triangle		$A = \frac{1}{2} \times b \times h$	$P = 2a + b$
6.	Equilateral triangle		$A = \frac{\sqrt{3}a^2}{4}$	$P = 3a$
7.	Right Triangle	 p – perpendicular b – base h – hypotenuse	$A = \frac{1}{2}\,bp$ Pythagoras Theorem $h^2 = p^2 + b^2$	$P = p + b + h$
8.	Parallelogram	 a = breadth b = base (length) h = altitude	$A = b \times h$	$P = 2(a + b)$

9.	Irregular Quadrilateral	 d = diagonal h_1 and h_2 are altitudes	$A = \frac{1}{2} \times d \times (h_1 + h_2)$	$P = p + q + r + s$
10.	Trapezium	 • a & b are two parallel sides • m & n are two non-parallel sides • h is the distance between two parallel sides	$A = \frac{1}{2}(a + b)h$	$P = a + b + m + n$
11.	Rhombus	 d_1 & d_2 are diagonals	$A = \frac{1}{2} d_1 \times d_2$	$P = 4a$
12.	Pathway Outside Rectangle	 l = length b = breadth a = width of the path way	Area of pathway, $A = (l + 2a)(b + 2a) - lb$	Perimeter of pathway, $P = l + b + 4a$
13.	Pathway Inside Rectangle	 l = length, b = breadth a = width of the pathway	Area of pathway, $A = lb - (l - 2a) \times (b - 2a)$	Perimeter of pathway $P = l + b - 4a$
14.	Semi-circle	 r = radius	$A = \frac{\pi r^2}{2}$	$P = \pi r + 2r$
15.	Sector of a circle & segment of a circle	 $OACBD$ is sector of circle	Area of sector $OACBD$ $= \frac{\theta}{360°} \times \pi r^2$ Area of segment ACB = Area of sector $OABD$ – Area of triangle AOB $= \frac{\theta}{360°} \times \pi r^2 - \frac{1}{2} r^2 \sin\theta$	Length of arc, (ACB) $l = \frac{\theta}{360°} \times 2\pi$ Perimeter of segment = length of the arc + length of segment $= \frac{\pi r\theta}{180°} + 2r \sin\frac{\theta}{2}$

16.	Ring		Area of ring, $A = \pi(r_2^2 - r_1^2)$	

➢ **Solid Figures (3*D*-figures)**

S.N.	Shape	Figure	Volume (cubic units)	Curved surface or lateral surface Area (sq units)	Total surface Area (sq. units)
1.	Cuboid	 Length of diagonal $= \sqrt{l^2+b^2+h^2}$ units	$V = l \times b \times h$	Area of four walls $= 2(l + b) \times h$ sq. units	$TSA = 2(lb + bh + lh)$
2.	Cube	 length of longest diagonal $= \sqrt{3}a$	$V = a^3$	Area of 4 walls $4a^2$	$TSA = 6a^2$
3.	Right Prism		V = Area of base × height	CSA = Perimeter of base × height	$TSA = CSA$ + 2(area of its tri-angular base)
4.	Right Circular Cylinder	 r = radius of base h = height	$V = \pi r^2 h$	$CSA = 2\pi rh$	$TSA = 2\pi r(r + h)$
5.	Right Circular Cone	 r = radius of base h = height l = slant height	$V = \frac{1}{3}\pi r^2 h$	$CSA = \pi rl$ where, $l = \sqrt{r^2+h^2}$	$TSA = \pi r(l + r)$
6.	Sphere		$V = \frac{4}{3}\pi r^3$	$CSA = 4\pi r^2$	$TSA = 4\pi r^2$

7.	Hemi-sphere		$V = \frac{2}{3}\pi r^3$	$CSA = 2\pi r^2$	$TSA = 3\pi r^2$
8.	Frustum of a Cone		$V = \frac{1}{3}\pi h$ $(r_1^2 + r_1 r_2 + r_2^2)$	$CSA = \pi l(r_1 + r_2)$, $l = \sqrt{h^2 + (r_1 - r_2)^2}$	$TSA = \pi l(r_1 + r_2) + \pi r_1^2 + \pi r_2^2$

➢ **Seating Arrangement**

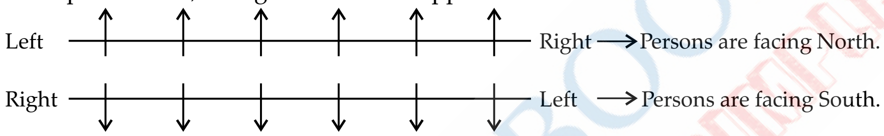
The arrangement of objects or people in logical manner is known as seating arrangement or sitting arrangement. Questions of seating arrangement are based on a set of information containing certain conditions. Seating arrangement are classified into following four categories :

(1) Linear Arrangement

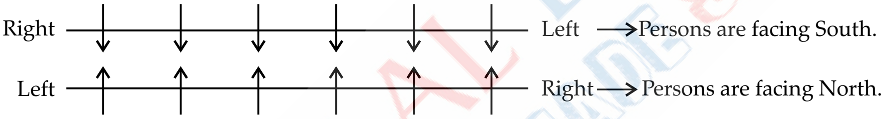
Here, the arrangement of the persons is linear *i.e.*, we have to arrange them in a single row. Here, generally a line is formed.

(2) Double Row Arrangement

Here, the arrangement is done between two groups of persons, one group in one row and the other group in other row or parallel row, facing each other or opposite.



Persons facing opposite direction



Persons facing each other

For example : Seven boys A, B, C, D, E, F and G are standing in a line.

(i) G is between A and E.

(ii) F and A have one boy between them.

(iii) E and C have two boys between them.

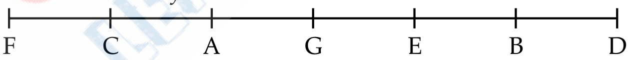
(iv) D is to the right of F.

(v) C and B have three boys between them.

(a) Who is second from left.

(b) Who is standing between A and F.

Sol. The correct position of all seven boys in a row is shown as below :

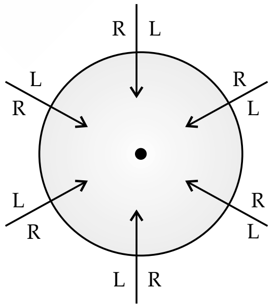


(a) Clearly, C is the second from the left.

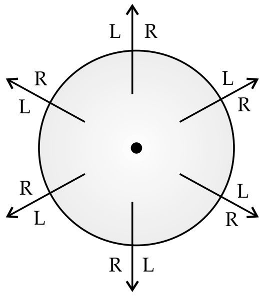
(b) C is standing between A and F.

(3) Circular Arrangement

In this arrangement, people are sitting around a circle facing toward or outside the centre.



When the persons are looking towards the centre, then the left hand side will be in clock-wise direction and right hand side will be in the anti-clock wise direction.



When the persons are looking away from the centre, then the right hand side will be in clock-wise direction and left hand side will be in the anti-clock wise direction

For example : Four ladies A, B, C and D and four gentleman E, F, G and H are sitting in a circle round a table facing each other

(i) Now two ladies or two gentleman are sitting side by side

(ii) C, who is sitting between G and E, is facing D.

(iii) F is between D and A and is facing G.

(iv) H is to the right of B.

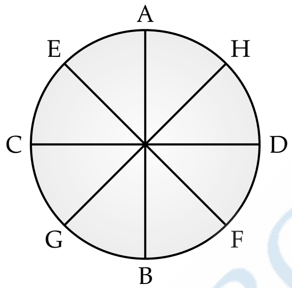
Then,

(a) Who is sitting to left To A ?

(b) E is facing whom ?

(c) Who are immediate neighbours of B ?

Sol. On the basis of the given information, the sitting arrangement of the persons will be shown as below :



(a) From the above figure, it is clear that F is sitting left to A.

(b) E is facing H.

(c) G and H are immediate neighbours of B.

(4) Rectangular or Square Arrangement

These arrangements are almost similar to the circular arrangements; the only difference is that the persons are sitting around a rectangular table.

UNIT – II: ALGEBRA

CHAPTER-3

SETS

Revision Notes

➢ **Definition of Set :**

The concept of set is fundamental in all branches of mathematics. In everyday life, we have to deal with collections and aggregates of objects of one kind or the other. For example, consider the following collections :

(i) The collection of odd numbers less than 17 *i.e.*, the numbers are 1, 3, 5, 7, 9, 11, 13.

(ii) The collection of all states of India.

(iii) The collection of roots of equation $x^2 + 5x + 6 = 0$ *i.e.*, – 2 and – 3.

Here, we note that each one of collections is a well-defined collection of objects.

A set is any **well-defined** class or collection of **objects**. By a well-defined collection, we mean that there exists a rule with the help of which, it is possible to tell whether the objects belongs or does not belong to the given collection. The objects in the set may be anything *i.e.*, numbers, people, mountains, rivers, etc. The objects constitutes the set are called **elements** or **member** of the set. We shall denote the sets by capital letters A, B, C, X, Y, Z etc. and their elements by small letters a, b, c, x, y, z etc. If an object x is a member of a set A, then we write $x \in A$, which may be read as 'x belongs to A' or 'x is a element of A'. On the other hand, an object x is not a member of a set A, then we write $x \notin A$.

Now, consider the following collection :

(i) All big cities of India.

(ii) All intelligent students of class XI of your school.

(iii) Five most renowned businessman of world.

None of the above collection is a set because the terms 'big cities', 'intelligent student' and 'most renowned' are unclear (or uncertain) and are not well defined.

➢ **Representation of sets :** A set often represented in two forms :

- **Roster or Tabular form :** In this form, all the elements of a set are listed, the elements are being separated by comma *i.e.* ',' and are enclosed within brackets {}.

  For example :

  $E = \{2, 4, 6, 8, 10\}$ is a set of even numbers ≤ 10.

  S = {February, April, June, September, November} is a set of months of a year having less than 31 days.

- **Set Builder form :** In this form, all elements of a set possess a single common property which is not possessed by any element outside the set. If A is a set consisting of elements x having property p, we write $A = \{ x : x$ has a property $p\}$. The colon ':' stands for the words 'such that' or sometimes it is used in place of the colon ':'.

  For example :

  $E = \{x : x$ is a even number and $x \leq 10 \}$

  $S = \{x : x$ is a month of a year having less than 31 days $\}$

➢ **Types of sets**

- **Empty Set :** A set which does not contain any element is called *empty set or null set or void set* and denoted by {} or Φ.

  For example :

  (i) The collection of even natural numbers less than zero.

  (ii) $\{x : x$ is an odd number divided by $2\}$.

  (iii) $\{x : x$ is a natural number satisfies $2x + 5 = 3\}$.

- **Singleton (or unit) set :** A set which contains only one element is called *singleton set*.

  For example :

  $\{0\}$, $\{x : 2x + 3 = 5\}$ and $\{x : x$ is the capital of the country$\}$ are singleton sets.

- **Finite and Infinite sets :** Sets which have finite number of elements are known as *finite sets* and sets which have infinite number of elements are known as *infinite sets*.

  For example :

  (i) $N = \{x : x$ is a natural number and $3 < x < 10$, it can also be written as $N = \{4, 5, 6, 7, 8, 9\}$. Thus, it is a finite set set as it contains fixed number of elements *i.e.* 6.

  (ii) $P = \{x : x$ is an even number and $x \geq 4$, it can also be written as $P = \{4, 6, 8, 10, \ldots\}$. Thus, it is an infinite as it contains infinite number of elements.

➢ **Cardinal Number (Order) of a Finite Set :** The number of different elements in a finite set A is called the *cardinal number* (or *order*) of set and denoted by $n(A)$ or o (A).

For example :

(i) $O = \{1, 3, 5, 7\}$ has 4 elements, thus its order is 4 and denoted as $n(O) = 4$.

(ii) $L = \{x : x$ is an alphabet present in NEED$\}$, can be written as $L = \{N, E, D\}$. Since, it has only three elements so $n(L) = 3$.

Note here repeated elements will be written once only.

➢ **Equivalent sets :** Two sets are said to be equivalent if they contain same number of elements.

For example :

(i) If $A = \{1, 2, 3, 4, 5\}$ and $B = \{p, q, r, s, t\}$, then $A \leftrightarrow B$, since $n(A) = n(B)$.

(ii) If $P = \{x : x$ is a letter in the word ALLOY$\}$ and $Q = \{x : x$ is a letter in LOYAL$\}$.

Here, $P = \{A, L, O, Y\}$ and $Q = \{L, O, Y, A\} \Rightarrow n(P) = n(Q) = 4$. Therefore, $P \leftrightarrow Q$.

➢ **Equal sets :** Two sets are said to be equal if they have same elements.

For example :

(i) $A = \{2, 4, 6\}$ and $B = \{2, 2, 4, 6, 6\}$, then $A = B$.

(ii) $P = \{1, 3, 5\}$ and $Q = \{x : x$ is an odd natural number and $1 \leq x \leq 5\}$, then $P = Q$.

➢ **Subset :**

A set A is said to be a subset of a set B, if every element of A is also an element of B. In symbols, we write $A \subset B$, and read as A is subset of B.

Thus, $A \subset B$ if $x \in A \Rightarrow x \in B$.

Also, if A is not a subset of B, we write $A \not\subset B$. Further, if A is contained in B, then B is said to be **superset** of A and denoted by $B \supset A$.

For example :

(i) If $A = \{1, 3, 5\}$ and $B = \{1, 2, 3, 4, 5\}$ then $A \subset B$ because each element of A is present in B.

(ii) If L is the set of letters present in SCHOOL and M is the set of letters present in SCHOLAR, then L = {S, C, H, O, L} and M = {S, C, H, O, L, A, R}. We say, $L \subset M$ as all elements of L are in M.

Also, we denote,

- Set of natural numbers, $N = \{1, 2, 3, 4, 5, ...\}$
- Set of whole numbers, $W = \{0, 1, 2, 3, 4,.......\}$
- Set of rational numbers, $Q = \{x : x$ is in the form of $\frac{p}{q}$, where p & q are integers and $q \neq 0\}$
- Set of integers, $I = \{....,-2, -1, 0, 1, 2, ...\}$
- Set of irrational numbers, $T = \{x : x \in I$ and $x \notin Q\}$
- Set of real numbers, R = $\{x : x$ is a rational or irrational number$\}$

We observe that, $N \subset I \subset Q \subset R$ and $T \subset R, Q \subset R, N \subset W$

➢ **Proper subset :** If A is any set and B is any non-empty set then A is known as proper subset of B, if every member of A is also a member of B and there is atleast one member in B which is not a member of A.

In above both the examples, both A and L are proper subsets of B and M, respectively.

Subsets of a set, number of subsets of a given set is given by 2^n, where n is no. of elements in a given set. If $A = \{a\}$, then subsets of A will be ϕ and $\{a\} \Rightarrow n(A) = 1$ and no. of subsets are $2^1 = 2$. If $A = \{a, b\}$, then subsets of A will be, ϕ, $\{a\}$, $\{b\}$, $\{a, b\} \Rightarrow n(A) = 2$ and no. of subsets are $2^2 = 4$.

Remark :

- Every set is a subset of itself.
- The empty set is a subset of every set.
- **Power set :** The set formed by all the subsets of a given set is known as power set of given set.

For example : If $A = \{a, b\}$ then power set of $A = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

➢ **Universal set :** It is a basic set, in particular context whose elements and subsets are relevant to the particular context *i.e.*, it is a set in which all elements under consideration in a given problem are contained or exist. Denoted by U or ξ.

For example :

(i) If $\{x : x \in W$ and $2 \leq x \leq 10\}$, then universal set will be $\{0, 1, 2, ...\}$.

(ii) The set of vowels in English alphabets, the universal set can be the set of all alphabets in English.

➢ **Venn Diagram**

Most of the ideas about the sets and the various relationships between them can be visualized by the mean of geometrical figures known as **Venn diagrams** (or **Venn-Euler diagrams**). Usually universal set (U or ξ) is denoted by a rectangle and its subsets by closed curves within the rectangle, such as circles, ovals (ellipses) etc.

For example :

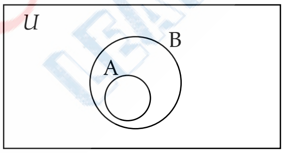


Figure 1

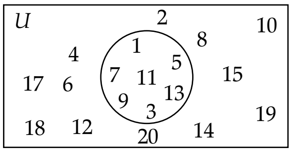


Figure 2

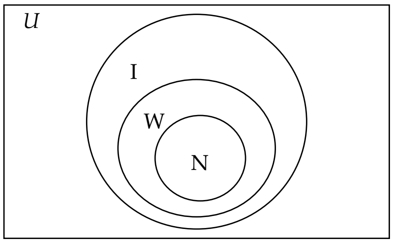


Figure 3

Fig. 1, represents $A \subset B$, fig. 2, all individual elements of A are marked inside the circle and Universal set is $\{1, 2, 3, ...20\}$ and fig. 3 represents $\boldsymbol{N \subset W \subset I}$.

➢ **Operation on sets**

➢ **Union of two sets :**

Set consisting of all the elements of the either / or / all the sets under consideration is known as union of sets. Denoted by inserting '$\cup$' between the sets. Symbolically, $A \cup B = \{x : x \in A$ or $x \in B$ or $x \in A$ and $x \in B\}$.

Diagrammatically, it is shown as shaded area

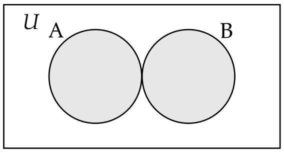


Figure 4

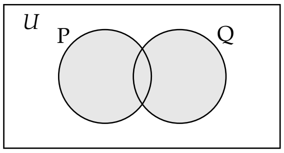


Figure 5

For example : If $A = \{2, 4, 6, 8\}$ and $B = \{1, 3, 5, 7, 9\}$, then $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and if $P = \{a, b, c\}$ and $Q = \{a, c, d, e\}$, then $P \cup Q = \{a, b, c, d, e\}$.

- **Properties of Union of Sets :**
  - **(i)** $A \cup B = B \cup A$ (Commutative property)
  - **(ii)** $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative Property)
  - **(iii)** $A \cup \phi = A$ (Identity property)
  - **(iv)** $A \cup A = A$ (Idempotent property)
  - **(v)** $U \cup A = U$

➢ **Intersection of two sets :** Set consisting of all the common elements of the sets under consideration is known as intersection of sets. Denoted by inserting '$\cap$' between the sets. Symbolically, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Diagrammatically, it is shown as shaded area

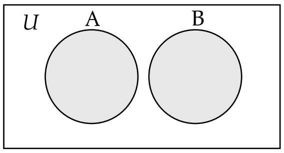


Figure 6

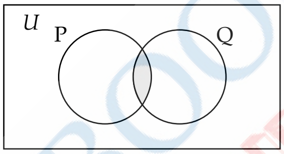


Figure 7

For example : If $A = \{2, 4, 6, 8\}$ and $B = \{1, 3, 5, 7, 9\}$, then $A \cap B = \phi$ (null set as no element is common) and if $P = \{a, b, c\}$ and $Q = \{a, c, d, e\}$ then $P \cap Q = \{a, c\}$.

- **Properties of Intersection of Sets :**
  - **(i)** $A \cap B = B \cap A$ (Commutative property)
  - **(ii)** $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative Property)
  - **(iii)** $A \cap \phi = \phi$, $U \cap A = A$ (Identity property)
  - **(iv)** $A \cap A = A$ (Idempotent property)
  - **(v)** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive Property)

➢ **Difference of two sets :**

If A and B are two sets, then $A - B$ is defined as the set which contains all the elements of A but not of B. Symbolically, $A - B = \{x : x \in A \text{ and } x \notin B\}$ and similarly, $B - A = \{x : x \in B \text{ and } x \notin A\}$.

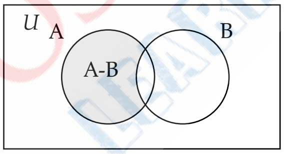


Figure 8

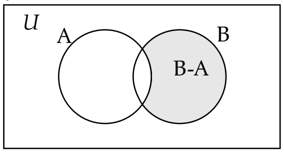


Figure 9

For example : If $A = \{a, b, c, d, e, f\}$ and $B = \{a, c, e, g, h\}$, then $A - B = \{b, d\}$ and $B - A = \{g, h\}$.

➢ **Complement of a set :**

If U is a universal set and A is any set then complement of A is the set that contains all elements of U but not of A. Denoted by A' or A^c.

Symbolically, we write, $A' = \{x : x \in U \text{ and } x \notin A\}$

Diagrammatically, it is represented as shaded area below

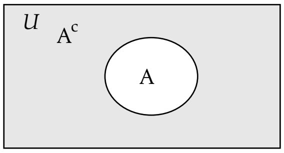


Figure 10

For example : If $U = \{1, 2, 3, \ldots.10\}$ and $A = \{2, 4, 6, 8\}$ then $A' = \{1, 3, 5, 7, 9, 10\}$.

➢ **Properties of Complement of Sets :**

(i) Law of complements : (a) $A \cup A' = U$ (b) $A \cap A' = \phi$

(ii) De- Morgan's law : (a) $(A \cup B)' = A' \cap B'$ (b) $(A \cap B)' = A' \cup B'$

(iii) $(A')' = A$

(iv) $U' = A$ and $\phi' = U$

- **Some Basic Results about Cardinal Numbers**

  If A and B are finite sets, then in counting the elements of $A \cup B$, the elements of $A \cap B$ are counted twice-once in counting the elements of A and second time in counting the elements of B. Hence, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

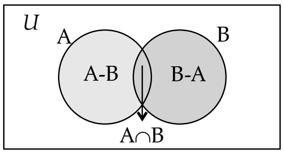


Figure 11

In particular, if $A \cap B = \phi$, then $n(A \cup B) = n(A) + n(B)$

Also, from fig.11, it is clear that

(i) $n(A \text{ only}) = n(A - B) = n(A \cup B) - n(B) - n(A \cap B)$

(ii) $n(B \text{ only}) = n(B - A) = n(A \cup B) - n(A) - n(A \cap B)$

(iii) $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$

(iv) $n(A') = n(U) - n(A)$, provided U is finite

Further, if A, B and C are finite sets, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Remark : Two sets are called disjoint if $A \cap B = \phi$ otherwise they are called **joint or overlapping sets.**

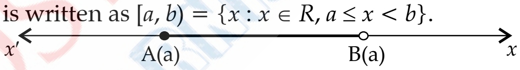
- **Intervals as subsets of R**
- **Finite intervals :**

  If a, and b be two (distinct) real numbers and ab less then the set of all real numbers lying in between a and b is said to be an open interval, denoted by (a, b). It is written as $(a, b) = \{x : x \in R, a < x < b\}$.

If the set of numbers lying between a and b including the numbers a and b is said to form closed interval, denoted by $[a, b]$. It is written as
$\{x : x \in R, a \le x \le b\}$.

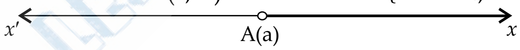
If the set of all real numbers lying between a and b and including the number b is said to form an open closed interval, denoted by $(a, b]$. It is written as $(a, b] = \{x : x \in R, a < x \le b\}$.

If the set of all real numbers lying between a and b and including the number a is said to form a closed open interval, denoted by $[a, b)$. It is written as $[a, b) = \{x : x \in R, a \le x < b\}$.

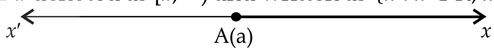


- **Infinite intervals :**

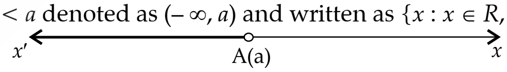
  The set of all real numbers $x > a$ denoted as (a, ∞) and written as $\{x: x \in R, x > a\}$



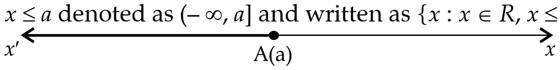
The set of all real numbers $x \ge a$ denoted as $[a, \infty)$ and written as $\{x : x \in R, x \ge a\}$



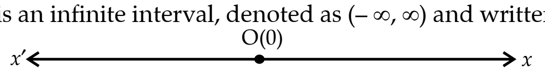
The set of all real numbers $x < a$ denoted as $(-\infty, a)$ and written as $\{x : x \in R, x < a\}$



The set of all real numbers $x \le a$ denoted as $(-\infty, a]$ and written as $\{x : x \in R, x \le a\}$



The set of all real numbers is an infinite interval, denoted as $(-\infty, \infty)$ and written as $\{x : x \in R\}$



Important Formulae

- $A \subset A \cup B$ and $A \cap B \subset A$
- If $A \subset B$, then $A \cup C \subset B \cup C$
- If $A \subset B$, then $A \cap C \subset B \cap C$
- $A - B = A \cap B'$ and $B - A = B \cap A'$
- $A \cap B = A$ if $A \subset B$ and $A \cup B = B$ if $A \subset B$
- $A - B = \phi$ if $A \subset B$
- $A \subset B$ if $B' \subset A'$
- $-\infty$ is smallest without any bound *i.e.* smaller than any negative real number.
- ∞ or $+\infty$ is largest without any bound *i.e.* greater than any positive real number.

CHAPTER-4

RELATIONS

Revision Notes

➢ **Cartesian Product**

For two non-empty sets A and B, the set of all ordered pairs (x, y), where $x \in A$ and $y \in B$ is called the Cartesian Product of A and B; symbolically, we write

$$A \times B = \{ (x, y) : x \in y \text{ and } y \in B\}$$

Thus, Cartesian Product of two sets represents the set which represents the coordinates of all the points in two dimensional space.

For example : If $A = \{1, 2, 3\}$ and $B = \{4, 5\}$, then

$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$ and

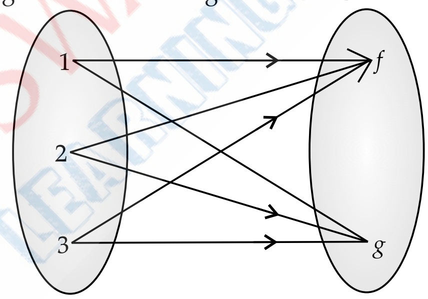
$B \times A= \{(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)\}$

Remark :

- If there are three sets A, B, C and $a \in A, b \in B$ and $c \in C$, then we form an ordered triplet (a, b, c). The set of all ordered triplets (a, b, c) is called the cartesian product of these sets A, B and C *i.e.*, $A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C$.
- If $n(A) = m$ and $n(B) = n$, then $n(A \times B) = m \times n$
- Two ordered pairs are equal, if and only if the corresponding first elements are equal and second element are also equal, *i.e.*, $(x, y) = (u, v)$ if and only if $x= u, y = v$.
- $A \times B \neq B \times A$, with condition when $A = B$, then $A \times B = B \times A$
- If A and B are finite sets, then
$$n(A \times B) = n(A) \times n(B) \text{ and } n(A \times B) = n(B \times A)$$
- If $A \times B = \phi$, when one or both of A, B are empty sets.
- If $A \times B \neq \phi$, if and only if $A \neq \phi$ and $B \neq \phi$.

➢ **Diagrammatic Representation of Cartesian Product of Two Sets**

In order to represent $A \times B$, by an arrow diagram, we first draw Venn diagrams representing set A and set B, one opposite to the other as shown in given figure and Writing the elements of sets. Now, we call draw line segments starting from each element of set A and terminating to each element of set B. For example: If $A = \{1, 2, 3\}$ and $B = \{f, g\}$, then following figure gives the arrow diagram of $A \times B$.



Relation and its Types

➢ Relation : A relation R from a non-empty set A to a non-empty set B is a subset of a Cartesian product set $A \times B$. The subset is derived by the relationship between the first element and the second element of the ordered pair in $A \times B$.

For example: Let $A=\{2, 3, 5, 9\}$ and $B = \{4, 6, 9, 15, 25\}$. There is a relation '*is a divisor of*' between the elements of the sets A and B. If we write R for the relation '*is a divisor of*', then we get

$$2 R 4, 2 R 6, 3 R 6, 3 R 9, 3 R 15, 5 R 15, 5 R 25, 9 R 9.$$

This can be write as a set of ordered pairs R where

$R = \{(2, 4), (2, 6), (3, 6), (3, 9), (3, 15), (5, 15), (5, 25), (9, 9)\}$, in roster form

and $R = \{ (x, y) : x \in A, y \in B, x \text{ is a divisor of } y\}$, in Set-Builder form

Thus, the relation '*is a divisor of*' from set A to the set B gives rise to a subset R of $A \times B$ *such that* $(x, y) \in R$ *i.e.*, if and only if x is a divisor of y.

Remark : If $n(A) = p$ and $n(B) = q$, then total number of possible relations form the set A to set $B = 2^{pq}$

➢ **Domain and Range of a Relation**

Let R be a relation from a set A to set B. Then, set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R, while the set of all second components or coordinates of the ordered pairs belonging to R is called the range of R.

Thus, domain of $R = \{a : (a, b) \in R\}$ and range of $R = \{b : (a, b) \in R\}$.

For example : Let A={1, 3, 4, 5, 7} and B = {2, 4, 6, 8} and a relation R *'is one less than'* from set A to the set B, then $R = \{(1, 2), (3, 4), (5, 6), (7, 8)\}$. Here,

Domain of $R = \{1, 3, 5, 7\}$ and Range of $R = \{2, 4, 6, 8\}$.

In the above example, note that range of $R = B =$ co-domain of R.

- **Types of Relations**
  - **Void Relation :** As $\phi \subset A \times A$, for any set A, so ϕ is a relation on A, called the empty or void relation.
  - **Universal Relation :** Since, $A \times A \subseteq A \times A$, so $A \times A$ is a relation on A, called the universal relation.
  - **Identity Relation :** The relation $I_A = \{(a, a) : a \in A\}$ is called the identity relation on A.

    For example : If $A = \{1, 2, 3, 4, 5\}$, then $I_A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$.
  - **Reflexive Relation :** A relation R is said to be reflexive relation, if every element of A is related to itself.

    Thus, $(a, a) \in R, \forall a \in A \Rightarrow R$ is reflexive.

    For example : If $A = \{1, 2, 3, 4\}$, then

    (i) Relation $R_1 = \{(1, 1), (2, 4), (3, 3), (4, 1), (4, 4)\}$ in A is not reflexive since $2 \in A$ and $(2, 2) \notin R_1$.

    (ii) Relation $R_2 = \{(1, 1), (1, 3), (2, 2), (3, 3), (4, 4), (3, 4)\}$ in A is reflexive since $(a, a) \in R_2$ and $\forall a \in A$.
  - **Symmetric Relation :** A relation R is said to be symmetric relation,

    iff $(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$ *i.e.*, $a R b \Rightarrow b R a, \forall a, b \in A \Rightarrow R$ is symmetric.

    For example : If $A = \{1, 2, 3, 4\}$, then $R = \{(1, 2), (3, 4), (2, 1), (3, 3)\}$. Here, we see that $(3, 4) \in R$ but $(4, 3) \notin R$. Therefore, R is not symmetric. On the other hand, the relation $R_1 = \{(1, 1), (1, 4), (4, 1)\}$ is a symmetric relation.
  - **Anti-Symmetric Relation :** A relation R is said to be anti-symmetric relation,

    iff $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b, a, b \in A$

    For example : Let N be the set of all natural numbers. Let R be a relation in N defined by "x is a divisor of y". Then R is anti-symmetric, since $a \mid b$ and $b \mid a \Rightarrow a = b$. If $a \neq b$, we cannot have both $a \mid b$ and $b \mid a$.
  - **Transitive Relation :** A relation R is said to be transitive relation,

    iff $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R, \forall a, b, c \in A$

    For example :

    (i) Let L be the set of all straight lines in a plane and R be a relation on L defined by "x is perpendicular to y". If straight line a is perpendicular to b and b is perpendicular to c, then a is parallel to c, *i.e.*, a is not perpendicular to c.

    Thus, $a R b$ and $b R c$ does not implies that $a R c$.

    (ii) Let N be the set of all natural numbers. Let R be a relation in N defined by "x is a less than y". Since, $a < b$ and $b < c \Rightarrow a < c$, therefore R is transitive.
  - **Equivalence Relation :**

    A relation R is said to be an equivalence relation, if it is simultaneously reflexive, symmetric and transitive on a set A.
  - **Partial Order Relation :** A relation R is said to be a partial order relation, if it is simultaneously reflexive, anti-symmetric and transitive on a set A.

    For example : Let N be the set of all natural numbers. Let R be a relation in N defined by "a divides b".

    (i) We have, $\forall a \in N$, a is a divisor of a *i.e.*, aRa. Therefore, R is reflexive.

    (ii) Again, if a is a divisor of b, then b cannot be a divisor of a, unless $a = b$. Thus, aRb and $bRa \Rightarrow a = b$. Therefore, R is anti-symmetric.

    (iii) Finally, a is a divisor of b and b is a divisor of c implies a is a divisor of c. Therefore R is transitive. Since, R is reflexive, anti-symmetric and transitive, therefore R is a partial order relation.

- **Inverse of a Relation:**

  Let A and B be two sets and let R be a relation from a set A to set B. Then the inverse of a relation R, denoted by R^{-1}, is a relation from set B to set A and is defined as

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

Clearly, $$(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$$

Also, Domain of R = Range of R^{-1} and Range of R = Domain of R^{-1}

For example : Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ be the two sets and let $R = \{(1, a), (1, c), (2, c), (2, d)\}$ be a relation from set A to set B. Then $R^{-1} = \{(a, 1), (c, 1), (c, 2), (d, 2)\}$ is a relation from B to A.

Also, $\text{Dom }(R) = \{1, 2\} = \text{Range }(R^{-1})$

and $\text{Range (R)} = \{a, c, d\} = \text{Dom }(R^{-1})$

CHAPTER-5

SEQUENCES AND SERIES

TOPIC-1

Arithmetic Progression (A.P.)

Revision Notes

➢ Sequences have many important applications in several spheres of human activities. When a collection of objects is arranged in a definite order such that it has an identification first member, second member, third member and so on, we say that the collection is listed in a sequence.
For example :
(i) The amount of money in a fixed deposit in a bank over a number of years occurs in a sequence.
(ii) The population of bacteria at different times forms a sequence.

➢ **Sequence :**
A set of number arranged in a particular order according to some definite rule (or rules) is called a **sequence**. Also, we define a sequence as a function whose domain is the set of natural numbers or some subset of the type {1, 2, 3, ... k}. Each member or number of the set is called a '*term*' of sequence.
We denote the terms of sequence by a_1, a_2, a_3, ..., etc. the subscript denotes the position of the term.
In view of the above a sequence in the set X can be regarded as a function $f : N \to X$ defined by

$$f(n) = t_n \ \forall \ n \in N$$

Domain of f is the set of natural number or some subset of it denoting the position of term. If its range denoting the value of terms is subset of R real numbers then it is called a **real sequence**.
A sequence is called **finite** or **infinite** according as the number of terms in it is finite or infinite.
For example :
(i) 3, 5, 7, 9, ..., 21 is a sequence because each term is obtained by adding 2 to the previous terms.
(ii) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}$, ... is a sequence because each term is obtained by multiplying the preceding term by $-\frac{1}{2}$.
Here, in example (i), the given sequence is finite whereas in example (ii), the given sequence in infinite.
The sequence 3, 5, 7, 9, ... can be written as $a_n = 2n + 1, n \in N$.
However, we expect a theoretical scheme or rule for generating the terms.
Moreover to define a sequence, we need not always have an explicit formula for the n^{th} term.
Remark :
The fibonacci sequence is given by
$a_1 = 1, a_2 = 1$ and $a_{n+1} = a_n + a_{n+1}, n \geq 2$
The terms of the sequence are : 1, 1, 2, 3, 5, 8, ...

➢ **Series :** If the terms of the sequence are connected by plus '+' signs, we get a series. If a_1, a_2, a_3 ...., a_n .... is a sequence, then the expression $a_1 + a_2 + a_3 + \ldots\ldots + a_n + \ldots\ldots$ is a series, The series is finite or infinite according as the given sequence is finite or infinite.
For example :
(i) 8 + 5 + 2 + (– 1) + (– 4) + ...
(ii) 1 + 4 + 9 + 16 + ...

- If a_n denotes the general term of a sequence then $a_1 + a_2 + a_3 + \ldots\ldots a_n$ is a series of n terms.
- In a series $a_1 + a_2 + a_3 + \ldots\ldots a_k + \ldots\ldots$, the sum of first n terms is denoted by S_n.

  Thus, $S_n = a_1 + a_2 + a_3 + \ldots\ldots a_n = \sum_{k=1}^{n} a_k$.
- If S_n denotes the sum of n terms of a sequence, then
  $S_n - S_{n-1} = (a_1 + a_2 + a_3 + \ldots\ldots + a_n) - (a_1 + a_2 + a_3 + \ldots\ldots + a_{n-1})$

$= a_n$

Thus, $a_n = S_n - S_{n-1}$

- **Progression :** It is not necessary that the terms of a sequence always follow a certain pattern or they are represented by some explicit formula for the n^{th} term. Those sequence whose terms follows certain patterns are called progressions. In this section, we shall study arithmetic progression defined as follows :
  - **Arithmetic Progression (A. P.) :** A sequence in which the difference of two consecutive terms is constant, is called **Arithmetic Progression,** *i.e.*, ($a_n - a_{n-1}$ = constant = d) for all $n \in N$. The constant difference, generally denoted by d is called the common difference.

    For Example :

    (i) 1, 4, 7, 10, ... is an A.P. whose first term is 1 and common difference is equal to $4 - 1 = 7 - 4 = 10 - 7 = 3$

    (ii) 11, 7, 3, –1, ... is an A.P. whose first term is 11 and common difference is equal to $7 - 11 = 3 - 7 = (-1) - 3 = -4$.

    (iii) 1, 2, 5, 9, 10, ... is not an A.P. because the common difference between the terms is not constant (or unique) *i.e.*, $2 - 1 \neq 5 - 2 \neq 9 - 5$.

- **General term of an AP :** Let a be the first term and d be the common difference of an A.P. Then its nth term (general term) is given by

$$T_n \text{ or } a_n = a + (n-1)d$$

For example : Show that the sequence 9, 12, 15, 18, ... is an A.P. Find its general term and 16th term.

Sol. We have $(12 - 9) = (15 - 12) = (18 - 15) = 3$. Therefore, the given sequence is an A.P. with common difference 3.

Here, first term = 9

$\therefore$ General term (nth term), $a_n = a + (n-1)d$
$= 9 + (n-1)3$
$= 3n + 6$

and 16^{th} term, $a_{16} = a + (16-1)\,d$
$= 9 + 15 \times 3$
$= 54$

  - **Last term of an A.P. :** If the last term of an AP consisting of n terms is denoted by l, then

$$l = a + (n-1)d$$

  - ***n*th term of an A.P. from the end :** Let a be the first term and d be the common difference of an AP having m terms. Then nth term from the end is $(m - n + 1)^{th}$ term from the beginning.

    $\therefore$ nth term from the end $= a_{m-n+1}$
    $= a + (m - n + 1 - 1)\,d$
    $= a + (m - n)d$

    Also, n^{th} term from the end $= a_m + (n-1)(-d)$

    [$\because$ Taking a_m as the first term and common difference equal to '$-d$']

    For example : Determine the number of terms in the A.P. 3, 7, 11, ..., 407. Also, find its 20^{th} term from the end.

    Sol. Here, first term = 3 and common difference = 4

    Let there be n terms in the given A.P. Then,

    $407 = n^{th}$ term $\Rightarrow 407 = 3 + (n-1) \times 4 \Rightarrow 4n = 408$ *i.e.*, $n = 102$.

    Now, 20^{th} term from the end = $[102 - 20 + 1]$th term from the beginning
    = 83rd term from the beginning
    $= 3 + (83 - 1) \times 4 = 331$

- **Sum of *n* terms of an A.P. :** Let 'a' be the first term and 'd' be the common difference of an A.P., then the sum S_n of n terms of A.P. is given by

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

or, $$S_n = \frac{n}{2}(a + l), \text{ where } l = \text{last term} = a + (n-1)d$$

For example : Find the sum of the series : 5 + 13 + 21 ..... + 181.

Sol. The terms of the given series form an A.P. with first term, $a = 5$ and common difference, $d = 8$. Let there be n terms in the given series. Then,

$$a_n = 181$$
$$\Rightarrow \quad a + (n-1)d = 181$$
$$\Rightarrow \quad 5 + (n-1)8 = 181$$
$$\Rightarrow \quad 8n = 184 \Rightarrow n = 23$$

Since, $$S_{23} = \frac{n}{2}(a + l)$$

$\therefore \quad S_{23} = \frac{23}{2}(5+181)$

$= 2139 \quad [\because a_n = l = 181]$

➢ **Arithmetic Mean :** When three numbers (terms) are in A.P., the middle term is said to be Arithmetic Mean (A.M.) between the other two.

Thus, if a, A, b are in A.P., the A is the A.M. between a and b

So, $\quad A - a = b - A$

[Since, their common difference are equal]

$\Rightarrow \quad 2A = a + b$

$$A = \frac{a+b}{2}$$

A .M. between two numbers = Half of their sum.

Remark : Sum of n A.M.'s between two terms is n times the single A.M. between them, *i.e.*,

Let $A_1, A_2, A_3, \ldots\ldots, A_n$ be the n, A.M.'s between two numbers a and b.

So, we can find the A.M.'s as

$$A_1 = a + d, d = \frac{b-a}{n+1}, A_2 = a + 2d = a + \frac{2(b-a)}{n+1}, \ldots\ldots, A_n = a + nd = a + \frac{n(b-a)}{n+1}$$

➢ **Properties of A.P. :**

(i) If a constant term is added or subtracted from each term of an A.P., then resulting sequence is also an A.P. with same common difference. *i.e.*, If a, b, c are in A.P., then $a \pm k, b \pm k, c \pm k$ are also in A.P.

(ii) If each term of given A.P. is multiplied or divide by a non-zero constant k, then the resulting sequence is also an A.P. with common difference kd or $\frac{d}{k}$, where d is the common difference of the given A.P.

(iii) In a finite A.P. the sum of terms equidistant from the beginning and end is always same and equal to the sum of first and last term *i.e.*, $a_k + a_{n-(k-1)} = a_1 + a_n$ for all $k = 1, 2, 3, \ldots\ldots, n-1$.

➢ **Some Important Results :**

- If three terms of A.P. are to be taken then we choose them as $a - d, a, a + d$.
- If four terms of A.P. are to be taken then we choose them as $a - 3d, a - d, a + d, a + 3d$.
- If five terms of A.P. are to be taken then we choose them as $a - 2d, a - d, a, a + d, a + 2d$.
- If the sum of first m terms of an A.P. is equal to the sum of first n terms, then sum of first $(m + n)$ terms is zero *i.e.*, if $S_m = S_n \Rightarrow S_{(m+n)} = 0$.
- If the sum of p terms of an A.P. is q and the sum of q terms is p, then the sum of $p + q$ terms is $-(p + q)$ *i.e.*, if $S_p = q$ and $S_q = p \Rightarrow S_{p+q} = -(p + q)$.

TOPIC-2
Geometric Progression (G.P.)

Revision Notes

A sequence (finite or infinite) of non-zero numbers is said to a **geometric progression** (G.P.),if the ratio of a term and the term preceding to it is always a constant quantity.

We can say that, a sequence $a_1, a_2, a_3 \ldots\ldots, a_n$ is called a geometric progression (geometric sequence), if it follows the relation $\frac{a_{k+1}}{a_k} = r$ (constant)

This non-zero constant ratio is generally denoted by r and is called **common ratio**.

For example : The sequence 4, 12, 36, 108, ... is a G.P., because $\frac{12}{4} = \frac{36}{12} = \frac{108}{36} = \ldots\ldots = 3$, which is constant.

Clearly, this sequence is a G.P. with first term 4 and common ratio 3.

➢ **Geometric series**

If $a_1, a_2, a_3, \ldots\ldots a_n, \ldots\ldots$ is a G.P., then the expression $a_1 + a_2 + a_3 + \ldots\ldots + a_n + \ldots\ldots$ is called a **geometric series.**

Note that the geometric series is finite or infinite according as the corresponding G.P. consists of finite or infinite number of terms.

- **General Term of a G.P.**

Consider a G.P. with first non-zero term 'a' and common ratio 'r', then G.P. can be written as a, ar, ar^2, ar^3, ... and so on.

Then, the **general term** or $\boldsymbol{n^{th}}$ **term** of G.P. is given by $T_n = a_n = ar^{n-1}$

Let term 'l' of a G.P. is same as the n^{th} term and is given by $l = ar^{n-1}$.

For example : Find the 20th term and the n^{th} term of the sequence $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}$, ...

Sol. The given sequence $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}$, ... is a G.P. with first term, $a = \frac{5}{2}$ and common ratio $r = \frac{1}{2}$.

$$\therefore \qquad a_{20} = ar^{20-1} = \frac{5}{2}\times\left(\frac{1}{2}\right)^{19} = \frac{5}{2^{20}}$$

and
$$T_n = a_n = ar^{n-1} = \frac{5}{2}\times\left(\frac{1}{2}\right)^{n-1} = \frac{5}{2^n}$$

- $\boldsymbol{n^{th}}$ **term from the end of a finite G.P.**

- If a, ar, ar^2, ... is a finite G.P. consisting of m terms, then the n^{th} term from the end

$$= (m - n + 1)\text{th term from the beginning}$$
$$= ar^{m-n+1-1}$$
$$= ar^{m-n}$$

- If a, ar, ar^2, ... is a finite G.P. with last term l, then nth term from the end $= l\left(\frac{1}{r}\right)^{n-1}$

For example : In G.P. $2\sqrt{2},\ 4$, ..., $128\sqrt{2}$, find the 4th term from the end.

Sol. Given G.P. is $2\sqrt{2},\ 4$, ..., $128\sqrt{2}$

Here, $a = 2\sqrt{2},\ r = \sqrt{2}$ and $l = 128\sqrt{2}$

$$\because \qquad n^{th}\text{ term from the end} = l\left(\frac{1}{r}\right)^{n-1}$$

$$\therefore \qquad 4^{th}\text{ term from the end} = 128\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)^{4-1}$$
$$= 128\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)^{3}$$
$$= \frac{128\sqrt{2}}{2\sqrt{2}}$$
$$= 64$$

Remark : Sometimes it required to select a finite number of terms in G.P. It is always convenient, if we select the terms in following manner :

(i) Three terms of G.P. are taken as $\frac{a}{r},\ a, ar$ with common ratio 'r'.

(ii) Four terms of G.P. are taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$ with common ratio 'r^2'.

(iii) Five terms of G.P. are taken as $\frac{a}{r^2}, \frac{a}{r}, a,\ ar,\ ar^2$ with common ratio 'r'.

- **Sum of first n terms of a G.P.**

- If a and r are the first term and common ratio of a G.P., respectively, then the sum of n terms of this G.P. is given by

$$S_n = \frac{a(1-r^n)}{1-r},\ \text{when } r < 1$$

and $$S_n = \frac{a(r^n - 1)}{r - 1}, \text{ when } r > 1$$

and $s_n = na$, when $r = 1$

- If l is the last term of G.P., then $l = ar^{n-1}$

  Then the sum of n terms of G.P., in given as

$$S_n = \frac{a - lr}{1 - r} \text{ when } r < 1$$

and $$S_n = \frac{lr - a}{r - 1} \text{ when } r > 1$$

➢ **Sum of an infinite G.P.**

The sum of an infinite G.P. with first term 'a' and common ratio 'r' ($-1 < r < 1$ *i.e.*, $|r| < 1$) is given by

$$S(\text{or } S_\infty) = \frac{a}{1 - r}$$

Note : If $r \geq 1$, then the sum of an infinite G.P. tends to infinity.

For example : Find the sum of n terms of the series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots\ldots$. Also, find the sum to infinity of the given series.

Sol. The given series is an infinite G.P. with first term, $a = 1$ and common ratio, $r = \frac{1}{3}$. Hence, the sum of first n terms is

$$S_n = a\left(\frac{1 - r^n}{1 - r}\right) \qquad \left[\because r = \frac{1}{3} < 1\right]$$

$$= 1\left[\frac{1 - \left(\frac{1}{3}\right)^n}{1 - \frac{1}{3}}\right]$$

$$= \frac{3}{2}\left(1 - \frac{1}{3^n}\right)$$

$$\text{sum to infinity, } S_\infty = \frac{a}{1 - r} = \frac{1}{1 - \frac{1}{3}}$$

$$= \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

➢ **Properties of G.P.**

(i) If all the terms of a G.P. is multiplied by the some non-zero constant, then it remains a G.P. with the same common ratio *i.e.*, if a, b, c are in G.P., then ak, bk, ck are also in G.P. where $k \neq 0$ and $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}$ are also in G.P. where $k \neq 0$

(ii) The reciprocals of the terms of a given G.P. form a G.P. *i.e.*, if $a_1, a_2, a_3, \ldots\ldots, a_n, \ldots\ldots$ are in G.P. with common ratio r, then $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots\ldots, \frac{1}{a_n}, \ldots\ldots$ are also in G.P. with common ratio $\frac{1}{r}$.

(iii) If each term of a G.P. be raised to the same power, the resulting sequence also form a G.P. *i.e.*, if $a_1, a_2, a_3, \ldots\ldots, a_n, \ldots\ldots$ are in G.P. with common ratio r, then $a_1^k, a_2^k, a_3^k, \ldots\ldots a_n^k, \ldots\ldots$ are also in G.P. with common ratio r^k.

(iv) In a finite G.P. the product of the terms equidistant form the beginning and the end is always same and is equal to the product of the first and last term.

(v) If $a_1, a_2, a_3, \ldots\ldots, a_n, \ldots\ldots$ is a G.P. of non-zero non-negative terms, then $\log a_1, \log a_2, \ldots\ldots, \log a_n, \ldots\ldots$ is an A.P. and vice-versa.

(vi) If $a_1, a_2, a_3, ...$ and $b_1, b_2, b_3, ...$ are two G.P., then $a_1b_1, a_2b_2, a_3b_3, ...$ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, ...$ are also in G.P.

(vii) If $a_1, a_2, a_3, ...$ are in A.P. ($a_i > 0\ \forall\ i$), then $x^{a_1}, x^{a_2}, x^{a_3}, ...$ are in G.P. ($\forall\ x > 0$).

(viii) If $a_1, a_2, a_3, ...$ an are in G.P., then $a_1a_n = a_2a_{n-1} = a_3a_{n-2} = ...$

➢ **Geometric Mean (G.M.)**

- If a, G, b are G.P., then G is called the geometric mean between the numbers a and b.

  Since a, G, b are in G.P., $\frac{G}{a} = \frac{b}{G} \Rightarrow G^2 = ab \Rightarrow G = \sqrt{ab}$ (conventionally, G is taken positive).

  Hence, the geometric mean of two positive numbers a and b is $G = \sqrt{ab}$.

  Remark : If a and b are two numbers of opposite signs, then geometric mean between them does not exist.
- If $G_1, G_2, G_3, ... G_n$ are the n numbers between a and b such that $a, G_1, G_2, G_3, ..., G_n, b$ is a G.P., then the numbers $G_1, G_2, G_3, ... G_n$ are called n **geometric mean** between a and b.

  Also, $G_k = a\left(\frac{b}{a}\right)^{\frac{k}{k+1}}, \ 1 \le k \le n$

  and the product of these n G.M.'s *i.e.*,

  $G_1 . G_2 . G_3 . ... G_n = (\sqrt{ab})^n = n^{\text{th}}$ power of single G.M. between a and b.

  For example : Insert 4 geometric means between 3 and 96. Also, show that their product is the 4$^{\text{th}}$ power of the G.M. between them.

  Sol. Let G_1, G_2, G_3, G_4 be four G.M.'s between 3 and 96 then 3, G_1, G_2, G_3, G_4, 96 are in G.P. Let r be the common ratio. Since, 96 is the 6$^{\text{th}}$ term,

  $$96 = 3r^5 \Rightarrow r^5 = 32 = 2^5 \Rightarrow r = 2$$

  $$\therefore \quad G_1 = ar = 3 \times 2 = 6,$$
  $$G_2 = ar^2 = 3 \times 2^2 = 12$$
  $$G_3 = ar^3 = 3 \times 2^3 = 24$$
  $$G_4 = ar^4 = 3 \times 2^4 = 48$$

  Hence, the four geometric means are 6, 12, 24, 48.

  Also, if G is G.M. of 3 and 96, then $G = \sqrt{3 \times 96} = \sqrt{288} = 12\sqrt{2}$

  Now, $G_1.G_2.G_3.G_4 = 6.12.24.48 = 12^4.2^2 = (12\sqrt{2})^4 = G^4$

➢ **Relationship between A.M. and G.M.**

Let a, b be any two (distinct) positive real numbers, then

$$A = \text{A.M. between } a \text{ and } b = \frac{a+b}{2}$$

and
$$G = \text{G.M. between } a \text{ and } b = \sqrt{ab}$$

$$\therefore \quad A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}(a + b - 2\sqrt{ab})$$

$$= \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 > 0 \quad [\because (\sqrt{a} - \sqrt{b})^2 \text{ is the square of a non-zero real numbers}]$$

$$\Rightarrow \quad A > G$$

Hence, if A and G are arithmetic and geometric means between any two different positive real numbers, then $A > G$.

In particular, $A = G$ iff $\frac{(\sqrt{a} - \sqrt{b})^2}{2} = 0$ *i.e.*, iff $a = b$.

Thus, in general, we have :

If A and G are arithmetic and geometric means between two positive real numbers, then $A \ge G$.

For example : The A.M. and G.M. between two positive numbers are 10 and 8 respectively find the numbers.

Sol. Let the two positive numbers be a and b. Then A.M. $= \frac{a+b}{2} = 10$ (given)

$$\Rightarrow \quad a + b = 20 \qquad ...(i)$$

and
$$\text{G.M.} = \sqrt{ab} = 8 \qquad \text{(given)}$$

$$\Rightarrow \quad ab = 64$$

$$\Rightarrow \quad a(20 - a) = 64 \qquad \text{[using (i)]}$$

$$\Rightarrow \quad a^2 - 20a + 64 = 0$$

$$\Rightarrow \quad (a - 4)(a - 16) = 0$$

$$\Rightarrow \quad a = 4, 16$$

Taking $a = 4$, we get $b = 20 - 4 = 16$

Taking $a = 16$, we get $b = 20 - 16 = 4$

Thus, the two numbers are 4, 16, or 16, 4

➢ **Arithmetico Geometric Series**

A series in which each term is formed by multiplying the corresponding forms of an A.P. and G.P. *i.e.*, called **arithmetico geometric series**.

The arithmetico-geometric series of n terms in given by

$$a + (a + d)r + (a + 2d)r^2 + \ldots\ldots + [a + (n - 1)d]r^{n-1} \qquad \text{...(i)}$$

Remark :

- When we substitute $r = 1$ in eq(i), the above arithmetico-geometric series is reduced in arithmetic series of n terms *i.e.*, $a + (a + d) + (a + 2d) + \ldots\ldots + [a + (n - 1)d]$
- When we substitute $d = 0$ in eq (i), the above arithmetico-geometric series is reduced in geometric series of n terms *i.e.*, $a + ar + ar^2 + \ldots\ldots + ar^{n-1}$

➢ **Sum of n terms of an Arithmetico-geometric series**

Let a, $(a + d)\,r$, $(a + 2d)\,r^2$, ... be the given sequence and let S_n be the sum of n terms.

Then,
$$S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{a+(n-1)r^n}{1-r}$$

- **Sum to infinity of an arithmetico-geometric series**

  If $|r| < 1$ and $n \to \infty$, then

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

Remark : It is difficult to remember the formula of sum of n terms of arithmetico-geometric series. So in the following example, we have given the procedure to solve this type of series.

For example : Sum to n term the series

$2 + 5x + 8x^2 + 11x^3 + \ldots\ldots\ |x| < 1$. Deduce the sum to infinity.

Sol. The given series in formed by multiplying corresponding terms of A.P. 2, 5, 8, 11 ...... and G.P. 1, x, x^2, x^3, ...

Hence, n^{th} term of given arithmetico-geometric series is

$$[2 + (n - 1)\,3].\,x^{n-1} = (3n - 1)x^{n-1}$$

Let
$$S_n = 2 + 5x + 8x^2 + 11x^3 + \ldots\ldots + (3n - 4)\,x^{n-2} + (3n - 1)x^{n-1}$$

$$x\,S_n = 2x + 5x^2 + 8x^3 + \ldots\ldots + \ldots\ldots + (3n - 4)\,x^{n-2} + (3n - 1)x^n$$

$$\Rightarrow \quad (1 - x)\,S_n = 2 + (3x + 3x^2 + 3x^3 + \ldots\ldots + 3x^{n-1}) - (3n - 1)\,x^n.$$

$$= 2 + 3x\left(\frac{1-x^{n-1}}{1-x}\right) - (3n-1)x^n$$

$$\Rightarrow \quad S_n = \frac{2}{1-x} + \frac{3x(1-x^{n-1})}{(1-x)^2} - \frac{(3n-1)x^n}{1-x}$$

To find the sum up to infinity *i.e.*, S_∞ we note that x^{n-1}, x^n, $n\,x^n \to 0$ as $n \to \infty$ because $|x| < 1$

$$S_\infty = \frac{2}{1-x} + \frac{3x}{(1-x)^2} = \frac{2+x}{(1-x)^2}$$

➢ **Sum to n Terms of Special series**

(i) Sum of the first n natural numbers :

$$\Sigma n = 1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}$$

(ii) Sum of the square of first n natural numbers :

$$\Sigma n^2 = 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(iii) Sum of cube of first n natural numbers :

$$\Sigma n^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

CHAPTER-6

PERMUTATIONS AND COMBINATIONS

TOPIC-1
Permutations

Revision Notes

- In daily life, we come across many problems of finding the number of ways of arranging or selecting objects. Suppose, one day you wanted to travel from Bengaluru to Prayagraj by train. If there is no direct train from Bengaluru to Prayagraj, but there are trains from Bengaluru to Itarsi and from Itarsi to Prayagraj. From the railway timetable you found that there are two trains from Bengaluru to Itarsi and three trains from Itarsi to Prayagraj. Now, the question arises! In how many ways can you travel from Bengaluru to Prayagraj '?' To answer this question, you may immediately start listing the all possible ways to travel from Bengaluru to Prayagraj. But, this method of listing the ways (arrangements) is very tedious and time consuming, because the number of possible ways in large.

  Here, in this topic, we shall learn some basic counting techniques which will enable to answer this type of problems with actually listing all the possible ways (arrangements).

- **Factorial :** The continued product of first n natural number is called n factorial or factorial n and is denoted by $\underline{|n}$ or $n!$.

  Thus, $\underline{|n}$ or $n! = 1 \times 2 \times 3 \times 4 \times 5 \times \ldots\ldots\ldots \times (n-1) \times n$

  $= n \times (n-1) \times \ldots\ldots\ldots\ldots \times 5 \times 4 \times 3 \times 2 \times 1$ (In reverse order)

  For Example : $3! = 1 \times 2 \times 3 = 6$, $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$ etc.

  Clearly, $n!$ is defined for positive integers only.

  - **Zero factorial :** According to the above definition, $\underline{|0}$ is meaning less. However, we define $\underline{|0} = 1$.

- **Fundamental Principles of Counting**

  In this section, we shall discuss two fundamental principles of counting *i.e.*, (i) Principle of Addition and (ii) Principle of Multiplication. These two principles form the base of this chapter.

  - **Fundamental Principle of Addition :** If there are two events such that they can occur independently in m and n ways, respectively. Then either of two events can be occur in $(m+n)$ ways.

    For example: In the class there are 10 girls and 8 boys. The teacher wants to select a boy or a girl to represent the class in annual day. In how many ways can the teacher make this selection?

    Sol. Here, the teacher can make the selection in the following ways:

    (i) select a boy out of 8 boys.

    and (ii) select a girl out of 10 girls.

    The first selection can be done in 8 ways where as the second selection can be done in 10 ways. Therefore, by the fundamental principle of addition either of the two selections can be done in $(8+10)=18$ ways. Hence, the teacher can make the selection of either a boy or a girl in 18 ways.

  - **Fundamental Principle of Multiplication :** If there are two events such that one of them can occur in m ways, and when it has been occurred in any of the m ways, second event can be occurred in n ways; then the two events in succession can be occurred in $m \times n$ ways.

    For example : A room has 6 doors. In how many ways a man can enter the room through one door and come out through a different door?

    Sol. Here, it is clear that, a man can enter the room through any one of the six doors so, there are 6 ways of entering into the room. After entering into the room the man can come out through any one of the remaining five doors, So, he can come out through a different door in 5 ways. Hence, the number of ways in which a man can enter a room through one door and come out through a different door $= 6 \times 5 = 30$

    Remark : The above principle can be extended for any finite number of events as stated below :

    If there are n events $E_1, E_2, E_3\ldots\ldots, E_n$ such that the event E_i can be occurred independently in m_i ways, where $i = 1, 2, \ldots\ldots n$. Then the total number of ways in which all the events can be occurred is $m_1 \times m_2 \times m_3 \times \ldots\ldots \times m_n$.

- **Permutation**

  A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.

For example : (i) If there are three objects say A,B and C, then the permutations of these objects taken two at a time are *AB, BC, BA, CB* and *CA*. So, number of permutations is $3! = 1 \times 2 \times 3 = 6$.

(ii) Write down all the permutation of the vowels *A, E, I, O, U* in English alphabets taking three at a time, and starting with *A*.

Sol. The permutation of vowels A, E, I, O, U taking three at a time, and starting with *A* are:

AEI, AIE, AEO, AOE, AEU, AUE, AIO, AOI, AIU, AUI, AOU, AUO clearly, there are 12 permutations,

Note : If should be noted that in permutations the order of arrangement is taken into account when the order is changed, a different permutation is obtained.

Thus, in example (i) AB and BA are different arrangements and in example (ii) AEI and AIE etc. are different permutations.

- **Notation of Permutation**

If r and n are the two positive integers such that $0 \le r \le n$, then the number of permutations of n different objects taken r at a time, repetition of objects not allowed, is denoted by ${}^nP_r$ or $P(n,r)$.

We have, $P(n,r) = {}^nP_r = \dfrac{n!}{(n-r)!}$, $0 \le r \le n$

Here,

(i) When $r = 0$, then ${}^nP_0 = \dfrac{n!}{(n-0)!} = \dfrac{n!}{n!} = 1$

i.e., the number of permutations of n different objects taken nothing at all is 1.

(ii) When $r = n$, then ${}^nP_n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = n!$ $[\because 0! = 1]$

i.e., the number of permutation of n different objects taken all at a time is $n!$.

For example :

(i) In how many ways three different rings can be worn in four fingers with at most one each finger ?

Sol. The total number of ways is same as the number of arrangements of 4 fingers, taken 3 at a time.

So, required number of ways $= {}^4P_3 = \dfrac{4!}{(4-3)!} = \dfrac{4!}{1!} = 4! = 1 \times 2 \times 3 \times 4 = 24.$

(ii) In how many ways can 6 persons stand in a queue ?

Sol. The number of ways in which 6 persons can stand in a queue is same as the number of arrangements of 6 different things taken all at a time.

Hence, the required number of ways $= {}^6P_6 = 6! = 720$.

- **Permutation when all objects are not different**

The number of all permutations of n objects taken all together, when p of the objects are alike of one kind, q of them alike of another kind, r of them alike of a third kind and the remaining all different is $\dfrac{n!}{p!\,q!\,r!}$

For example 1 : In how many ways can 4 red, 3 yellow and 2 green discs can be arranged in a row if discs of the same colour are indistinguishable ?

Sol. In total, we have $4 + 3 + 2 = 9$ discs of which 4 are of one kind (red), 3 of another kind (yellow) and 2 of another kind (green).

The number of ways of arranging these discs in a row are

$$= \frac{9!}{4!\,3!\,2!} = \frac{9\times8\times7\times6\times5\times4!}{4!\times6\times2} = 1260$$

For example 2 : How many words, can be formed out of the letters of the word 'OBEDIENCE' so that vowels and consonants occur together ?

Sol. The word 'OBEDIENCE' has 5 vowels – three E's, one O and one I; it has four different consonants : B, D, N, C.

Considering 5 vowels as a block and 4 consonants as another block. The two block can be arranged in 2! ways.

Now, with the block of vowels, 5 vowels can be arranged in $\dfrac{5!}{3!}$ ways. Also, within the blocks of consonants, 4 different consonants can be arranged in 4! ways.

By the multiplication principle of counting, the required number of words formed :

$$= 2! \times \frac{5!}{3!} \times 4!$$
$$= 2 \times 5 \times 4 \times 4 \times 3 \times 2 \times 1$$
$$= 960$$

➢ **Some Properties of Permutation**

(i) ${}^nP_1 = n$

(ii) ${}^nP_{n-1} = n!$

(iii) ${}^nP_r = {}^{n-1}P_{r-1} = n(n-1)\,{}^{n-2}P_{r-2} = n(n-1)(n-2){}^{n-3}P_{r-3}$

(iv) ${}^{n-1}P_r + r.\,{}^{n-1}P_{r-1} = {}^nP_r$

(v) $\dfrac{{}^nP_r}{{}^nP_{r-1}} = n - r + 1$

TOPIC-2
Combinations

Revision Notes

Suppose you are painting your house. If a particular shade or colour is not available, you may be able to create it by, mixing different colours and shades. While creating new colours this way, the order of mixing is not important. It is the combination or choice of colours that determine the new colours; but not the order of mixing.

In the above example, you need to find out the number of choices in which the mixing of colours can be done. Such types of problems comes under the combinations :

"Each of the groups or selections which can be made by taking some or all of a number of objects without reference to the order of each group is called a **combination**."

For example :

(i) The combinations *i.e.*, selections which can be made from the letters A, B, C by taking 2 at a time are AB, AC, BC.

(ii) The combinations *i.e.*, selections which can be made from the letters A, B, C, D by taking 2 at a time are AB, AC, AD, BC, BD, CD.

Here note that, AB and BA represent the same selection because each of these selections contain same letters A and B.

➢ **Notation**

Let r and n be two positive integers and $0 \le r \le n$, then the number of combinations of n different objects taken r at a time is denoted by ${}^nC_r$ or $C(n, r)$.

Remark : The difference between a permutation and a combination of objects is that 'order' does matter in a permutation, while it does not matter in the case of a combination.

➢ **Combinations when All Objects are Different**

The number of all combinations of n distinct objects, taken r at a time is given by

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

- When $r = n$, then ${}^nC_n = \dfrac{n!}{n!(n-n)!} = \dfrac{n!}{n!0!} = 1$ $\quad [\because 0! = 1]$

- When $r = 0$, then ${}^nC_0 = \dfrac{n!}{0!(n-0)!} = \dfrac{n!}{n!} = 1$

Thus, ${}^nC_n = {}^nC_0 = 1$

For example : From a class of 36 students, 3 students are to be selected to take part in a competition. How many such selections can be made ?

Sol. The required number of selections $= {}^{36}C_3$

$$= \frac{36!}{3!(36-3)!}$$

$$= \frac{36!}{3!\ 33!}$$

$$= \frac{36 \times 35 \times 34}{1 \times 2 \times 3}$$

$$= 7140.$$

Properties of ^nC_r or $C(n, r)$

(i) For $0 \le r \le n$, we have $^nC_r = {}^nC_{n-r}$

(ii) Let n and r be non-negative integers such that $r \le n$. Then,

$$^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

(iii) If $1 \le r \le n$, then

$$n\ {}^{n-1}C_{r-1} = (n-r+1)\ {}^nC_{r-1}$$

(iv) Let n and r be non-negative integers such that $1 \le r \le n$. Then,

$$^nC_r = \frac{n}{r} . {}^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)}\ {}^{n-2}C_{r-2}$$

(v) $^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$.

(vi) The number of selections from 'n' different objects, taking at least one $= {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots {}^nC_n = 2^n - 1$.

➢ **Relation Between Permutation and Combination**

We know that,

$$^nC_r = \frac{n!}{r!(n-r)!} = \frac{1}{r!}\left(\frac{n!}{(n-r)!}\right) = \frac{1}{r!}\ {}^nP_r$$

Hence, $\quad {}^nP_r = r!\ {}^nC_r$

For example : If $^nP_r = 720$ and $^nC_r = 120$, find r.

Sol. we know that,

$$^nC_r = \frac{^nP_r}{r!}$$

$$\Rightarrow \quad 120 = \frac{720}{r!}$$

$$\Rightarrow \quad r! = \frac{720}{120} \Rightarrow r! = 6 \Rightarrow r! = 3! \Rightarrow r = 3.$$

UNIT – III: MATHEMATICAL REASONING

CHAPTER-7

LOGICAL REASONING

Revision Notes

Logical reasoning (verbal reasoning) refers to the ability of a candidate to understand and logically work through concepts and problems expressed in words. It checks the ability to extract and work with the meaning, information and implication from the bulk of the text. Logical reasoning is the process of using a rational, systematic series of steps based on sound mathematical procedures and given statements to arrive at conclusion.

➢ **Coding-Decoding**

The term **Coding-Decoding** primarily relates with message sent in secret form which cannot be understood by others easily. Coding refers a rule or method used to hide the actual meaning of a word or group of words and decoding means the method of making out the actual message that is disguised in coding. The person who transmit the code or signal, is called the sender and the person who receives it, is called the receiver.

In questions on coding-decoding, a word (basic word) is coded in a particular way and the candidates are asked to code other word in the same way. Questions of coding-decoding are designed to test the candidates ability to understand the rule used for the coding and then translate it quickly to find out the coding for the given word.

The following table helps to remember the positions of English alphabets in forward or backward order.

Order of the English Alphabet

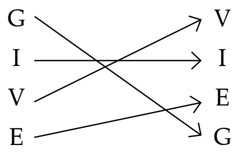
Verification of Truth of statement	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
Alphabets	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Backward Order Position	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

- **Type 1 : Letter Coding**

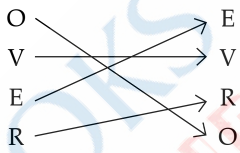
  In this type, we deal with questions, in which the letters of a word are replaced by certain other letters according to a specific pattern/rule to form a code.

  For example : In a certain code language 'GIVE' is written as 'VIEG' and 'OVER' is written as 'EVRO'. How will 'DISK' be written in that same code ?

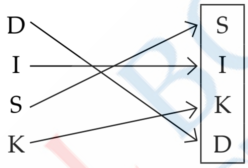
  Sol. Here,



and



  Similarly,



  Hence, DISK can be written as SIKD.

- **Type 2 : Direct Letter Coding**

  In the direct letter coding system, the code letters occur in the same sequence as the corresponding letters occur in the words. This is basically a **direct substitution method**.

  For example : In certain coding system, 'SHEEP' is written as 'GAXXR' and 'BLEAT' as 'HPXTN'. How can 'SLATE' be written in that coding system ?

  Sol. Using direct letter coding method, we have

S ⟶ G
H ⟶ A
E ⟶ X
E ⟶ X
P ⟶ R

and

B ⟶ H
L ⟶ P
E ⟶ X
A ⟶ T
T ⟶ N

  Similarly, using the direct codes

S ⟶ G
L ⟶ P
A ⟶ T
T ⟶ N
E ⟶ X

- **Type 3 : Number/Symbol Coding**

  In this type of questions, either numerical code values are assigned to a word or alphabetical code letters are assigned to the numbers.

  For example : If 'WORK' is coded as '4-12-9-16', then how will you code 'WOMAN' ?

  Sol. We have,

W ⟶ 4
O ⟶ 12
R ⟶ 9
K ⟶ 16

  Here, each letter is coded by the numerical value obtained by 'Backward Order Position' according to the English alphabets.

Hence, WOMAN can be coded as :

W $\longrightarrow$ 4
O $\longrightarrow$ 12
M $\longrightarrow$ 14
A $\longrightarrow$ 26
N $\longrightarrow$ 13

i.e., WOMAN is coded as : '4 - 12 - 14 - 26 - 13'.

- **Type 4 : Deciphering Message Word Coding/Numerical Coding**

  In this type of questions to analyse such codes, some message bearing a common word/numeral are picked up. The common code word/numeral will represent that word/code. Proceeding similarly by picking up all possible combinations of two, the entire message can be decoded and the codes for every individual word/numeral can be found.

  For example : In a certain code language, '786' means 'study very hard', '958' means 'hard work pays' and '645' means 'study and work'. How can you code 'very' ?

  Sol.

  7 $\boxed{8}$ 6 $\longrightarrow$ study very $\boxed{\text{hard}}$ ...(i)

  9 5 $\boxed{8}$ $\longrightarrow$ $\boxed{\text{hard}}$ work pays ...(ii)

  6 4 5 $\longrightarrow$ study and work ...(iii)

  In the first and second, the common code digit is '8' and the common word is 'hard'. So, '8' means 'hard. In the first and third statements, the common code digit is '6' and the common word is 'study'. So, '6' means 'study'.

  From eqs (i) and (ii), 8 $\longrightarrow$ hard
  From eqs (i) and (iii), 6 $\longrightarrow$ study
  Hence, very $\longrightarrow$ 7

- **Type 5 : Substituting Coding**

  In this type, some particular words are assigned with certain substituted names.

  For example : If 'white' is called 'blue', 'blue' is called 'red', 'red' is called 'yellow', 'yellow' is called 'green', 'green' is called 'black', 'black' is called 'violet' and 'violet' is called 'orange', then what is the colour of human blood ?

  Sol. We know that, the colour of human blood is 'red' and given that 'red' is called 'yellow'. So, the colour of human blood will be 'yellow'.

➢ **Odd Man Out**

In odd man out problems all the items given in the question except one follow certain pattern or belongs to a group. This item which is different and doesn't belong to that group will be the odd man out (*i.e.*, answer).

These type of problems are classified into following three categories given as follows—

(i) Alphabet Classification
(ii) Word Classification
(iii) Number Classification

- **Alphabet Classification**

  In this type, a group of messed letters is kept together. The pattern in which they are grouped is to be analysed and we have to find which group out of this have the same pattern or relationship between the letters. The choice which doesn't follow that pattern will be the right answer.

  For example : Find the odd man out :

  (a) ZW **(b)** TQ **(c)** SP **(d)** NL **(e)** PM

  Sol. (a) $Z \xrightarrow{-3} W$ **(b)** $T \xrightarrow{-3} Q$ **(c)** $S \xrightarrow{-3} P$ **(d)** $N \xrightarrow{-2} L$ **(e)** $P \xrightarrow{-3} M$

  Clearly, the answer will be NL.

- **Word Classification**

  In this type, different items which belong to common properties like places, parts of speech, professions etc. are present. The one which doesn't match to that category will be the odd one.

For example : Find out the pair which is different :

(a) Cow and Buffalo
(b) Cock and Hen
(c) Horse and Mare
(d) Peacock and Peahen

Sol. From the above, the second one is the feminine of the first one except the first pair so, the answer will be (a).

- **Number Classification**

  Number classification means a group of numbers which follows the same pattern. In this case, the numbers may belong to a particular Set *i.e.*, they may be odd, even, prime, rational, cubes, squares, coded binary digits etc. One choice will not follow the rule and that will be the answer.

  For example : Find the odd one out :

  (a) 121
  (b) 253
  (c) 286
  (d) 372

  Sol. Here, the correct answer is (d), because in all other numbers, the sum of the first and last digit equals to the middle digit. *i.e.*, **(a)** $1 + 1 = 2$, **(b)** $2 + 3 = 5$, **(c)** $2 + 6 = 8$ but **(d)** $3 + 2 \neq 7$.

➢ **Blood Relations (Family Relations)**

This topic involves the analysis of information showing blood relationship among members of a family. In the questions, a chain of relationship is given in the form of information and on the basis of these informations relation between any two members of the chain is asked.

- **Some common Terms Related to Blood Relations**

  Meaning of some terms often used in questions on family relationship are given below :

  (a) Parent — Mother or father
  (b) Child — Son or Daughter (even if an adult)
  (c) Sibling — Brother or Sister
  (d) Spouse — Husband or Wife

- **Basic Relationship (Aunt, Uncle, Nice and Nephew)**

  (a) **Uncle :** Father's brother, Mother's brother, Father's sister's husband, Mother's sister's husband.
  (b) **Aunt :** Father's sister, Mother's sister, Father's brother's wife, Mother's brother's wife.
  (c) **Niece :** Brother's or sister's daughter.
  (d) **Nephew :** Brother's or sister's son.

- **Relationship involving the Term 'in-law'**

  Any relationship term ending with in-law indicates that the relationship is by marriage and not by blood. In other words in-law will be a blood relative of the spouse.

  (a) **Father-in-law, Mother-in-law, Son-in-law and Daughter-in-law**

  ♦ Father-in-law is the father of spouse, mother-in-law is the mother of spouse. If parent get divorced and remarry, their new spouses are called stepparents, not mother-in-law and father-in-law.

  ♦ The husband of daughter is son-in-law, the wife of son is daughter-in-law. If spouse has children from a previous marriage, those are called step children, not sons-in-law or daughters-in-law. The person is their step-father or step mother, not father-in-law or mother-in-law.

  (b) **Brother-in-law and sister-in-law**

  Brother-in-law and sister-in-law each have two or three meanings as follows :

  ♦ Sister-in-law could be

  (i) The sister of spouse, or (ii) The wife of brother, or (iii) The wife of spouse's brother.

  ♦ Similarly, Brother-in-law could be

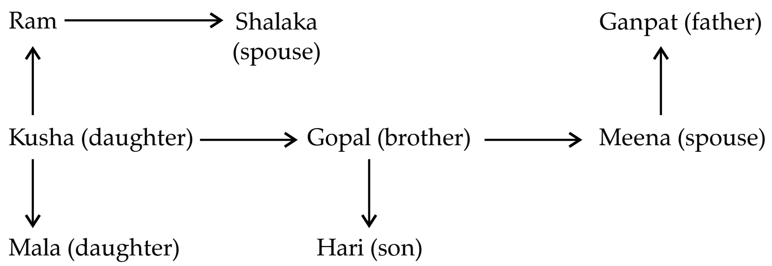
  (i) The brother of spouse, or (ii) The husband of sister, or (iii) The husband of spouse's sister.

- **Relationships involving the terms 'Great' and 'Grand'**

  The relationships of the second generation are prefixed with the word **Grand.** For example, for a person, the first generation below him/her would be that of his/her child/children. The next/second generation would be the children of the children who would be called **Grand Children** of the person. The next/third generation would be called **Great Grand Children** of that person. The next/fourth generation relationship are called **Great Great Grand.** For example, son of Great Grand Son is Great Great Grand Son.

  For example : 'Ram' is the father of 'Kusha' but 'Kusha' is not his son. 'Mala' is the daughter of 'Kusha'. 'Shalaka' is the spouse of 'Ram'. 'Gopal' is the brother of 'Kusha'. 'Hari' is the son of 'Gopal'. 'Meena' is the spouse of 'Gopal'. 'Ganpat' is the father of 'Meena'. Who is the grand daughter of 'Ram'?

Sol.



Here, we can see 'Mala' is the daughter of 'Kusha' and 'Ram is the father of 'Kusha'. So, 'Mala' is the grand-daughter of 'Ram.

- **Summary of Some Common Relationships**

  The following table helps you to understand the summary of some common relationships.

	Relation	Commonly Used Terms
1.	Grand father's or Grand mother's only son	Father
2.	Grand father's or Grand mother's only daughter-in-law	Mother
3.	Father's father or Mother's father	Grand father
4.	Father's mother or Mother's mother	Grand mother
5.	Father's brother or Mother's brother	Uncle
6.	Father's sister or Mother's sister	Aunt
7.	Son's wife	Daughter-in-law
8.	Daughter's Husband	Son-in-law
9.	Husband's or wife's sister	Sister-in-law
10.	Husband's or wife's brother	Brother-in-law
11.	Brother's wife	Sister-in-law
12.	Brother's or Sister's Son	Nephew
13.	Brother's or Sister's Daughter	Niece
14.	Uncle's or Aunt's son or daughter	Cousin
15.	Sister's Husband	Brother-in-law
16.	Brother's wife	Sister-in-law
17.	Grand Son's or grand daughter's daughter	Great Grand Daughter
18.	Grand son's or grand daughter's Son	Great Grand son

➢ **Syllogism**

The term 'Syllogism' is used to denote that form of reasoning where conclusion is drawn from two or three statements. This is undoubtedly the most important part of logical reasoning. To understand syllogism, we must know the following terminology used in logic.

- **Proposition :** A **Proposition** is also known as a **premises,** is a grammatical sentence which comprises a subject, a predicate and a copula. Subject is that which affirms or denies a fact. Predicate is a term which states something about subject and copula establishes relationship between subject and predicate. *e.g.,*

  (i) All $\underbrace{\text{Jars}}_{\text{Subject}}$ $\underbrace{\text{are}}_{\text{Copula}}$ $\underbrace{\text{blue}}_{\text{Predicate}}$

  (ii) No $\underbrace{\text{Car}}_{\text{Subject}}$ $\underbrace{\text{is}}_{\text{Coupla}}$ $\underbrace{\text{red}}_{\text{Predicate}}$

- **Qualifiers :** Premises usually start with the words :

  All, No, Some and Some-not.

  (i) The word 'All', has its synonyms which gives the same meaning—Every, Any, Each.

  (ii) The word 'some' is also replaced by Many, Few, Most of, More, A little, etc.

- **Classification of Proposition/Premise :**

  Proposition has been classified on the basis of quality and quantity. Quality denotes whether the proposition is affirmative or negative. Quantity represents whether the proposition is universal or particular. The four-fold classification of proposition can be summarized as under :

Symbol	Proposition	Quantity	Quality
A	All A are B	Universal	Affirmative
E	No A is B	Universal	Negative
I	Some A are B	Particular	Affirmative
O	Some A are not B	Particular	Negative

♦ **Distribution :** In Proposition a term (either subject or predicate) is said to be distributed if the quantity of that term is well defined. For example, in A type proposition (All A are B), the quantity of subject A is defined. In E type proposition (No A is B), the quantity of subject and predicate both are defined. In I-type proposition (Some A are B) the quantity of neither subject nor predicate is defined. In O type proposition (some A are not B) the quantity of predicate is defined.

1. Universal Affirmative or A-Type Proposition (All A are B)

In this proposition, entire category, represented by subject is distributed in the category as represented by predicate.

e.g. : (i) All girls are beautiful.
(ii) All bulbs are tigers.

2. Universal Negative or E-Type Proposition (No A is B)

In this type of proposition both subject and predicate are denial of each other. (A) and (B) have nothing in common and hence both subject and predicate are distributed.

e.g. : (i) No professor is lazy.
(ii) No boxes are bangles.

3. Particular Affirmative or I-Type Proposition (Some A are B)

In this type, subject and predicate have something in common. This implies that in I-type neither subject nor predicate is distributed.

e.g. : (i) Some girls are smart.
(ii) Some girls are cats.

4. Particular Negative or O-Type Proposition (Some A are Not B)

In O-type proposition some of the category represented by (A) subject is not (B), which means that a section of A is denied with the entire category of B. It is therefore, deduced that in O-type proposition only predicate is distributed.

e.g. : (i) Some are not orange flowers.
(ii) Some fans are not white.

● **Inference for Conclusions is Syllogism**

In Syllogism there are two major types of inferences, these are :

(a) Immediate Inference : If an inference is drawn from any one of the statement without taking into consideration the other statement, it is known as immediate inference. Here, we have given below the immediate inferences drawn from the two statements.

e.g.,

Statements : (i) Some tigers are cows.
(ii) No cow is goat.

Conclusion : (I) Some cows are tigers [From statement (i)]
(II) No goat is cow [From statement (ii)]

The following table comprises the valid immediate inference rules for each type of propositions.

Type	Proposition	Valid Immediate Inference
A	All A are B	Some B are A Some A are B No B is A
E	No A is B	Some A are not B Some B are not A
I	Some A are B	Some B are A
O	Some A are not B	No inference

(b) Mediate Inference : In mediate inference conclusion is drawn from the two statements connected by the middle term. Hence, in mediate inference middle term will be missing.

e.g.,

Statements : (i) Some tables are trees.
(ii) No tree is door.

Conclusion : (I) Some tables are not trees.
(II) No table is a door.

In the above conclusions, we see that the middle term 'tree' is missing and hence these conclusions are mediate inferences. So, it is clear from the above conclusions that Immediate inference is drawn from single statement while the mediate inference is drawn from two statements.

➢ **Rules for Mediate Inference**

Rule 1 : The middle term must be distributed at least once in the premises, other-wise inference is uncertain or doubtful.

e.g.,

Statements : (i) All mangoes are chairs.
(ii) Some chairs are tables.

Conclusions : (I) All mangoes are tables.
(II) Some tables are mangoes.
(III) No mango is a table.

Sol. The term chair is common to both the statements and hence is the middle term. Statement (i) is of A-type proposition and in A-type proposition, only subject is distributed, hence chair being the predicate in the statement I is not distributed, Likewise chair is not distributed in the statement (ii), because statement (ii) is of I-type in which neither subject nor predicate is distributed. Since, middle term 'chair' is not distributed in any one of the statement, therefore, as per rule 1 no mediate inference can be drawn. Thus, none of the conclusions following statements above is a valid inference.

Rule 2 : No term can be distributed in the conclusion unless it is distributed in the premises.

e.g.,

Statements : (i) Some ships are aeroplanes.
(ii) All aeroplanes are jets.

Conclusions : (I) All ships are jets.
(II) No ships are jets
(III) Some ships are jets.
(IV) All jets are ships.

Sol. From the statements, we observe that middle term is distributed in statement II and as per rule (1), mediate inference can be drawn from such pair of statements.

Now, we have to test each conclusion for rule 2 to find which conclusion is not valid conclusion.

In conclusion (I), term 'ships' is distributed but it is not distributed in statement (i). Hence, it is not valid conclusion.

In conclusion (II), both terms 'ships' and 'jets' are distributed which are not distributed in statements (i) and (ii). Hence, it is not valid conclusion.

In conclusion (III), neither ships nor jets are distributed and hence rule 2 cannot be checked. But this is the only valid mediate inference drawn from above statements (refer rule 3).

In conclusion (IV), the term jet is distributed which is not distributed in statement (ii). Hence, conclusion (iv) is not valid.

Rule 3 : If one premises is particular, conclusion is particular.

e.g.,

Statements : (i) Some ships are aeroplanes.
(ii) All aeroplanes are jets.

Conclusions : (I) All ships are jets.
(II) No ships is jets.
(III) Some ships are jets.

Sol. As per this rule, Statement (i) is particular, therefore, conclusion (III) is the valid inference. Moreover, it does not violate rules (1) and (2).

Rule 4 : If one of the statements is negative, conclusion will always be negative.

e.g.,

Statements : (i) Some cups are jars.
(ii) No jar is kettle.

Conclusions : (I) Some cups are kettles.
(II) Some cups are not kettles.
(III) All kettles are cups.
(IV) No cup are kettles.

Sol. As per rule 4, if one of the premises is negative, conclusion will always be negative. From the conclusions, we see that conclusion (II) and (IV) are negative. So, these may be valid inferences. But for being valid they must not violate any of the rules described earlier.

Conclusion (IV) violates rule 2 because cups is distributed in conclusion but cups is not distributed in the statement I. This conclusion also violates rule 3, which stipulates if one of the premises is particular, conclusion will be particular but conclusion (IV) is universal.

Conclusion (II) complies with rule 2 as term kettle which is distributed in the conclusion is also distributed in the statement (ii) and conclusion (II) being of particular quantity, also complies with rule 3.

Rule 5 : If both the premises are negative, no conclusion can be drawn.

e.g.,

Statements : (i) No car is chair.
(ii) No chair is cat.

Conclusions : (I) Some cars are cats.
(II) No car is cat.

Sol. Since, both statements are negative, we cannot draw any valid mediate inference from such pair of statements. Therefore, no conclusion follows.

Rule 6 : If both the premises are particular, no conclusion follows.

e.g.,

Statements : (i) Some bullets are guns.
(ii) Some guns are tanks.

Conclusions : (I) Some bullets are tanks.
(II) No bullet is a tank.

Sol. For the above example this rule is a mere reiteration of rule 1 which states that if middle term is not distributed atleast once in the premises, no conclusion follows. Therefore, none of the conclusions (I) and (II) is valid according to Rule 6.

Rule 7 : Complementary pair of conclusions

Sometimes where drawing mediate inference from given statements, we have to select complementary pair of conclusions where neither of the conclusions is definitely true but a combination of both a complementary pair.

e.g.,

Statements : (i) Some cups are books.
(ii) Some books are black.

Conclusions : (I) Some cups are black
(II) No cup is black.

Sol. It is very clear from the above example that none of the conclusions is definitely true as per rule 1 because middle term is not distributed in the statements. However, both of the conclusions form complementary pair. Hence, either conclusion (I) or follows from the statements.

➢ **Logical Arrangement of Words**

In this topic of reasoning, a sequence is formed from the words given in the question in such a way that final arrangement of the words describes step by step completion situation.

For example : Arrange the following words in a logical and meaningful order.

1. Travel, 2. Destination, 3. Payment, 4. Berth/seat number, 5. Reservation, 6. Availability of berth/seat for reservation.

Sol. From the above words, it is very clear that in order to perform a journey. First destination is defined, secondly availability of berth is known, which follows payment for reservation. As a result berth is allotted and travelling is performed.

Hence, the correct, logical and meaningful arrangement of the given words is 2, 6, 3, 5, 4, 1.

➢ **Verification of Truth of Statement**

In this topic, we shall deal with the statement, which is tested on the basis of given options for its truthfulness.

For example : A bulb always has ....

(a) Filament (b) Light (c) Glass (d) Current

Sol. From the above options given below the question, we see that a bulb has all four components. But again the most proximate is filament.

Rest of the other option are remote. Hence, option (a) is our answer.

UNIT – IV CALCULUS

CHAPTER-8

DOMAIN AND RANGE OF FUNCTIONS

TOPIC-1
Functions

Revision Notes

A function in mathematics is an expression, rule or law that defines relationship between one variable to another variable. A function is a special kind of relation. In this chapter we shall introduce the concept of a function as a correspondence between two sets.

➢ **Functions**

Let A and B are two non-empty sets. A relation f from a set A to a set B, *i.e.*, a subset of $A \times B$, is said to be a function from A to B, if

(i) for each $a \in A$, there exists $b \in B$ such that $(a, b) \in f$

(ii) $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$

Thus, a non-void subset f of $A \times B$ is a function from A to B, if each element appears in some ordered pair in f and no two ordered pairs in f have same first element.

If f is a function from a set A to set B, then we write,

$$f : A \to B \text{ or } A \xrightarrow{f} B$$

and it is read as f is a function from A to B.

If $(a, b) \in f \Rightarrow f(a) = b$.

Here, b is the **image** of a under f and a is called the **pre-image** of b under f.

For example :

(i) Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ and f_1, f_2 and f_3 are three subsets of $A \times B$ as given below :

$$f_1 = \{(1, 2), (2, 3), (3, 4)\}$$
$$f_2 = \{(1, 2), (1, 3), (2, 3), (3, 4)\}$$
$$f_3 = \{(1, 3), (2, 4)\}$$

Then, f_1 is a function from A to B but f_2 and f_3 are not functions from A to B. f_2 is not a function from A to B because $1 \in A$ has two images 2 and 3 in B and f_3 is not a function from A to B because $3 \in A$ has no image in B.

(ii) Suppose, $f = \{(n, n^2) \mid n \text{ is a positive integer}\}$ is a given relation, then it is a function, because the square of any positive integer is unique *i.e.*, every element in the domain has unique image.

(iii) Suppose, $g = \{(4, 6), (3, 9), (-11, 6), (3, 11)\}$ is a given relation, then it is not a function, because 3 has two images 9 and 11.

- **Main features of a function**

Let f be a function from X to Y, then

(i) to every $x \in X$, there exists a unique element $y \in Y$ such that $y = f(x)$

(ii) no element of X can have more than one images in Y

(iii) there may be elements of Y which are not associated with any element of X.

(iv) distinct elements of X may have some image in Y

(v) function f is determined when $f(x)$ is known for all $x \in X$

Remark :

Let A and B be two non-empty finite sets such that $n(A) = p$, and $n(B) = q$, then number of functions from A to B $= q^p$.

➢ **Domain, Co-domain and Range of a Function**

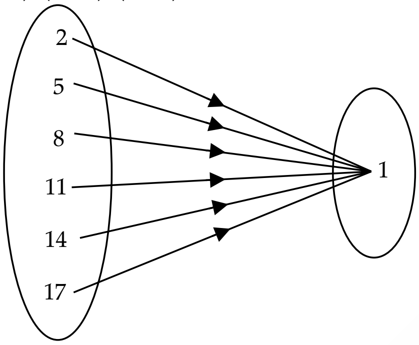
If f is a function from A to B and each element of A corresponds to one and only one element of B, whereas every element in B need not be the image of some x in A. Then, the set A is called the **domain** of function f and the set B is called the **co-domain** of f. The subset of B containing the images of elements of A is called the **range** of the function.

Thus, if a function f is expressed as the set of ordered pairs, then the domain of f is the **set of all first elements** of ordered pairs and the range of f is **the set of second elements** of ordered pairs of f. *i.e.,*

$$\text{Domain of } f = \{a : (a, b) \in f\}$$
$$\text{Range of } f = \{b : (a, b) \in f\}$$

For example :

Given $R = \{(2, 1), (5, 1), (8, 1), (11, 1), (14, 1), (17, 1)\}$



It is clear from the above diagram, the given relation is a function because to each first element of the ordered pair, there corresponds exactly one second element.

$\therefore$ Domain = {2, 5, 8, 11, 14, 17} and Range = {1}

- **Real Functions**

Real valued functions :

A function $f : A \to B$ is called a real valued function, if B is a subset of R (set of all real numbers). If A and B both are subsets of R, then f is called a real function.

For example : $f : R \to R$ given by $f(x) = x^2 + x + 1$ or, $f : A \to B$ given by $f(x) = \dfrac{x-1}{x^2-4}$ etc.

- **Domain and Range of Real Function**

If f is a real function, then :

The set of all possible real numbers x for which $f(x)$ is a real number is called the **domain** of the function f, it is usually denoted by D_f. Thus,

$$D_f = \{x : x \in R, f(x) \in R_f\}$$

The set of images of $f(x)$ for all $x \in D_f$ is called the **range** of f, it is usually denote by R_f. Thus,

$$R_f = \{f(x) ; \text{for all, } x \in D_f\}$$

For example : Find domain and range of the function $f(x)$ given by $f(x) = \dfrac{x-2}{3-x}$.

Sol. Clearly, $f(x)$ is defined for all x satisfying $3 - x \neq 0$ *i.e.,* $x \neq 3$

Hence, $D_f = R - \{3\}$

Range of f : Let $y = f(x)$, *i.e.,*
$$y = \frac{x-2}{3-x}$$
$$\Rightarrow \quad 3y - xy = x - 2$$
$$\Rightarrow \quad x(y + 1) = 3y + 2$$
$$\Rightarrow \quad x = \frac{3y+2}{y+1}$$

Clearly, x assumes real values for all y except $y + 1 = 0$ *i.e.,* $y = -1$.

Hence, $R_f = R - \{-1\}$

Remark

1. A function does not exist if its domain is empty.

2. If the domain of a real valued function is given, then we are not to find the admissible subset of R.

3. The distinctive property of a function is that, for a given element x of domain, $f(x)$ is uniquely determined element of codomain.

4. For any real function,

$f : D \to R$ and $n \in N$, we define

$$\underbrace{(f\,f\,f........f)(x)}_{n \text{ times}} = \underbrace{f(x)f(x)....f(x)}_{n \text{ times}} = \{f(x)\}^n, \forall x \in D$$

➢ **Algebra of Real Functions**

Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions with domain D_1 and D_2, respectively. Then, algebraic operations as addition, subtraction, multiplication and division of two real functions are given below.

(i) Addition of two real functions : The sum function $(f + g)$ is defined by

$$\boxed{(f + g)(x) = f(x) + g(x), \forall\ x \in D_1 \cap D_2}$$

The domain of $(f + g)$ is $D_1 \cap D_2$.

(ii) Subtraction of two real functions : The difference function $(f - g)$ is defined by

$$\boxed{(f - g)(x) = f(x) - g(x),\ \forall x \in D_1 \cap D_2}$$

The domain of $(f - g)$ is $D_1 \cap D_2$.

(iii) Multiplication of two real functions : The product function (fg) is defined by

$$\boxed{(fg)(x) = f(x).g(x),\ \forall x \in D_1 \cap D_2}$$

The domain of (fg) is $D_1 \cap D_2$.

(iv) Quotient of two real functions : The quotient function is defined by

$$\boxed{\frac{f}{g}(x) = \frac{f(x)}{g(x)},\ \forall x \in D_1 \cap D_2 - [x : g(x) \neq 0]}$$

The domain of $\left(\frac{f}{g}\right)$ is $D_1 \cap D_2 - [x : g(x) \neq 0]$

(v) Multiplication of a real function by a scalar : The scalar multiple function cf is defined by

$$\boxed{(cf)(x) = c.f(x),\ \forall x \in D_1}$$

where, c is scalar (real number).

The domain of cf is D_1.

➢ **Equal Functions**

Two functions f and g are said to be equal iff

(i) domain of f = domain of g

(ii) codomain of f = codomain of g

(iii) $f(x)=g(x)$ for every x belonging to their common domain.

If two functions f and g are equal, then we write $f = g$.

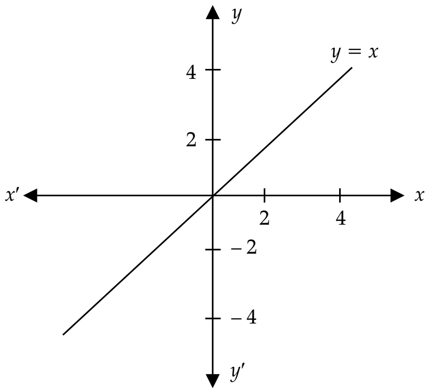
For example : Let $A = \{1, 2\}$, $B = \{3, 6\}$ and given $f : A \to B$ given by $f(x) = x^2 + 2$ and $g : A \to B$ given by $g(x) = 3x$. Then, we observe that f and g have same domain and codomain. Also, we have $f(1) = 3 = g(1)$ and $f(2) = 6 = g(2)$. Hence, $f = g$.

TOPIC-2
Various Functions and their Graphs

Revision Notes

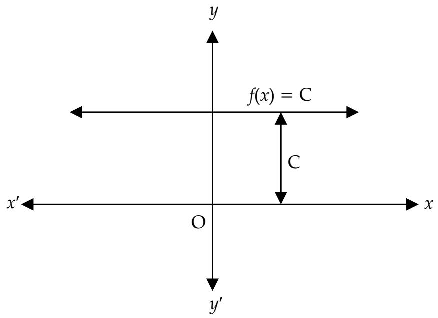
➢ **Graphs of Functions**

- **Identity function :** Let R be the set of real numbers. A real valued function f is defined as $f : R \to R$ by $y = f(x) = x$ for each value of $x \in R$. Such a function is called the identity function. Its graph is a straight line passing through the origin and inclined at an angle of 45° with x-axis.



Here, Domain = R and Range = R.

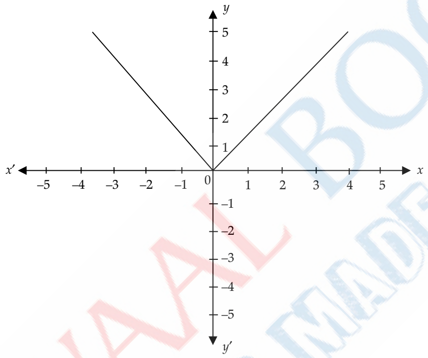
- **Constant function :** The function $f : R \to R$ defined by $f(x) = C$ for each $x \in R$ is called constant function. (where C is a constant). Its graph is a straight line parallel to x-axis at a distance $|C|$ units from it.



Here, Domain = R and Range = $\{C\}$

- **Modulus function :** The function $f : R \to R$ defined by $f(x) = |x|$ for each $x \in R$ is called modulus function or absolute valued function.

i.e., $$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

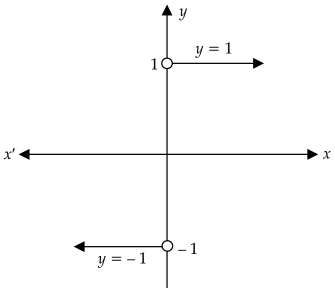


Here, Domain = R and Range = $R^+ \cup \{0\} = \{x : x \in R; x \geq 0\}$

- **Signum function :** Let R be the set of real numbers, then the function $f : R \to R$ defined by

$f(x) = \dfrac{|x|}{x} = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ is known as signum function. Signum function is denoted by *sgn x*.

Here, Domain = R and Range = $\{-1, 0, 1\}$



Properties of Modulus function

(i) For any real number x, we have $\sqrt{x} = |x|$

(ii) If a, b are positive real numbers

$$x^2 \leq a^2 \Leftrightarrow |x| \leq a \Leftrightarrow -a \leq x \leq a$$
$$x^2 \geq a^2 \Leftrightarrow |x| \geq a \Leftrightarrow x \leq -a \text{ or } x \geq a$$
$$x^2 < a^2 \Leftrightarrow |x| < a \Leftrightarrow -a < x < a$$
$$x^2 > a^2 \Leftrightarrow |x| > a \Leftrightarrow x < -a \text{ or } x > a$$

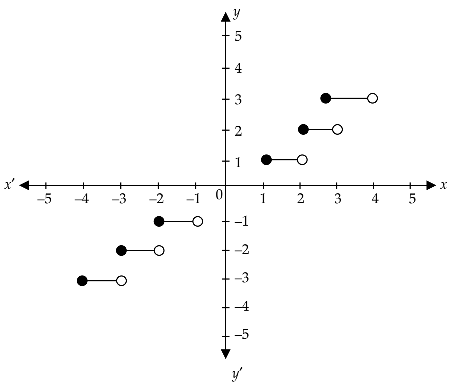
- **Greatest integer function :** The function $f : R \to R$ defined by $f(x) = [x]$, $x \in R$ assumes the value of the greatest integer, less than or equal to x, such a function is called the greatest integer function.

From the definition of $[x]$, we have

$[x] = -1$, for $-1 < x \leq 0$ and for $0 \leq x < 1$.

$= 1$, for $1 \leq x < 2$

$= 2$, for $2 \leq x < 3$



Here, Domain = R and Range = Z

Properties of greatest integer function

If n is an integer and x is a real number between n and $n + 1$, then

(i) $[-n] = -[n]$

(ii) $[x + k] = [x] + k$ for any integer

(iii) $[-x] = -[x] - 1$

(iv) $[-x] = -[x] + 1$, where $x \in R - Z$

(v) $[x] + [-x] = \begin{cases} -1, & \text{if } x \notin Z \\ 0, & \text{if } x \in Z \end{cases}$

(vi) $[x] - [-x] = \begin{cases} 2[x]+1, & \text{if } x \notin Z \\ 2[x], & \text{if } x \in Z \end{cases}$

Some examples of greatest integer function are : $[5] = 5$, $[-5] = -5$, $[0] = 0$, $\left[\frac{14}{3}\right] = 4$, $[\sqrt{2}] = 1$, $[\sqrt{-3}] = -2$, $\left[-\frac{9}{2}\right] = -5$ and $[-\pi] = -4$ etc.

- **Exponential function :** If a is any positive real number, then the function f defined by $f(x) = a^x$; $a \neq 1$ is called the exponential function.

  Here, Domain $= R$ and Range $= (0, \infty)$

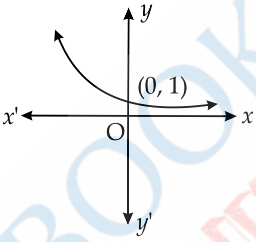
  Some properties of exponential functions are:

  (i) $a^0 = 1$

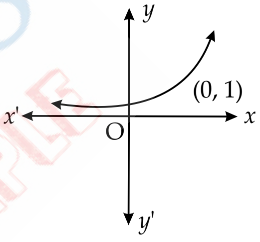
  (ii) $a^x . a^y = a^{x+y}$ for all $x, y \in R$

  (iii) $(a^x)^y = a^{xy}$ for all $x, y \in R$

  (iv) $a^{-x} = \frac{1}{a^x}$ for all $x \in R$



$0 < a < 1$

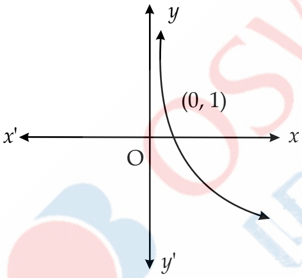


$a > 1$

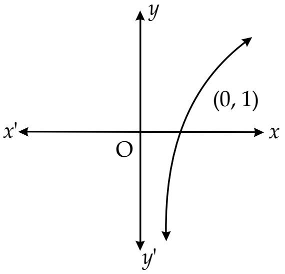
- **Remark :** Natural exponential function, $f(x) = e^x$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... \infty, 2 < e < 3$$

- **Logarithmic function :** If a is any positive real number, $a \neq 1$, and then x is any positive real number, then logarithmic function with base a is denoted by symbol $\log_a x$ and is defined as $y = f(x) = \log_a x$ if and only if $x = a^y$.



$f(x) = \log x \ (a > 1)$



$f(x) = \log x \ (0 < a < 1)$

Here, Domain $= (0, \infty)$ and Range $= R$

Remark : Natural logarithmic function $f(x) = \log e^x$ or $\ln x$.

➢ **Polynomial function :** The function that can be expressed in the form of a polynomial is called a polynomial function. A polynomial is generally represented as $P(x)$. The highest power of the variable of $P(x)$ is known as its degree.

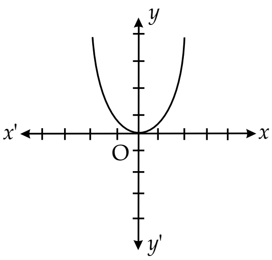
A polynomial function in degree in n can be written as $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots\ldots\ldots + a_2 x^2 + a_1 x + a_0$, where $a_0, a_1, a_2, \ldots\ldots, a_n$ are all real numbers and n is non-negative integer.

The domain of a polynomial function is entire real numbers (R) *i.e.*, Domain = R

If $a_n \neq 0$, then P is called a polynomial function of degree n. If $n=1$, it is called a linear function; if $n=2$, it is called a quadratic function and if $n=3$, it is called a cubic function etc.

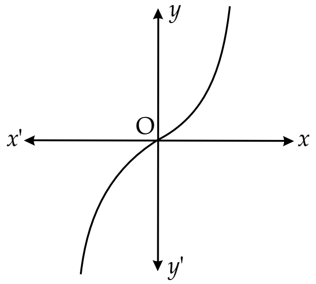
Graphs of some polynomial functions are given below :

Quadratic Polynomial Function: $f : R \to R$, $f(x) = x^2$



Domain = R and Range = $[0, \infty)$

Cubic Polynomial Function: $f : R \to R, f(x) = x^3$



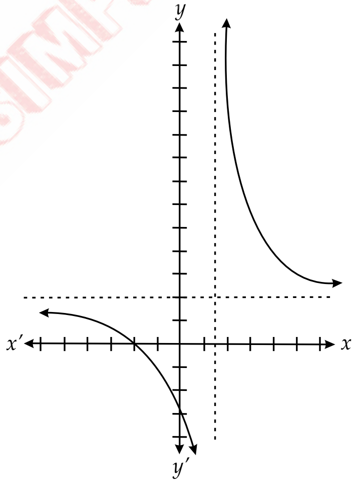
Domain = R and Range = R

- **Rational function :** A rational function is any function where one polynomial function is divided by another. It's "rational" because the two polynomials form a ratio (*i.e.*, a fraction). In other words, a function which can be expressed as the quotient of two polynomial functions *i.e.*, $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are the polynomials and $h(x) \neq 0$.

For example : $f(x) = \dfrac{6x+12}{4-3x}$ with Domain = R – $\left\{\dfrac{4}{3}\right\}$

$f(x) = \dfrac{x^2+5}{x+2}$ with Domain = R – {–2}

Graph of rational function $f(x) = \dfrac{6x+12}{4-3x}$ is as follows :



Remarks :

(i) All polynomials are rational functions, because every polynomial can be written as a quotient of itself divided by 1.

(ii) Proper rational function is a ratio of functions where the degree of the numerator is less than the degree of the denominator.

CHAPTER-9

CONCEPTS OF LIMITS AND CONTINUITY AND DIFFERENTIATION OF DERIVATIVES

TOPIC-1
Limit and Continuity

Revision Notes

- The invention of calculus was one of the most far reaching events in the history of mathematics. It is that part of mathematics mainly deals with the study of change in the value of a function as the value of the variable in the domain change. In this chapter, we shall study the concept of limit and derivatives of real functions, study of algebra of limits and derivatives and will find limits and derivatives of algebraic, trigonometric and logarithmic functions.
- **Definition of Limit**

  Limit is the degree of closeness to any value or the approaching term. Let $y = f(x)$ be a function of x. If at $x = a$, $f(x)$ takes indeterminate form, then we consider the value of the function which is very near to a. If these values tend to a definite unique number as x tends to a then the unique number, so obtained is called the limit of $f(x)$ at $x = a$ and is written as $\lim\limits_{x \to a} f(x)$.

OR

If $f(x)$ approaches to a real number l, when x approaches to a *i.e.*, if $f(x) \to l$ when $x \to a$, then l is called the limit of the function $f(x)$. In symbolic form, it can be written as :

$$\lim_{x\to a} f(x) = l$$

For Example : If $f(x) = 2x + 5$, then

$$\lim_{x\to 1}(2x+5) = 2.1 + 5 = 7,$$

is true, because if you substitute numbers x close to 1 in $f(x)$ the result will be close to 7.

➢ **Left hand and right hand limit**

A real number l_1, is the left hand limit of function $f(x)$ at $x = a$, if the value of $f(x)$ can be made as close as l_1, at point closed to a and on the left of a. Symbolically,

$$\text{L.H.L} = \lim_{x\to a^-} f(x) = l_1$$

A real number l_2 is the right hand limit of function $f(x)$ at $x = a$, if the values of $f(x)$ can be made as close as l_2 at points closed to a and on the right of a symbolically,

$$\text{R.H.L.} = \lim_{x\to a^+} f(x) = l_2$$

- **Method to find left hand and right hand limit**

Step 1 : For left hand limit, write the given function as $\lim_{x\to a^-} f(x)$ and for right hand limit, write the given function as $\lim_{x\to a^+} f(x)$.

Step II : For left hand limit, put $x = a - h$ and change the limit $x \to a^-$ by $h \to 0$. Then limit obtained in step I is $\lim_{h\to 0} f(a-h)$.

For right hand limit, put $x = a + h$ and change the limit $x \to a^+$ by $h \to 0$. Then, Limit obtained in step I is $\lim_{h\to 0} f(a+h)$.

Step III : Simplify the result obtained in step II *i.e.*, $\lim_{h\to 0} f(a-h)$ or $\lim_{h\to 0} f(a+h)$.

➢ **Existence of limit**

If the right hand limit and left hand limit coincide, then we say that limit exists and their common value is called the limit of $f(x)$ at $x = a$ and is denoted by $\lim_{x\to a} f(x)$.

Remarks :

- $x \to a^-$ is read as x tends to 'a' from left and it means that x is very close to 'a' but it is always less than a.
- $x \to a^+$ is read as x tends to 'a' from right and it means that x is very close to 'a' but it is always greater than 'a'.
- $x \to a$ is read as x tends to 'a' and it means that x is very close to a but it is not equal to 'a'.
- Left hand limit and right hand limit of a constant function is the constant itself

  e.g., $\lim_{x\to 1^-} 3 = 3$, $\lim_{x\to 3^+} 4 = 4$.

For example :

Find $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} \dfrac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Sol. L.H.L.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{-x} = \lim_{x\to 0^-} (-1) = -1$$

R.H.L.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x}{|x|} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$$

Since $\lim_{x\to 0-} f(x) \neq \lim_{x\to 0+} f(x)$

$\therefore \lim_{x\to 0} f(x)$ does not exist.

➢ **Difference between the Value of the Function at a Point and the Limit at a Point**

Let $f(x)$ be a function and let a be a point. Then, we have the following possibilities :

- **$\lim_{x \to a} f(x)$ exists but $f(a)$ (the value of $f(x)$ at $x = a$) does not exists.**

  Consider a function, $f(x) = \dfrac{x^2 - 9}{x - 3}$

  Here, it is clear that $f(3)$ does not exist *i.e.*, the function $f(x)$ is not defined at $x = 3$, because it attains the form $\dfrac{0}{0}$.

  But $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 6$. Hence, $\lim_{x \to 3} f(x)$ exists.

- **The value of $f(a)$ exists but $\lim_{x \to a} f(x)$ does not exists.**

  Consider a function, $f(x) = [x]$

  Here, $f(x) = [x]$ = greatest integer less than or equal to x. at $x = k$, where k is an integer.

  But,

  (L.H.L. at $x = k$) = $\lim_{x \to k^-} f(x) = \lim_{h \to 0} f(k-h) = \lim_{h \to 0} [k-h] = \lim_{h \to 0} k - 1 = k - 1$

  ($\because k - 1 < k - h < k$, $\therefore [k - h] = k - 1$)

  (R.H.L. at $x = k$) = $\lim_{x \to k+} f(x) = \lim_{h \to 0} f(k+h) = \lim_{h \to 0} [k+h] = \lim_{h \to 0} k = k$

  ($\because k < k + h < k + 1$, $\therefore [k + h] = k$)

  Clearly, $\lim_{x \to k^-} f(x) \neq \lim_{x \to k^+} f(x)$. Hence, $\lim_{x \to k} f(x)$ does not exist.

- **$\lim_{x \to a} f(x)$ and $f(a)$ both exist but are unequal.**

  Consider a function, $f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, & x \neq 3 \\ 3, & x = 3 \end{cases}$

  Here, $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 6$. So, $\lim_{x \to 3} f(x)$ exists and equal to 6. It is also clear from the given function the value of $f(3)=3$.

  Thus, $\lim_{x \to 3} f(x)$ and $f(3)$ both exist but are unequal.

- **$\lim_{x \to a} f(x)$ and $f(a)$ both exist and are equal.**

  Consider a function, $f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, & x \neq 3 \\ 6, & x = 3 \end{cases}$

  Here, $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 6$. So, $\lim_{x \to 3} f(x)$ exists and equal to 6. It is also clear from the given function the value of $f(3) = 6$.

  Thus, $\lim_{x \to 3} f(x)$ and $f(3)$ both exist and are equal.

➢ **Algebra of limits**

Let 'f' and 'g' be two real functions with common domain D, such that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exists, then,

(i) Limit of sum of two functions is sum of the limits of the function *i.e.*,

$$\lim_{x \to a} (f + g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

(ii) Limit of difference of two functions is difference of the limits of the function *i.e.*,

$$\lim_{x \to a} (f - g)(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

(iii) Limit of product of two functions is product of the limits of the function *i.e.*,

$$\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$$

(iv) Limit of quotient of two functions is quotient of the limits of the function *i.e.*,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}, \text{ where } \lim_{x\to a} g(x) \neq 0.$$

(v) Limit of product of a constant and on function is the product of that constant and limit of the function *i.e.*,

$$\lim_{x\to a}\{c.f(x)\} = c\lim_{x\to a} f(x), \text{ where '}c\text{' is a constant.}$$

➢ **Indeterminate Forms**

A mathematical expression is said to be indeterminate or meaningless if it is not definitively or precisely determined. In other words, the **indeterminate** form is a mathematical expression that we cannot be able to determine the original value even after the substitution of the limits. For example, a limit of the form $\frac{0}{0}$ *i.e.*, $\lim\limits_{x\to 0}\frac{f(x)}{g(x)}$, where $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} g(x) = 0$, is indeterminate since the value of the overall limit actually depends on the limiting behaviour of the combination of the two functions $\left(e.g., \lim\limits_{x\to 0}\frac{x}{x} = 1, \lim\limits_{x\to 0}\frac{x^2}{x} = 0\right)$. There are seven indeterminate forms involving 0, 1, and ∞ :

$$\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 1^{\infty}, 0^0, \infty^0.$$

Evaluation of Limits

There are several methods for calculating the limit of the functions.

(i) Direct Substitution Method : Direct substitution is just what the name implies : you directly substitute a given value into a limit.

Consider the following limits :

(a) $\lim\limits_{x\to a} f(x)$ **(b)** $\lim\limits_{x\to a}\frac{g(x)}{h(x)}$

If $f(a)$ and $\frac{g(a)}{h(a)}$ exist and are fixed real numbers, then we say that

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a}\frac{g(x)}{h(x)} = \frac{g(a)}{h(a)}$$

In other words, if the direct substitution of the point, to which the variable tends to, we obtain a fixed real number, then the number obtained is the limit of the function or we obtained value of function.

For example : If $f(x) = (x-2)^2(2x-3)$, then find limit of $f(x)$ at $x = 2$.

Sol.

$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2}(x-2)^2(2x-3)\\ &= \lim_{x\to 2}(2-2)(4-3)\\ &= 0\times 1\\ &= 0\end{aligned}$$

Alter

$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2}[(x-2)^2(2x-3)]\\ &= \lim_{x\to 2}[(x^2-4x+4)(2x-3)]\\ &= \lim_{x\to 2}[2x^3-8x^2+8x-3x^2+12x-12]\\ &= 2\lim_{x\to 2}(x^3) - 11\lim_{x\to 2}(x^2) + 20\lim_{x\to 2}(x) - 12\\ &= 2.8 - 11.4 + 20.2 - 12\\ &= 16 - 44 + 40 - 12\\ &= 0.\end{aligned}$$

(ii) Factorization Method : When the function is of the form $\frac{f_1(x)}{f_2(x)}$, then we factorize the numerator and denominator. Then, we cancel out the common factors of the numerator and the denominator and then substitute the value of x.

For example : $\lim\limits_{x\to 2}\dfrac{3x^2-x-10}{x^2-4}$.

Sol.
$$\lim_{x\to 2}\frac{3x^2-x-10}{x^2-4} = \lim_{x\to 2}\frac{3x^2-6x+5x-10}{x^2-4}$$
$$= \lim_{x\to 2}\frac{(3x+5)(x-2)}{(x+2)(x-2)}\ (x\neq 2)$$
$$= \lim_{x\to 2}\frac{(3x+5)}{x+2} \qquad [x\neq 2]$$
$$= \frac{\lim\limits_{x\to 2}(3x+5)}{\lim\limits_{x\to 2}(x+2)}$$
$$= \frac{3.2+5}{2+2} = \frac{11}{4}.$$

- **Some Important Formulae**

  Some factorization formulae which we use in finding limit of a function are—

  (i) If $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

  (ii) $a^2 - b^2 = (a - b)(a + b)$

  (iii) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

  (iv) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

  (v) $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a + b)(a - b)$.

(iii) Variable Substitution Method : In this method, we put $x = a + h$ in the function $f(x)$. Here, h is a very small number ($\neq 0$). Therefore, if $x \to a$, then $h \to 0$. Now, simplify the numerator and denominator, and cancel out their common factors ($\because h \neq 0$). Then, the required limit is obtained by putting $h = 0$ in the resulting function.

For example : Find the value of $\lim\limits_{x\to a}\dfrac{x^2-a^2}{x-a}$.

Sol. Let $x = a + h$, so that when $x \to a$, then $h \to 0$. Therefore,
$$\lim_{x\to a}\frac{x^2-a^2}{x-a} = \lim_{h\to 0}\frac{(a+h)^2-a^2}{(a+h)-a}$$
$$= \lim_{h\to 0}\frac{(a^2+2ah+h^2)-a^2}{h}$$
$$= \lim_{h\to 0}\frac{2ah+h^2}{h}$$
$$= \lim_{h\to 0}\frac{h(2a+h)}{h}$$
$$= \lim_{h\to a}(h+2a) = 0 + 2a = 2a.$$

(iv) Rationalization Method : This method is particularly used when either numerator or denominator or both involve expression consisting of square root $(\sqrt{\ })$ and substituting the value of x in the rational expression takes form $\frac{0}{0}, \frac{\infty}{\infty}$ etc.

For example :

Evaluate : $\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

Sol. $\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$
$$= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$
$$= \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}.$$

(v) Standard Substitution Method :

For any positive integer n,

$$\lim_{x\to a} \frac{x^n - a^n}{x-a} = n\,a^{n-1}$$

- The result $\lim_{x\to a} \frac{x^n - a^n}{x-a} = n\,a^{n-1}$ is also true for any rational number 'n' and positive 'a'.

For example : Find $\lim_{x\to 0} \frac{(x+1)^5 - 1}{x}$.

Sol. Let $x + 1 = y$

where $x \to 0, y \to 1$

$$\therefore \lim_{y\to 1} \frac{y^5 - 1}{y-1} \qquad \left[\lim_{y\to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$$

$= 5(1)^{5-1} = 5.$

➢ **Limit of Polynomial Function**

Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$ be a polynomial function.

then,

$$\lim_{x\to a} f(x) = \lim_{x\to a} [a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n]$$

$$= a_0 + a_1 \lim_{x\to a} x + a_2 \lim_{x\to a} x^2 + \ldots\ldots + a_n \lim_{x\to a} x^n$$

$$= a_0 + a_1a + a_2a^2 + \ldots\ldots + a_na^n$$

➢ **Limit at Infinity**

Let a be a constant and x be a variable, then fraction $\frac{a}{x}$ increases as x decreases, and by making x as small as possible, we can make $\frac{a}{x}$ as large as possible; thus when x tends to zero, $\frac{a}{x}$ has no finite value. This is usually by saying "when x tends to zero, the limit is infinite." Symbolically this is expressed as follows :

$$\lim_{x\to 0} \frac{a}{x} = \infty$$

Again, then fraction $\frac{a}{x}$ can be made as small as possible by sufficiently increasing x *i.e.*, we can make $\frac{a}{x}$ approximate to zero as we please taking x large enough. This is usually by saying "when x tends to infinity, the limit of $\frac{a}{x}$ is zero." Symbolically this is expressed as follows :

$$\lim_{x\to \infty} \frac{a}{x} = 0$$

- **Algorithm to find the limit of the function at infinity**

Step 1 : Write down the given expression in the form of rational function, *i.e.*, $\frac{f(x)}{g(x)}$, if it is not so.

Step 2 : If k is the highest power of x in numerator and denominator both, then divide each term in numerator and denominator by x^k.

Step 3 : Use the result $\lim_{x\to\infty} \frac{a}{x^k} = 0$ and $\lim_{x\to\infty} a = a$, where $n > 0$.

For example : Evaluate : $\lim\limits_{x\to\infty}\sqrt{x}(\sqrt{x+c}-\sqrt{x})$.

Sol. The given expression is of the form $\infty - \infty$. So, we first write it in rational form $\frac{f(x)}{g(x)}$.

We have,

$$\lim_{x\to\infty}\sqrt{x}(\sqrt{x+c}-\sqrt{x})$$

$$= \lim_{x\to\infty}\frac{\sqrt{x}(\sqrt{x+c}-\sqrt{x})\sqrt{x}(\sqrt{x+c}+\sqrt{x})}{\sqrt{x}(\sqrt{x+c}+\sqrt{x})}$$

$$= \lim_{x\to\infty}\frac{x(x+c-x)}{\sqrt{x}(\sqrt{x+c}+\sqrt{x})}$$

$$= \lim_{x\to\infty}\frac{cx}{\sqrt{x}(\sqrt{x+c}+\sqrt{x})} \qquad \left(\text{Form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty}\frac{c}{\left(\sqrt{1+\frac{c}{x}}+1\right)} \qquad \text{(Divide Numerator and Denominator by } x\text{)}$$

$$= \frac{c}{(\sqrt{1+0}+1)} = \frac{c}{2}. \qquad \left(\frac{1}{\infty}=0\right)$$

Remark :
- The domain of exponential function $f(x) = e^x$ is $(-\infty, \infty)$ and its range is $(0, \infty)$.
- The domain of logarithmic function $f(x) = \log_e x$ is $(0, \infty)$ and its range is $(-\infty, \infty)$.

➢ **Standard Limits of Trigonometric, Exponential and Logarithmic Functions**
- $\lim\limits_{x\to 0}\sin x = 0$
- $\lim\limits_{x\to 0}\cos x = 1$
- $\lim\limits_{x\to 0}\tan x = 0$
- $\lim\limits_{x\to c}\sin x = \sin c$
- $\lim\limits_{x\to c}\cos x = \cos c$
- $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$
- $\lim\limits_{x\to 0}\frac{\tan x}{x} = 1$
- $\lim\limits_{x\to a}\frac{\sin(x-a)}{(x-a)} = 1$
- $\lim\limits_{x\to a}\frac{\tan(x-a)}{(x-a)} = 1$
- $\lim\limits_{x\to 0}\frac{1-\cos x}{x} = 1$
- $\lim\limits_{x\to 0}e^x = 1$
- $\lim\limits_{x\to c}e^x = e^c$
- $\lim\limits_{x\to 0}a^x = a, a > 0, a \neq 1$
- $\lim\limits_{x\to c}a^x = a^c, a > 0, a \neq 1$
- $\lim\limits_{x\to c}\log x = \log c, c > 0$
- $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$
- $\lim\limits_{x\to 0}\frac{\log(1+x)}{x} = 1$
- $\lim\limits_{x\to 0}\frac{e^x-1}{x} = 1$
- $\lim\limits_{x\to 0}\frac{a^x-1}{x} = \log a, a > 0, a \neq 1$
- $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$
- $\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$

➢ **Continuity**
- A function $f(x)$ is said to be continuous at a point $x = a$ of its domain if, $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = f(a)$, where $\lim\limits_{x\to a^-} f(x)$ is **Left Hand Limit** of $f(x)$ at $x = a$ and $\lim\limits_{x\to a^+} f(x)$ is **Right Hand Limit** of $f(x)$ at $x = a$. Also $f(a)$ is the value of function $f(x)$ at $x = a$.

- A function $f(x)$ is continuous at $x = a$ (say) if, $f(a) = \lim_{x \to a} f(x)$ *i.e.*, a function is continuous at a point in its domain if the **limit value of the function** at that point **equals** the value of the function at the same point.
- For a continuous function $f(x)$ at $x = a$, $\lim_{x \to a} f(x)$ can be directly obtained by evaluating $f(a)$.
- A function which is not continuous is said to be a discontinuous function.

➢ **Algebra of Continuous Functions**

Let f and g be two real functions, continuous at $x = a$. Let α be a real number. Then,

(i) $f + g$ is continuous at $x = a$

(ii) $f - g$ is continuous at $x = a$

(iii) αf is continuous at $x = a$

(iv) fg is continuous at $x = a$

(v) $\frac{1}{f}$ is continuous at $x = a$, provided that $f(a) \neq 0$

(vi) $\frac{f}{g}$ is continuous at $x = a$, provided that $g(a) \neq 0$

➢ **Some Useful Expansions**

Sometimes, the following expansions are useful in evaluating the limits. Students are advised to remember these expansions.

1. $(1+x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \ldots\ldots$

2. $e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \ldots\ldots$

3. $e^{-x} = 1 - x + \frac{x^2}{\underline{|2}} - \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} - \ldots\ldots$

4. $a^x = 1 + x(\log_e a) + \frac{x^2}{\underline{|2}}(\log_e a)^2 + \frac{x^3}{\underline{|3}}(\log_e a)^3 + \ldots\ldots$

5. $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\ldots, |x| < 1.$

6. $\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots\ldots, |x| < 1.$

7. $\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \ldots\ldots$

8. $\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \ldots\ldots$

9. $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots\ldots$

TOPIC-2
Differentiation

Revision Notes

➢ In calculus, **differentiation** is the process of determining the **derivative of a function** at any point regarding the independent variable and can be applied to measure the function per unit change in the independent variable.

Let $y = f(x)$ be a function of x. Here, x is the independent variable and y is the dependent variable. Then, the rate of change of "y" per unit change in "x" is given by $\frac{dy}{dx}$.

When a function is denoted as $y = f(x)$, the derivative is indicated by the following notations.

1. $D(y)$ or $D[f(x)]$ is called Euler's notation.

2. $\frac{dy}{dx}$ is called Leibniz's notation.

3. $f'(x)$ is called Lagrange's notation.

- **Derivative at a Point**

Suppose f is a real valued function and 'a' is a point in its domain. Then, Derivative of f at a is defined by $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$, provided this limit exists.

The derivative of $f(x)$ at a is denoted by $f'(a)$.

For example : Find derivative of function $f(x) = 10x$ at $x = 2$.

Sol. Given, $f(x) = 10x$

By the definition,

$$\begin{aligned} f'(2) &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0}\frac{10(2+h)-10(2)}{h} \\ &= \lim_{h\to 0}\frac{20+10h-20}{h} \\ &= \lim_{h\to 0}\frac{10h}{h} \\ &= \lim_{h\to 0}10 = 10. \end{aligned}$$

Remark :

- Derivative of f at $x = a$ is also given by substituting $x = a$ in $f'(x)$ and it is denoted by $\left.\dfrac{d}{dx}f(x)\right|_a$ *or* $\left.\dfrac{d}{dx}\right|_a$ or $\left(\dfrac{df(x)}{dx}\right)_{x=a}$.

- **Geometrical meaning of Derivative at a point**

Geometrically, derivative of a function at a point x = the slope of tangent to the curve $y = f(x)$ at the point $c, f(c)$.

$$\begin{aligned} \text{Slope of tangent at } P &= \lim_{x\to c}\frac{f(x)-f(c)}{x-c} \\ &= \left\{\frac{df(x)}{dx}\right\}_{x=c} \text{ or } f'(c) \end{aligned}$$

- **Derivative using First Principle**

Suppose f is a real valued function, the function defined by $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$, wherever the limit exists and is defined to be the derivative of f and is denoted by $f'(x)$. This definition of derivative is called the first principle of derivative. Thus, $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

$f'(x)$ is also denoted by $\dfrac{d}{dx}[f(x)]$ or if $y = f(x)$, then it is denoted by $\dfrac{dy}{dx}$ and referred to as derivative of $f(x)$ or y with respect to x.

For example : Find derivative of function $f(x) = \dfrac{1}{x^2}$ using first principle.

Sol. Given,

$$f(x) = \frac{1}{x^2}$$

$\therefore$

$$\begin{aligned} \frac{df(x)}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\left\{\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right\} \\ &= \lim_{h\to 0}\frac{1}{h}\left\{\frac{x^2-x^2-2xh-h^2}{x^2(x+h)^2}\right\} \end{aligned}$$

$$= \lim_{h\to 0}\frac{1}{h}\left\{\frac{-h(2x+h)}{x^2(x+h)^2}\right\}$$

$$= \lim_{h\to 0}\frac{-2x-h}{x^2(x+h)^2} \qquad [x \neq 0]$$

$$= \frac{-2x}{x^2.x^2}$$

$$= \frac{-2}{x^3}.$$

- **Algebra of Derivatives of Functions**

Let f and g be two function such that their derivatives are defined in a common domain. Then,

(i) Derivative of sum of two functions is sum of the derivatives of the functions.

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

or
$$(u+v)' = u' + v'$$

(ii) Derivative of difference of two function is difference of the derivative of the functions.

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x)-\frac{d}{dx}g(x)$$

or
$$(u-v)' = u' - v'$$

(iii) Derivative of product of two functions is given by the following product rule.

$$\frac{d}{dx}[f(x).g(x)] = \left[\frac{d}{dx}f(x)\right]g(x)+f(x)\left[\frac{d}{dx}g(x)\right]$$

or
$$(u.v)' = u'.v + v.u'$$

(iii) Derivative of quotient of two functions is given by the following quotient rule.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{[g(x)]^2},\ g(x)\neq 0$$

or
$$\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$$

- **Differentiation using Chain Rule**

This rule plays a major role in finding the differentiation of two or more composite functions. If a function $z = f(y)$ and $y = g(x)$, then the chain rule for differentiation is defined as,

$$\frac{dz}{dx} = \frac{dz}{dy}\times\frac{dy}{dx}$$

OR

Derivative of z with respect to x = (Derivative of z with respect to y)× (Derivative of y with respect to x)

Remark :

This chain rule can be extended further as :

If $z = f(u)$, $u = g(v)$ and $x = v(x)$, then the chain rule for differentiation is defined as,

$$\frac{dz}{dx} = \frac{dz}{du}\times\frac{du}{dv}\times\frac{dv}{dx}$$

OR

Derivative of z with respect to x = (Derivative of z with respect to u) × (Derivative of u with respect to v) × (Derivative of v with respect to x)

For example : Find derivative of function $y = (x^2 + x + 1)^4$ with respect to x.

Sol. Given, $y = (x^2 + x + 1)^4$

Put $(x^2 + x + 1) = u$

Then, $y = u^4$ and $u = (x^2 + x + 1)$

$\therefore$ $\frac{dy}{du} = 4u^3$ and $\frac{du}{dx} = 2x+1$

Now, $$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$$

$\Rightarrow$ $$\frac{dy}{dx} = 4u^3 \times (2x + 1) = 4(x^2 + x + 1)^3 (2x + 1).$$

- **Differentiation of Parametric Functions**

Sometimes x and y are given as functions of a single variable e.g. $x = f(t)$ and $y = g(t)$ are two functions of a single variable. In such cases, x and y are called parametric functions or parametric equations and t is called the parameter.

To find $\frac{dy}{dx}$ in case of parametric functions, we first obtain relationship between x and y by eliminating the parameter t and then we differentiate with respect to x. but, it not always convenient to eliminate the parameter. Therefore, $\frac{dy}{dx}$ can also be obtained by the following formula:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dt}{dx}}$$

For example : Find the value of $\frac{dy}{dx}$ for the following functions which are expressed in the parametric form : $x = 3t^3, y = 3t^4 + 5$.

Sol. Given, $x = 3t^3, y = 3t^4 + 5$

$\therefore$ $$\frac{dx}{dt} = 9t^2 \text{ and } \frac{dy}{dt} = 12t^3$$

We know that,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{12t^3}{9t^2} = \frac{4t}{3}.$$

- **Differentiation of Implicit Functions**

If the variables x and y are connected by a relation of the form $f(x, y) = 0$ and it is not possible to express y as the function of x in the form $y = \phi(x)$, then y is said to be an implicit function of x. To find $\frac{dy}{dx}$ in such case, we differentiate both sides of the given relation with respect to x, keeping in mind that the derivative of $\phi(y)$ with respect to x is $\frac{d\phi}{dy}.\frac{dy}{dx}$.

For example : If $x^2+2xy + y^3 = 42$, find $\frac{dy}{dx}$.

Sol. Given, $x^2 + 2xy + y^3 = 42$

Differentiating both sides of this respect to x, we get

$$\frac{d}{dx}(x^2)+2\frac{d}{dx}(xy)+\frac{d}{dx}(y^3) = \frac{d}{dx}(42)$$

$\Rightarrow$ $$2x+2\left(x\frac{dy}{dx}+y\right)+3y^2\frac{dy}{dx} = 0$$

$\Rightarrow$ $$2x+2y+\frac{dy}{dx}(2x+3y^2) = 0$$

$\Rightarrow$ $$\frac{dy}{dx}(2x+3y^2) = -2(x+y)$$

$\Rightarrow$ $$\frac{dy}{dx} = -\frac{2(x+y)}{(2x+3y^2)}.$$

- **Logarithmic Differentiation**

The method of differentiating functions of the forms $[f(x)]^{g(x)}$ by first taking logarithms and then differentiating is called logarithmic differentiation. We use logarithmic differentiation in situations where it is easier to differentiate the logarithm of a function than to differentiate the function itself. This approach allows calculating derivatives of power, rational and some irrational functions in an efficient manner.

For example : Differentiate $(\cos x)^x$ with respect to x.

Sol. Let, $y = (\cos x)^x$

Then, $y = e^{\log(\cos x)^x} = e^{x \log \cos x}$

$$\Rightarrow \quad \frac{dy}{dx} = e^{x\log\cos x} \times \frac{d}{dx}(x\log\cos x)$$

$$\Rightarrow \quad \frac{dy}{dx} = (\cos x)^x\{\log\cos x\frac{d}{dx}(x) + x\frac{d}{dx}(\log\cos x)\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\cos x)^x\{\log\cos x + x\frac{1}{\cos x}(-\sin x)\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\cos x)^x\{\log\cos x - x\tan x\}.$$

- **Differentiation of a Function with respect to another Function**

Let $u = f(x)$ and $v = g(x)$ be two functions of x. Then, to find the derivative of $f(x)$ with respect to $g(x)$ *i.e.*, to find $\frac{du}{dv}$, we use the following formula

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

Thus, to find the derivative of $f(x)$ with respect to $g(x)$, we first differentiate both with respect to x and then divide the derivative of $f(x)$ with respect to x by the derivative of $g(x)$ with respect to x.

For example : Differentiate x^x with respect to $x \log x$.

Sol. Let $u = x^x$ and $v = x \log x$. Then,

$$u = x^x = e^{\log x^x} = e^{x\log x} \text{ and } v = x\log x.$$

$$\Rightarrow \quad \frac{du}{dx} = e^{x\log x} \times \frac{d}{dx}(x\log x) \text{ and } \frac{dv}{dx} = x\times\frac{1}{x} + 1\times\log x$$

$$\Rightarrow \quad \frac{du}{dx} = x^x(1+\log x) \text{ and } \frac{dv}{dx} = 1+\log x$$

$$\Rightarrow \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{x^x(1+\log x)}{1+\log x} = x^x.$$

- **Some Important Derivatives**

  - If $f(x) = c$, then $\frac{d}{dx}(c) = 0$ for all $x \in R$
  - $\frac{d}{dx}(1) = 0$
  - $\frac{d}{dx}(x^n) = nx^{n-1}$
  - $\frac{d}{dx}(x) = 1$
  - $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$, c is a constant
  - $\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}$
  - If $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots\ldots\ldots\ldots + a_1x + a_0$,
  then $f'(x) = na_n x^{n-1} + (n-1)\, a_{n-1} x^{n-2} + (n-2)\, a_{n-2} x^{n-3} + \ldots\ldots\ldots\, a_1$

- $\frac{d}{dx}(a^x) = a^x \log_e a$
- $\frac{d}{dx}(\log_e x) = \frac{1}{x}$
- $\frac{d}{dx}(e^x) = e^x$

TOPIC-3
Instantaneous Rate of Change

Revision Notes

➢ The derivative tell us the rate of change of one quantity compared to another at particular instant or point (So, we call it "instantaneous rate of change"). This concept has many applications in electricity, economics, fluid flow, population modelling, queueing theory and so on.

Let $y = f(x)$ be a function of x. Let Δy be the change in corresponding to small change Δx in x. Then, $\frac{\Delta y}{\Delta x}$ represents the change in y due to a unit change in x. In other words, $\frac{\Delta y}{\Delta x}$ represents the average rate of change of y with respect to x as x changes from x to $x + \Delta x$.

As $\Delta x \to 0$, the limiting value of this average rate of change of y with respect to x in the interval $[x, x + \Delta x]$ becomes the instantaneous rate of change of y with respect to x.

Thus, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ = Instantaneous rate of change of y with respect to x

$\Rightarrow$ $\frac{dy}{dx}$ = Rate of change of y with respect to x $\left[\because \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}\right]$

The word "instantaneous" is often dropped.

Hence, $\frac{dy}{dx}$ represents the rate of change of y w.r.t x for a definite value of x.

Remark :

- The value of $\frac{dy}{dx}$ at $x = x_0$ *i.e.*, $\left(\frac{dy}{dx}\right)_{x=x_0}$ represent the rate of change of y w.r.t x at $x = x_0$.
- If $x = f(t)$ and $y = g(f)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, provided $\frac{dx}{dt} \neq 0$

  Thus, the rate of change of y w.r.t. 'x' can be calculated by using the rate of change of y and that of x each w.r.t. 't'.

UNIT – V: PROBABILITY

CHAPTER-10

PROBABILITY INTRODUCTION

Revision Notes

➢ In daily life, we use words having similar meaning as the word probability. For instance, there is possibility of rain today; perhaps I may pass the examination this year; there is an expectation of good crop this year, etc. All these sentences suggest uncertainty. The mathematical measure of this uncertainty is called **probability**. It has found a very extensive applications in development of physical sciences. In 1993, A.H. Kolmogrov, a Russian Mathematician, tried successfully to relate the theory of probability with the set theory. We explain some terms before actually discussing the probability.

- **Experiment**

An operation which can produce some well-defined outcomes, is known as experiment. There are two types of experiments. These are :

(i) Deterministic experiment and

(ii) Random experiment.

- **Random experiment**

An experiment conducted repeatedly under the identical conditions does not give necessarily the same result every time, then the experiment is called random experiment. For example : rolling an unbiased die, drawing a card from a well shuffled pack of cards, etc.

- **Outcomes and Sample Space**

A possible result of a random experiment is called its outcome. The set of all possible outcomes in a random experiment is called sample space and is denoted by S *i.e.*, sample space = {All possible outcomes}.

Each element of a sample space is called a sample point or an event point.

For example : When we throw a die, then possible outcomes of this experiment are 1, 2, 3, 4, 5 or 6.

and sample space, S = {1, 2, 3, 4, 5, 6}.

- **Event**

A subset of the sample space associated with a random experiment is called an event, generally denoted by 'E'.

An event associated with a random experiment is said to occur, if any one of the elementary events associated to it is an outcome of the experiment.

For example : Suppose a die is thrown, then we have the sample space S = {1, 2, 3, 4, 5, 6}. Then,

E = {2, 3, 4} is an event.

Also, if the outcome of experiment is 4. Then, we say that event E has occurred.

- **Type of events :**

On the basis of the element in an event, events are classified into the following types :

(i) Simple event : If an event has only one sample point of the sample space, it is called a simple (elementary) event.

For example : Let a die is thrown, then sample space,

$$S = \{1, 2, 3, 4, 5, 6\}$$

Then, $A = \{4\}$ and $B = \{6\}$ are simple events.

(ii) Compound event : If an event has more than one sample point of the sample space, then it is called compound event.

For example : On rolling a die, we have the sample space.

$$S = \{1, 2, 3, 4, 5, 6\}$$

Then, $E = \{2, 4, 6\}$, F = the event of getting an odd number are compound events.

(iii) Sure event : The event which is certain to occur is said to be the sure event. The whole sample space 'S' is a sure or certain event, since it is a subset of itself.

For example : On throwing a die, we have sample space,

$$S = \{1, 2, 3, 4, 5, 6\}$$

Then, E = Event of getting a natural number less than 7, is a sure event, since $E = \{1, 2, 3, 4, 5, 6\}$ = S.

(iv) Impossible event : The event which has no element is called an impossible event or null event. The empty set 'ϕ' is an impossible event, since it is a subset of sample space S.

For Example : On throwing a die, we have the sample space

$$S = \{1, 2, 3, 4, 5, 6\}$$

Then E = event of getting a number less than 1, is an impossible event, since $E = \phi$.

(v) Equally likely events : Events are called equally likely when we do expect the happening of one event in preference to the other.

(vi) Independent Events : If two events E and F are such that the occurrence of one of them does not depend on the occurrence of other, then the events E and F are called **independent events**. But if the events E and F are not independent of each other then the events E and F are called **dependent events.**

For example : If on tossing a coin the event of *head* coming up be E, and on tossing the coin again the event of tail coming up be F, then E and F are independent of each other, because head coming up first time does not affect the coming of tail next time. Thus, here E and F are independent events.

Consider another example, if on drawing a card from a pack of 52 cards, the event of it being a red card be E and without replacing the drawn card into the pack another card be drawn and the event of its being a red be F, then the events E and F are not independent, because at the time of drawing the second there will be 51 cards in the pack. Clearly, the chances of the first and second card being red will be different.

(vii) Mutually exclusive events : Two events are said to be mutually exclusive, if the occurrence of any one of them excludes the occurrence of the other event *i.e.*, they cannot occur simultaneously.

Thus, two events E_1 and E_2 are said to be mutually exclusive, if $E_1 \cap E_2 = \phi$.

For example : In throwing a die, we have the sample space

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let, E_1 = Event of getting even numbers = $\{2, 4, 6\}$

and E_2 = Event of getting odd numbers = $\{1, 3, 5\}$

Then, $E_1 \cap E_2 = \phi$

So, E_1 and E_2 are mutually exclusive events.

In general, events $E_1, E_2, \ldots\ldots\ldots, E_n$ are said to be mutually exclusive, if they are pair-wise disjoint, *i.e.*, if $E_1 \cap E_2 = \phi \forall ij$.

(viii) Exhaustive events : A set of events is said to be exhaustive, if the performance of the experiment always results in the occurrence of at least one of them.

Let $E_1, E_2, \ldots\ldots\ldots E_n$ be n subset of a sample space S. Then, events $E_1, E_2, \ldots\ldots\ldots E_n$ are exhaustive events, if $E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n = S$.

For example : Consider the experiment of throwing a die. Then,

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let E_1 = event of getting a number less than 3.

E_2 = event of getting an odd number.

E_3 = event of getting a number greater than 3.

Then, $E_1 = \{1, 2\}, E_2 = \{1, 3, 5\}, E_3 = \{4, 5, 6\}$

Thus, $E_1 \cup E_2 \cup E_3 = S$. Hence, E_1, E_2, E_3 are exhaustive events.

➢ **Algebra of events**

Let A and B be two events associated with a sample space S, then—

(i) Complementary event : For every E, there corresponds another event E' called the complementary event of E, which consists of those outcomes that do not correspond to the occurrence of E. E' is also called the 'event E''.

For example : In tossing three coins, the sample space is

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

Let E = {THT, TTH, HTT} = the event of getting only one head.

Then, Complement event E' = {HHT, HTH, THH, TTT, HHH}.

(ii) The event *A* OR *B* : The event 'A or B' is same as the event $A \cup B$ and it contains all those element which are either in event A or in B or in both. Thus,

$$A \text{ or } B = A \cup B = \{x : x \in A \text{ or } x \in B\}$$

(iii) The event *A* and *B* : The event A and B is same as the event '$A \cap B$' and it contains all those elements which are both in A and B. Thus,

$$A \text{ and } B = A \cap B = \{x : x \in A \text{ and } x \in B\}$$

(iv) The event *A* but not *B* : The event A but not B is same as the event. $A - B = (A \cap B')$ and it contains all those elements which are in A but not in B.

Thus, A but not in $B = A - B = \{x : x \in A \text{ and } x \notin B\}$.

➢ **Some Important Facts**

- A sample space is called a discrete sample space, if S is a finite set.
- We can define as many events as there are subsets of a sample space. Thus, number of events of a sample space S is 2^n, where 'n' is the number of elements in S.
- Elementary events associated with a random experiments are also known as in decomposable events.
- All events other than elementary events and impossible events associated with a random experiment are called compound events.
- For any event E, associated with a sample space S, E' = not E = S – E = $\{\omega : \omega \in S \text{ and } \omega \notin E\}$.

- Simple events of a sample space are always mutually exclusive.
- If $E_1 \cap E_2 = \phi$ for $i \neq j$ *i.e.*, events E_i and E_j are pair-wise disjoint and $E_1 \cup E_2 \cup E_n = S$, then events $E_1, E_2, \ldots, E_n$ are called mutually exclusive and exhaustive events.
- Complementary events E and E' are always mutually exclusive.

➢ **Probability of occurrence of an event.**

A numerical value that conveys the chance of occurrence of an event, when we perform an experiment, is called the probability of that event. The different approaches of probability are :

(i) Statistical Approach to Probability : In statistical approach, probability of an event 'A' is the ratio of observed frequency to the total frequency.

i.e.,
$$P(A) = \frac{\text{Number of observed frequencies}}{\text{Total frequency}}$$

(ii) Classical approach to Probability : To obtain the probability of an event, we find the ratio of the number of outcomes favourable to the event to the total number of equally likely outcomes. This theory is known as classical theory of probability or theoretical probability.

i.e.,
$$P(A) = \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}}$$

(iii) Axiomatic Approach to Probability : Let 'S' be the sample space of a random experiment. The probability P is a real valued function whose domain is the power set of S and range is the interval [0, 1] satisfying the following axioms.

(a) For any event E, $P(E) \geq 0$.

(b) $P(S) = 1$.

(c) If E and F are mutually exclusive, then

$$(E \cup F) = P(E) + P(F)$$

Let 'S' be a sample space containing outcomes.

$$E_1, E_2, \ldots E_n \text{ i.e., } S = \{E_1, E_2, \ldots E_n\}$$

Then, from the axiomatic approach to probability, we have—

(i) $0 \leq P(E_i) \leq 1, \forall E_i \in S$.

(ii) $P(E_1) + P(E_2) + \ldots\ldots + P(E_n) = 1$.

(iii) For any event A, $P(A) = \sum_{i=1}^{n} P(E_i), E_i \in A$.

➢ **Probability of Equally Likely Outcomes**

The outcomes of a random experiment are said to be equally likely, if the chance of occurrence of each outcome is same.

Let the sample space of an experiment is—

$S = \{s_1, s_2 \ldots\ldots\ldots, s_n\}$. Also, let the all the outcomes are equally likely. *i.e.*, $P(s_i) = p \ \forall \ s_i \in S, 0 \leq p \leq 1$.

By Axiomatic approach to probability,

$$\sum_{i=1}^{n} P(s_i) = 1.$$

$$\Rightarrow \quad \underbrace{p + p + \ldots + p}_{n \text{ times}} = 1$$

$$\Rightarrow \quad np = 1$$

$$\Rightarrow \quad p = \frac{1}{n}$$

$$\therefore \quad P(s_i) = \frac{1}{n} = 1, 2, \ldots, n.$$

➢ **Addition Rule of Probability**

If A and B are two events associated with a random experiment, then—

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

or $$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

It is known as addition law of probability for two events A and B.

When A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

When A and B are mutually exclusive and exhaustive events, then $P(A \cup B) = P(A) + P(B) = 1$.

- **Probability of the Event *A* or *B* or *C***

  If A, B and C are three events associated with a random experiment, then—

  $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$.

- **Probability of Independent :**

  Let E and F be the two events associated with the same random experiment. If E and F are independent events, the probability of these events are given by

  $$P(E \cap F) = P(E) \,.\, P(F)$$

  Remark :

  Sometimes, there is a confusion between independent events and mutually exclusive events. Term 'independent' is defined in terms of 'probability of events' whereas mutually exclusive is defined in term of events (subset of sample space). Moreover, mutually exclusive events never have an outcome common, but independent events, may have common outcome.

- **Probability of Complementary Event**

  Let E be an event and $(\bar{E})$ be its complementary event.

  Then, $$P(\bar{E}) = 1 - P(E).$$

  Remarks :

  - For any two events A and B, $A \leq B \Rightarrow P(A) \leq P(B)$
  - For an event A, $0 \leq P(A) \leq 1$
  - Probability of an impossible or null event is zero *i.e.*, $P(\phi) = 0$.
  - If A, B and C are mutually exclusive events, *i.e.*, $A \cap B = \phi$, $B \cap C = \phi$, $C \cap A = \phi$ and $A \cap B \cap C = \phi$, then $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

- **Important Formulae**

  - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ *i.e.*, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
  - $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$
  - $P(\bar{A} \cap B) = P(\text{only } B) = P(B - A) = P(B \text{ but not } A) = P(B) - P(A \cap B)$
  - $P(A \cap \bar{B}) = P(\text{only } A) = P(A - B) = P(A \text{ but not } B) = P(A) - P(A \cap B)$
  - $P(\bar{A} \cap \bar{B}) = P(\text{neither } A \text{ nor } B) = 1 - P(A \cup B) = P(\overline{A \cup B})$
  - $P(A \cap B) \leq P(A)$ and $P(A \cap B) \leq P(B)$

- **Events and Symbolic Representations**

Verbal description of the event	Equivalent Set
A or B (occurrence of atleast on A or B)	$A \cup B$ or $A + B$
A and B (simultaneous occurrence of both A and B)	$A \cap B$ or $A - B$
A but not B (A occurs but B does not)	$A \cap \bar{B}$ or $A - B$
Neither A nor B	$\bar{A} \cap \bar{B}$ or U $-$ $(A \cap B)$ [U-Universal Set]
Exactly one of A and B	$(A \cap \bar{B}) \cup (\bar{A} \cap B)$
Atleast one of A, B and C	$A \cup B \cup C$
All three of A, B and C	$A \cap B \cap C$
Exactly two of A, B and C	$(A \cap B \cap \bar{C}) \cup (A \cap \bar{B} \cap C) \cup (\bar{A} \cap B \cap C)$

CHAPTER-11

PROBABILITY APPLICATIONS

Revision Notes

In the previous chapter, we have studied some basic terms related to probability, addition theorem of probability etc. In continuation of these, we will introduce the concept of conditional probability which will be useful in obtaining multiplication rule of probability, total law of probability and Bayes' theorem in this chapter.

➢ **Conditional Probability**

By the conditional probability, we mean the probability of occurrence of event A, when event B has already occurred. The 'conditional probability of event A, when event B has already occurred' is sometimes also called as probability of event A w.r.t. B.

Let A and B be two events associated with a random experiment. Then, $P(A/B)$ denotes the conditional probability of occurrence of A given that B has already occurred. Here, $P(B) \neq 0$.

Similarly, $P\left(\frac{B}{A}\right)$, when $P(A) \neq 0$ is defined as the conditional probability of occurrence of event B, when A has already occurred.

For example : Let there be a bag containing 5 white and 4 red balls. Two balls are drawn from bag one after the other with replacement.

Consider the following events.

A = Drawing a white ball in first draw

B = Drawing a red ball in second draw

Then, $P\left(\frac{B}{A}\right)$ = Probability of drawing a red ball in second draw given that a white ball has already drawn in the first draw.

$\Rightarrow$ $P\left(\frac{B}{A}\right)$ = Probability of drawing a red ball from a bag containing 4 white and 4 red balls

$$\Rightarrow \quad P\left(\frac{B}{A}\right) = \frac{4}{8} = \frac{1}{2}$$

- Conditional probability is obtained by using the following formula :

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) \neq 0$$

and

$$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0$$

- Also, if A and B are independent events, then

$$P(A \cap B) = P(A).P(B)$$

Hence,

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{P(A).P(B)}{P(B)} = P(A)$$

and

$$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{P(A).P(B)}{P(A)} = P(B)$$

➢ **Properties of Conditional Probability**

- If A and B are two events associated with sample space S, then $0 \leq P\left(\frac{A}{B}\right) \leq 1$.

- If A is an event associated with the sample space S of a random experiment, then

$$P\left(\frac{S}{A}\right) = P\left(\frac{A}{A}\right) = 1$$

- If A and B are two events associated with a random experiment and S is the sample space. If C is an event such that $P(C) \neq 0$, then

$$P\left[\frac{(A \cup B)}{C}\right] = P\left(\frac{A}{C}\right) + P\left(\frac{B}{C}\right) - P\left[\frac{(A \cap B)}{C}\right]$$

In particular, if A and B are mutually exclusive events, then $P(A \cap B) = 0$.

$$\therefore \qquad P\left[\frac{(A \cup B)}{C}\right] = P\left(\frac{A}{C}\right) + P\left(\frac{B}{C}\right)$$

- If A and B are two events associated with a random experiment, then

(i) $P\left(\frac{\bar{A}}{B}\right) = P\frac{(\bar{A} \cap B)}{P(B)}, \; P(B) \neq 0$

(ii) $P\left(\frac{A}{\bar{B}}\right) = P\frac{(A \cap \bar{B})}{P(\bar{B})}, \; P(\bar{B}) \neq 0$

(iii) $P\left(\frac{\bar{A}}{\bar{B}}\right) = P\frac{(\bar{A} \cap \bar{B})}{P(\bar{B})}, \; P(\bar{B}) \neq 0$

(iv) $P\left(\frac{A}{B}\right) + P\left(\frac{\bar{A}}{B}\right) = 1, \; P(B) \neq 0$

For example : A fair die is rolled. Consider the events
$A = \{1, 3, 5\}$, $B = \{2, 3\}$ and $C = \{2, 3, 4, 5\}$. Then,
$n(A) = 3$, $n(B) = 2$, $n(C) = 4$, $n(S) = 6$

Also, $P(A) = \frac{3}{6} = \frac{1}{2}$, $P(B) = \frac{2}{6} = \frac{1}{3}$, $P(C) = \frac{4}{6} = \frac{2}{3}$

$P(A \cap B) = \frac{1}{6}$, $P(A \cap C) = \frac{2}{6} = \frac{1}{3}$, $P(B \cap C) = \frac{2}{6} = \frac{1}{3}$

$P(A \cap B \cap C) = \frac{1}{6}$ and $P(A \cup B) = \frac{4}{6} = \frac{2}{3}$

Hence,
$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}$$

$$P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

$$P\left(\frac{A}{C}\right) = \frac{P(A \cap C)}{P(C)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$$P\left(\frac{C}{A}\right) = \frac{P(A \cap C)}{P(A)} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

$$P\left(\frac{B}{C}\right) = \frac{P(B \cap C)}{P(C)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$$P\left[\frac{(A \cap B)}{C}\right] = \frac{P[(A \cap B) \cap C]}{P(C)} = \frac{P(A \cap B \cap C)}{P(C)} = \frac{\frac{1}{6}}{\frac{2}{3}} = \frac{1}{4}$$

$$P\left[\frac{(A \cup B)}{C}\right] = P\left(\frac{A}{C}\right) + P\left(\frac{B}{C}\right) - P\left[\frac{(A \cap B)}{C}\right]$$
$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}.$$

➢ **Multiplication Theorems on Probability**

Theorem 1 : If A and B are two events associated with a random experiment, then

$$P(A \cap B) = P(B).P\left(\frac{A}{B}\right), \text{ provided } P(B) \neq 0$$

and
$$P(A \cap B) = P(A).P\left(\frac{B}{A}\right), \text{ provided } P(A) \neq 0$$

Theorem 2 : (Extension of Multiplication Theorem) If $A_1, A_2, A_3, \ldots\ldots, A_n$ are n events associated with a random experiment, then

$$P(A_1 \cap A_2 \cap A_3 \cap \ldots\ldots \cap A_n) = P(A_1).P\left(\frac{A_2}{A_1}\right).P\left(\frac{A_3}{A_1 \cap A_2}\right) \ldots\ldots P\left(\frac{A_n}{A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_{n-1}}\right)$$

where, $P\left(\frac{A_i}{A_1 \cap A_2 \ldots \cap A_{i-1}}\right)$ represents the conditional probability of the occurrence of event A_i, given that A_1, $A_2 \ldots\ldots A_{i-1}$ have already occurred.

Particular case : If A, B and C are three events associated with a random experiment, then

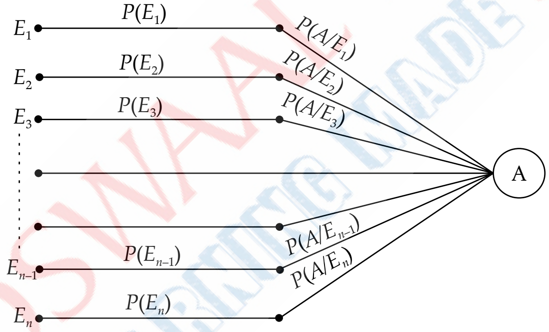
$$P(A \cap B \cap C) = P(A).P\left(\frac{B}{A}\right).P\left(\frac{C}{A \cap B}\right)$$

➢ **The Law of Total Probability**

Let S be the sample space and let $E_1, E_2, \ldots.. E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or .... or E_n then

$$P(A) = P(E_1).P\left(\frac{A}{E_1}\right) + P(E_2).P\left(\frac{A}{E_2}\right) + \ldots + P(E_n).P\left(\frac{A}{E_n}\right)$$

The tree diagram of Law of Total Probability is shown below :



For example : One bag contains 4 white and 5 black balls. Another bag contains 6 white and 7 black balls. A ball is transferred from first bag to second bag and then a ball is drawn from the second bag. Find the probability that the ball drawn is white.

Sol. White ball from second bag can be drawn in two ways.

(1) By transferring a white ball from first bag to second bag and then drawing a white ball from second bag.

(2) By transferring a black ball from first bag to second bag and then drawing a white ball from second bag.

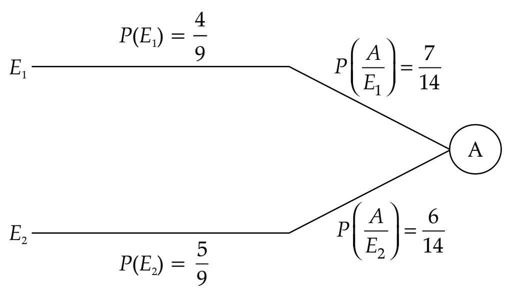
Both the above ways are mutually exclusive.

Let, E_1 = a white ball is transferred from I bag to II bag

E_2 = a black ball is transferred from I bag to II bag

Since, the first bag contains 4 white and 5 black balls

$\therefore P(E_1) = \frac{4}{9}$ and $P(E_2) = \frac{5}{9}$



If E_1 has already occurred *i.e.*, a white ball has already transferred from first bag to the second bag, then the second bag contains 7 white and 7 black balls.

Therefore, $$P\left(\frac{A}{E_1}\right) = \frac{7}{14}$$

Similarly, if E_2 has already occurred *i.e.*, a black ball has already transferred from first bag to the second bag, then second bag contains 6 white and 8 black balls.

Therefore, $$P\left(\frac{A}{E_2}\right) = \frac{6}{14}$$

Now, by the law of total probability, we get

Probability that the ball is drawn white *i.e.*,

$$\begin{aligned} P(\text{Getting a white ball}) &= P(A) \\ &= P(E_1).P\left(\frac{A}{E_1}\right) + P(E_2).P\left(\frac{A}{E_2}\right) \\ &= \frac{4}{9}\times\frac{7}{14}+\frac{5}{9}\times\frac{6}{14} \\ &= \frac{28}{126}+\frac{30}{126} = \frac{58}{126} = \frac{29}{63} \end{aligned}$$

➢ **Bayes' Theorem**

Let S be the sample space and let $E_1, E_2, ..., E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A be any event which occurs with E_1 or E_2 or E_3 or ... or E_n then

$$P\left(\frac{E_i}{A}\right) = \frac{P(E_i).P\left(\frac{A}{E_i}\right)}{\sum_{i=1}^{n} P(E_i).P\left(\frac{A}{E_i}\right)},\ i = 1, 2, ..., n$$

In particular, $$P\left(\frac{E_1}{A}\right) = \frac{P(E_1).P\left(\frac{A}{E_1}\right)}{P(E_1).P\left(\frac{A}{E_1}\right)+P(E_2).P\left(\frac{A}{E_2}\right)+P(E_3).P\left(\frac{A}{E_3}\right)}$$

Remark :

- Bayes' theorem is also known as the formula for the probability of causes.
- If $E_1, E_2, E_3, ..., E_n$ form a partition of S and A be any event, then

  $$P(A) = P(E_1).P\left(\frac{A}{E_1}\right)+P(E_2).P\left(\frac{A}{E_2}\right)+...+P(E_n).P\left(\frac{A}{E_n}\right) \qquad \left[\because P(E_i \cap A) = P(E_i).P\left(\frac{A}{E_i}\right)\right]$$

- The probabilities $P(E_1), P(E_2), ..., P(E_n)$ which are taken before the experiment takes place are called **prior probabilities** and $P\left(\frac{A}{E_1}\right), P\left(\frac{A}{E_2}\right) ... P\left(\frac{A}{E_n}\right)$ are called **posterior probabilities.**

For example : A company has two plants of manufacturing scooters. Plant I manufactures 70% of the scooters and plant II manufactures 30%. At Plant I, 80% of the scooters are rated as of standard quality and at Plant II, 90% of the scooters are rated as of standard quality. A scooter is chosen at random and is found to be of standard quality. What is the probability that it has come from plant II ?

Sol. Let E_1, E_2 and A be the following events :

E_1 = Plant I is chosen

E_2 = Plant II is chosen

A = Scooter is of standard quality

Then, $$P(E_1) = \frac{70}{100},\ P(E_2) = \frac{20}{100}$$

and $$P\left(\frac{A}{E_1}\right) = \frac{80}{100},\ P\left(\frac{A}{E_2}\right) = \frac{90}{100}$$

By Bayes' Theorem,

Probability that a standard quality scooter has chosen from plant II, *i.e.*,

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2).P\left(\frac{A}{E_2}\right)}{P(E_1).P\left(\frac{A}{E_1}\right)+P(E_2).P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{30}{100}\times\frac{90}{100}}{\frac{70}{100}\times\frac{80}{100}+\frac{30}{100}\times\frac{90}{100}}$$

$$= \frac{27}{56+27} = \frac{27}{83}.$$

UNIT – VI: DESCRIPTIVE STATISTICS

CHAPTER-12

DATA INTERPRETATION

TOPIC-1
Central Tendency and Measures of position

Revision Notes

➢ **Measures of position**

There are three measures of position given as follows :

- **Quartiles :** It is a partition measure, The quartiles divide the distribution in four parts. There are three quartiles denoted by Q_1, Q_2, and Q_3 divides the frequency distribution to four equal parts. The **first quartile, Q_1** or **lower quartile** has 25% of the items of the distribution below it and 75% of the items are greater than it. The **second quartile, Q_2 or median** has 50% of items below it and 50% of observations above it. The **third quartile, Q_3** or **upper Quartile** has 75% of the items of the distribution of the items of the distribution below it and 25% of the items above it.

25% of data	Q_1 ↓	25% of data	Q_2 ↓ ↑ Median = Q_2	25% of data	Q_3 ↓	25% of data

The upper and lower quartile (Q_1 and Q_3) are used to calculate the inter quartile range.

i.e., Interquartile Range = $Q_3 - Q_1$

(a) For ungrouped data

If the data set consist of n items and arranged in ascending order, then

$$Q_1 = \left(\frac{n+1}{4}\right)^{th} \text{ item, } Q_2 = \left(\frac{n+1}{2}\right)^{th} \text{ item and } Q_3 = 3\left(\frac{n+1}{4}\right)^{th} \text{ item}$$

For example : Compute the first and third quartiles of the data of the marks obtained by ten students in an examination :

25, 18, 30, 8, 15, 5, 10, 35, 40, 45

Sol. Arranging the data in ascending order

5, 8, 10, 15, 18, 25, 30, 35, 40, 45

$$Q_1 = \left(\frac{n+1}{4}\right)^{th} \text{ item} = \left(\frac{10+1}{4}\right)^{th} \text{ item} = (2.75)^{th} \text{ item}$$

$$= 2^{nd} \text{ item } + 0.75\,(3^{rd} \text{ item} - 2^{nd} \text{ item})$$

$$= 2^{nd} \text{ item} + \frac{3}{4} (3^{rd} \text{ item} - 2^{nd} \text{ item})$$

$$= 8 + \frac{3}{4} (10 - 8)$$

$$= 8 + \frac{3}{4} \times 2 = 8 + \frac{3}{2} = 8 + 1.5 = 9.5$$

and $$Q_3 = 3\left(\frac{n+1}{4}\right)^{th} \text{ item}$$

$$= 3\left(\frac{10+1}{4}\right)^{th} \text{ item} = 3 \times (2.75)^{th} \text{ item}$$

$$= (8.25)^{th} \text{ item}$$

$$= 8 \text{ item} + 0.25\,(9^{th} \text{ item} - 8^{th} \text{ item})$$

$$= 35 + \frac{1}{4} (40 - 35)$$

$$= 35 + \frac{5}{4} = 35 + 1.25 = 36.25$$

Hence, first qurtile, $Q_1 = 9.5$ and third quartile, $Q_3 = 36.25$.

(b) For Grouped Data

In case of the grouped data,

$$Q_1 = l_1 + \frac{\left(\frac{N}{4}\right) - c}{f_1} \times i \text{ and } Q_3 = l_3 + \frac{3\left(\frac{N}{4}\right) - c}{f_3} \times i$$

where, $N = \Sigma f$ = total of all frequency values,

l_1 = lower limit of the first quartile class,

l_3 = lower limit of the third quartile class,

f_1 = frequency of the first quartile class,

f_3 = frequency of third quartile class,

i = width of the quartile class,

c = cumulative frequency preceding the quartile class.

For example :

The marks secured by group of student in their internals :

Class	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60
Frequency	4	3	2	1	5

Sol.

Class	**Frequency** f	**Cumulative frequency** (*c.f.*)
10 – 20	4	4
20 – 30	3	7
30 – 40	2	9
40 – 50	1	10
50 – 60	5	15

Here, $\frac{N}{4} = \frac{15}{4} = 3.75$ which lies in 10 - 20

Q_1 Lies in the group 10 - 20

$$Q_1 = l_1 + \frac{\frac{N}{4} - c}{f_1} \times i$$

$$= 10 + \frac{(3.75 - 0)}{4} \times 10$$

$$= 10 + 9.38 = 19.38 \text{ (Approx.)}$$

$$Q_3 = \frac{3N}{4} = \frac{45}{4} = 11.25$$

Therefore Q_3 lies in the group 50 - 60

$$Q_3 = l_3 + \frac{3\left(\frac{N}{4}\right) - c}{f_3} \times i$$

$$= 50 + \frac{(11.25 - 10)}{5} \times 10 = 50 + 2.5 = 52.5.$$

- **Deciles :**

  Deciles are the partition values which divide the arranged data into ten equal parts. There are nine deciles *i.e.,* $D_1, D_2, D_3, \ldots D_9$ and 5$^{\text{th}}$ decile is same as median or Q_2 because it divides the data in two equal parts.

  (a) For ungrouped Data

  If the data set consist of n items and are arranged in ascending order, then nine Deciles ($D_1, D_2, D_3, \ldots D_9$) are calculated.

  From $$D_r = \frac{r(n+1)}{10} \text{ th value}$$

  where, $r = 1, 2, 3, \ldots. 9$

  (b) For Grouped Data

  In case of grouped data,

  $$D_r = l + \left(\frac{\frac{rn}{10} - c}{f}\right)$$

  where, $r = 1, 2, 3, \ldots. 9$

  c = Cumulative frequency of class preceding the r^{th} decile class

  f = frequency of the r^{th} decile class

  i = width of the r^{th} decile class

  l = lower limit of the r^{th} decile class

  N = sum of all frequencies

- **Percentile :** Percentiles are the values which divide the arranged data into hundred equal parts. There are 99 percentiles *i.e.,* $P_1, P_2, P_3, \ldots. P_{99}$. The 50$^{\text{th}}$ percentile divides the series into two equal parts and $\boldsymbol{P_{50} = D_5 = Q_2}$ **= median.**

  Similarly, the value of $Q_1 = P_{25}$ and value of $Q_3 = P_{75}$

  (a) For Ungrouped Data

  If the data set consist of n items and are arranged in ascending order, then 99 percentiles ($P_1, P_2, \ldots\ldots P_{99}$) are calculated from

  $$P_r = \frac{r(n+1)}{100} \text{ th value}$$

  where, $r = 1, 2, \ldots., 99$

  (b) For Grouped Data

  In Case of grouped data,

  $$P_r = l + \left(\frac{\frac{rN}{100} - c}{f}\right) \times i$$

  where, $r = 1, 2, \ldots., 99$

  c = cumulative frequency of class preceding the r^{th} percentile class

f = frequency of the r^{th} percentile class
i = width of the r^{th} percentile class
l = lower limit of the r^{th} percentile class
N = sum of all frequencies

For example :

The marks of 9 students in a test were 13, 17, 20, 5, 3, 3, 18, 15, and 20. Find the 6^{th} decile and P_{62}.

Sol. First we have to arrange the data in ascending order :

3, 3, 5, 13, 15, 17, 18, 20, 20

Here, $n = 9$

$$6^{th} \text{ decile, } D_6 = \frac{6(9+1)}{10} \text{th value}$$
$$= 6^{th} \text{ value} = 17$$

and
$$P_{62} = \frac{62(9+1)}{100} \text{th value}$$
$$= 6.2^{th} \text{ value}$$
$$= 6^{th} \text{ value} + 0.2\,(7^{th} \text{ value} - 6^{th} \text{ value})$$
$$= 17 + 0.2(18 - 17)$$
$$= 17.2.$$

Merits of Quartiles, Deciles and percentiles :

- These positional values can be directly determined in case of open end class intervals.
- These positional values can be calculated easily in absence of some data.
- These are help in the calculation of measure of skewness.
- These are not affected very much by the extreme items.
- These can be located graphically.

Demerits of Quartiles, Deciles and Percentiles :

- These values are not easily understood by a common man.
- These values are based on all the observations of a series.
- These values cannot be computed if items are not given in ascending or descending order.
- These values have less sampling stability.

TOPIC-2
Measures of Dispersion

Revision Notes

➤ **Range and Quartile deviation**

The measure of central tendency are not sufficient to give complete information about the given data. The following example clarifies this :

Consider that the run scored by two batsman in their last ten matches are as follows :

Match	1	2	3	4	5	6	7	8	9	10
Batsman B_1	30	92	0	64	42	80	30	6	116	70
Batsman B_2	52	45	49	50	53	53	58	60	57	53

After calculating the mean and median for both, we get

	Batsman B_1	**Batsman B_2**
Mean	53	53
Meadian	53	53

Here, we can observe that the mean and median of the runs scored by the batsmen B_1 and B_2 are same, *i.e.*, 53. Now, at this situation, on the basis of the mean and median can we say something about the performance of two players *i.e.*, which is better than other. The answer is no, we can't say about the performance on the basis of mean and median.

At this stage, the central tendencies are not enough to give sufficient information or complete information about a given data. Here, we need the measures of dispersion.

Variability is another factor which required to studied under statistics. The single number that describes variability is called measure of dispersion. It is the measure of scattering of data about central tendency.

The following are the measures of dispersion :

1. Range, 2. Quartile Deviation, 3. Mean Deviation, 4. Standard Deviation.

(1) Range : A range is the most common and easily understandable measure of dispersion. It is the difference between the largest and smallest observations in the data set.

$$\text{Range } (R) = L - S$$

- **For Grouped Data**

The grouped frequency distribution of values in the data set, the range is the difference between the upper class limit of the last class interval (L) and the lower class limit of the first class interval (S).

- **Coefficient of Range**

The relative range of the range is called the coefficient of range.

$$\text{Coefficient of Range} = \frac{L-S}{L+S}\times 100$$

For example :

(1) Given marks of Sameer and Suresh as follows :

Sameer = 79, 62, 40, 5

Suresh = 60, 45, 52, 42

For Sameer, range = 79 – 5 = 74

For Suresh, range = 60 – 42 = 18

Thus, range of sameer > range of Suresh

So, the scores are **scattered** or **dispersed** in case of Sameer while for Suresh, these are closed to each other. The range of data gives the rough idea of variability or scatterness but does not tell about the dispersion data of the data from the measure of central tendency.

(2) Calculate range and the coefficient from the following distribution :

x	10-15	15-20	20-25	25-30
frequency	**4**	**10**	**16**	**8**

Sol. Here, lower class limit of the first class interval, $S = 10$

and upper class limit of the last class interval, $L = 30$

$\therefore$ Range $= L - S = 30 - 10 = 20$

$$\text{Coefficient of range} = \frac{L-S}{L+S}\times 100 = \frac{30-10}{30+10}\times 100 = \frac{20}{40}\times 100 = 0.5\times 100 = 50$$

- **Merits of Range**

(i) It is the simplest of the measure of dispersion.

(ii) Easy to calculate.

(iii) Easy to understand.

(iv) Independent of change of origin.

- **Demerits of Range**

(i) It is based on two extreme observations. Hence, get affected by fluctuations.

(ii) A range is not a reliable measure of dispersion.

(iii) Dependent on change of scale.

(2) Quartile Deviation

In previous topic, we have studied about quartiles. The difference between third quartile and first quartile is known as **Inter quartile range**. The inter quartile range is based upon middle 50% values in a distribution and is therefore, not affected by extreme values. Half of interquartile range is called **Quartile deviation (Q.D.)**. Thus,

$$\text{Interquartile range} = Q_3 - Q_1$$

and
$$\text{Quartile Deviation (Q.D.)} = \frac{Q_3 - Q_1}{2}$$

Q.D. is also known as semi-interquartile range

$$\text{Coefficient of quartile deviation} = \frac{Q_3 - Q_1}{Q_3 + Q_1} \times 100$$

Merits of Quartile Deviation

- All the drawbacks of Range by Quartile Deviation.
- It uses half of the data.
- Independent of change of origin.
- The best measure of dispersion for open-end classification.

Demerits of Quartile Deviation

- It ignores fifty percent of data.
- Dependent of change of scale.
- Not a reliable measure of dispersion.

For example :

Calculate the Quartile Deviation of the following observations :

20, 25, 29, 30, 35, 39, 41, 48, 51, 60 and 70

Sol. For Q.D., we need to calculate values of Q_3 and Q_1. Also, the given data is already arranged in ascending order.

$$Q_1 = \left(\frac{n+1}{4}\right)^{\text{th}} \text{item} = \left(\frac{11+1}{4}\right)^{\text{th}} \text{item} = 3^{\text{rd}} \text{ item} = 29$$

and
$$Q_3 = 3\left(\frac{n+1}{4}\right)^{\text{th}} \text{item} = 3\left(\frac{11+1}{4}\right)^{\text{th}} \text{item} = 9^{\text{th}} \text{ item} = 51$$

$\therefore$
$$\text{Inter Quartile Range} = Q_3 - Q_1 = 51 - 29 = 22$$

Thus,
$$\text{Quartile Deviation} = \frac{1}{2}(Q_3 - Q_1) = \frac{1}{2} \times 22 = 11.$$

(3) Mean Deviation

Mean deviation is an important measure of deviation, which depend upon the deviations of the observations from a central tendency. It is defined as the arithmetic mean of the absolute deviations of all the values taken from a central value or a fixed number a. The mean deviation from 'a' is denoted by M.D. (a) and is defined by :

$$\text{M. D.}(a) = \frac{\text{Sum of absolute values of deviations from '}a\text{'}}{\text{Number of observations}}$$

(a) For Ungrouped Data

Let $x_1, x_2 \ldots\ldots x_n$ be a observations. Then mean deviation about mean ($\bar{x}$) or median (M) can be found by the formula.

$$\text{Mean Deviatiom} = \frac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n} \text{ or } \frac{\sum_{i=1}^{n} |x_i - \text{M}|}{n}, \text{ where } n \text{ is the number of observations}$$

For example : Find the mean deviation for the series 3, 5, 7, 8, 12.

Sol. Let M be the arithmetic mean of the given series, Then, $M = \dfrac{3+5+7+8+12}{5} = \dfrac{35}{5} = 7$

Thus, mean deviation from Mean (M.D.) $= \dfrac{|3-7|+|5-7|+|7-7|+|7-8|+|7-12|}{5}$

$= \dfrac{4+2+0+1+5}{5} = \dfrac{12}{5} = 2.4.$

(b) For Grouped Data

(i) For discrete frequency distribution : Let the data have 'n' distinct values $x_1, x_2, \ldots, x_n$ and their corresponding frequencies are $f_1, f_2, \ldots, f_n$ respectively. Then, this data can be represented in the tabular form as

x_i	x_1	x_2	x_3	...	x_n
f_i	f_1	f_2	f_3	...	f_n

and is called discrete frequency distribution. There, mean deviation about mean or median is given by $\dfrac{\sum_{i=1}^{n} f_i |x_i - A|}{N}$, where $N = \sum_{i=1}^{n} f_i$ = total frequency, and A = mean or median.

For example : Calculate the mean deviation from median for the following table :

Wages (in ₹)	20	18	16	14	12	10	8	6
Frequency	2	4	9	18	27	25	14	1

Sol. To find median and mean deviation, we prepare the following table :

Variable x	**Frequency** f	**Cumulative frequency**	$d = x - M$	$\|d\|$	$f\|d\|$
20	2	2	8	8	16
18	4	6	6	6	24
16	9	15	4	4	36
14	18	33	2	2	36
12	27	60	0	0	0
10	25	85	–2	2	50
8	14	99	–4	4	56
6	1	100	–6	6	6
	$\Sigma f = 100$				$\Sigma f\|d\| = 224$

Here, $N = \Sigma f = 100$, so that

Median Number $= \dfrac{N+1}{2} = \dfrac{100+1}{2} = 50.5$

which comes under the cumulative frequency 60. Therefore,

Median, M = variable against cumulative frequency 60

= 12

Hence, Mean Deviation from median $= \dfrac{\Sigma f|d|}{\Sigma f} = \dfrac{224}{100} = 2.24.$

(ii) For continuous frequency distribution : A continuous frequency distribution is a series in which the data is classified into different class intervals without gaps along with their respective frequencies.

Mean deviation about mean $(\bar{x})$, *i.e.*,

M.D. $(\bar{x}) = \dfrac{1}{N}\sum_{i=1}^{n} f_i |x_i - \bar{x}|$, where' x_i' s' are the mid-point of the intervals and

Also, mean $(\bar{x}) = \frac{1}{N}\sum_{i=1}^{n} f_i x_i$

Mean deviation about median (M), *i.e.,*

M.D. $(M) = \frac{1}{N}\sum_{i=1}^{n} f_i |x_i - M|$, where, x_i's are the mid-point of the intervals and $\sum_{i=1}^{n} f_i = N$. Also, median

$M = l + \frac{\frac{N}{2} - c.f.}{f} \times h$, where l, f, h and $c.f.$ are lower limit, the frequency, the width of median class and cumulative frequency of class just preceding the median class.

(c) Shortcut (Step-deviation) Method for calculating the Mean deviation about Mean : This method is used to manage large data. In this method, we take an assumed mean, which is in the middle or just close to it, in the data we denote the new variable by u_i and is defined by $u_i = \frac{x_i - a}{h}$, where a is the assumed mean and h is the common factor or length of class interval the mean $\bar{x}$ by step deviation method is given by

$$\bar{x} = a + \frac{\sum_{i=1}^{n} f_i u_i}{N} \times h$$

➢ **Coefficient of Mean Deviation**

The ratio between deviation and mean in called coefficient of mean deviation.

Thus, coefficient of Mean Deviation $= \frac{\text{Mean Deviation}}{\text{Used mean/median}} \times 100.$

For example :

The following table gives the marks in mathematics of 50 students of a class. Find mean deviation from median and coefficient of deviation :

Class Marks	0-10	10-20	20-30	30-40	40-50
Frequency	5	8	15	16	6

Sol. Calculation of Median : The cumulative frequency table for the given table is as follows :

Class Marks	Frequency	Cumulative Frequency
0-10	5	5
10-20	8	13
20-30	15	28
30-40	16	44
40-50	6	50
	$\Sigma f = 50$	

Here, $N = \Sigma f = 50$

Therefore, median number $= \frac{N}{2} = \frac{50}{2} = 25$

Therefore, Median $(M) = l + \left(\frac{\frac{N}{2} - c.f.}{f}\right) \times h$

Here, $l = 20, h = 10, f = 15, c.f. = 13, N = 50$

$\therefore$ Median,
$$M = 20+\left(\frac{\frac{50}{2}-13}{15}\right)\times 10$$
$$= 20+\frac{10}{15}\times 12 = 28.$$

Calculation of Mean Deviation : For this required table is as follows (here M = 28)

Class Marks	Mid-value x	Frequency f	$d = x - M$	$\lvert d\rvert$	$f\lvert d\rvert$
0-10	5	5	–23	23	115
10-20	15	8	–13	13	104
20-30	25	15	–3	3	45
30-40	35	16	7	7	112
40-50	45	6	17	17	102
		$\Sigma f = 50$			$\Sigma f\lvert d\rvert = 478$

Hence, Mean deviation $= \dfrac{\Sigma f\,|d|}{\Sigma f} = \dfrac{478}{50} = 9.56$

and coefficient of deviation $= \dfrac{\text{Mean Deviation}}{\text{Median}}\times 100$

$$= \frac{9.56}{28}\times 100 = 0.3414\times 100 = 341.4.$$

Merits of Mean Deviation

- It is well defined.
- It is easy to understand and calculate the mean deviation.
- It is based upon all term-values.
- It can be calculated from any mean.
- It is less affected by limiting values.
- It provides relative importance to all the terms.

Demerits of Mean Deviation

- It is mathematically incorrect due to considering each deviation as positive.
- Its mathematical investigation is not possible.
- It's calculation can be done from any mean. Hence it is indefinite.

➢ **Utility of Mean Deviation :** It is more used in economical, social and commercial areas. The study of income and money distribution etc., is done using mean deviation.

TOPIC-3
Variance and Standard Deviation

Revision Notes

➢ **Variance**

The mean of squares of deviations from mean is called the variance and it is denoted by the symbol 'σ^2'.
The variance of 'n' observations $x_1, x_2, \ldots, x_n$ is given by :

$$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}$$

$$\text{Coefficient of variance} = \frac{\sigma}{\bar{x}} \times 100$$

For example : Find the variance of the following data :

6, 8, 10, 12, 14, 16, 18, 20, 22, 24

Sol. Here,

$$\text{mean, } \bar{x} = \frac{6+8+10+12+14+16+18+20+22+24}{10} \qquad (\because n = 10)$$

$$= \frac{1}{10} \times \frac{10}{2}(6+24) = \frac{30}{2} = 15$$

Variable x	Deviation $= x - \bar{x}$	(Deviation)^2 $= (x-\bar{x})^2$
6	–9	81
8	–7	49
10	–5	25
12	–3	9
14	–1	1
16	1	1
18	3	9
20	5	25
22	7	49
24	9	81
$n = 10$		$\Sigma(x-\bar{x})^2 = 330$

Hence,

$$\text{Var}(x) = \frac{1}{n}\Sigma(x-\bar{x})^2$$

$$= \frac{1}{10} \times 330$$

$$= 33.$$

➢ **Standard Deviation (S.D.)**

Standard Deviation is the square root of the arithmetic mean of the squares of deviations from mean and it is denoted by the symbol σ. It is a absolute measure of dispersion. It is the best measure of variation and most commonly used measure.

OR

The square root of variance, is called standard deviation *i.e.*, $\sqrt{\sigma^2} = \sigma$. It is also known as root mean square deviation.

- **Significance of deviation**
  - **(i)** It the deviation is zero, it means there is no deviation at all observations are equal to mean.
  - **(ii)** If deviation is small, this indicates that the observation are close to the mean.
  - **(iii)** If the derivation is large, there is a high degree of dispersion of the observation from the mean.

➢ **Variance and Standard deviation of ungrouped data**

Variance of n observation $x_1, x_2, ... x_n$ is given by :

$$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n} = \frac{\sum_{i=1}^{n}x_i^2}{n} - \left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^2$$

and,

$$\text{Standard Deviation, } \sigma = \sqrt{\text{variance}} = \sqrt{\sigma^2}$$

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}} \text{ or } \sqrt{\frac{\sum_{i=1}^{n}x_i^2}{n} - \left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^2}$$

$$= \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n} - \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2}$$

Considering the previous example, we can calculate the value of standard deviation as follows :

Since, variance, $\sigma^2 = 33$

Thus, Standard Deviation $= \sqrt{\text{variance}}$

$= \sqrt{33}$

$= 5.745$

- **Variance and Standard deviation of Grouped data**

(i) For discrete frequency distribution

Let the discrete frequency distribution be $x_1, x_2, x_3, ... x_n$ and $f_1, f_2, f_3, ... f_n$. Then by direct method :

$$\text{Variance } (\sigma^2) = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2$$

$$= \frac{1}{N}\sum f_i x_i^2 - \left(\frac{\Sigma f_i x_i}{N}\right)^2$$

and

$$\text{Standard deviation } (\sigma) = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2}$$

$$= \frac{1}{N}\sqrt{N\sum f_i x_i^2 - \left(\sum f_i x_i\right)^2}$$

where

$$N = \sum_{i=1}^{n} f_i$$

$$\text{By short cut method, variance } (\sigma^2) = \frac{1}{N}\sum_{i=1}^{n} f_i d_i^2 - \left(\frac{\sum f_i d_i}{N}\right)^2$$

and

$$\text{standard deviation } (\sigma) = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i d_i^2 - \left(\frac{\sum f_i d_i}{N}\right)^2}$$

Where $d_i = x_i - a$, deviation from assumed mean and a = assumed mean.

For example : Find the standard deviation from the following table :

Marks	8	10	12	14	16
Frequency	4	7	8	7	4

Sol. Calculation of Arithmetic Mean

Marks x	**Frequency** f	**Product** fx
8	4	32
10	7	70
12	8	96
14	7	98
16	4	64
Total	$\Sigma f = 30$	$\Sigma fx = 360$

$$\therefore \quad \text{Mean } \bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{360}{30} = 12$$

Calculation of Standard Deviation $\sigma(\bar{x} = 12)$ **:**

Marks x	**Frequency** f	**Deviation** $x - \bar{x}$	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
8	4	–4	16	64
10	7	–2	4	28
12	8	0	0	0
14	7	2	4	28
16	4	4	16	64
Total	$\Sigma f = 30$			$\Sigma f(x - \bar{x})^2 = 184$

Now, variance $\sigma^2 = \frac{\Sigma f(x - \bar{x})^2}{\Sigma f}$

$$= \frac{184}{30} = 6.133 \text{ (Approx.)}$$

Thus, standard deviation, $\sigma = \sqrt{\text{variance}}$

$$= \sqrt{6.133} = 2.476 \text{ (Approx.)}$$

(ii) For Continuous frequency distribution

Direct method : If there is a frequency distribution of n classes and each class defined by its mid-point x_i with corresponding frequency f_i then the variance and standard deviation are :

$$\text{Variance } (\sigma^2) = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2$$

$$\text{and Standard deviation } (\sigma) = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2}$$

or,

$$\sigma^2 = \frac{1}{N^2}\left[N\sum f_i x_i^2 - \left(\sum f_i x_i\right)^2\right]$$

$$\sigma = \frac{1}{N}\sqrt{N\sum f_i x_i^2 - \left(\sum f_i x_i\right)^2}$$

Step-Deviation (short-cut method) : Sometimes the values of mid points x_i of different classes in a continuous distribution are large and so the calculation of mean and variance becomes tedious and time consuming. For this we use the step-deviation method. Here,

$$\text{Variance } \sigma^2 = h^2\left[\frac{1}{N}\sum_{i=1}^{n} f_i u_i^2 - \left(\frac{\sum_{i=1}^{n} f_i u_i}{N}\right)^2\right]$$

$$\text{and Standard deviation } (\sigma) = h\sqrt{\left[\frac{1}{N}\sum_{i=1}^{n} f_i u_i^2 - \left(\frac{\sum_{i=1}^{n} f_i u_i}{N}\right)^2\right]}$$

where, $u_i = \frac{x_i - a}{h}$, a = assumed mean and h = width of class interval.

For example :

Find the mean and variance for the following frequency distribution :

Classes	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
Frequencies	5	8	15	16	6

Sol.

Classes	Mid-value x_i	Frequency (f_i)	$u_i = \frac{x_i - 25}{10}$	$f_i u_i$	u_i^2	$f_i u_i^2$
0 – 10	5	5	–2	–10	4	20
10 – 20	15	8	–1	–8	1	8
20 – 30	25	15	0	0	0	0
30 – 40	35	16	1	16	1	16
40 – 50	45	6	2	12	4	24
Total		**$N = 50$**		**$\Sigma f_i u_i = 10$**		**$\Sigma f_i u_i^2 = 68$**

Here, $$u_i = \frac{x_i - a}{h} = \frac{x_i - 25}{10} \quad [\because a = 25, h = 10]$$

$$\bar{x} = a + hu_i$$

$$= 25 + 10 \times \frac{\sum f_i u_i}{\sum f_i}$$

$$= 25 + 10 \times \frac{10}{50} = 27$$

Variance, $$\sigma^2 = h^2 \left[\frac{1}{N} \sum_{i=1}^{n} f_i u_i^2 - \left(\frac{\sum_{i=1}^{n} f_i u_i}{N} \right)^2 \right]$$

$$= (10)^2 \left[\frac{1}{50} \times 68 - \left(\frac{10}{50} \right)^2 \right]$$

$$= \frac{100}{50} \left(68 - \frac{100}{50} \right)$$

$$= 2(68 - 2) = 2 \times 66 = 132$$

Hence, mean = 27 and variance = 132.

- **Analysis of frequency distribution**

  The mean deviation and standard deviation have the same units in which the data is given. The measures of dispersion are unable to compare the variability of two or more series which are measured in different units. So we require those measures which are independent of units.

  The measures of variability which is independent of units, is called coefficient of variation denoted as CV and it is given by

$$\text{C.V.} = \frac{\sigma}{\bar{x}} \times 100$$

  where $\bar{x}$ and σ are respectively the mean and standard deviation of the data. For comparing the variability of two series, we calculate the co-effecient of variations for each series.

- **Comparison of two frequency distributions with same mean**

  Let us consider two frequency distributions with standard deviations σ_1 and σ_2 and having same mean $\bar{x}$, then

$$\text{C.V. (1st distribution)} = \frac{\sigma_1}{\bar{x}} \times 100$$

and $$\text{C.V. (2nd distribution)} = \frac{\sigma_2}{\bar{x}} \times 100$$

$$\frac{\text{C.V. (1st distribution)}}{\text{C.V. (2nd distribution)}} = \frac{\frac{\sigma_1}{\bar{x}} \times 100}{\frac{\sigma_2}{\bar{x}} \times 100} = \frac{\sigma_1}{\sigma_2}$$

Thus, two C.V.'s can be compared on the basis of values of σ_1 and σ_2. Thus, if two series have equal means, then the series with greater standard deviation (or variance) is said to be more variable than the other. The other series with lesser value of the standard deviation is said to be more consistent than the other.

For example :

Two teams A and B scored goals in a football season as follows :

Number of goals scored in a match		0	1	2	3	4
Number of matches	Team A	27	9	8	5	4
	Team B	17	9	6	5	3

Tell which team is more consistent ?

Sol. Calculation of coefficient of deviation for team A :

Let assumed mean, $a = 2$

Number of goals, x	Number of matches, f	Deviation $d = x - a$	fd	fd^2
0	27	–2	–54	108
1	9	–1	–9	9
2	8	0	0	0
3	5	1	5	5
4	4	2	8	16
Total	$\Sigma f = 53$		$\Sigma fd = -50$	$\Sigma fd^2 = 138$

$\therefore$ Standard Deviation $\sigma = \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2}$

$$= \sqrt{\frac{138}{53} - \left(\frac{-50}{53}\right)^2}$$

$$= \sqrt{2.6038 - 0.8836}$$

$$= \sqrt{1.7202} = 1.31$$

Also, mean $(\bar{x}) = a + \frac{\Sigma fd}{\Sigma f} = 2 - \frac{50}{53} = 2 - 0.94 = 1.06$

$\therefore$ Coefficient of deviation $= \frac{\sigma}{\bar{x}} = \frac{1.31}{1.06} = 1.236.$

Calculation of coefficient of deviation for team B :

Let assumed mean, $a = 2$.

Number of goals, x	Number of matches, f	Deviation $d = x - A$	fd	fd^2
0	17	–2	–34	68
1	9	–1	–9	9
2	6	0	0	0
3	5	1	5	5
4	3	2	6	12
Total	$\Sigma f = 40$		$\Sigma fd = -32$	$\Sigma fd^2 = 94$

$\therefore$ Standard deviation, $\sigma = \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2}$

$$= \sqrt{\frac{94}{40} - \left(\frac{-32}{40}\right)^2}$$

$$= \sqrt{2.35 - 0.64}$$

$$= \sqrt{1.71} = 1.307$$

Also, mean, $\bar{x} = a + \frac{\Sigma fd}{\Sigma f} = 2 - \frac{32}{40} = 2 - 0.8 = 1.2$

$\therefore$ Coefficient of deviation $= \frac{\sigma}{\bar{x}} = \frac{1.307}{1.2} = 1.089$

Since, the coefficient of deviation for team B is less than that for team A, hence team B is more consistent.

Merits of standard Deviation

(i) It is based upon all the values of series of given data.

(ii) It is a clear and finite measure of dispersion.

(iii) It is used in higher mathematical calculations due to being mathematically correct.

(iv) It is useful in comparison of variation of different groups.

Demerits of standard Deviation

(1) Its calculation is difficult.

(2) It is much effected by limiting values as we take squares of deviation in this.

Remark : Standard Deviation of first n natural numbers is $\sqrt{\frac{n^2-1}{12}}$.

TOPIC-4
Skewness and Kurtosis

Revision Notes

A fundamental task many in many statistical analysis is to characterize the **location** and **variability** of a data set. A further characterization of data includes **skewness** and **kurtosis**. Skewness and Kurtosis both are the measure of shape of a distribution.

➢ **Moments**

Moment are the arithmetic means of first, second, third and so on *i.e.*, r^{th} power of the deviation taken from either mean or an arbitrary point of distribution. In other words, moments are statistical measures that give certain characteristics of the distribution. Generally, in any frequency distribution, four moments are obtained which are known as first, second, third and fourth moments. These four moments describe the information about mean, variance, skewness and kurtosis. Moments can be classified in raw and central moments.

- **Raw moments**

  Raw moments can be defined as the arithmetic mean of various powers of deviations taken from origin or arbitrary origin. The r^{th} raw moment is denoted by μ'_r, $r = 1, 2, 3, ...$, then the first four raw moments are given by

Raw Moments	Raw data $d = x - A$	Discrete Data $d = x - A$	Continuous Data $d = \frac{x-A}{i}$
μ'_1	$\frac{\Sigma d}{n}$	$\frac{\Sigma fd}{N}$	$\frac{\Sigma fd}{N} \times i$
μ'_2	$\frac{\Sigma d^2}{n} = \sigma$	$\frac{\Sigma fd^2}{N} = \sigma$	$\frac{\Sigma fd^2}{N} \times i^2 = \sigma$
μ'_3	$\frac{\Sigma d^3}{n}$	$\frac{\Sigma fd^3}{N}$	$\frac{\Sigma fd^3}{N} \times i^3$
μ'_4	$\frac{\Sigma d^4}{n}$	$\frac{\Sigma fd^4}{N}$	$\frac{\Sigma fd^4}{N} \times i^4$

where, n = number of observations

N = sum of all frequencies

i = width of the class

A = arbitrary origin

Here note that, when $A = 0$ then raw moments are taken around **origin** and when A is any **number**, then we say raw moments are taken around **arbitrary origin** or **arbitrary number**.

For example : Find the first four moments about 5 for 7, 8, 6, 5.

Sol.

x	$(x - A)$	$(x - A)^2$	$(x - A)^3$	$(x - A)^4$
7	2	4	8	16
8	3	9	27	81
6	1	1	1	1
5	0	0	0	0
Total	$\Sigma(x - A) = 6$	$\Sigma(x - A)^2 = 14$	$\Sigma(x - A)^3 = 36$	$\Sigma(x - A)^4 = 98$

Here, given $A = 5$ and $n = 4$

Therefore

$$\mu'_1 = \frac{\Sigma(x - A)}{n} = \frac{6}{4} = 1.5$$

$$\mu'_2 = \frac{\Sigma(x - A)^2}{n} = \frac{14}{4} = 3.5$$

$$\mu'_3 = \frac{\Sigma(x - A)^3}{n} = \frac{36}{4} = 9$$

$$\mu'_4 = \frac{\Sigma(x - A)^4}{n} = \frac{98}{4} = 24.5$$

- **Central Moments**

Central moments can be defined as the arithmetic mean of various power of deviation taken from the mean of the distribution. The r^{th} central moment is denoted by μ_r, $r = 1, 2, 3, \ldots\ldots$

Central moments	Raw data	Discrete Data	Continuous data $d' = (x - \bar{x}) \times \frac{1}{i}$
μ_1	$\frac{\Sigma(x - \bar{x})}{n} = 0$	$\frac{\Sigma f(x - \bar{x})}{N} = 0$	$\frac{\Sigma fd'}{N} \times i$
μ_2	$\frac{\Sigma(x - \bar{x})^2}{N} = \sigma^2$	$\frac{\Sigma f(x - \bar{x})^2}{N} = \sigma^2$	$\frac{\Sigma fd'^2}{N} \times i^2 = \sigma^2$
μ_3	$\frac{\Sigma(x - \bar{x})^3}{n}$	$\frac{\Sigma f(x - \bar{x})^3}{N}$	$\frac{\Sigma fd'^3}{N} \times i^3$
μ_4	$\frac{\Sigma(x - \bar{x})^4}{n}$	$\frac{\Sigma f(x - \bar{x})^4}{N}$	$\frac{\Sigma fd'^4}{N} \times i^4$

For example : Calculate the first four central moments from the following data :

x	0	1	2	3	4	5	6	7	8
f	10	20	30	40	50	60	70	80	90

Sol.

x	f	fx	$d = x - \bar{x}$ $\bar{x} = 5.33$	fd	fd^2	fd^3	fd^4
0	10	0	– 5	– 50	250	– 1250	6250
1	20	20	– 4	– 80	320	– 1280	5120
2	30	60	– 3	– 90	270	– 810	2430
3	40	120	– 2	– 80	160	– 320	640
4	50	200	– 1	– 50	50	– 50	50
5	60	300	0	0	0	0	0

6	70	420	1	70	70	70	70
7	80	560	2	160	320	640	1280
8	90	720	3	270	810	2430	7290
	$N = 450$	$\Sigma fx = 2400$	$\Sigma fd = -9$	$\Sigma fd = 150$	$\Sigma fd^2 = 2250$	$\Sigma fd^3 = -570$	$\Sigma fd^4 = 23130$

$$\text{Mean, } \bar{x} = \frac{\Sigma fx}{N} = \frac{2400}{450} = 5.33 \approx 5$$

$$\mu'_1 = \frac{\Sigma fd}{N} = \frac{150}{450} = 0.33$$

$$\mu'_2 = \frac{\Sigma fd^2}{N} = \frac{2250}{450} = 5$$

$$\mu'_3 = \frac{\Sigma fd^3}{N} = \frac{-570}{450} = -1.266$$

$$\mu'_4 = \frac{\Sigma fd^4}{N} = \frac{23130}{450} = 51.4$$

➢ **Relationship between Raw Moments and Central Moments**

Relationship between moments about arithmetic mean and moments about an origin are given below :

$$\mu_1 = \mu'_1 - \mu'_1 = 0$$
$$\mu_2 = \mu'_2 - (\mu'_1)^2 = \sigma^2$$
$$\mu_3 = \mu'_3 - 3\mu'_2\,\mu'_1 + 2(\mu'_1)^3$$
$$\mu_4 = \mu'_4 - 4\mu'_3\,\mu'_1 + 6\mu'_2\,(\mu'_1)^2 - 3(\mu'_1)^4$$

Note :

- The first raw moment about origin is mean whereas the first central moment is zero.
- The second raw moment and central moments are mean square deviation and variance, respectively.
- The third and fourth moments are useful in measuring skewness and kurtosis.

➢ **Skewness :**

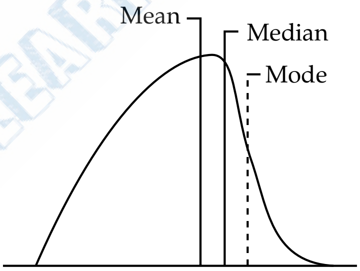
We know that, measure of dispersion tells about the variation of the data set. On the other hand, Skewness tells about the direction of variation of data set.

"Skewness means the symmetry or lack of symmetry of a data. A distribution or data, is symmetric, if it looks the same to the left and right of the centre point".

➢ **Types of Skewness :**

- **Positive skewness :** Skewness is said to be positive when the tail of the curve of the frequency distribution elongates more on the right. Also, skewness is positive if mean, median and mode of the frequency distribution satisfy the condition :

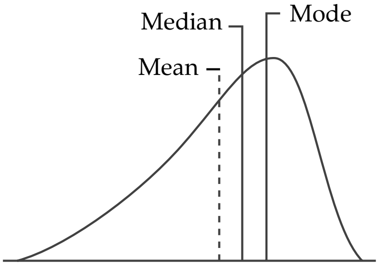
Mean < Median < Mode



Positive skew

- **Negative Skewness :** Skewness is negative when the tail of the curve of the frequency distribution elongates more on the left, Also, skewness is negative if mean, median and mode of the frequency distribution satisfy the condition

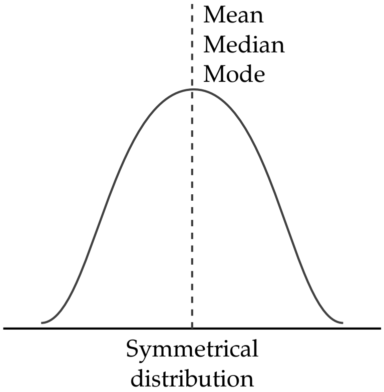
Mean > Median > Mode



Negative skew

- If the curve of the frequency distribution is symmetrical, the skewness is **zero**. In this case, we have the relation.

Mean = Median = Mode



➢ **Difference between variance and Skewness :**

The following two difference between variance and skewness should be taken into consideration :

(i) Variance tells about the amount of variability while skewness gives the direction of variability.

(ii) In business and economic series, measures of variation have greater practical application than measures of skewness. However, in medical and life science field measures of skewness have greater practical applications than variance.

➢ **Measures of Skewness :**

Measures of skewness help us to know to what degree and in which direction (positive or negative) the frequency distribution has a departure from symmetry. Measures of skewness can be the both absolute as well as relative.

- **Absolute measures of Skewness :**

  Following are the absolute measure of skewness :

  (1) Skewness (S_k) = Mean – Median

  (2) Skewness (S_k) = Mean – Mode

  (3) Skewness (S_k) = $(Q_3 - Q_2) - (Q_2 - Q_1)$

  For comparing two series, we do not calculate these absolute measures. We can calculate **relative measures** which are called **coefficient of skewness.** Coefficient of skewness are pure numbers independent of units of measurements.

- **Relative Measures of Skewness :**

  If the values of skewness is obtained in ratio or percentages, it is called **relative or coefficient of skewness**. When skewness is presented in ratios or percentages, comparison becomes easy. The following are important methods of measuring relative skewness.

  (1) **Karl Pearson's Coefficient of Skewness**

  This method is most frequently used for measuring skewness. The formula for measuring coefficient of skewness is given by

$$S_k = \frac{\text{Mean} - \text{Mode}}{\sigma}$$

  ♦ The value of this coefficient would be zero in a symmetrical distribution.

  ♦ If mean is greater than mode, coefficient of skewness would be positive otherwise negative.

  ♦ If $S_k > 0$, the distribution is positively skewed and if $S_k < 0$, it is negatively skewed.

  ♦ The value of the Karl Pearson's coefficient of skewness usually lies between ± 1 for moderately skewed distribution.

  ♦ If mode is not well defined, coefficient of skewness can be calculated with help of empirical relation between mean, median and mode i.e., Mean – Mode = 3(Mean – Median). Thus, Karl Pearson's coefficient skewness is defined in terms of median as

$$S_k = \frac{3(\text{Mean} - \text{Median})}{\sigma}; -3 \le S_k \le 3$$

  For example : For a distribution Karl Person's coefficient of skewness is 0.64, standard deviation is 13 and mean is 59.2. Find mode and median.

Sol. Given, $S_k = 0.64$, $\sigma = 13$ and Mean $(\bar{x}) = 59.2$

We know that, $$S_k = \frac{\text{Mean} - \text{Mode}}{\sigma}$$

$$\Rightarrow \quad 0.64 = \frac{59.20 - \text{Mode}}{13}$$

$$\Rightarrow \quad \text{Mode} = 59.20 - 8.32 = 50.88$$

Now $$\text{Mode} = 3\text{ Median} - 2\text{ Mean}$$

$$\Rightarrow \quad 50.88 = 3(\text{Median}) - 2(59.2)$$

$$\Rightarrow \quad \text{Median} = \frac{50.88 + 118.4}{3} = \frac{169.28}{3} = 56.42. \text{ (Approx.)}$$

(2) Bowley's Coefficient of Skewness

This method is based on quartiles. The formula for calculating coefficient of skewness is given by

$$S_k = \frac{(Q_3 - Q_2) - (Q_2 - Q_1)}{(Q_3 - Q_1)}$$

or $$S_k = \frac{Q_3 - 2Q_2 + Q_1}{Q_3 - Q_1}$$

Since, we know that Q_2 = Median (M_d). So, we can define the above formula as :

$$S_k = \frac{Q_3 + Q_1 - 2\text{ Median}}{Q_3 - Q_1}$$

If S_k would be zero if, it is a symmetrical distribution. If the value is greater than zero, it is positively skewed and if the value is less than zero, it is negatively skewed distribution. It will take value between + 1 and – 1.

For example :

Calculate the coefficient of skewness by Bowley's method for the given set of data.

Monthly Salary	1000 – 1200	1200 – 1400	1400 – 1600	1600 – 1800	1800 – 2000	2000 – 2200	2200 – 2400	2400 – 2600	2600 – 2800
No. of Employees	5	14	23	50	52	25	22	7	2

Monthly Salary	Frequency	Cumulative Frequency
1000 – 1200	5	5
1200 – 1400	14	19
1400 – 1600	23	42
1600 – 1800	50	92
1800 – 2000	52	144
2000 – 2200	25	169
2200 – 2400	22	191
2400 – 2600	7	198
2600 – 2800	2	200
Total	200	

Q_1 has $\frac{N}{4}$ observations or 50 observations below it. It is in the class 1600 – 1800.

$$\therefore \quad Q_1 = l + \frac{\left(\frac{N}{4} - C\right)}{f_1} \times i$$

$$= 1600 + \frac{(50 - 42)}{50} \times 200$$

$$= 1632$$

Q_2 (Median) has $\frac{N}{2}$ observations or 100 observations. So, it lies in the class 1800 – 2000.

$$Q_2 = M_d = l + \left(\frac{\frac{N}{2} - C}{f_2}\right) \times i$$

$$= 1800 + \left(\frac{100 - 92}{52}\right) \times 200$$

$$= 1830.77 \text{ (Approx.)}$$

Q_3 has $\frac{3N}{4}$ observations or 150 observations below it. It lies in the class 2000 – 2200

$$Q_1 = l + \left(\frac{\frac{3N}{4} - C}{f_3}\right) \times i$$

$$= 2000 + \left(\frac{150 - 144}{25}\right) \times 200$$

$$= 2048$$

$\therefore$ Coefficient of $S_k = \frac{Q_3 + Q_1 - 2M_d}{Q_3 - Q_1}$

$$= \frac{2048 + 1632 - (2 \times 1830.77)}{2048 - 1632}$$

$$= \frac{3680 - 3661.54}{416} = \frac{18.46}{416} = 0.045$$

(3) Kelly's coefficient of skewness

The coefficient of skewness proposed by kelly is based on percentiles and deciles. The formula for calculating the coefficient of skewness is given by

Based on Percentiles

$$S_k = \frac{(P_{90} - P_{50}) - (P_{50} - P_{10})}{(P_{90} - P_{10})}$$

$$= \frac{P_{90} - 2P_{50} + P_{10}}{P_{90} - P_{10}}$$

Where, P_{90}, P_{50} and P_{10} are 90^{th}, 50^{th} and 10^{th} percentiles

Based on Deciles

$$S_k = \frac{D_9 - 2D_5 + D_1}{D_9 - D_1}$$

Where, D_9, D_5 and D_1 are 9^{th}, 5^{th} and 1^{st} deciles

➢ **Measure of Skewness based on Moments :**

The measure of skewness based on moments is denoted by β_1 and is given by

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3},$$ where μ_2 and μ_3 are second and third central moments.

$$\gamma_1 = \pm\sqrt{\beta_1} = \sqrt{\frac{\mu_3^2}{\mu_2^3}} = \frac{\mu_3}{(\sqrt{\mu_2})^3} = \frac{\mu_3}{\sigma^3} \qquad (\because \mu_2 = \sigma^2)$$

Interpretation :

- If $\gamma_1 = 0$, there is no skewness in the distribution.
- If $\gamma_1 > 0$, there is a positive skewness in the distribution.
- If $\gamma_1 < 0$, there is a negative skewness in the distribution.

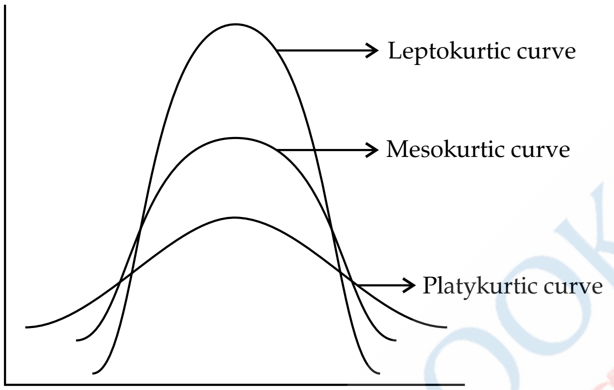
For example : Find β_1 for the following data $\mu_1 = 0$, $\mu_2 = 8.76$ and $\mu_3 = -2.91$.

Sol. $\because \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(-2.91)^2}{(8.76)^3} = \frac{8.47}{672.24} = 0.0126.$

- **Kurtosis**

  If we have a knowledge of measures of central tendency, dispersion and skewness, even then we cannot get a compute idea of a distribution. In addition to these measures, we need to know another measure to get the complete idea about the shape of distribution which can be studied with the help of kurtosis. Prof. Karl Pearson has called it the "convexity of a curve" Kurtosis gives a measure of flatness of distribution.

  The degree of Kurtosis of a distribution is measured relative to that of a normal curve. The curves with **greater peakness** than the normal curve are called **leptokurtic**. The curves which are **more flat** than the normal curve are called **Platykurtic.** The normal curve is called 'Mesokurtic'. The following figure describe the three different curves mentioned above.



- **Measures of Kurtosis.**

(1) Karl Pearson's Measure of Kurtosis

For calculating the Kurtosis, the second and fourth central moments of variable are used. For this, following formula give by Karl Pearson is used :

$$\beta_2 = \frac{\mu_4}{\mu_2} \text{ or } \gamma_2 = \beta_2 - 3$$

where, μ_2 = second order central moment of distribution

μ_4 = Fourth order central moment of distribution

Interpretation :

- If $\beta_2 = 3$ or $\gamma_2 = 0$, then curve is said to be mesokurtic.
- If $\beta_2 < 3$ or $\gamma_2 < 0$, then curve is said to be platykurtic.
- If $\beta_2 > 3$ or $\gamma_2 > 0$, then curve is said to be leptokurtic.

(2) Kelly's Measure of Kurtosis

Kelly has given a measure of Kurtosis based on percentiles.

The formula is given by

$$\beta_2 = \frac{P_{75} - P_{25}}{P_{90} - P_{10}}$$

where, P_{75}, P_{25}, P_{90} and P_{10} are 75^{th}, 25^{th}, 90^{th} and 10^{th} percentiles of dispersion, respectively.

Interpretation :

- If $\beta_2 > 0.26315$, then the distribution is platykurtic.
- If $\beta_2 < 0.26315$, then the distribution is leptokurtic.

For example : First four moments about mean of a distribution are 0, 2.5, 0.7 and 18.75. Find coefficient of skewness and kurtosis.

Sol. We have, $\mu_1 = 0$, $\mu_2 = 25$, $\mu_3 = 0.7$ and $\mu_4 = 18.75$

Therefore,

$$\text{skewness, } \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(0.7)^2}{(2.5)^2} = 0.031$$

$$\text{Kurtosis, } \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{18.75}{(2.5)^2} = \frac{18.75}{6.25} = 3$$

As β_2 is equal to 3, so the curve is mesokurtic.

CHAPTER-13

STATISTICS APPLICATIONS

Revision Notes

➢ **Percentile Rank and Quartile Rank**

When ranking numbers, such as test scores or height of a student, it can be helpful to conceptualize one rank in relation to another. For example, you might want to know if you scored higher or lower than the rest of your class or if your height is longer or shorter than the other student in your class. One way to conceptualize a ranking system is through the use of percentiles.

- **Percentile rank** is a usual term mainly used in statistics which is arrived from percentile. Percentile (also referred to as centile) is the percentage of scores ranges between 0 to 100, which is less than or equal to the given set of distribution.
- Percentiles divide any distribution into 100 equal parts. This is pre–dominantly used for the interpretation of scores with the different ranges across various test.
- Percentile Rank (PR) is arrived base on the total number of ranks and the number of ranks below and above percentile.

(I) Percentile Rank for Individual and Discrete series

(i) To identify percentile rank (PR) of score *X*, out of *N*, where *X* is included

Percentile Rank (PR) = $\left[\dfrac{M+(0.5\times R)}{Y}\right]\times 100$

where, M = Number of ranks below X
R = Number of ranks equals X
Y = Total number of ranks

For example :

Assume there are 20 students in a class. All are participated in quiz contest. Mukesh and Radha are two among those 20 students and both are ranking as 10^{th} among 20. Calculate the percentile of Mukesh.

Sol.(1) A total number of ranks will be the same as the total no. of students in this case. So, $Y = 20$

(2) Percentile rank needs to be calculated for Mukesh who is at 10^{th} rank. Hence, $X = 10$

(3) Identifying the count the same 10^{th} rank. In this case, it will be Mukesh and Radha. So, $R = 2$

(4) Count the ranks that are less than 10, which will be considered as M, Hence, $M = 9$

(5) Apply formula :

$$PR = \left[\frac{M+(0.5\times R)}{Y}\right]\times 100$$

$$= \left[\frac{9+(0.5\times 2)}{20}\right]\times 100$$

$$= \frac{10}{20}\times 100 = 50\%$$

Hence, the percentile rank for Mukesh will be 50%. Mukesh standing as 50^{th} percentile in his class.

(ii) To identify percentile rank (PR) of score X out of Y, where X is not included

Percentile Rank (PR) = $\dfrac{M}{Y}\times 100$

where, M = number of rank at X
Y = Total number of ranks

For example :

There are 9 Students attended Aptitude test in a class. Scores of the students are 7, 6, 4, 25, 20, 21, 20, 19, 6. Identify the percentile rank for score 7.

Sol. (1) Arrange the score data set in ascending order

4, 6, 6, 7, 19, 20, 20, 21, 25

(2) Add ranking for the ordered score :

Score	4	6	6	7	19	20	20	21	25
Ranking	1	2	3	4	5	6	7	8	9

(3) A total number of ranks will be the same as the total number of students in this case. So, $Y = 9$

(4) Percentile rank needs to be calculated for score 7, which is at 4^{th} rank. Hence, $X = 4$

(5) Count the ranks at 4 which will be considered as M. Hence, $M = 4$.

(6) Here, there is no score same as X *i.e.* there is no repeated value same as 7. Thus, here $R = 0$ and we apply formula

$$PR = \frac{M}{Y} \times 100$$

$$i.e., \quad PR = \frac{4}{9} \times 100 = 44\% \text{ (Approx)}$$

This concludes score 7 is 44% of the scores.

(II) Percentile Rank for Continuous grouped series

When the given data set is in the form of continuous series, the following formula is used to calculate percentile rank,

$$PR = \frac{100}{N} + \left(c.f. + \frac{X-l}{i} \times f\right)$$

where, X = score for which PR to be calculated
$c.f.$ = cumulative frequency of class preceding to class having X
i = class interval
l = lower limit having X
f = frequency of $C.\ I.$ having X
N = Total frequency

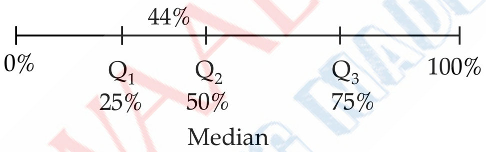
- **Quartile Rank**

As we know, quartiles divides the data set into four equal part. In addition, every 25th percentile is known as one quartile. Out of 100, the 25th percentile is known as 1st quartile. 50th percentile is known as 2nd quartile or median, the 75th percentile is called as 3rd quartile.

Quartile ranking can calculated with the help of percentile ranking.

For example : In the previous example, we have calculated the percentile rank, $PR = 44\%$

Now, according to quartile scale, 44% lies between



25% to 50% as shown above. So, the quartile ranking of score 7 is Q_2 or second quartile.

- **Correlation**

Consider the heights of husbands and the wives at the time of marriage. If the height of the bridegroom is represented by x in general, and that the bride by y, then to each marriage there corresponds a pair of values (x_1, y_1) of the variables x and y. Now, our object is to discover whether there is any connection between stature of husband (x) and stature of wife (y). Do tall men tend on the average to wed all women, or do we find tall men choosing short women for wives just about as often as they choose tall women ? Then we try to find out a relation between x and y. **Whenever two variables x and y are so related that a change in one is accompanied by a change in other in such a way that an increase (decrease) in one is accompanied by an increase or decrease (or decrease or increase), in the other, then variables are said to be correlated.**

Variates may be heights and weights of students in a class, amount of fertiliser per hectare and the yield of grain per hectare on the number of different plots, the age and yields of milk in cow, records of rainfall and fields of crops or relation between income and expenditure etc.

There are two types of correlation :

(i) Positive Correlation : If two variables move in the same direction *i.e.*, an increase (or decrease) on the part of one variable introduces an increase (or decrease) on the part of the other variable, then the two variables are known to be **positively correlated.** As for example, height and weight, yield and rainfall, profit and investment etc, are positively correlated.

(ii) Negative Correlation : If two variables move in the opposite direction *i.e.*, an increase (or a decrease) on the part of one variable results a decrease (or increase) on the part of the variable, then the two variables are known to have a **negative correlation.** The price and demand of an item, the profits of Insurance Company and the number of claims it has to meet etc, are examples of variables having negative correlation.

Remark :

- The two variables are known to be uncorrelated if the movement on the part of one variable does not produce any movement of the other variable in a particular direction. For example : Shoe size and intelligence are uncorrelated.

- **Measures of Correlation**

There are mainly three measures of correlation :

(i) Scatter Diagram (ii) karl Pearson's Coefficient of Correlation and (iii) Spearman's Rank Correlation Coefficient. Here, we are going two discuss the last two measures of correlation.

(I) Karl Pearson's Coefficient of Correlation

This is the best method for finding correlation between two variables provided the relationship between the two variables is linear. This method is also known as **product moment correlation coefficient.** Pearson's correlation coefficient may be defined as the **ratio of covariance** between the two variables to the **product of the standard deviations** of the two variables.

If the two variables are denoted by x and y and of the corresponding bivariates data are (x_i, y_i) for $i = 1, 2, 3, ...,$ n, then the coefficent of correlation between x and y due to Karl Pearson, is given by :

$$r = r_{xy} \text{ or } f(x, y) = \frac{\text{Cov}(x,y)}{\sqrt{Var\ x}.\sqrt{Var\ y}} = \frac{\text{Cov}(x,y)}{\sigma_x.\sigma_y}$$

where,

$$\text{Cov}(x, y) = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{N} = \frac{\Sigma xy}{N} - \bar{x}.\bar{y}$$

$$\sigma_x = \sqrt{\frac{\Sigma(x-\bar{x})^2}{N}} = \sqrt{\frac{\Sigma x^2}{N} - \bar{x}^2}$$

$$\sigma_y = \sqrt{\frac{\Sigma(y-\bar{y})^2}{N}} = \sqrt{\frac{\Sigma y^2}{N} - \bar{y}^2}$$

- If $x - \bar{x}$, $y - \bar{y}$ are small fractions, we use

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2}\sqrt{\Sigma(y-\bar{y})^2}} \quad ....(1)$$

- If x, y are small numbers, we use

$$r = \frac{\Sigma xy - \frac{1}{N}\Sigma x \Sigma y}{\sqrt{\Sigma x^2 - \frac{1}{N}(\Sigma x)^2}\sqrt{\Sigma y^2 - \frac{1}{N}(\Sigma y)^2}} \quad ....(2)$$

- If x, y are large numbers, we use assumed mean A and B and $u = x - A$, $v = y - B$

$$r = \frac{\Sigma uv - \frac{1}{N}\Sigma u \Sigma v}{\sqrt{\Sigma u^2 - \frac{1}{N}(\Sigma u)^2}\sqrt{\Sigma v^2 - \frac{1}{N}(\Sigma v)^2}} \quad ....(3)$$

For example :

Find Karl Pearson's coefficient of correlation between X and Y for the following :

x	5	4	3	2	1
y	4	2	10	8	6

Sol. Here, $N = 5$ and x, y are small numbers, so here we use formula **(1)**

x	x^2	y	y^2	xy
5	25	4	16	20
4	16	2	4	8
3	9	10	100	30
2	4	8	64	16
1	1	6	36	6
$\Sigma x = 15$	$\Sigma x^2 = 55$	$\Sigma y = 30$	$\Sigma y^2 = 220$	$\Sigma xy = 80$

$$\therefore \qquad r = \frac{\Sigma xy - \frac{1}{N}\Sigma x \Sigma y}{\sqrt{\Sigma x^2 - \frac{(\Sigma x)^2}{N}}\sqrt{\Sigma y^2 - \frac{(\Sigma y)^2}{N}}}$$

$$= \frac{80 - \frac{1}{5} \times 15 \times 30}{\sqrt{55 - \frac{(15)^2}{5}}\sqrt{220 - \frac{(30)^2}{5}}}$$

$$= \frac{-10}{\sqrt{10}\sqrt{40}} = \frac{-10}{20} = -0.5.$$

- **Properties of Correlation Coefficient**

(i) r has no unit and $-1 \le r \le 1$.
(ii) A negative value of r indicates an inverse relation.
(iii) If r is positive the two variables move in the same direction.
(iv) If $r = 0$, the two variables are uncorrelated *i.e*, there is no linear relation between them.
(v) If $r = 1$ or $r = -1$, the correlation is perfect.
(vi) The magnitude of r is unaffected by change of origin and change of scale. *i.e.*, $r_{xy} = r_{uv}$

(II) Spearman's Rank Correlation Coefficient

When we need finding correlation between two qualitative characteristics, say, beauty and intelligence, we take recourse to using rank correlation coefficient. Rank correlation can also be applied to find the level of agreement (or disagreement) between two judges so far as assessing a qualitative characteristic is concerned. As compared to product moment correlation coefficient, rank correlation is easier to compute, it can also be advocated to get a first hand impression about the correlation between a pair of variables.

Spearman's rank correlation coefficient is given by

$$r_S = 1 - \frac{6\Sigma d^2}{n(n^2-1)}$$

where,

d = difference between ranks of corresponding x and y
n = number of pairs of values (x,y) in the data

- When the rank are **repeated,** the Spearman's rank correlation coefficient formula is given by

$$r_S = 1 - \frac{6\left[\Sigma d^2 + \frac{(m_1{}^3 - m_1)}{12} + \frac{(m_2{}^3 - m_2)}{12} + ...\right]}{n(n^2-1)}$$

where, m_1, m_2, ..., are the number of repetitions of ranks and $\frac{m_1{}^3 - m_1}{12}$ ...... their corresponding correlation factors.

For example :

Let 5 students be ranked in Maths and Physics as

Student	A	B	C	D	E
Maths	1	2	3	4	5
Physics	1	2	3	4	5

Here, we can see that $d = 0$ for each pair, so $r = +1$

Now, Let the same 5 students be ranked in Maths and Sports are

Student	A	B	C	D	E
Maths	1	2	3	4	5
Sports	5	4	3	2	1

Then, we have differences $d = -4, -2, 0, 2, 4$. So, $\Sigma d^2 = 40$, and $r_S = 1 - \frac{6(40)}{5(25-1)} = -1$

Thus, we can observe that $r_S = +1$, when ranks are in complete agreement but in the same direction. Also, $r_S = -1$ when ranks are incomplete agreement but in opposite direction otherwise, r_S varies between -1 and $+1$.

Now, let us assume that 5 students are give marks as follows in Maths and English :

Student	A	B	C	D	E
Maths	90	80	70	60	50
English	90	70	80	60	50

To find Spearman's rank correlation, we have

Students	Maths marks	Rank in Maths (x)	English marks	Rank in English (y)	Difference in rank $d = x - y$	d^2
A	90	1	90	1	0	0
B	80	2	70	3	–1	1
C	70	3	80	2	1	1
D	60	4	60	4	0	0
E	50	5	50	5	0	0

Here, $n = 5, \Sigma d^2 = 2$

$$\therefore r_S = 1 - \frac{6(2)}{5(5^2 - 1)} = +0.9$$

Now, let us assume that the marks in History and Science are :

Student	A	B	C	D	E
History	70	70	60	50	40
Science	80	80	80	60	60

Here, rank of A and B for History is same, so we use average rank for A and B for History as $\frac{1+2}{2} = 1.5$. Similarly, rank for Science for A,B,C is same, so $\frac{1+2+3}{3} = 2$. Also, rank for Science for D and E is $\frac{4+5}{2} = 4.5$. so we have

Student	Rand in History x	Rank in Science y	Difference $d = x - y$	d^2
A	1.5	2	–0.5	0.25
B	1.5	2	–0.5	0.25
C	3	2	1	1
D	4	4.5	–0.5	0.25
E	5	4.5	0.5	0.25

Here, $n = 5, \Sigma d^2 = 2$ and we have to add 3 corrections due to three ties (2 in y, of 3 and 2 terms and one in x, of 2 terms). So,

$$r_S = 1 - \frac{6\left[2 + \frac{1}{12}(3^3 - 3) + \frac{1}{12}(2^3 - 2) + \frac{1}{2}(2^3 - 2)\right]}{5(5^2 - 1)}$$

$$= 1 - \frac{6(2 + 2 + 0.5 + 0.5)}{120} = 1 - \frac{30}{120} = 1 - 0.25 = 0.75$$

Thus, $r_S = 0.75$ for repeated ranks in this case.

UNIT – VII: BASICS OF FINANCIAL MATHEMATICS

CHAPTER-14

BASIC CONCEPTS OF FINANCIAL MATHEMATICS

TOPIC-1
Interest, Present Value and Future Value

Revision Notes

- People earn money for spending it on housing, food, clothing, education, entertainment etc. Sometimes extra expenditures have also to be met with. For example, there might be a marriage in a family; one may want to buy house, one may want to set up his or her business, one may want to buy a car and so on. Some people can manage to put aside some money for such expected and unexpected expenditures. But most of people have to borrow money for such contingencies.
  Money can borrowed from friends or money lenders or Banks. If you can arrange a loan from a friend it might be interest free but if you borrow money from lenders or Banks, you will have to pay some charge periodically for using money of money lenders or Banks. This charge is called **Interest.**
- **Time Value of Money :** Time value of money means that the value of a unity of money is different in different time periods. The sum of money received after some time in future is less valuable than it is today. In other words, the present worth of rupees received after some time will be less than a rupee received today. Since a rupee received today has more value rational investor would prefer current receipts to future receipts. If they postpone there receipts they will certainly charge some money *i.e.*, interest.
- **Interest :** Interest is a price paid by a borrower for the use of a lender's money. If you borrow (or land) some money from (or to) a person for a particular period you would pay (or receive) more money that your initial borrowing (or lending). This excess money paid (or receive) is called **interest.**
  For example : Suppose you borrow (or lend) ₹ 5000 for a year and you pay (or receive) ₹ 55,000 after one year in the difference between initial borrowing (for lending) ₹ 50,000 and end payment (or receipts) ₹ 55,000 *i.e.*, ₹ 5000 is the amount of interest you paid (or earned).
- **Principal :** Principal is initial value of lending (or borrowing). If you invest your money, the value of initial investment is also called **principal.**
  For example : (i) Suppose you borrow (or lend) ₹ 50,000 from a person for one year. ₹ 50,000 in this example is the 'Principal'.
  (ii) Suppose you deposit ₹ 20,000 in your bank account for one year, then here ₹ 20,000 is the 'Principal'.
- **Rate of Interest :** The rate at which the interest is charged for a defined length of time (period) for use of principal, generally on yearly basis is known to be the **rate of interest.** Rate of interest is usually expressed as percentages.
  For example : Suppose you invest ₹ 20,000 in your bank account for one year with the interest rate 5% per annum. It means you would earn ₹ 5 as interest every ₹ 100 of principal amount in a year. (per annum (*p.a.*) means for a year).
- **Accumulated Amount (or Balance) :** Accumulated amount is the final value of an investment. It is the sum of principal and interest earned.
  For example : Suppose you deposit ₹ 50,000 in your bank for one year with a interest rate of 5% *p.a.*, you would earned interest of ₹ 2500 after one year. After one year you will get ₹ 52,500 (principal + interest), ₹ 52,500 is **accumulated amount** or **amount** or **balance.**
- **Method of Analysis :** The concept of time value of money helps in arriving at the comparable value of the different rupee amount arising at different points of time into equivalent values of particular point of time (present or future). This can be done by either –
  (a) Compounding : The present money to future data *i.e.*, finding out future values of present money, or
  (b) Discounting : Future money to the present date *i.e.*, finding out present value of future money

Present Date		Future Date
Present Cash Flow or Present Money	Compounding → ← discounting	Future Cash Flow or Future money

➢ **Simple Interest :** Simple interest is the interest computed on the principal for the entire period of borrowing. It is calculated on the outstanding principal balance and not on the interest previously earned. It means no interest paid on interest earned during the term of loan.

Simple Interest can be computed by applying following formulas :

- $$I = Pit$$
- $$\begin{aligned} A &= P + I \\ &= P + Pit \\ &= P(1 + it) \end{aligned}$$
- $$I = A - P$$

Here,

A = Accumulated amount (final value of an investment)
P = Principal (initial value of an investment)
i = Annual interest rate in decimal
I = Amount of Interest
t = Time in years

For example : To buy a furniture for a new apartment, Aneesha borrowed ₹ 5000 at 8% simple interest for 2 years. How much interest will she pay ?

Sol. Required interest amount is given by

$$I = P \times i \times t$$

Given, $P = ₹\,5000,\ i = 8\% = \frac{8}{100} = 0.08$ and $t = 2$ years

So, $I = 5000 \times 0.08 \times 2 = ₹\,800$

Thus, total interest she will pay is ₹ 800.

➢ **Compound Interest :** Simple interest is normally used for loans or investment of a year or less. For the longer periods compound interest is used. In compound interest, the interest is added (or compounded) to the principal sum so that interest is paid on the whole amount. Under this method, if the interest for the first year is left in the account, the interest for the second year is calculated on the whole amount, so far accumulated.

For example :

Suppose you deposit ₹ 50,000 in ICICI bank for 2 years at 7% *p.a.* compounded annually. Interest will be calculated in the following way :

Interest for 1st year

$$\begin{aligned} I &= Pit \\ &= 50{,}000 \times \frac{7}{100} \times 1 = ₹\,3500 \end{aligned}$$

Interest for 2nd year :

For calculating interest for second year principal would not be the initial deposit. Principal for calculating interest for second year will be the initial deposit plus interest for first year,

Therefore principal for calculating interest for second year would be

$$\begin{aligned} &= ₹\,50{,}000 + ₹\,3500 \\ &= ₹\,53{,}500 \end{aligned}$$

$\therefore$ Interest for the 2nd year $= ₹\,53{,}500 \times \frac{7}{100} \times 1$

$= ₹\,3745$

Total interest = interest for 1st year + interest for 2nd year

$= ₹\,(3500 + 3745) = ₹\,7245$

This interest is ₹ 245 more than the simple interest on ₹ 50,000 for two year at 7% *p.a.*

- **Conversion Period :** In the above example, the interest was calculated on yearly basis *i.e.*, the interest was compounded annually. However, in practice it is not necessary that interest be compounded annually. For example; in banks the interest is often compounded twice in a year (half yearly of semi annually) *i.e.*, interest is calculated and added to the principal after every six months. In some financial institutions interest is compounded quarterly *i.e.*, four times in a year. **The period at the end of which the interest is compounded is compounded is called conversion period.**

Typical conversion periods are given in the following table :

Conversion Period	Description	Number of conversion period in a year
1 day	Compounded daily	365
1 month	Compounded monthly	12
3 months	Compounded quarterly	4
6 months	Compounded semi annually	2
12 months	Compounded annually	1

- **Formula for Compound Interest :**
The accrued amount (compound amount) A_n on a principal P after n conversion periods at r(in decimal) rate of interest per conversion period is given by

$$A_n = P(1+i)^n$$

where, $$i = \frac{\text{Annual rate of interest}}{\text{Number of conversion periods per years}}$$

$$\text{Interest} = A_n - P = P(1+i)^n - P$$
$$= P[(1+i)^n - 1]$$

n is total conversions *i.e.*, $t \times$ no. of conversion per year.

For example : Compute the compound interest on ₹ 4000 for 1½ years at 10% per annum compounded half yearly.

Sol. Here, principal P = ₹ 4000. Since, the interest is compounded half-yearly the number of conversion periods in $1\frac{1}{2}$ years are 3. Also, the rate of interest per conversion period (6 months) is $10\% \times \frac{1}{2} = 5\%$ (0.05 in decimal)

Thus, A_n(in₹) is given by

$$A_n = P(1+i)^n$$

or, $$A_3 = 4000\,(1+0.05)^3 = ₹\,4630.50$$

The compound interest is therefore ₹ (4630.50 – 4000)

= ₹ 630.50.

➢ Effective Rate of Interest

If interest is compounded more than once a year the effective interest rate for a year exceeds the per annum interest rate. In other words, the actual interest rate or simple interest rate that is equivalent to compound interest is called **effective interest rate.**

For example :

Suppose you invest ₹ 10,000 for a year at the rate of 6% per annum compounded semi-annually. Effective interest rate for a year will be more than 6% per annum since interest is being compounded more than once in a year. For computing effective rate of interest first we have to compute the interest.

Using formula of compound interest

$$A_n = P\,(1+i)^n$$

Here, $$P = ₹\,10{,}000,\ i = 6\% \times \frac{1}{2} = 3\% = 0.03,\ n = 2$$

$\therefore$ $$A_n = 10{,}000(1+0.03)^2$$
$$= 10{,}000 \times 1.0609 = ₹\,10{,}609$$

$$\text{Interest} = A_n - P = 10{,}609 - 10{,}000 = ₹\,609$$

We can compute effective rate of interest by following formula

$$I = PEt$$

where,
I = Amount of interest
E = Effective rate of interest in decimal
t = Time period
P = Principal amount

Putting the values, we have

$$609 = 10{,}000 \times E \times 1$$

$\Rightarrow$ $$E = \frac{609}{10{,}000} = 0.0609 \text{ or } 6.09\%$$

Thus, "if we compound the interest more than once a year effective interest rate for the year will be more than actual (nominal or stated) interest rate per annum, But if interest is compound annually effective interest rate for the year will be equal to actual interest rate per annum."

The effective interest rate can be computed directly by following formula :

$$E = (1+i)^n - 1$$

where, E is effective interest rate
i = actual interest rate
n = number of conversion period

Effective interest rate for above example can be calculated using direct formula as :

$$E = (1+0.03)^2 - 1$$
$$= (1.03)^2 - 1$$
$$= 1.0609 - 1$$
$$= 0.0609 = 6.09\%$$

- **Equivalence of Interest Rates :**

Two rates are said to be equivalent if, for the same initial investment and over the same time interval (one full year,for example), the final value of the investment, calculated with the two interest rate, is equal. Consider a case where we deposit an amount P in a bank offering a quarterly interest rate i_{quart}. At he end of the year *i.e.,* 4 quarters, the value of investment will be:

$$P\ (1 + i_{quart})^4$$

Now, suppose that another bank offers an annual interest rate i_{ann}. At the end of one full year, the value of investment will be :

$$P\ (1 + i_{ann})^1$$

According to definition, the rate i_{quart} and i_{ann} are equivalent if

$$P\ (1 + i_{quart})^4 = P(1 + i_{ann})^1$$

$$\Rightarrow \quad (1 + i_{quart})^4 = (1 + i_{ann})^1$$

Note that the initial investment value is of no importance. The relation allows us to pass from a quarterly interest rate to an equivalent annual interest rate.

For example :

A bank A offers you an (effective) annual interest rate of 6%, the bank B offers an interest rate of 1.5%. per quarter

Which of these two banks offers the best return ?

Sol. Bank B offers a quarterly rate 1.5%, the equivalent annual interest rate (or effective rate) for this interest can be obtained by the relation

$$(1 + i_{quart})^4 = 1 + i_{ann}$$

$$(1.015)^4 = 1 + i_{ann}$$

$$i_{ann} = (1.015)^4 - 1 = 0.06136 = 6.136\%$$

Bank B, therefore offers a better return with (effective) annual interest rate of 6.136% than bank A.

Alternative solution :

Bank A offers an effective interest rate 6%. The quarterly equivalent is obtained from the relation :

$$(1 + i_{quart})^4 = 1 + i_{ann}$$

$$(1 + i_{quart})^4 = 1 + 0.06 = 1.06$$

$$[(1 + i_{quart})^4]^{1/4} = (1.06)^{1/4}$$

$$1 + i_{quart} = (1.06)^{1/4}$$

Taking *log* both sides,

$$\log(1 + i_{quart}) = \frac{1}{4} \log(1.06)$$

$$\Rightarrow \quad \log(1 + i_{quart}) = \frac{1}{4} \times 0.0253$$

$$\Rightarrow \quad \log(1 + i_{quart}) = 0.006325$$

Taking antilog both sides.

$$1 + i_{quart} = \text{antilog}(0.006325)$$

$$\Rightarrow \quad 1 + i_{quart} = 1.01467$$

$$\Rightarrow \quad i_{quart} = 0.01467 = 1.46\%$$

This rate is inferior to the quarterly interest rate offered by bank B and we arrive at the same conclusion.

The reasoning we just made applies to all interest rate conversions. A periodic interest rate can always be converted given that the rate equivalence relation is respected.

Interest Equivalence Relation

$$(1 + i_{ann})^1 = (1 + i_{bian})^2 = (1 + i_{quart})^4 = (1 + i_{mon})^{12}$$

where,

i_{ann} – annual interest rate

i_{bian} – bi-annually or semi-annual interest rate

i_{quart} – quarterly interest rate

i_{mon} – monthly interest rate

- **Future Value (F.V.) :** Future value is the cash value of an investment at some time in the future. It is tomorrow's value of today's money compounded at the rate of interest.

For example : Suppose you invest ₹ 1000 in fixed deposit that pays you 7% per annum as interest. At the end of first year you will have ₹ 1070. This consist of the original principal of ₹ 1000 and interest earned of ₹ 70. ₹ 1070 is the future value of ₹ 1000 invested for one year at 7%. We can say that ₹ 1000 today is worth ₹ 1070 in one year's time if the interest rate is 7%.

We know that,

$$A = P(1 + i)^n$$

where, A = Accumulated amount

n = no. of conversion period

i = rate of interest per conversion period in decimal

P = principal

Future value of a single cash flow can be computed by above formula. Replace A by future value (F.V.) and P by single cash flow (C.F.), therefore

$$\text{F.V.} = \text{C.F.}(1 + i)^n$$

For example : You invest ₹ 3000 in a two year investment that pay you 12% per annum. Calculate the future value of the investment.

Sol. We know that

$$\text{FV} = \text{C.F.}(1 + i)^n$$

Here, $\text{C.F.} = ₹\ 3000$

$$i = 12\% = \frac{12}{100} = 0.12$$

$$n = \text{time period} = 2 \text{ years}$$

$$\therefore \quad \text{F.V.} = 3000(1 + 0.12)^2$$

$$= 3000(1.12)^2$$

$$= 3000 \times 1.2544$$

$$= ₹\ 3763.20$$

- **Present Value (P.V.) :**

We have learned that future value is tomorrow's value of today's money compounded at some interest rate.

Present Value is today's value of tomorrow's money discounted at the interest rate. Future and present value related to each other infact they are reciprocal of each other.

For example : Suppose we invest ₹ 1000 for two years at 7% per annum, we will get ₹ 1144.90 offer two years. It means ₹ 1144.90 is the future value of today's ₹ 1000 at 7% and ₹ 1000 is the present value of ₹ 1144.90 where time period is two years and rate of interest is 7% per annum.

We can get the present value of a cash flow (inflow or outflow) by applying compound interest formula.

The present value P.V. of the amount A_n due to n conversion period at the rate i per interest period may be obtained by solving for P the below given. equation

$$A_n = \text{P.V.}(1 + i)^n$$

$$i.e,\ \text{P.V.} = \frac{A_n}{(1+i)^n}$$

For example : What is the present value of ₹ 1 to be received after two years compounded annually at 10% interest rate ?

Sol. Here, $A_n = ₹\ 1$

$$i = 10\% = \frac{10}{100} = 0.1$$

$$n = 2$$

$$\text{Present Value, (P.V.)} = \frac{A_n}{(1+i)^n}$$

$$= \frac{1}{(1+0.1)^2}$$

$$= \frac{1}{(1.1)^2} = \frac{1}{1.21} = 0.8264 \approx ₹\ 0.83$$

Thus, ₹ 0.83 shall grow to ₹ 1 after 2 years at 10% interest rate compounded annually.

- **Discounting :** Future money to the present date involves finding out present value of future money. Discounting is the opposite of compounding and hence, Present Value of future cash flow is given by the following formula :

$$\text{Present Value (P.V.)} = \frac{FV_1}{(1+i)^1} + \frac{FV_2}{(1+i)^2} + \frac{FV_3}{(1+i)^3} + \dots + \frac{FV_n}{(1+i)^n}$$

where, 1, 2, 3, 4, ... n represent future time periods and FV = cash flow arising at those future point of time and i denotes the discount rate/rate of interest.

Note : Present value of a single cash flow : Present Value of single cash flow is given by the formula $\text{P.V.} = \dfrac{\text{C.F.}}{(1+i)^n}$.

- **Net present Value (NPV) :** The net present value or net present worth is the difference the present value of cash in flows and the present value of cash outflows over a period of time that occurs as the result of undertaking an investment
  - If present value of cash inflow > present value of cash outflow NPV is positive and the project is acceptable.
  - If present value of cash inflow = Present value of cash outflow NPV is zero and the project is acceptable.
  - If present value of cash inflow < Present value of cash outflow NPV is negative and the project is not acceptable.

For example : A friend needs ₹ 500 now, and will pay you back ₹ 570 in a year. Is that a good investment when you can get 10% elsewhere ?

Sol. Cash outflow : ₹ 500 now

You invested ₹ 500 now

So, P.V. = – ₹ 500

Cash inflow : ₹ 570 next year

$$\text{P.V.} = ₹\frac{570}{(1+0.10)^1} = \frac{570}{1.10} = ₹\ 518.18$$

The net amount is

Net Present Value = ₹ (518.18 – 500) = ₹ 18.18

So, at 10% interest, that investment is worth ₹ 18.18.

(It is a good investment)

TOPIC-2
Annuities

Revision Notes

- **Annuity**

In many cases you must have noted that your parents have to pay equal amount of money regularly like every month or every year. For example payment of life insurance premium, rent of your house (if you stay in rented house), payment of housing loan, vehicle loan etc. In all these cases they pay a constant amount of money regularly. Time period between two consecutive payments may be one month, one quarter or one year.

Sometimes some people received a fixed amount of money regularly like pension, rent of house etc. In all these cases annuity comes into the picture. When we pay (or receive) a fixed amount of money periodically over a specified period of time, it is know as **annuity.** In other words, **Annuity** can be defined as a sequence of periodic payments (or receipts) regularly over a specified period of time.

There is a special kind of annuity also that is called **Perpetuity.** It is one where the receipt or payment takes place forever. Since, the payment is forever we cannot compute a future value of perpetuity. However, we can compute the present value of the perpetuity.

To be called annuity a series of payments (or receipts) must have following features :

(i) Amount paid (or received) must be constant over the period of annuity and

(ii) Time interval between two consecutive payments (or receipts) must be the same.

The annuity is described through the following parameters :

1. **Annuity Component :** the value of each separate payment.
2. **Annuity Term :** Time from the beginning of the annuity till the end of its last period (interval)
3. **Annuity interval :** time interval between payments.
4. **Interest Rate :** The rate used for accumulation or discounting of payments of which the annuity is composed.

- **Types of Annuities**

(1) **Annual Annuity :** An annuity is called annual if its period equals one year.

(2) **p-due Annuity :** An annuity is called *p*-due, if its period is less than a year and the number of annual payments is *p*.

(These annuities are **discrete** since their payments are coordinates with discrete time points. There are **continuous** annuities when the payment stream is characterized with the continuous function.)

(3) **Constant and variable Annuities :** Annuities may be constant and variable. An annuity is constant if all its payments are equal and do not change in time. If amounts of payment depend on time, the annuity is variable.

(4) **Annuity Regular :** In annuity regular, first payment/receipt take place at he end of first period. Consider the following table :

Year End	Payment in ₹
I	5000
II	5000
III	5000
IV	5000
V	5000
VI	5000

We can see that first payment/receipt takes place at the end of the first year, therefore it is an annuity regular. It is also known as Ordinary Annuity.

(5) Annuity Due/Annuity Immediate : When the first receipts or payment is made today (at the beginning of the annuity) it is called annuity due or annuity immediate. Consider the following table :

In the beginning of	Payment in ₹
I year	5000
II year	5000
III year	5000
IV year	5000
V year	5000
VI year	5000

We can see that first receipt or payment is made in the beginning of the first year. This type of annuity is called annuity due or annuity immediate.

- **Future value of an Annuity Regular**

If C.F. be the periodic payments, (cash flow in each period), the future value F.(n, i) of the annuity is given by

$$\text{Future value (F.V.) of an Annuity} = \text{C.F.}\left[\frac{(1+i)^n - 1}{i}\right]$$

Here, i = interest rate per period (in decimal)

n = number of periods

For example : Find the future value of an annuity of ₹ 500 made annually for 3 years at interest rate of 14% compounded annually .

Sol. Here, annual payment C.F. = ₹ 500

$n = 3, i = 14\% = \frac{14}{100} = 0.14$

Future value of the annuity

$$= 500\left[\frac{(1+0.14)^3 - 1}{0.14}\right]$$

$$= 500\left[\frac{(1.14)^3 - 1}{0.14}\right]$$

$$= 500\left(\frac{1.4815 - 1}{0.14}\right)$$

$$= \frac{500 \times 0.4815}{0.14}$$

$$= \frac{240.75}{0.14}$$

$$= ₹\ 1719.64$$

- **Present Value of an Annuity Regular**

The present value of an annuity of n payments at end of consecutive interest periods with compounded at rate of interest i per period is

$$\text{Present value of an Annuity Regular} = C.F. \times \frac{[(1+i)^n - 1]}{i(1+i)^n}$$

i.e., In other words,

$$\text{Present value (P.V.) of an Annuity Regular} = \frac{\text{Future value(F.V.)of Annuity Regular}}{(1+i)^n}$$

Consequently,

$$\text{C.F.} = \frac{\text{Present value of an Annuity Regular(P.V.)}}{P\,(n,i)}$$

Where

$$P(n, i) = \frac{[(1+i)^n - 1]}{i(1+i)^n}$$

is useful in problems of amortization.

Amortization : A loan with fixed rate of interest is said to be amortization if entire principal and interest are paid over equal periods of time by way of sequence of equal payments.

For example : Suppose your dad purchases a car for ₹ 5,50,000. He gets a loan of ₹ 5,00,000 at 15% p.a. from a bank and balance 50,000 he pays at the time of purchase. Your dad has to pay whole amount of loan in 2 equal installments with interest starting from the end of every 6 months.

Sol. Here, we have to calculate how much money has to be paid at the end of every six months. we can compute equal installment by following formula.

$$\text{C.F.} = \frac{\text{P.V.}}{P(n, i)}$$

Here,

$$\text{P.V.} = ₹\,5{,}00{,}000$$

$$n = 2$$

$$i = \frac{15\%}{2} = \frac{5}{2 \times 100} = \frac{0.15}{2} = 0.075$$

$$P(n, i) = \frac{(1+i)^n - 1}{i(1+i)^n}$$

$$P(2, 0.075) = \frac{(1+0.075)^2 - 1}{0.075(1+0.075)^2}$$

$$= \frac{(1.075)^2 - 1}{0.075(1.075)^2}$$

$$= \frac{1.1556 - 1}{0.075 \times 1.1556} = \frac{0.1556}{0.08667} = 1.7953$$

$\therefore$

$$\text{C.F.} = \frac{500000}{1.7953} = 278{,}504.985$$

Therefore, your dad will have to pay 2 equal installments of ₹ 278,504.9 at the end of every six months.

➢ **Applications**

(i) Sinking funds : If funds credited for a specified purpose by way of sequence of periodic payments over a time period at a specified interest rate. Interest is compounded at the end of every period. If you make payment of PMT at the end of every period, the future value after t years or n = mt periods will be

$$FV = PMT\frac{(1+i)^n - 1}{i}$$

For example : How much amount is required to be invested every years so as to accumulate ₹ 300000 at the end of 3 years, if interest is compounded annually at 10% ?

Sol. Here, given $F.V. = 300000$, $i = 10\% = \dfrac{10}{100} = 0.1$, $n = 3$

$\therefore$

$$PMT = \text{F.V.}\frac{i}{(1+i)^n - 1}$$

$$= 300000\left\{\frac{0.1}{(1+0.1)^3 - 1}\right\}$$

$$= 300000 \times \frac{0.1}{0.331}$$

$$= 300000 \times 0.3021$$

$$= ₹\, 90{,}634.44$$

This can also be calculated by the formula of future value of annuity regular.

We know that, $$A(n, i) = A\left[\frac{(1+i)^n - 1}{i}\right]$$

$$300000 = A\left[\frac{(1+0.1)^3 - 1}{0.1}\right]$$

$$300000 = A \times \frac{0.331}{0.1}$$

$$A = \frac{300000}{3.31} = 90{,}634.44$$

(ii) Leasing : Leasing is a financial arrangement under which the owner of the asset (lessor) allows the user of the asset (lessee) to use the asset for a defined period of time (lease period) for a consideration (lease rental) payable over a given period of time. This is the kind of taking an asset on rent. How can we decide whether a lease agreement is favourable to lessor or lessee, it can be understand by following example.

A company is considering proposal of purchasing a machine either by making full payment of ₹ 4000 or by leasing it for 3 years at an annual rate of ₹ 1250. Which course of action is preferable if the company can borrow money at 14% compounded annually.

Sol. The present value, *P.V.* of annuity is given by

$$P.V. = \text{C.F. } P(n, i)$$

$$= 1250\, P(3, 0.14)$$

Now, $$P(n, i) = \frac{(1+i)^n - 1}{i(1+i)^n}$$

$\therefore$ $$P(3, 0.14) = \frac{(1+0.14)^3 - 1}{0.14(1+0.14)^3}$$

$$= \frac{1.4815 - 1}{0.14 \times 1.4815}$$

$$= \frac{0.4815}{0.2074}$$

$$= 2.3216$$

$\therefore$ $$P.V. = 1250 \times 2.316$$

or $$P.V. = ₹\, 2902$$

Which is less than the purchase price and consequently leasing is preferable.

(iii) Capital Expenditure (Investment Decision) : Capital expenditure means purchasing an asset (which results in outflows of money) today in anticipation of benefits (cash inflow) which would flow across the life of investment. For taking investment decision, we compare the present value of cash outflow and present value of cash inflows. If present value of cash inflows is greater than present value of cash outflows decision should be in the favour of investment. We have already discussed this concept with the help of example in Net present value.

(iv) Valuation of Bond : A bond is a debt security in which the issuers owes the holder a debt and is obliged to repay the principal and interest. Bond are generally issued for a fixed term longer than one year.

Present value of bond = Present value of Interest Payments + Present value of Maturity payments

$$BV = \sum_{i=1}^{n}\left\{\frac{C}{(1+i)^n}\right\} + \frac{F}{(1+i)^n}$$

where, BV = the present value or the market value of the bond

C = cash flow from the interest of the bond

n = number of periods

F = face value of bond

i = rate of interest (in decimal)

For example : An investor intends purchasing a three year ₹ 1000 par value bonding having nominal interest rate of 10%. At what price the bond may be purchased now if it matures at par and the investor requires a rate of return of 14% ?

Sol. Present value of Bond = Present value of Interest Payments + Present value of Maturity payments

or
$$BV = \sum_{i=1}^{n}\left\{\frac{C}{(1+i)^n}\right\} + \frac{F}{(1+i)^n}$$

Here,
$$C = 1000 \times 10\% = 1000 \times \frac{10}{100} = 100$$
$$i = 14\% = \frac{14}{100} = 0.14$$
$$F = 1000$$

$\therefore$
$$BV = \frac{100}{(1+0.14)^1} + \frac{100}{(1+0.14)^2} + \frac{100}{(1+0.14)^3} + \frac{1000}{(1+0.14)^3}$$
$$= 100 \times 0.87719 + 100 \times 0.769467 + 100 \times 0.674972 + 1000 \times 0.674972$$
$$= 87.719 + 76.947 + 67.497 + 674.97$$
$$= 907.125$$

Thus, the purchase value of bond is ₹ 907.125.

CHAPTER-15

APPLICATION OF TAXATION AND BILLS

TOPIC-1
Tax

Revision Notes

- **Tax :** In order to run the government and manage the affairs of a state, money is required. So, the government imposes taxes in many forms on the incomes of individuals and companies. It is the financial charge (fee) imposed by the Government on income, commodity or activity. Taxes are considered to be the "cost of living in a society". Taxes in India are levied by the Central Government and the State Governments to meet common welfare expenditure of society.

  The reason for levy of taxes is that they constitute the basic source of revenue to the Government. Revenue so raised is utilized for meeting the expenses of Government like defence, provision of education, health-care, infrastructure facilities like roads, dams etc. Taxes in India can be categorized as direct and indirect taxes.

- **Types of Taxes :**

Particulars	Direct Tax	Indirect Tax
Levied on	▪ **Income/wealth** of the person.	▪ **Price of Goods or Services**
Shifting of burden	▪ There is no shifting of burden. ▪ Direct Taxes are **directly** borne by the taxpayer.	▪ Tax burden is shifted to subsequent buyer/user. ▪ Thus whole **burden** falls on **final consumer.**
Time of Collection	▪ Collected on yearly basis.	▪ Collected at the time of sale/purchase of goods or rendering of services.
Examples	▪ Income tax, Tax on undisclosed foreign Income or Assets.	▪ GST, Custom duty.

➢ **Concept of Financial Year (FY), Previous Year (PY) & Assessment Year (AY)**

FY	▪ Financial year means a year starting on 1st April and ending on 31st March.
PY [Sec 3]	▪ FY in which the **income is earned** is called "Previous Year". ▪ PY means the Financial Year **immediately preceding the AY.**
AY [Sec 2(9)]	▪ The year in which **income is assessed to tax** is called Assessment Year. ▪ AY 2019-20 will commence on 1.4.2019 and will end on 31.3.2020. ▪ Thus Income earned during PY 2018-19 will be assessed/taxed in AY 2019-20.

For Example :

1. A is running a business from 1993 onwards. Determine the previous year for the assessment year 2020-21.

Sol. The previous year will be 1.4.2019 to 31.3.2020.

2. A chartered accountant sets up his profession on 1st July, 2019. Determine the previous year for the assessment year 2020-21.

Sol. The previous year will be from 1.7.2019 to 31.3.2020.

➢ **Levy of Income-tax**

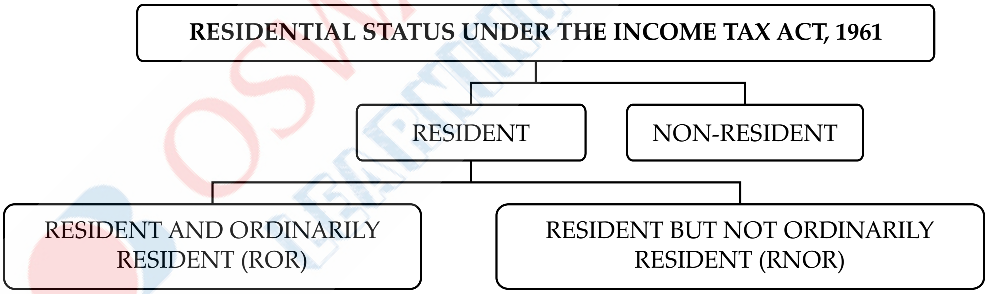
Income-tax is a tax levied on the total income of the previous year of every person. Everyone who earns or gets an income in India is subject to income tax. (Yes, be it a resident or a non-resident of India). Your income could be salary, pension or could be from a savings account that's quietly accumulating a 4% interest. Even, winners of 'Kaun Banega Crorepati' have to pay tax on their prize money.

Taxpayers in India, for the purpose of income tax include :

- **Individuals :** Individual means only a Natural Human Being (Male/Female/Minor/Unsound Mind)
- Hindu Undivided Family (HUF)
- Association of Persons(AOP) and Body of Individuals (BOI)
- Firms
- Companies

(I) Total Income and Tax Payable

Generally, the word 'Income' covers receipts in the shape of money or money's worth which arise with certain regularity. Income-tax is levied on an assessee's total income. Such total income has to be computed as per the provisions contained in the Income-tax Act, 1961.



Note : Assessee means a person by whom any tax or any other sum of money is payable.

Let us go step by step to understand the procedure for computation of total income of an individual for the purpose of levy of income-tax :

Step 1 : Determination of residential status

The residential status of a person has to be determined to ascertain which income is to be included in computing the total income. The residential status as per the Income-tax Act, 1961 can be classified as under –

INDIVIDUAL	
BASIC CONDITIONS	An Individual is **Resident** in India if he satisfies **ANY ONE** of the following Basic Conditions: **(a)** He has been in India for **total** period of **182 days** or more during **PY** OR **(b)** **(i)** He has been in India for at least **60 days** in the relevant **PY** AND **(ii)** He has been in India for at least **365 days** during **Last 4 PYs.** **NOTE :** → Individual satisfy **ANY 1** Condition → **Resident** [Additional Conditions]. → If **Both** conditions are **NOT** satisfied → **Non-Resident** □.

EXCEPTIONS	Following Individuals will be Resident only if Period of Stay during PY is 182 days or more. [2nd Condition → NA in the following cases] **(i)** **Indian Citizen** who leaves India **during PY** as a **Member of Crew of Indian ship** or for **employment outside** India; **(ii)** **Indian Citizen** or **Person of Indian Origin** who comes on **visit to India** in PY. [Such Person must be engaged in employment/business o/s India]
ADDITIONAL CONDITION	**Resident Individual can be ROR or RNOR :** To determine whether Individual is ROR/ RNOR, we need to check 2 Additional conditions. **(i)** His **Total Stay** in India in **Last 7 years is 730 days** or more **AND** **(ii)** He is a **Resident** in **Any 2 out of Last 10 years** **NOTE :** If an Individual Satisfy: → **Both** Additional conditions → He is **ROR.** → **None or one** of the Additional conditions → He is **RNOR**.

For Example : During the previous year 2018-19, X, a foreign citizen, stayed in India for just 69 days. Determine his residential status for the assessment year 2019-20 on the basis of the following information:

(i) During 2015-16, X was present in India for 366 days.

(ii) During 2012-13 and 2011-12, X was in Japan for 359 and 348 days respectively and for the balance period in India.

Sol. (i) To determine whether Mr. X is resident or not

- He is resident for previous year 2018-19 as he satisfies the second condition as he was here during the previous year for 69 days and in the preceding 4 years for 366 days.

(ii) To determine whether he is ordinarily resident or not

- He should satisfy both of the additional conditions.

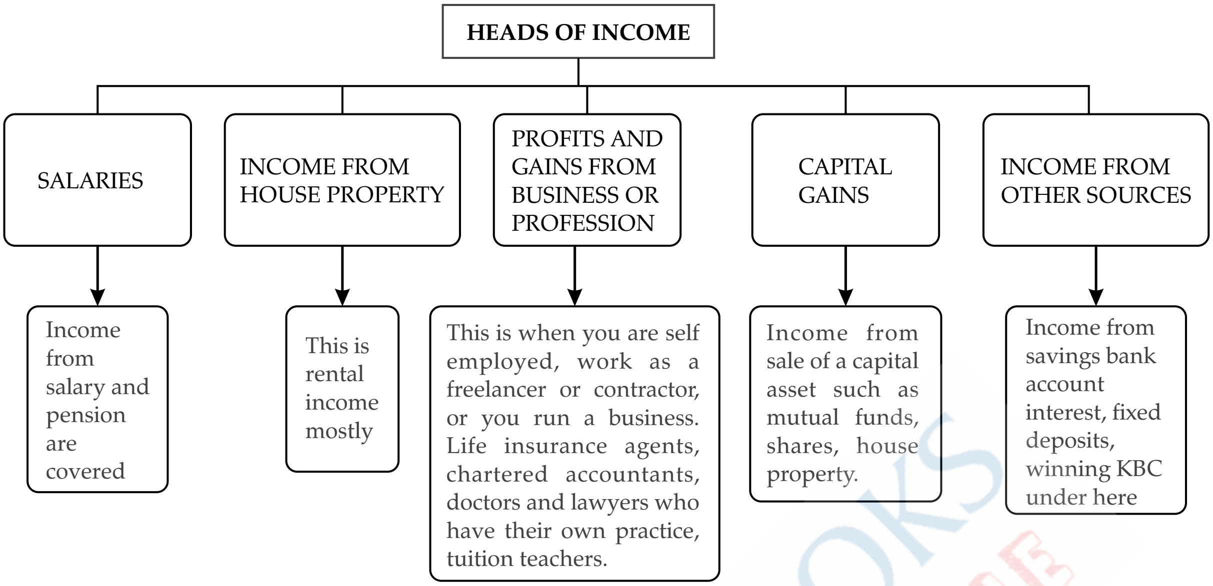
Year	Stay in India	Whether resident or non-resident
2017-18	Nil	Non-resident
2016-17	Nil	Non-resident
2015-16	366 days	Resident
2014-15	Nil	Non-resident
2013-14	Nil	Non-resident
2012-13	7 days	Non-resident
2011-12	17 days	Non-resident
2010-11	Nil	Non-resident
and earlier years	Nil	Non-resident

He was in India for less than 730 days in the 7 preceding previous years. He is also non-resident in 9 out of 10 previous years preceding the previous year. Hence he is "resident but not ordinarily resident".

Step 2 : Classification of income under different heads

A person may earn income from different sources. For example, a salaried person earns income by way of salary. He also gets interest from bank savings account/fixed deposit. Apart from this, if he has invested in shares, he would be earning dividend and when he sells these shares, he may earn profit on such sale. If he owns a residential property which he has let out, he would earn rental income.

Under the Income-tax Act, 1961, for computation of total income, all income of a tax payer are classified into five different heads of income. These are shown below –For simpler classification, the Income Tax Department breaks down income into five heads :



Step 3– Computation of income under each head

Income is to be computed in accordance with the provisions governing a particular head of income.

Exemptions: There are certain incomes which are wholly exempt from income-tax e.g. agricultural income. Such types of incomes have to be excluded and will not form part of total income.

Also, some incomes are partially exempt from income-tax e.g. House Rent Allowance, Education Allowance. These incomes are excluded only to the extent of the limits specified in the Act. The balance income over and above the prescribed exemption limits would enter computation of total income and have to be classified under the relevant head of income.

EXEMPT INCOMES : SUCH INCOME DO NOT FORM PART OF TOTAL INCOME

Section	Particulars
10(1)	Agricultural Income.
10(2)	Share received by the member from the Income of HUF.
10(2A)	Share of profit of a partner in the Income of a firm.
10(4)	(i) Interest on Notified Securities & Bonds held by a NR. (ii) Interest on **NR (External) Account** in any bank in India as per the rules of FEMA. (iii) Interest on Saving Certificate to NR.
10(6)	Remuneration to certain Individuals who are not Citizens of India.
10(7)	Allowances/Perqs paid o/s India to **Indian Citizen** for services rendered o/s India.
10(10BB)	Payment under Bhopal Gas Leak Disaster (Processing of Claims) Act, 1985.
10(10BC)	Compensation received or receivable on account of any disaster.
10(10CC)	Tax on non-monetary perquisites paid by employer.
10(10D)	Amount received under a Life Insurance Policy.
10(11)	Withdrawal of Accumulated balance from Provident Fund.
10(11A)	Interest & withdrawals from Sukanya Samriddhi Account
10(12)/(13)	Payments from RPF/Any payment from Approved SAF.
10(15)	Interest, premium or bonus on specified investments issued by CG/SG.
10(16)	Scholarships granted to meet the Cost of Education.
10(17)	Daily & constituency allowance, etc, received by MPs & MLAs.
10(17A)	Award or Reward.

10(18)	Pension received by certain awardees/any member of their families.
10(22D)	Income of notified Mutual Funds.
10(26)	Income of a member of Scheduled Tribe residing in certain specified areas.
10(26AAA)	Income of a Sikkemese Individual.
10(30)/(31)	Tea board Subsidy & Other Subsidies.
10(34)/(35)	Dividend from shares/units of MF to be exempt in the hands of the shareholders.
10(34A)	Income arising to a shareholder on account of buys back of unlisted shares.
10(37)	CG on compensation on compulsory acquisition of urban agricultural land.
10(43)	Amount received by Individual as Loan under Reverse Mortgage.
10(45)	Notified Allowance or Perquisite paid to Chairman/Member of UPSC.

Deductions : There are deductions and allowances prescribed under each head of income. For example, while calculating income from house property, municipal taxes and interest on loan are allowed as deduction. Similarly, deductions and allowances are prescribed under other heads of income. These deductions etc. have to be considered before arriving at the net income chargeable under each head.

Step 4 : Clubbing of income of spouse, minor child etc.

In case of individuals, income-tax is levied on a slab system on the total income. The tax system is progressive *i.e.*, as the income increases, the applicable rate of tax increases. Some taxpayers in the higher income bracket have a tendency to divert some portion of their income to their spouse, minor child etc. to minimize their tax burden.

In order to prevent such tax avoidance, clubbing provisions have been incorporated in the Act, under which income arising to certain persons (like spouse, minor child etc.) have to be included in the income of the person who has diverted his income for the purpose of computing tax liability.

Step 5 : Set-off or carry forward and set-off of losses

An assessee may have different sources of income under the same head of income. He may have profit from one source and loss from the other. For instance, an assessee may have profit from his textile business and loss from his printing business. This loss can be set-off against the profits of textile business to arrive at the net income chargeable under the head "Profits and gains of business or profession".

Similarly, an assessee can have loss under one head of income, say, Income from house property and profits under another heads of income, say, profits and gains of business or profession. There are provisions in the Income-tax Act, 1961 for allowing inter head adjustment in certain cases.

However, there are also restrictions in certain cases, like business loss is not allowed to be set-off against salary income. Further, losses which cannot be set-off in the current year due to inadequacy of eligible profits can be carried forward for set-off in the subsequent years as per the provisions contained in the Act. Generally, brought forward losses under a particular head cannot be set-off against income under another head *i.e.*, brought forward business loss cannot be set-off against income from house property of the current year.

Step 6 : Computation of Gross Total Income (GTI)

- The income computed under each head, after giving effect to the clubbing provisions and provisions for set-off and carry forward and set-off of losses, have to be aggregated to arrive at the gross total income.
- The process of computing GTI is depicted here under :
  Add income computed under each head → Apply clubbing provisions → Apply the provisions for set-off and carry forward of losses

Step 7 : Deductions from Gross Total Income

Certain deductions are allowable from gross total income to arrive at the total income. These deductions contained in Chapter VI-A can be classified as :

- **Deduction in respect of certain payments**

Section	Nature of Payment/Deposit
80C	Payment of life insurance premium, tuition fees of children, deposit in public provident fund, repayment of housing loan etc.
80D	Medical insurance premium paid by an individual/HUF for the specified persons/ contribution to CGHS etc.
80E	Payment of interest on educational loan taken for self or relative

- **Deduction in respect of certain incomes**

Section	Nature of Income

80QQB	Royalty income of authors of certain books other than textbooks
80RRB	Royalty on patents.
80RRB	Royalty on patents.

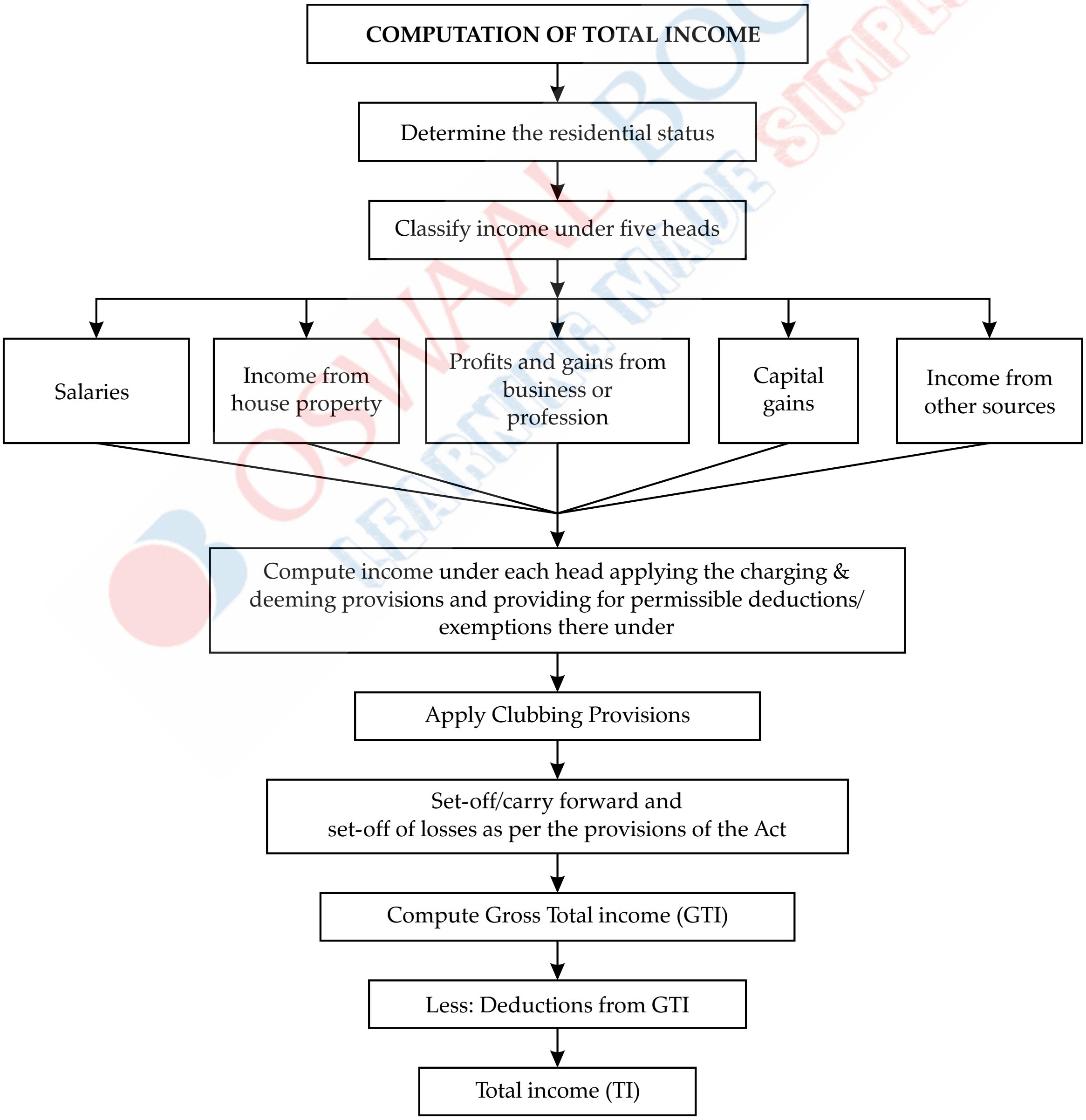
- **Deduction in respect of other incomes**

Section	Nature of Income
80TTA	Interest on savings account with a bank, co-op-society and post office.
80TTB	Interest on deposit with a bank, co-op-society and post office in case of senior citizens

- **Other Deductions**

Deduction under section 80U in case of a person with disability. These deductions are allowable subject to satisfaction of the conditions prescribed in the relevant sections. There are limits in respect of deduction under certain sections. The payments/incomes are allowable as deduction subject to such limits. For example, the maximum deduction under section 80RRB is ₹ 3 lakhs.

The income arrived at, after claiming the above deductions from the Gross Total Income is known as the Total Income. It should be rounded off to the nearest multiple of ₹ 10. The process of computation of total income is shown here under :



Step 9 : Application of the rates of tax on the total income

Each of these tax payers is taxed differently under the Indian income tax laws. While firms and Indian companies have a fixed rate of tax of 30% of profits, the individual, HUF, AOP and BOI tax payers are taxed based on the income slab they fall under. People's incomes are grouped into blocks called tax brackets or tax slabs and each tax slab has a different tax rate. Income tax slabs for individuals for FY 2019-20 (AY 2020-21) is given in the following table :

Income Tax Slab	Tax
when the total income does not exceed ₹ 2,50,000	NIL
when the total income exceeds ₹ 2,50,000 but does not exceed ₹ 5,00,000	5% of the amount by which the total income exceeds ₹ 2,50,000
when the total income exceeds ₹ 5,00,000 but does not exceed ₹ 10,00,000	₹ 12,500 + 20% of the amount by which the total income exceeds ₹ 5,00,000
when the total income exceeds ₹ 10,00,000	₹ 1,12,500 + 30% of the amount by which the total income exceeds ₹ 10,00,000

Table 1 : Income tax slabs for Individuals (below the age of 60 years), which includes residents as well as non-residents for FY 2019-20 (AY 2020-21)

Income Tax Slab	Tax
where the total income does not exceed ₹ 3,00,000	NIL
where the total income exceeds ₹ 3,00,000 but does not exceed ₹ 5,00,000	5% of the amount by which the total income exceeds ₹ 3,00,000
where the total income exceeds ₹ 5,00,000 but does not exceed ₹ 10,00,000	₹ 10,000 + 20% of the amount by which the total income exceeds ₹ 5,00,000
where the total income exceeds ₹ 10,00,000	₹ 1,10,000 + 30% of the amount by which the total income exceeds ₹ 10,00,000

Table 2 : Income tax slabs for senior citizens (being resident individuals of the age of 60 years or more but less than 80 years)

Income Tax Slab	Tax
when the total income does not exceed ₹ 5,00,000	NIL
when the total income exceeds ₹ 5,00,000 but does not exceed ₹ 10,00,000	20% of the amount by which the total income exceeds ₹ 5,00,000
when the total income exceeds ₹ 10,00,000	₹ 1,00,000 + 30% of the amount by which the total income exceeds ₹ 10,00,000

Table 3 : Income tax slabs for resident individuals of the age of 80 years or more at any time during the previous year

Note :

- Assessment is the procedure by which the income of an assessee is determined by the Assessing Officer. It may be by way of a normal assessment or by way of reassessment of an income previously assessed.

For Example : Mr. X of age 35 has a total income of ₹ 12,00,000 comprising of his salary income and interest on fixed deposit. Compute his tax liability.

Sol. Tax liability :

First	₹ 2,50,000	- Nil	
Next	₹ 2,50,000 – ₹ 5,00,000	- @ 5% of ₹ 2,50,000	= ₹ 12,500
Next	₹ 5,00,000 – ₹ 10,00,000	-@20% of ₹ 5,00,000	= ₹ 1,00,000
Balance *i.e.,*	₹ 12,00,000 – ₹ 10,00,000	-@30% of ₹ 2,00,000	= ₹ 60,000
			Total = ₹ 1,72,500

Step 10 : Surcharge / Rebate under section 87A (where total income ≤ ₹ 5,00,000) / Surcharge (where total income > ₹ 50,00,000)

Surcharge : Surcharge is an additional tax payable over and above the income- tax. Surcharge is levied as a percentage of income-tax. In case where the total income of an individual/HUF/AOP/BOI/Artificial Juridical Person :

Total Income of individual/ HUF/ AOP/ BOI/ Artificial Juridical Person	Surcharge
> ₹ 50 lakhs ≤ ₹ 1 crore	10% of income-tax
₹ 1 crore ≤ ₹ 2 crore	15% of income-tax
₹ 2 crore ≤ ₹ 5 crore	25% of income-tax
>₹ 5 crore	37% of income-tax

Rebate under section 87A : In order to provide tax relief to the individual tax payers who are in the 5% tax slab, section 87A provides a rebate from the tax payable by an assessee, being an individual resident in India, whose total income does not exceed ₹ **5,00,000**. The rebate shall be equal to the amount of income - tax payable on the total income for any assessment year or an amount of ₹ **12,500,** whichever is less.

Level of Total Income	Surcharge	Rebateu/s87A
≤ ₹ 5,00,000	Not applicable	Income - tax on total income or ₹ 12,500, whichever is less
₹ 5,00,000 ≤ ₹ 50,00,000	Not applicable	Not applicable
₹ 50,00,000 ≤ ₹ 1,00,00,000	10% of income tax	Not applicable
₹ 1,00,00,000 ≤ ₹ 2,00,00,000	15% of income tax	Not applicable
₹ 2,00,00,000 ≤ ₹ 5,00,00,000	25% of income tax	Not applicable
> ₹ 5,00,00,000	37% of income tax	Not applicable

Marginal relief : Marginal relief is available in case of such persons having a total income exceeding ₹50 lakh *i.e.*, the total amount of income-tax payable (together with surcharge) on such income should not exceed the amount of income tax payable on total income of ₹ 50 lakh by more than the amount of income that exceeds ₹ 50 lakh.

For Example :

Compute the tax liability of Mr. A (aged 42), having total income of ₹ 51 lakhs for the Assessment Year 2020-21. Assume that his total income comprises of salary income, Income from house property and interest on fixed deposit. Ignore cess.

Sol. Computation of tax liability of Mr. A for the A.Y. 2020-21

(A) Tax payable including surcharge on total income of ₹ 51,00,000	
₹ 2,50,000 – ₹ 5,00,000 @5%	₹ 12,500
₹ 5,00,000 – ₹ 10,00,000 @20%	₹ 1,00,000
₹ 10,00,000 – ₹ 51,00,000 @30%	₹ 12,30,000
Total	₹ 13,42,500
Add : Surcharge @ 10%	₹ 1,34,250
	₹ 14,76,750
(B) Tax Payable on total income of ₹ 50 lakhs (₹ 12,500 + ₹ 1,00,000 + ₹ 12,00,000)	₹ 13,12,500
(C) Excess tax payable (A)-(B)	₹ 1,64,250
(D) Marginal Relief (₹ 1,64,250 – ₹ 1,00,000, being the amount of income in excess of ₹ 50,00,000)	₹ 64,250
(E) Tax payable (A)-(D)	₹ 14,12,500

Step 11 : Health and Education Cess (HEC) on income-tax

The income-tax is to be increased by health and education cess @4% on income- tax plus surcharge/ minus rebate under section 87A, wherever applicable. This cess is payable by all assessees who are liable to pay income-tax irrespective of their level of total income.

Total Tax Liability of an individual	=	Tax on total income at applicable rates	+	Surcharge, at applicable rates, if total income > ₹ 50 lakhs,	+	HEC@4%
			–	Rebateu/s 87A, if total income ≤ ₹ 5lakh		

Note : In the above example, if cess is included, then

Tax Payable = ₹ 14,76,750 – ₹ 64,250 = ₹ 14,12,500 + 4% HEC = 14,69,000.

Step 12 : Advance Tax and Tax Deducted at source

Although the tax liability of an assessee is determined only at the end of the year, tax is required to be paid in advance in four installments on the basis of estimated income *i.e.*, on or before 15th June, 15th September, 15th December and 15th March. However, residents opting for presumptive taxation scheme can pay advance tax in one installment on or before 15th March instead of four installments. In certain cases, tax is required to be deducted at source from the income by the payer at the rates prescribed in the Income-tax Act, 1961 or the Annual Finance Act. Such deduction should be made either at the time of accrual or at the time of payment, as prescribed by the Act.

For example, in the case of salary income, the obligation of the employer to deduct tax at source arises only at the time of payment of salary to the employees. However, in respect of other payments like, fees for professional services, fees for technical services, interest payable to residents, the person responsible for paying is liable to deduct tax at source at the time of credit of such income to the accounts of the payee or at the time of payment, whichever is earlier. Such tax deducted at source has to be remitted to the credit of the Central Government through any branch of the RBI, SBI or any authorized bank.

Step 13 : Tax Payable/Tax Refundable

After adjusting the advance tax and tax deducted at source, the assessee would arrive at the amount of net tax payable or refundable. Such amount should be rounded off to the nearest multiple of ₹ 10. The assessee has to pay the amount of tax payable (called self-assessment tax) on or before the due date of filing of the return. Similarly, if any refund is due, assessee will get the same after filing the return of income.

(II) Return of Income

The Income-tax Act, 1961 contains provisions for filing of return of income. Return of income is the format in which the assessee furnishes information as to his/her total income and tax payable. The format for filing of returns by different assessees is notified by the CBDT(Central Board of Direct Taxes). The particulars of income earned under different heads, gross total income, deductions from gross total income, total income and tax payable by the assessee are required to be furnished in the return of income. In short, a return of income is the declaration of income by the assessee in the prescribed format.

➢ **Goods and Services Tax (GST)**

GST is known as the Goods and Services Tax. It is an indirect tax which has replaced many indirect taxes in India such as the excise duty, VAT, services tax, etc. The Goods and Service Tax Act was passed in the Parliament on 29th March 2017 and came into effect on 1st July 2017.

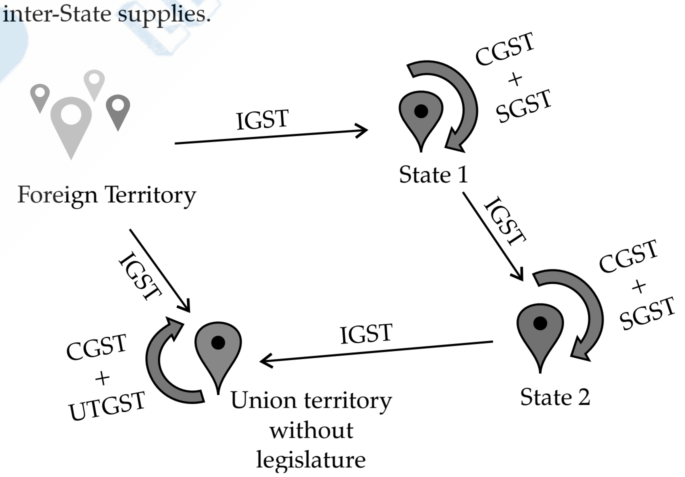
In other words, Goods and Service Tax (GST) is levied on the supply of goods and services. Goods and Services Tax Law in India is a **comprehensive, multi-stage, destination-based tax** that is levied on every **value addition.** GST is a single domestic indirect tax law for the entire country. In order to address the complex system in India, the Government introduced 4 types of GST which are given below.

(i) CGST (Central Goods and Service Tax) : levied and collected by Central Government.

(ii) SGST (State Goods and Service Tax) : levied and collected by State Governments/Union Territories with Legislatures.

(iii) UTGST (Union Territory Goods and Services Tax) : levied and collected by Union Territories without Legislatures, on intra-State supplies of taxable goods and/or services.

(iv) IGST (Integrated Goods and Services Tax) : Inter-State supplies of taxable goods and/or services are subject to Integrated Goods and Services Tax (IGST). IGST is the total sum of CGST and SGST/UTGST and is levied by Centre on all inter-State supplies.



- **Intra-state means :** Supply within the same state.

  In case of intra-state sale of goods/services, or both

  If GST rate is 18%, then

  $$\text{CGST} = 9\% \text{ of Sale price}$$

  SGST 9%of Sale price

IGST = 0

- **Inter-state means :** Supply from one state to another state.
  In case of inter-state of goods or services or both
  If GST rate is 18%, then
  IGST = 18% of Sale price
- Discount is never allowed on amount including GST.
  E.g. : Mr. Naresh is a manufacturer in Agra (U.P.) who sold goods worth ₹ 10,000 to Mr. Dinkar in Delhi. We see here, it is an interstate transaction; therefore IGST will be applicable here.
  ∴ Rate of interest for the GST = 18%
  Now IGST charge by the central Government
  $$= \frac{18}{100} \times 1,000$$
  = ₹ 1800
  Hence, the cost of the product = ₹ (10,000 + 1800) = ₹ 11,800
- By the Government, GST rates are 0%, 5%, 12%, 18% and 28% applicable.

TOPIC-2
Bill

Revision Notes

➢ A **bill** is an invoice that one of your suppliers will give you, and which, sooner or later, you will have to pay. A bill referred to as a 'purchase invoice' or a 'supplier invoice'. We have a number of days (normally between 7 and 30) to pay the bill to our supplier. This time lag is the supplier 'giving you credit', in other words we don't have to pay the bill straight away. E.g.: Your accountant gives you a bill for ₹ 6000, dated 1st January with 30 days payment terms. You pay that bill on 1st February.
The accountant has given you a month's credit.

➢ **Electricity Bill**
It is somewhat vital to understand the electricity bill. Each customer's electricity bill is determined by three elements – the quantity of electricity used, the tariff for the category of customer, and the Fuel Surcharge Cost Adjustment Factor or fuel surcharge as it is commonly called. It is very important to understand the electricity bill and it's components to make plans for energy savings. Our electricity bills are quite a lot of information to give us a good insight into our electricity consumption patterns. A good understanding of various components can help one plan for money saving exercise.

➢ **Tariff / Category**
Tariff and Category determine the rate structure applicable to the bill. Electricity pricing sometimes referred as electricity tariff. e.g.,: UPPCL (Uttar Pradesh Power Corporation Limited) has broadly categorized its tariff on the basis of voltage level.

- Low Medium Voltage (LMV).
- High Voltage (HV).

LMV Consumers are further sub-categorized in 10 parts, called - LMV1, LMV2,... LMV10. Similarly, HV consumers are also sub-categorized in 4 parts, called - HV1, HV2, HV3 and HV4. Based on the Area, such as Urban or Rural, Connection type such as Metered or Unmetered, Billing units, such as kWh, all these tariffs were divided into Supply Type.
Category in the bill determines if the connection is commercial or industrial. Different rates/slabs are applicable for different tariff codes and thus it is important to validate that the right tariff code is applied on the electricity bill.
Type of Supply & Connected Load (Fixed charges of each State/DISCOM) : Connected (or Sanctioned) load is the total pool of supply that is given to a meter. This is calculated in kW (or Kilo-Watts). This is the permissible total peak kW given to a meter based on the appliances connected to the meter. This is not your actual energy consumption and only impacts fixed charges on your electricity bill.
Connected load is determines if the connection will be a single-phase or three-phase. If the actual load is more than the sanctioned load, then it will impact the fixed charges for that month and some DISCOM imposes a penalty of increased fixed charges for the incremental increase of the actual load drawn. Every DISCOM is having a method of calculating the load to be sanctioned to the applicant and varies widely such as:

- constructed area
- assessment based on the load of the connected appliances and using redundancy factor
- based on the unit consumed in kWh and so on

- **Units Consumed (Unit rates of each State/Discom) :** Units consumed is the number of kWh (Kilo-Watt-Hour) consumed in a month. 1 kWh is equivalent to keeping a 100 Watts bulb on for 10 hrs. This information is calculated by finding the difference between meter readings of two consecutive months. This is the total monthly consumption by all the appliances that are connected to the meter. This is the value that need to come down in order to reduce the electricity bill. An observation of consumption history can give an indicator of the appliances having higher electricity consumption (typically Air Conditioners increase consumption in summers). In case you want to check the likely energy consumption in units (kWh) based on the information of appliances connected in your house and average daily consumption.
  - **Units of Electricity Use**

  The watt (W) is a unit of electrical power, which is the rate at which we use energy. We pay the electric company for the use of energy. A kilowatt (kW) is equal to 1000 watts: kilo is a prefix that means 1000. (Note that k is the normal prefix for kilo, W is the normal abbreviation for Watts and h is the normal abbreviation for hours). A watt-hour (Wh) and a kilowatt-hour (kWh) are units of energy, where 1000 Wh = 1 kWh. The equation relating energy and power is:

  $$\text{Energy } (E) = \text{Power } (P) \times \text{Time } (t)$$

  e.g.: If a 75 W bulb (power) is turned on for 5 hours (time), it will use 375 Wh or 0.375 kWh of energy (energy = power × time).

  The cost for each unit of energy we use is measured in ₹ per kWh. The typical cost per unit of energy is ₹ 4.5 kWh, in India. Using the equation below, we can determine the cost of our energy use.

  Cost of energy use = Energy used × Cost per unit of energy :

  Using our example above, the cost of using a 75 W bulb for 5 hours is :

  Cost of energy use = 0.375 kWh × ₹ 4.5 kWh = ₹ 1.6875.

- **Tariff Structure :** It is very important to note the tariff structure on your bill, as this is the best indicator of how the bill can be reduced. Typically for residential and SMB commercial connections, the structure is slab based (unlike industrial connections where the units charged at a high flat rate). The intent behind the slab structure is to reward low energy users and charge extra to those who have the high consumption. The slabs are based on the "Units Consumed" that we discussed earlier. As the number of units consumed increase, energy charge changes and also the fixed costs associated (single-phase) with the slab increases.

  e.g.: Tariff structure of torrent power corporation for LMV 1 is given in the following table:

UPERC Tariff Schedule (Effective from 12-September-2019)		
LMV-1	**DOMESTIC LIGHT, FAN & POWER**	
(b) (1)	Supply at Single point for bulk loads	
	Fixed Charge	₹ 110 /kW/month
	Energy Charge (Main City)	₹ 7.00 /kWh
(D) (1)	Life line consumer	Load-1kW; Consumption-100 kh/Month
	Fixed Charge	₹ 50 /kW/month
	Energy Charge (upto 100 Units)	₹ 3.00 /kWh
(C) (1)	Other metered Domestic Consumers (For All Loads)	
	Fixed Charge	₹ 110 /kW/month
	Energy Charge Upto 150 kWh/month	₹ 5.50 /kWh
	Energy Charge 151-300 kwh/month	₹ 6.00 /kWh
	Energy Charge 301-500 kwh/month	₹ 6.50 /kWh
	Energy Charge 501st unit onwards	₹ 7.00 /kWh

- **Fuel Adjustment Charge (FAC) :** As you can see in the tariff structure above, there is a FAC rate applicable at each slab. This is the additional cost of power incurred due to fuel price increments during a year. Fuel in most cases is Coal. As per the study, after 2011, the production rates coal will decline, reaching 1990 levels by the year 2037, and reaching 50% of the peak value in the year 2047. So, invariably FAC will increase till alternate sources of electricity are not developed to a state where they can generate electricity that cheap. So, electricity costs will surely increase in future.
- **Electricity Duty/Taxes (applicable taxes of each State) :** Every state is having an Electricity (Duty) Act wherein the applicable tax for different tariff structure is defined and one can check the same in his electricity bill. Understanding the elements of electricity bill mentioned above can help you understand your electricity bill and will also help you to plan your electricity consumption reduction project. Two things that should be targeted are Units Consumed and Connected Load. Reduce the two and your electricity bill will surely come down along with the impact of taxes and Fuel Surcharge.

➢ **Electricity Bill calculation procedure :** For Domestic Category

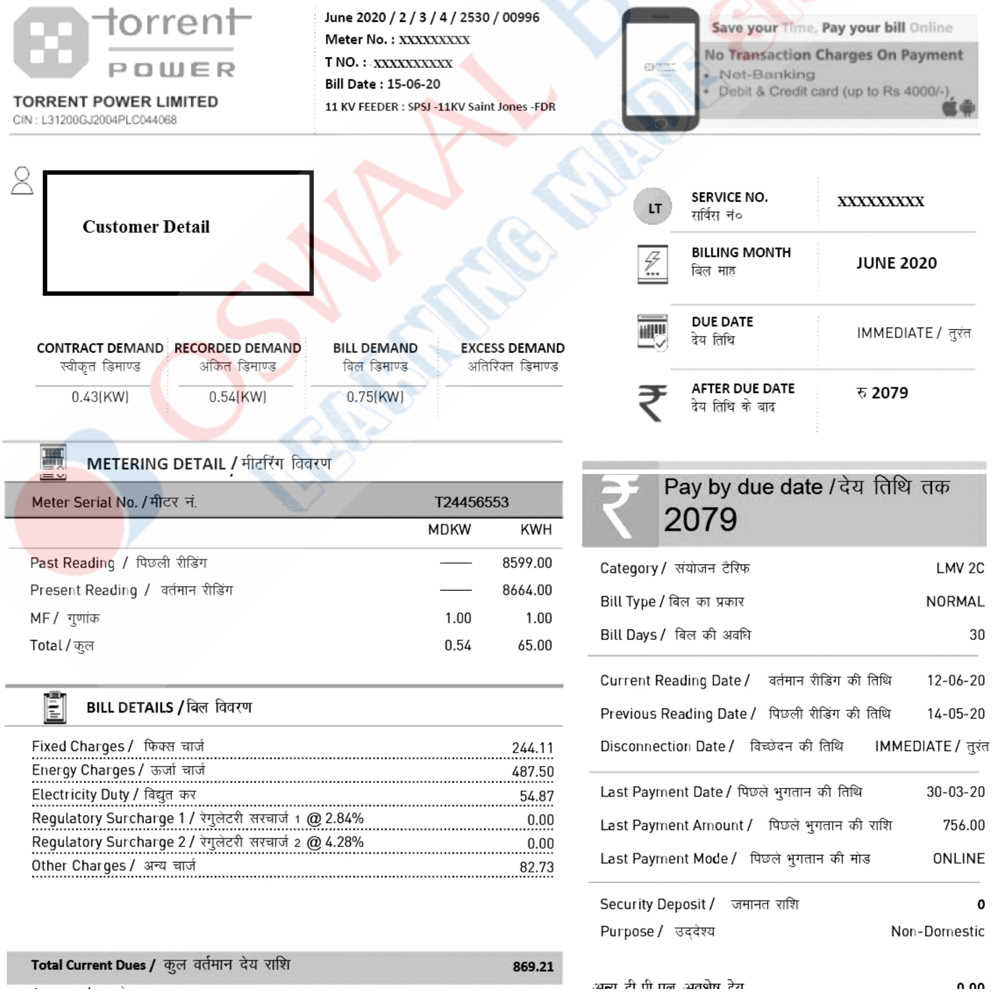
A domestic consumer can calculate the bill using the different tariff for different slabs given in the tariff table given on the back of the bill.

e.g., A consumer wants to calculate the bill for 450 units for a period of 1 months, the amount will be :

(A) Energy Charge Calculation :			
Unit Slabs	**Periods of Bill (in Months)**	**Tariff Rates**	
Upto 150 units	1	= 150 units × 5.50	= ₹ 825.00
Next 151-300 units	1	= 150 units × 6.00	= ₹ 900.00
Next (301-450) units	1	= 150 units × 6.50	= ₹ 975.00
		Total Energy Charges comes out to be	= ₹ 2700.00
(B) Fixed Charges :			= ₹ 110.00
		Total Charges	= ₹ 2810

Note : We have taken tariff schedule from the table UPERC Tariff Schedule (Effective from 12-September-2019) mentioned above.

➢ **A sample of an electricity bill of torrent power given as below :**



torrent POWER
TORRENT POWER LIMITED
CIN : L31200GJ2004PLC044068

June 2020 / 2 / 3 / 4 / 2530 / 00996
Meter No. : XXXXXXXXX
T NO. : XXXXXXXXXX
Bill Date : 15-06-20
11 KV FEEDER : SPSJ -11KV Saint Jones -FDR

Save your Time, Pay your bill Online
No Transaction Charges On Payment
• Net-Banking
• Debit & Credit card (up to Rs 4000/-)

Customer Detail

SERVICE NO. / सर्विस नं०	XXXXXXXXX
BILLING MONTH / बिल माह	JUNE 2020
DUE DATE / देय तिथि	IMMEDIATE / तुरंत
AFTER DUE DATE / देय तिथि के बाद	रु 2079

CONTRACT DEMAND / स्वीकृत डिमाण्ड	RECORDED DEMAND / अंकित डिमाण्ड	BILL DEMAND / बिल डिमाण्ड	EXCESS DEMAND / अतिरिक्त डिमाण्ड
0.43(KW)	0.54(KW)	0.75(KW)	

METERING DETAIL / मीटरिंग विवरण

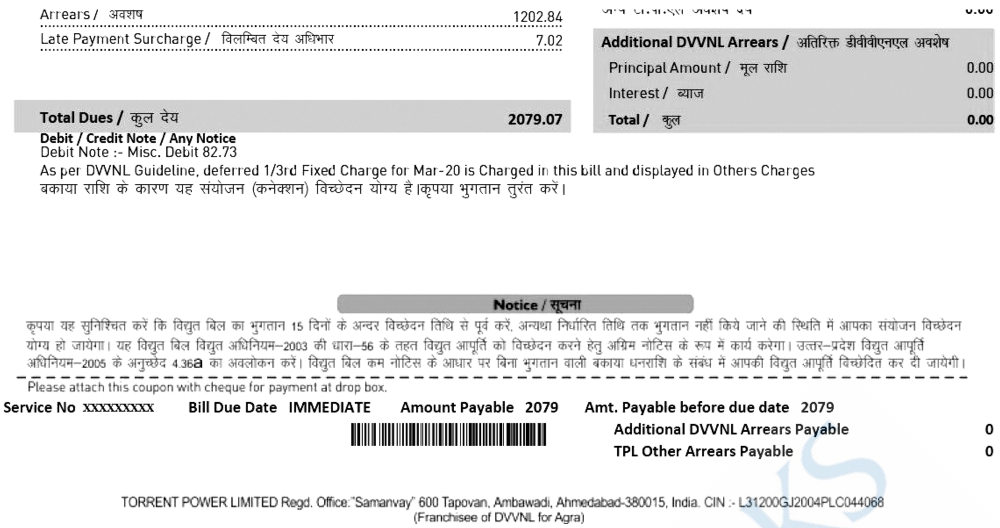
Meter Serial No. / मीटर नं.	T24456553	
	MDKW	KWH
Past Reading / पिछली रीडिंग	—	8599.00
Present Reading / वर्तमान रीडिंग	—	8664.00
MF / गुणांक	1.00	1.00
Total / कुल	0.54	65.00

BILL DETAILS / बिल विवरण

Fixed Charges / फिक्स चार्ज	244.11
Energy Charges / ऊर्जा चार्ज	487.50
Electricity Duty / विद्युत कर	54.87
Regulatory Surcharge 1 / रेगुलेटरी सरचार्ज 1 @ 2.84%	0.00
Regulatory Surcharge 2 / रेगुलेटरी सरचार्ज 2 @ 4.28%	0.00
Other Charges / अन्य चार्ज	82.73
Total Current Dues / कुल वर्तमान देय राशि	**869.21**

Pay by due date / देय तिथि तक
2079

Category / संयोजन टैरिफ	LMV 2C
Bill Type / बिल का प्रकार	NORMAL
Bill Days / बिल की अवधि	30
Current Reading Date / वर्तमान रीडिंग की तिथि	12-06-20
Previous Reading Date / पिछली रीडिंग की तिथि	14-05-20
Disconnection Date / विच्छेदन की तिथि	IMMEDIATE / तुरंत
Last Payment Date / पिछले भुगतान की तिथि	30-03-20
Last Payment Amount / पिछले भुगतान की राशि	756.00
Last Payment Mode / पिछले भुगतान की मोड	ONLINE
Security Deposit / जमानत राशि	0
Purpose / उद्देश्य	Non-Domestic



Arrears / अवशेष	1202.84
Late Payment Surcharge / विलम्बित देय अधिभार	7.02
Total Dues / कुल देय	**2079.07**

Additional DVVNL Arrears / अतिरिक्त डीवीवीएनएल अवशेष	
Principal Amount / मूल राशि	0.00
Interest / ब्याज	0.00
Total / कुल	**0.00**

Debit / Credit Note / Any Notice
Debit Note :- Misc. Debit 82.73
As per DVVNL Guideline, deferred 1/3rd Fixed Charge for Mar-20 is Charged in this bill and displayed in Others Charges
बकाया राशि के कारण यह संयोजन (कनेक्शन) विच्छेदन योग्य है।कृपया भुगतान तुरंत करें।

Notice / सूचना

कृपया यह सुनिश्चित करें कि विद्युत बिल का भुगतान 15 दिनों के अन्दर विच्छेदन तिथि से पूर्व करें, अन्यथा निर्धारित तिथि तक भुगतान नहीं किये जाने की स्थिति में आपका संयोजन विच्छेदन योग्य हो जायेगा। यह विद्युत बिल विद्युत अधिनियम–2003 की धारा–56 के तहत विद्युत आपूर्ति को विच्छेदन करने हेतु अग्रिम नोटिस के रूप में कार्य करेगा। उत्तर–प्रदेश विद्युत आपूर्ति अधिनियम–2005 के अनुच्छेद 4.36a का अवलोकन करें। विद्युत बिल कम नोटिस के आधार पर बिना भुगतान वाली बकाया धनराशि के संबंध में आपकी विद्युत आपूर्ति विच्छेदित कर दी जायेगी।

Please attach this coupon with cheque for payment at drop box.

Service No xxxxxxxxx **Bill Due Date IMMEDIATE** **Amount Payable 2079** **Amt. Payable before due date 2079**

Additional DVVNL Arrears Payable **0**
TPL Other Arrears Payable **0**

TORRENT POWER LIMITED Regd. Office:"Samanvay" 600 Tapovan, Ambawadi, Ahmedabad-380015, India. CIN :- L31200GJ2004PLC044068
(Franchisee of DVVNL for Agra)

- **Water Bill :** The amount one must pay to use water and sewage services each month. Normally, water and sewage is provided by a municipality, but this is not always the case. Water bills are usually based upon one's usage, such that those who use more water are charged more.
- **The water bill invoice** is provided by a company that supplies water on a residential and/or commercial basis. A customer that receives their water supply from such a company will receive a water bill invoice complete with the charges for the company's services and the amount owed for said services. From time to time, people forget to pay their utility bills, so the customer might see a summary of past due charges that must be paid on the next billing period. Payments not received on time could result in interest charges or additional fees. The water bill invoice will show the total amount due and the date upon which payment must be received.
- **CONSUMER CLASSIFICATION**

  In the water tariff, there will be generally four categories of consumers:-

  I. Domestic (Category –I) :

  There will be domestic category which will be mainly applicable to the residential and such other uses.

  II. Commercial / Industrial Category (Category –II)) :

  Industrial category has been merged with commercial category to make both categories come under one broad category known as Commercial / Industrial.

  All other category of consumers which are not covered either in DOMESTIC CATEGORY (CATEGORY-I) OR MIXED USE CATEGORY (CATEGORY-IA) will be treated in this category.

  III. Mixed use category (Category -I A) :

  A new category has been introduced which will be applicable to such premises where a part of the premises under residential use is also used for commercial purposes provided the water use is for non intensive purposes.

  e.g.: residences having some portion under uses such as *kirana* shop, stationery shop, barbar shop etc will be covered under the mixed use category. However, such uses as dhaba, tea shop, sweetmeat shop etc where water use by nature of trade activity is intensive, commercial / industrial category (Category II) will be applicable.

  IV. Govt. Institutions (Category - IIA) :

  Government offices / Institutions, Govt. schools, Govt. aided schools, etc. fall under this special category under the broad classification of Commercial / Industrial Category and they will be eligible for remission on their total monthly bills provided they adopt the water harvesting and waste water recycling.

  There will be following generally three components of water / sewerage bill :

  Fixed Water Charge : With a fixed water charge, the consumer pays a monthly water bill, which is the same independently of the volume consumed. In absence of a water metering system, a fixed water charge is the only possible tariff structure.

  Sewerage maintenance charge : This charge is levied for the maintenance of sewerage system and is charged according to volumetric consumption of water.

  Service charge : Service charge under the domestic category which is presently linked with the built up area of the property, that is, whether the covered area is more than or less than 200 sq.meters, and this has been now delinked from the area concept. Instead, under the new tariff it will be linked with the consumption slab for all categories of consumers including the domestic category.

  e.g: Delhi Jal Board water tariff rates are summarized in the following tables:

CATEGORY - I (DOMESTIC CONNECTIONS)

Monthly Consumption (in Kilolitre)	Service Charge (in ₹)	Volumetric Charge (Per kl in ₹)
Upto 20	146.41	5.27
20-30	219.62	26.36
> 30	292.82	43.93
Plus Sewer Maintenance Charge : 60% of water volumetric charge		

CATEGORY - II (NON-DOMESTIC CONNECTIONS-COMMERCIAL/INDUSTRIAL)

Monthly Consumption (in Kilolitre)	Service Charge (in ₹)	Volumetric Charge (Per kl in ₹)
0-06	146.41	17.57
06-15	292.82	26.35
15-25	585.64	35.14
25-50	1024.87	87.85
50-100	1171.28	140.56
>100	1317.69	175.69
Plus Sewer Maintenance Charge : 60% of water volumetric charge		

Note : k*l* = kiloliter and 1k*l* = 1000 litres.
For Example : Suppose for a domestic connection monthly consumption of water is 20 kilolitres, then find the water bill for a month.
Sol.: Since, given connection is domestic, so it's water bill is calculated according to category I.

Volumetric Charge = ₹ 20 × 5.27 = ₹ 105.4
Service Charge = ₹ 146.41
Sewage Charges = 60% of Volumetric Charges = 105.4 × 60% = ₹ 63.24
Amount of water bill for the given month = ₹ 105.4 + 146.41 + 63.24 = ₹ 315.05

UNIT – VIII: COORDINATE GEOMETRY

CHAPTER-16

STRAIGHT LINES

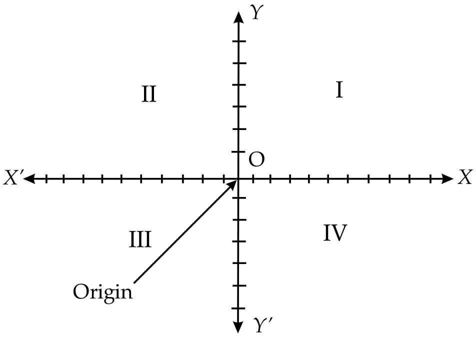
TOPIC-1
Two Dimensional Geometry and Slope of a Line

Revision Notes

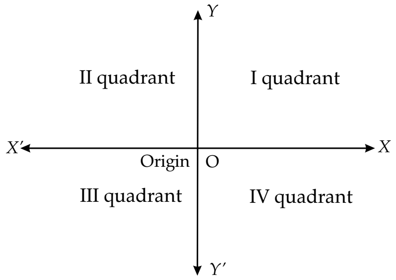
➢ Coordinate Geometry is the branch of Mathematics which deals with the study of geometry by means of algebra. In coordinate geometry, we represent a point in a plane by ordered pair of real numbers. Let us recapitulate some basic concepts of coordinate geometry studied in previous classes.

➢ **Coordinate Axes and Plane**

- The position of a point in a plane is fixed by selecting the axes or reference which are formed by two number lines intersecting each other at right angle. The horizontal number line is called X-axis and vertical number line is called Y-axis.



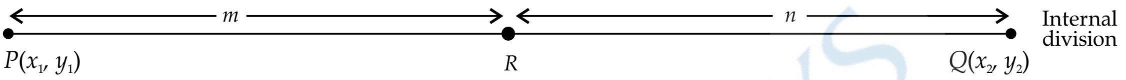
- The point of intersection of these two lines is called the origin. The intersection of X-axis and Y-axis divide the plane into four equal parts. These four parts are called quadrants, Each part is (¼th) of the whole portion. These are numbered, I, II, III and IV anticlockwise from OX. Thus, the plane consists of the axes and four quadrants is known as XY-plane or equation plane or coordinate plane and the axes are know as co-ordinate axes. These axes are also known as rectangular axes and are perpendicular to each other.



➢ Section Formulae

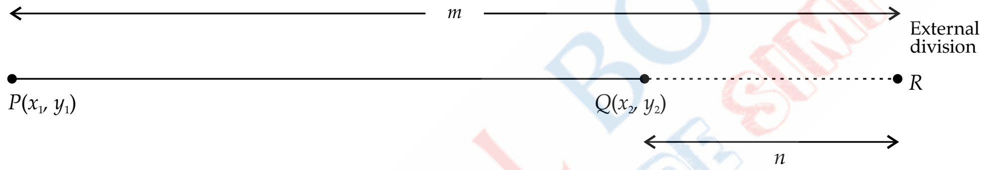
- If a point R divide the line segments joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ internally in the ratio $m : n$, then its coordinates are

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$



- If the division is external, then the co-ordinates of R are

$$\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$$

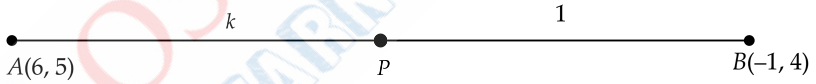


For example : Find the ratio in which the point P whose abscissa is 3 divides the line joining of $A(6, 5)$ and $B(-1, 4)$. Hence, find the coordinates of P.

Sol. Let point P divide the line segment AB in the ratio $k : 1$.

By section formula,

Coordinates of P are $\left(\frac{-k+6}{k+1}, \frac{4k+5}{k+1}\right)$



But given, abscissa of point P is 3, therefore,

$$\frac{-k+6}{k+1}=3 \Rightarrow -k+6=3k+3 \Rightarrow k=\frac{3}{4}.$$

Thus, the required ratio is $\frac{3}{4}:1$ *i.e.*, $3 : 4$ internally.

Now, coordinates of point P are $\left(3, \frac{4\left(\frac{3}{4}\right)+5}{\left(\frac{3}{4}\right)+1}\right)$ *i.e.*, $\left(3, \frac{32}{7}\right)$.

➢ Distance Formulae

- Distance between two points $P(x_1, y_1)$ and $Q(x_1, y_1)$ is $\left|\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\right|$ or $\left|\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\right|$.
- If the points are $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$, then distance between them is $\left|\sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)}\right|$.
- Distance of a point $P(x_1, y_1)$, from origin is $\sqrt{x_1^2+y_1^2}$.

For example : If the distance between the points $(a, -2)$ and $(5, 1)$ is 5 units, find the value(s) of a.

Sol. The distance between the points $(a, -2)$ and $(5, 1)$ is given by

$$d = \left|\sqrt{(5-a)^2 + [1-(-2)]^2}\right|$$

$$\Rightarrow \quad d = \left|\sqrt{(5-a)^2 + 3^2}\right|$$

Since, given $d = 5$ units

$$\therefore \quad 5 = \left|\sqrt{(5-a)^2 + 3^2}\right|$$

$$\Rightarrow \quad 25 = (5-a)^2 + 3^2 \qquad \text{(on squaring both sides)}$$

$$\Rightarrow \quad 25 = (5-a)^2 + 9$$

$$\Rightarrow \quad 16 = (5-a)^2$$

$$\Rightarrow \quad 5 - a = \pm 4$$

$$\Rightarrow \quad a = 1, 9$$

Thus, the required values of a are 1 and 9.

➢ **Area Formulae**

- Area of ΔABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is given by

$$\Delta = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

For example : If the vertices of a triangle are $(1, k)$, $(4, -3)$ and $(-9, 7)$ and its area is 15 sq. units, find the value(s) of k.

Sol. Area of triangle formed by the given points.

$$\Delta = \frac{1}{2}|1(-3-7) + 4(7-k) + (-9)(k+3)| = 15 \qquad \text{(given)}$$

$$\Rightarrow \quad |-13k - 9| = 30$$

$$\Rightarrow \quad |13k + 9| = 30$$

$$\Rightarrow \quad 13k + 9 = \pm 30$$

$$\Rightarrow \quad 13k = -39, 21$$

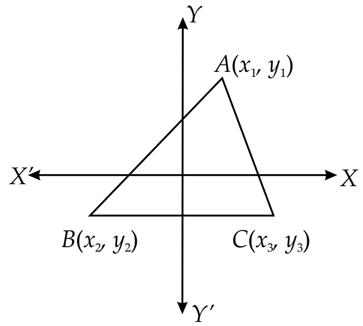
$$\Rightarrow \quad k = -3, \frac{21}{13}.$$

- If $\Delta = 0$, then the points A, B, C are collinear (*i.e*,. they lie in a same straight line).
- Area of quadrilateral $ABCD$ with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ is

$$\Delta = \frac{1}{2}|(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)|$$

- Area of trapezium formed by joining the vertices $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ is

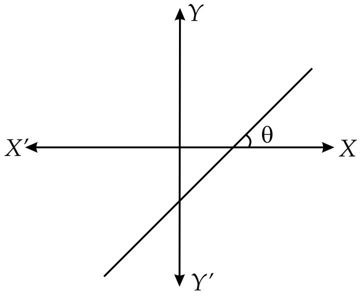
$$\Delta = \left|\frac{1}{2}(y_1 + y_2)(x_1 - x_2) + (y_3 + y_1)(x_3 - x_1) + (y_2 + y_3)(x_3 - x_2)\right|$$



➢ **Equation of the locus of a point**

- The equation of the locus of a point is the relation which is satisfied by the coordinates of every point on the locus of the point.
- The slope of a line 'l' is the tangent of the angle made by the line in the anti-clockwise direction with the positive X-axis.

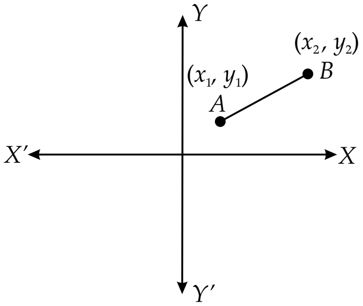
i.e., slope, $\quad m = \tan\theta,$



Where 'm' represents slope and 'θ' is the angle made by the line with positive X-axis.

- **Slope of a line joining two points**
  - The slope 'm' of a line segment AB joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$, and making angle 'θ' with positive X-axis, is given by

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

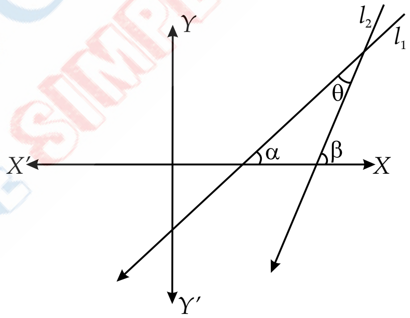


- **Angle between two lines**
  - Let l_1 and l_2 be two lines which make angles α and β respectively with positive X-axis. Then, their slopes are $m_1 = \tan\alpha$ and $m_2 = \tan\beta$.
  - Let 'θ' be the angle between l_1 and l_2, then

$$\theta = \tan^{-1}\left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

and
$$\theta = \pi - \tan^{-1}\left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

For example : Find the angle between the X-axis and the line joining the points (3, –1) and (4, –2).



Sol. Here, m_1 = slope of X-axis = 0 and m_2 = slope of line joining points (3, –1) and (4, –2) = $\frac{-2-(-1)}{4-3} = -1$

Let θ be acute angle between the lines, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-1-0}{1+(0)(-1)}\right|$$

$$\Rightarrow \quad \tan\theta = \left|\frac{-1}{1}\right| = 1$$

$$\Rightarrow \quad \tan\theta = \tan 45^\circ$$

$$\Rightarrow \quad \theta = 45^\circ$$

Hence, the acute angle between the lines is 45°.

  - If two lines are parallel, then angle between them is 0°.

$$\therefore \quad \text{Slope} = \tan\theta = \tan 0^\circ = 0$$

$$\Rightarrow \quad \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = 0$$

$$\Rightarrow \quad m_1 = m_2$$

Thus, two lines are parallel if and only if their slopes are equal *i.e.*, if $m_1 = m_2$.

  - If two lines are perpendicular, then angle between them is 90°. Therefore, slope = tan 90° = ∞

$$\Rightarrow \quad \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \infty$$

$$\Rightarrow \quad 1 + m_1 m_2 = 0$$

$$\Rightarrow \quad m_1 m_2 = -1$$

Thus, two lines are perpendicular, if and only if their slopes m_1 and m_2 satisfy $m_1 m_2 = -1$ or $m_1 = -\frac{1}{m_2}$.

- **Collinearity of three points**
  - Three points A, B, C in XY- plane are collinear *i.e.*, they lie on the same line if and only if slope of AB = slope of BC.

For example : If the three points (h, 0), (a, b) and (0, k) lie on a line, show that $\frac{a}{h} + \frac{b}{k} = 1$.

Sol. Since, the given points are collinear, we have slope of line joining points (h, 0) and (a, b) = slope of line joining points (a, b) and (0, k).

$$\Rightarrow \quad \frac{b-0}{a-h} = \frac{k-b}{0-a}$$

$\Rightarrow \quad (a-h)(k-b) = -ab$

$\Rightarrow \quad ak - ab - hk + bh = -ab$

$\Rightarrow \quad ak + bh = hk$

$\Rightarrow \quad \frac{a}{h} + \frac{b}{k} = 1.$ (Dividing both sides by hk)

Remarks :

- If a point R trisect the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$, then it will divide PQ in the ratio 2 : 1 or 1 : 2.
- If a point R is the mid-point of the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$, then the coordinates of R are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.
- If G is the centroid of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then co-ordinates of $G = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.
- The centroid divides each median of the triangle in the ratio 2 : 1.
- If the origin $O(0, 0)$ is shifted to another point (α, β), then the co-ordinates of point $P(x, y)$ are changed to $P'(x-\alpha, y-\beta)$.
- The angle 'θ', which a line makes with positive direction of X-axis, is called inclination angle of the line.
- The slope of Y-axis (or any line parallel to it) is, $m = \tan 90° = \infty$, which is not defined.
- Let 'θ' be the angle between two lines. Then

  (i) If $\tan\theta$ is positive, then θ will be an acute angle.

  (ii) If $\tan\theta$ is negative, then θ will be an obtuse angle.

TOPIC-2

Various Forms of Equations of Straight Line

Revision Notes

➢ **Horizontal and Vertical lines**

- The equation of a horizontal line (*i.e.*, any line parallel to X-axis) is $y = a$ or $y = -a$. If the line lies above X-axis, then 'a' is positive and if the line lie below the X-axis, then 'a' is negative.
- The equation of a vertical line (*i.e.*, any line parallel to Y-axis) is $x = b$ or $x = -b$. If line lie to the left of Y-axis, then 'b' is negative and if the line lie to the right of Y-axis, then 'b' is positive.

  For example : Find equation of straight line parallel to X-axis at a distance 5 units below it.

  Sol. We know that, equation of straight line parallel to X-axis is $y = a$. Here, $a = -5$.

  $\therefore$ The equation of the required line is $y = -5$ or $y + 5 = 0$.

  Note : (i) The equation of X-axis is $y = 0$.

  (ii) The equation of Y-axis is $x = 0$.

➢ **Point-slope form**

- The equation of the straight line having slope 'm' and passing through the point $P_0(x_0, y_0)$ is $y - y_0 = m(x - x_0)$.
- The equation of straight line passing through origin *i.e.*, (0, 0) and having slope m is

  $$y - 0 = m(x - 0)$$

  i.e., $$y = mx.$$

  For example : Find the equation of the line passing through the point (– 4, 3) with slope $\frac{1}{2}$.

  Sol. Using point slope form, the equation of the line is

  $$y - 3 = \frac{1}{2}[x - (-4)] \qquad \left[\because \text{ Given, } m = \frac{1}{2} \text{ and point } (x_0, y_0) = (-4, 3)\right]$$

  $\Rightarrow \quad 2(y-3) = x + 4$

  $\Rightarrow \quad 2y - 6 = x + 4$

  $\Rightarrow \quad x - 2y + 10 = 0.$

➢ **Slope intercept form**

- If a line 'L' has slope 'm' and make an intercept 'c' on Y-axis, then the equation of the line is $y = mx + c$.
- If the line passes through the origin, then its equation becomes $y = mx$.
- If a line 'L' has slope 'm' and make an intercept 'd' on x-axis, the equation of the line is $y = m(x - d)$.

For example : Find the equations of the lines for which $\tan\theta = \frac{1}{2}$, where θ is the inclination of line and

(i) Y-intercept is $-\frac{3}{2}$ (ii) X-intercept is 4.

Sol. Here, slope of the line $m = \tan\theta = \frac{1}{2}$

(i) Given, Y-intercept is $-\frac{3}{2}$ *i.e.*, $c = \frac{-3}{2}$

Using slope-intercept form *i.e.*, $y = mx + c$, we get

$$y = \frac{1}{2}x + \left(\frac{-3}{2}\right)$$

$$\Rightarrow \quad 2y = x - 3 \text{ or } x - 2y - 3 = 0.$$

(ii) Given, X-intercept is 4 *i.e.*, $d = 4$

Using point - intercept form *i.e.*, $y = m(x - d)$, we get

$$y = \frac{1}{2}(x - 4) \qquad \text{(STD form)}$$

$$\Rightarrow \quad 2y = x - 4 \text{ or } x - 2y - 4 = 0.$$

➢ **Two points form**

- The equation of a line passing through the points (x_1, y_1) and (x_2, y_2) is given by $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

For example : Find the equation of a straight line through (–1, 1) and (2, – 4).

Sol. Using two-point form, the equation of line is

$$y - 1 = \frac{-4-1}{2-(-1)}[x - (-1)]$$

$[\because (x_1, y_1) = (-1, 1) \text{ and } (x_2, y_2) = (2, -4)]$

$$\Rightarrow \quad y - 1 = \frac{-5}{3}(x+1)$$

$$\Rightarrow \quad 3y - 3 = -5x - 5$$

$$\Rightarrow \quad 5x + 3y + 2 = 0.$$

➢ **Intercept form**

- The equation of a line which cuts-off intercepts a and b on the X-axis and Y-axis respectively, is given by

$$\frac{x}{a} + \frac{y}{b} = 1$$

Here, a is called X-intercept and b is called Y-intercept.

- **Convention for the signs of intercepts**

(i) X-intercept is considered positive, if it is measured to the right of the origin and negative, if measured to the left of the origin.

(ii) Y-intercept is considered positive, if it is measured above the origin and negative, if measured below the origin.

For example : Find the equation of the line, which makes intercepts 2 and –3 on the X- and Y-axes, respectively.

Sol. Here, $a = 2$ and $b = -3$. By intercept form of straight line, we get

$$\frac{x}{2} + \frac{y}{-3} = 1$$

$$\Rightarrow \quad -3x + 2y = -6$$

$$\Rightarrow \quad 3x - 2y - 6 = 0.$$

Remarks :

- A line which passes through origin makes zero intercepts on axes.
- A horizontal line has no X-intercept and a vertical line has no Y-intercept.
- The intercepts on X-axis and Y-axis are usually denoted by a and b, respectively. But if only Y-intercept is considered, then it is usually denoted by 'c' and if only X-intercept is considered, then it is usually denoted by 'd'.

➢ **Normal (perpendicular) form**

- The equation of the straight line upon which the length of perpendicular from the origin is p and this perpendicular makes an angle 'α' with the positive direction of X-axis is $x \cos \alpha + y \sin \alpha = p$.
- Slope of the line $= \dfrac{-1}{\tan\alpha} = -\dfrac{\cos\alpha}{\sin\alpha}$

For example : Find the equation of the line whose perpendicular distance from the origin is 4 and the angle between X-axis and the perpendicular is 15°.

Sol. Here, $p = 4$ and $\alpha = 15°$

Using normal form, the equation of the line is given by

$$x \cos 15° + y \sin 15° = 4$$

$$\Rightarrow \quad x \times \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) + y\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) = 4$$

[From trigonometry, $\cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ and $\sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$]

$$\Rightarrow \quad \left(\sqrt{3}+1\right)x + \left(\sqrt{3}-1\right)y = 8\sqrt{2}.$$

$$\left(\sqrt{3}+1\right)x + \left(\sqrt{3}-1\right)y - 8\sqrt{2} = 0$$

TOPIC-3
General Equation of a Line and Family of Straight Lines

Revision Notes

➢ **General Equation of a line**

- An equation of the form $Ax + By + C = 0$, where A, B and C are real constants and at least one of A or B is non-zero, is called general linear equation or general equation of a line.

➢ **Various forms of $Ax + By + C = 0$**

The general equation of a line can be reduced into various forms, which are as follows :

- **Slope-intercept form :** If $B \neq 0$, then $Ax + By + C = 0$ can be written as

$$y = -\frac{A}{B}x - \frac{C}{B}$$

or $\quad y = mx + c,$

where $\quad m = -\dfrac{A}{B}$ and $c = -\dfrac{C}{B}$.

For example : Reduce $6x + 3y - 5 = 0$ into slope-intercept form and find the slope and y-intercept.

Sol. Given equation is, $\quad 6x + 3y - 5 = 0$

$\therefore \quad 3y = -6x + 5$

$\Rightarrow \quad y = -2x + \dfrac{5}{3}$

Hence, slope $= -2$ and y-intercept $= \dfrac{5}{3}$ (on comparing with $x = mx + c$)

- **Intercept form :** If $C \neq 0$, then $Ax + By + C = 0$ can be written as $\dfrac{x}{-\frac{C}{A}} + \dfrac{y}{-\frac{C}{B}} = 1$ or $\dfrac{x}{a} + \dfrac{y}{b} = 1$, where $a = -\dfrac{C}{A}$ and $b = -\dfrac{C}{B}$.

- **Normal form :** The normal form of the equation $Ax + By + C = 0$ can be written as $x \cos \alpha + y \sin \alpha = p$, where

$$\cos \alpha = \pm\frac{A}{\sqrt{A^2+B^2}}, \sin \alpha = \pm\frac{B}{\sqrt{A^2+B^2}} \text{ and } p = \frac{C}{\sqrt{A^2+B^2}}$$

For example : Find the normal form of the equation $y - 2 = 0$.

Sol. The normal form of the equation $y - 2 = 0$ is given by

$$y - 2 = 0$$

$$\Rightarrow \quad 0.x + 1.y = 2$$

$$\Rightarrow \quad \frac{0.x}{\sqrt{0^2+1^2}} + \frac{1.y}{\sqrt{0^2+1^2}} = \frac{2}{\sqrt{0^2+1^2}}$$

$$\Rightarrow \quad x.\cos 90° + y.\sin 90° = 2$$

Here, $p = 2$ and $\alpha = 90°$.

➢ **Angle between two lines, having general equations**

Let $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ be the general equations of two lines.

Then, the slope of the lines are $m_1 = -\dfrac{A_1}{B_1}$ and $m_2 = -\dfrac{A_2}{B_2}$

Let 'θ' be the angle between the two lines, then

$$\tan\theta = \pm\left(\frac{m_2 - m_1}{1 + m_1m_2}\right) = \pm\left(\frac{-\frac{A_2}{B_2} + \frac{A_1}{B_1}}{1 + \frac{A_1}{B_1}.\frac{A_2}{B_2}}\right)$$

For acute angle, we take $\theta = \tan^{-1}\left|\dfrac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}\right|$

For obtuse angle, we take $\theta = \pi - \tan^{-1}\left|\dfrac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}\right|$

➢ **Condition for two lines to be parallel and perpendicular**

Let $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ be the general equations of two lines.

Then, their slopes are, $m_1 = -\dfrac{A_1}{B_1}, m_2 = -\dfrac{A_2}{B_2}$

The lines are parallel, if $m_1 = m_2$

$$\Rightarrow \quad -\frac{A_1}{B_1} = -\frac{A_2}{B_2}$$

$$\Rightarrow \quad \frac{A_1}{B_1} = \frac{A_2}{B_2}$$

$$\Rightarrow \quad A_1B_2 = A_2B_1$$

and the lines are perpendicular, if $m_1m_2 = -1$

$$\Rightarrow \quad \left(-\frac{A_1}{B_1}\right)\left(-\frac{A_2}{B_2}\right) = -1$$

$$\Rightarrow \quad A_1A_2 + B_1B_2 = 0$$

➢ **Equations of perpendicular and parallel lines through a given point**

Let the general equation of a line be $Ax + By + C = 0$.

Then, the slope of the line is $m_1 = -\dfrac{A}{B}$.

Let, m_2 be the slope of perpendicular line, then we have

$$m_1m_2 = -1$$

$$\Rightarrow \quad -\frac{A}{B}m_2 = -1$$

$$\Rightarrow \quad m_2 = \frac{B}{A}$$

∴ Equation of a line passing through (x_1, x_2) and perpendicular to $Ax + By + C = 0$ is

$$y - y_1 = \frac{B}{A}(x - x_1)$$

$$\Rightarrow \quad Bx - Ay + (Ay_1 - Bx_1) = 0$$

Again, let m_2' be the slope of parallel, then we have

$$m_1 = m_2'$$

$$\Rightarrow \quad m_2' = -\frac{A}{B}$$

$\therefore$ Equation of a line parallel to the line $Ax + By + C = 0$ and passing through (x_1, y_1) is

$$y - y_1 = -\frac{A}{B}(x - x_1)$$

$$\Rightarrow \quad Ax + By - (Ax_1 + By_1) = 0.$$

➢ **Condition for collinearity of three points**

Three points A, B and C are said to be collinear, if $AB + BC = CA$, or $AC + CB = AB$, or $AB + AC = BC$

➢ **Condition for concurrency of three lines**

Three or more straight lines are said to be concurrent if they pass through a common point (*i.e.*,) they meet at the same point.

To check concurrency of three straight lines, we follow the following two rules

- If the equation of three lines are $A_1x + B_1y + C_1 = 0$, $A_2x + B_2y + C_2 = 0$, $A_3x + B_3y + C_3 = 0$ and if three constants L, M and N can be found such that $L(A_1x + B_1y + C_1)$, $M(A_2x + B_2y + C_2)$ and $N(A_3x + B_3y + C_3) = 0$ identically (*i.e.*, $= 0\ \forall$ values of x and y), then the three straight lines meet in a point.

or,

- If the point of intersection of two lines satisfies the equation of the third line, then the three lines are concurrent.

➢ **Equation of family of lines passing through the point of intersection of two lines**

Let the two intersecting lines L_1 and L_2 be given by

$$L_1 : A_1x + B_1y + C_1 = 0 \quad \text{...(i)}$$

and

$$L_2 : A_2x + B_2y + C_2 = 0 \quad \text{...(ii)}$$

Then, the equation of the family of lines passing through the intersection of lines (i) and (ii) is

$$A_1x + B_1y + C_1 + \lambda(A_2x + B_2y + C_2) = 0 \quad \text{...(iii)}$$

where λ is an arbitrary constant called parameter.

Equation (iii) represents a family of lines and a particular member of this family can be obtained by substituting a value of λ.

➢ **Distance of a point from a line**

The distance of a point from a line is the length of perpendicular drawn from the point to the line.

Let $L : A_1x + B_1y + C_1 = 0$ be a line, whose perpendicular distance from the point $P(x_1, y_1)$ is d. Then

$$d = \left|\frac{A_1x_1 + B_1y_1 + C_1}{\sqrt{A^2 + B^2}}\right|$$

➢ **Distance between two parallel lines**

The distance between two parallel lines

$$y = mx + c_1$$

and

$$y = mx + c_2,$$

is given by

$$d = \left|\frac{c_1 - c_2}{\sqrt{1 + m^2}}\right|$$

If the lines are given in general form i.e., $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$, then the distance between them is

$$d = \left|\frac{c_1 - c_2}{\sqrt{A^2 + B^2}}\right|$$

For example : Find the distance between parallel lines $15x + 8y - 34 = 0$ and $15x + 8y + 31 = 0$.

Sol. Given lines are :

$$15x + 8y - 34 = 0$$

$$\Rightarrow \quad y = -\frac{15}{8}x + \frac{34}{8}$$

and

$$15x + 8y + 31 = 0$$

$$\Rightarrow \quad y = -\frac{15}{8}x - \frac{31}{8}$$

Here, $m = -\frac{15}{8}$, $c_1 = \frac{34}{8}$ and $c_2 = \frac{-31}{8}$

$\therefore$ Distance between the lines is

$$d = \frac{|c_1 - c_2|}{\sqrt{1+m^2}} = \frac{\left|\frac{34}{8} - \left(\frac{-31}{8}\right)\right|}{\sqrt{1+\left(\frac{-15}{8}\right)^2}}$$

$$= \frac{\frac{65}{8}}{\frac{\sqrt{64+225}}{8}} = \frac{65}{8} \times \frac{8}{\sqrt{289}} = \frac{65}{17}$$

Hence, the distance between given parallel lines is $\frac{65}{17}$ units.

Remarks :

- In general, we consider the equation of a line perpendicular to the $Ax + By + C_1 = 0$ as $Bx - Ay + K = 0$. The value of 'K' can be evaluated by substituting any particular point (a, b) which lies on the line.
- In general, we consider the equation of a line parallel to line $A_1x + B_1y + C_1 = 0$ as $A_1x + B_1y + K = 0$. The value of 'K' can be evaluated by substituting any particular point (a, b) which lies on the line.
- If $A = 0, B \neq 0$, then $Ax + By + C = 0$, which reduces to $y = -\frac{C}{B}$, which is the equation of horizontal line (*i.e.*, parallel to X,-axis)
- If $A = C = 0, B \neq 0$, then $Ax + By + C = 0$, reduces to $y = 0$, which is the equation of X-axis.
- If $A \neq 0, B = C = 0$, then $Ax + By + C = 0$, reduces to $x = 0$, which is the equation of Y-axis.
- If $A \neq 0, B \neq 0, C = 0$, then $Ax + By + C = 0$, reduces to $y = -\frac{A}{B}x$, which is the equation of a straight line passing through the origin.

CHAPTER-17
PARABOLA

Revision Note

PARABOLA

➢ **Conic Section**

A conic section is the locus of a point which moves in a plane in such a way that the ratio of its distance from a fixed point and a fixed line is a constant. Then in the adjacent figure :

(i) The fixed point is called **focus** and is denoted by S.

(ii) The fixed straight line is called **directrix**.

(iii) The constant ratio is called **eccentricity** and is denoted by 'e'.

(iv) The line passing through foci and perpendicular to directrix is called the axis of the conic.

The fix point S in figure is foci, ZK is fix straight line (directrix) and P is variable (moving) point. If P moves such that

$$\frac{PS}{PM} = e \text{ (fix ratio)} = \text{constant} = \text{ecentricity}$$

then locus of point P is called conic

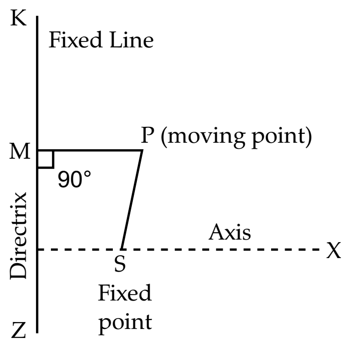


Fig. 1

- **Particular cases of a conic**

(i) If $e = 1$, then conic is a **parabola.**

(ii) If $e < 1$, then conic is an **ellipse.**

(iii) If $e > 1$, then conic is a **hyperbola.**

➢ **Parabola**

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point is always equal to its distance from a straight line in the same.

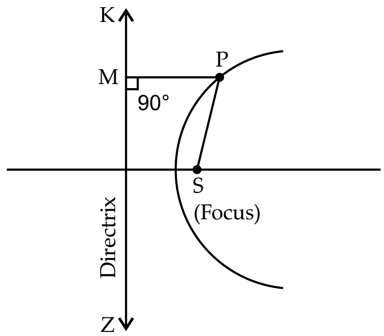


Fig. 2

Here, the fixed point 'P' is called the **focus** and the fixed straight line is called the **directrix** of the parabola. The line through the focus and perpendicular to the directrix is the axis of parabola. The point on the axis midway between focus and directrix is called the **vertex** of the parabola. Let S be the focus and ZK be the **directrix** and let P be any point on the parabola. Then, by definition

$$\frac{SP}{PM} = e = 1$$

$\Rightarrow$ $$SP = PM$$

where PM is the length of perpendicular from P on the directrix ZK.

For example : Find the equation of the parabola whose focus is $(-3, 0)$ and directrix is $x + 5 = 0$

Sol. Let $P(x, y)$ be any point on the parabola having its focus at $S(-3, 0)$ and directrix as the line $x + 5 = 0$

Then, $SP = PM$, where PM is the length of perpendicular from P on the directrix.

$\Rightarrow$ $SP^2 = PM^2$

$\Rightarrow$ $(x+3)^2 + (y-0)^2 = \left|\frac{x + 0.y + 5}{\sqrt{1+0}}\right|^2$

$\Rightarrow$ $y^2 = 4x + 16$, which is the required equation of parabola

➢ **Equation of parabola in its standard form**

The standard equation of parabola is

$$y^2 = 4ax$$

For the parabola $y^2 = 4ax$,

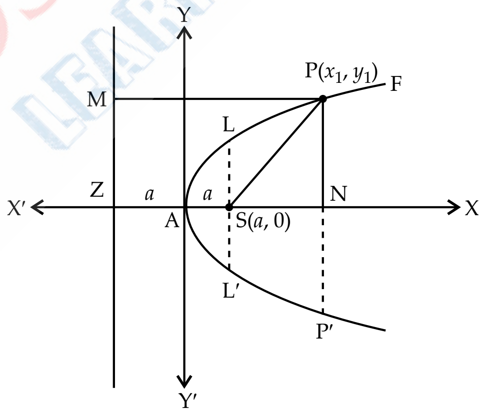


Fig. 3

(i) Coordinates of vertex, A are $(0, 0)$

(ii) Coordinates of focus, S are $(a, 0)$

(iii) Equation of directrix, ZM is $x = -a$ *i.e.*, $x + a = 0$

(iv) Equation of axis, ZX is $y = 0$ (*i.e.*, x-axis)

(v) Equation of tangent (YY') on the vertex is $x = 0$ (*i.e.*, y-axis)

Remark : It is clear that in parabola represented by $y^2 = 4ax$

- **Vertex** is the intersection point of straight lines $x = 0$ and $y = 0$.
- **Focus** is the intersection point of straight lines $x = a$ and $y = 0$.

Some terms related to parabola

(i) Focal Distance : The distance of any point $P(x, y)$ on the parabola from foci S *i.e.*, SP is called focal distance of point P (see figure 1). For parabola $y^2 = 4ax$,

The focal distance of point P is $SP = PM$ (By definition of parabola)

$$= NZ$$
$$= ZA + AN$$
$$= a + x_1$$

(ii) Foal chord : The chord of the parabola which passes through foci S of the parabola is called focal chord of the parabola.

(iii) Latus rectum : The chord of the parabola which passes through foci S and perpendicular on axis of the parabola is called latus-rectum of the parabola. In the figure 3, LSL' is latus rectum.

For parabola $y^2 = 4ax$,

(a) Length of latus-rectum, $LSL' = 2SL = 2.(2a) = 4a$

(b) Equation of latus rectum is $x = a$

(c) Coordinates of end points of latus-rectum LSL' are $(a, -2a)$,

Remark : Length of latus rectum $= 4a$

$$= 2(2a)$$
$$= 2.(SZ)$$
$$= 2.\text{(length of perpendicular from foci on directrix)}$$

(iv) Double ordinate : The intercepted portion of a straight line perpendicular on axis of the parabola is called double ordinate of the parabola. In figure, PNP' is a double ordinate of parabola. In other words, a chord passing through point P perpendicular to the axis of the parabola is called the double ordinate (PNP') through the point P.

➢ **Tracing of the parabola $y^2 = 4ax, a > 0$**

The equation of parabola $y^2 = 4ax$ can be written as

$$y = \pm 2\sqrt{ax} \quad ...(i)$$

We observe the following :

(i) Symmetry : From equation (i), it is clear that for every positive value of x, there are two equal values of y which are opposite in signs. Therefore, curve (parabola) is **symmetrical** relative to x-axis.

(ii) Region : For every negative value of x, the values of y are imaginary. Therefore, there is no point on the curve whose x-coordinate is negative, *i.e.*, no porition of curve is in left side of y-axis.

(iii) Origin : If $y = 0$, the value of x is zero, therefore x-axis meets the curve only at origin. Again, if $x = 0$, then $y^2 = 0$ gives two roots of the equation and every root is zero. Therefore, curve (parabola) touches y-axis at origin.

(iv) Portion occupied : By equation $y = \sqrt{2ax}$, it is clear that y increases as x increases. When x tends to infinity ($x \to \infty$), then y also tends to infinity ($y \to \infty$). Therefore, curve (parabola) goes upto infinity in the right side along x-axis. As the curve is symmetrical about x-axis, therefore curve is same in both sides of x-axis.

With the help of the above points and by joining some convenient points on the parabola, the general shape of the parabola $y^2 = 4ax$ is obtained as shown in Fig. 3.

For example : Find the vertex, focus, axis, directrix and latus-rectum of the parabola, $y' = x^2 - 2x + 3$. Also, draw rough sketch.

Sol. The given equation is

$$y = x^2 - 2x + 3$$
$$\Rightarrow \quad y = (x^2 - 2x + 1) + 2$$
$$\Rightarrow \quad y - 2 = (x - 1)^2$$
$$\Rightarrow \quad (x - 1)^2 = y - 2$$
$$\Rightarrow \quad Y^2 = X \quad ...(i)$$

where $Y = x - 1$ and $X = y - 2$ ...(ii)

Comparing eq. (i) with standard equation $y^2 = 4ax$, we get

(1) The **latus-rectum** of the given parabola, $4a = 1 \Rightarrow a = \frac{1}{4}$

(2) The **vertex** of the parabola is the point of intersection of lines $Y = 0$ and $X = 0$.

i.e., $x - 1 = 0$ and $y - 2 = 0$ or $x = 1$ and $y = 2$

Hence, the coordinates of the vertex of the given parabola are (1, 2).

(3) The **focus** of the parabola is the point of intersection of lines $Y = 0$ and $X = a$

i.e., $x - 1 = 0$ and $y - 2 = a$

or, $x = 1$ and $y = a + 2$

or, $x = 1$ and $y = \frac{1}{4} + 2 = \frac{9}{4}$

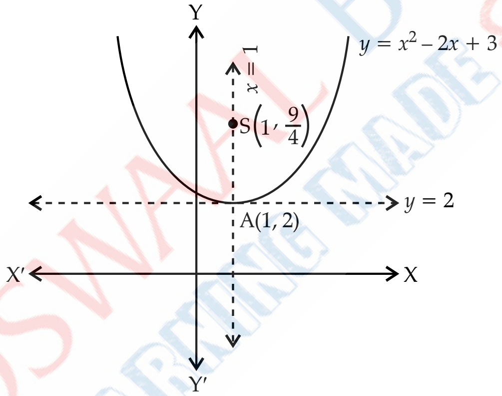
Hence, the coordinates of the focus of the given parabola are $\left(1, \frac{9}{4}\right)$

(4) The **axis** of the parabola is given by the equation

$$X = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$$

(5) The **directrix** of the parabola is given by the equation

$$X + a = 0 \Rightarrow (y - 2) + \frac{1}{4} = 0 \Rightarrow y = \frac{7}{4}$$



➢ **Other forms of Parabola**

(i) $y^2 = -4ax, a > 0$

$y^2 = -4ax$, represents a parabola whose concavity and axis both are on negative side of x-axis *i.e.*, left side.

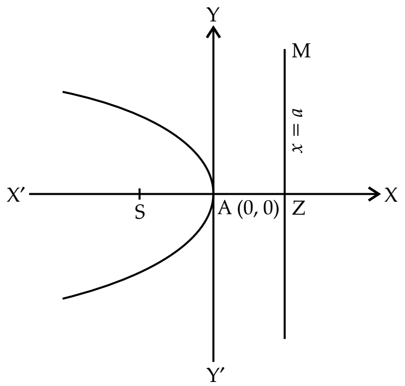
For the parabola, $y^2 = -4ax$:

(a) Coordinates of vertex A are (0, 0)

(b) Coordinates of focus S are $(-a, 0)$.

(c) Equation of directrix is $x = a$ *i.e.*, $x - a = 0$

(d) Equation of axis is $y = 0$ (*i.e.*, x-axis)



(ii) $x^2 = 4ay, a > 0$

Equation $x^2 = 4ay$ represents a parabola whose concavity and axis both are in positive side of y-axis *i.e.*, upper side.

Also, directrix is parallel to x-axis.

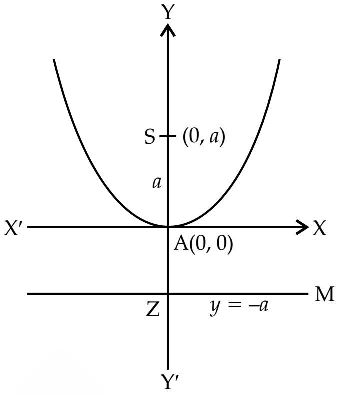
For the parabola $x^2 = 4ay$

(a) Coordinates of vertex A are (0, 0).

(b) Coordinates of focus S are $(0, a)$.

(c) Equation of directrix is $y = -a$ *i.e.*, $y + a = 0$

(d) Equation of axis is $x = 0$ (*i.e.*, y-axis)



(iii) $x^2 = -4ay, a > 0$

Equation $x^2 = -4ay$, represents a parabola whose concavity and axis both are in negative side of y-axis *i.e.*, downside and directrix is parallel to x-axis.

For the parabola, $x^2 = -4ay$

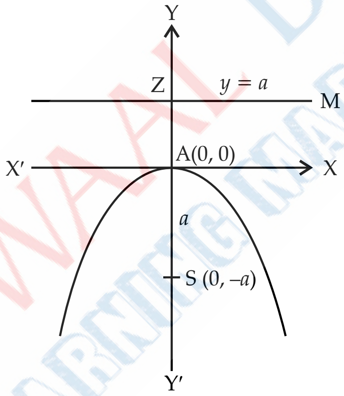
(a) Coordinates of vertex A are (0, 0)

(b) Coordinates of focus S are $(0, -a)$.

(c) Equation of directrix is $y = a$ *i.e.*, $y - a = 0$

(d) Equation of axis is $x = 0$ (*i.e.*, y-axis)

Remark : We can draw all above type of parabolas by following the steps mentioned in "Tracing of Parabola $y^2 = 4ax, a > 0$."



➤ **Equation of parabola for some particular cases**

- Equation of the parabola whose axis is the x-axis and directrix is the y-axis is

$$y^2 = 4a(x - a)$$

- Equation of parabola whose focus is origin, axis is x-axis and latus rectum is y-axis is

$$y^2 = 4a(x + a)$$

- Equation of parabola whose vertex is at point $A(h, k)$ and its latus rectum is of length $4a$, is

(i) $(y - k)^2 = 4a(x - h)$ or $(y - k)^2 = -4a(x - h)$ according as its axis is parallel to OX or OX'.

(ii) $(x - h)^2 = 4a(y - k)$ or $(x - h)^2 = -4a(y - k)$ according as its axis is parallel to OY or OY'.

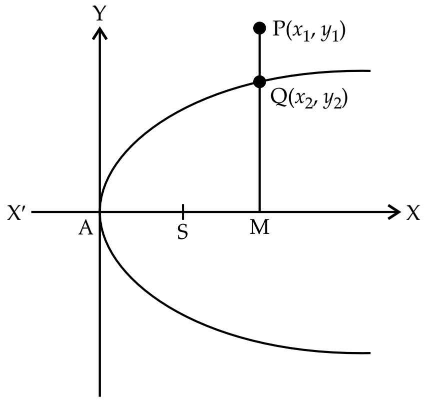
➤ **Position of a point relative to a parabola**

Position of any point $P(x_1, y_1)$ relative to given parabola $y^2 = 4ax$ is :

(i) If $y_1^2 - 4ax_1 > 0$, then point $P(x_1, y_1)$ lies outside the parabola.

(ii) If $y_1^2 - 4ax_1 = 0$, then point $P(x_1, y_1)$ lies on the parabola.

(iii) If $y_1^2 - 4ax_1 < 0$, then point $P(x_1, y_1)$ lies inside the parabola.



- **Parametric equations of parabola**

The equations $x = at^2$ and $y = 2at$ represents the parabola $y^2 = 4ax$. These are called parametric equations of the parabola $y^2 = 4ax$ and 't' is any arbitrary variable called parameter.

Thus, $(at^2, 2at)$ is any point on the parabola $y^2 = 4ax$.

Similarly,

- Parametric coordinates of parabola $y^2 = -4ax$ are $(-at^2, 2at)$ and parametric equations of parabola $y = -4ax$ are $x = -at^2$ and $y = 2at$.
- Parametric coordinates of parabola $x^2 = 4ay$ are $(2at, at^2)$ and parametric equation of parabola $x^2 = 4ay$ are $x = 2at$ and $y = at^2$.
- Parametric coordinates of parabola $x^2 = -4ay$ are $(2at, -at^2)$ and parametric equations of parabola $x^2 = -4ay$ are $x = 2at$ and $y = -at^2$.

For example : Write the parametric equations of the parabola $y^2 = 12x$.

Sol. The given equation of parabola is of the form $y^2 = 4ax$

On comparing the equation, we get

$$4a = 12 \Rightarrow a = 3$$

Therefore, the parametric equations of the given parabola are $x = 3t^2$ and $y = 6t$.

- **Main facts about four types of parabolas**

Parabola	Vertex	Focus	Latus rectum	Co-ordinate of L-R	Aixs	Directrix	Symmetry
$y^2 = 4ax$	(0, 0)	$(a, 0)$	$4a$	$(a_1 \pm 2a)$	$y = 0$	$x = -a$	x-axis
$y^2 = -4ax$	(0, 0)	$(-a, 0)$	$4a$	$(-a_1 \pm 2a)$	$y = 0$	$x = a$	x-axis
$x^2 = 4ay$	(0, 0)	$(0, a)$	$4a$	$(\pm 2a, a)$	$x = 0$	$y = -a$	y-axis
$x^2 = -4ay$	(0, 0)	$(0, -a)$	$4a$	$(\pm 2a, -a)$	$x = 0$	$y = a$	y-axis

Parabola with y^2 term	Parabola with x^2 term
1. Symmetrical about x-axis	**1.** Symmetrical about y-axis
2. Axis is along the x-axis	**2.** Axis is along the y-axis
3. It open right handed when co-efficient of 'x' is positive and left handed when co-efficient of 'x' is negative.	**3.** It opens upwards if co-efficient of 'y' is positive and downwards if co-efficients of 'y' is negative.

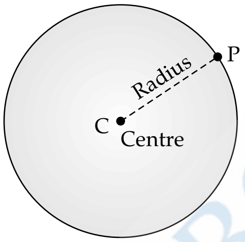
CHAPTER-18

CIRCLES

TOPIC-1
Equations of Circle in Different Forms

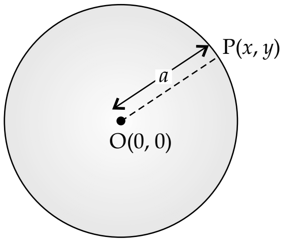
Revision Notes

- A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point is equal to a given distance. The fixed point is called the **centre** of the circle and the given distance is called the **radius** of the circle.



- **Equation of a circle whose centre is at origin**

Let $O(0, 0)$ be the centre of the circle and a (> 0) be its radius. Let $P(x, y)$ be a point in the plane, then P lies on the circle iff $OP = a$



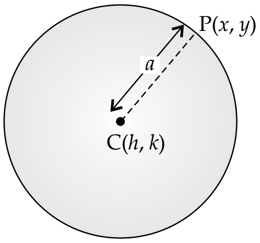
i.e., $$\sqrt{(x-0)^2+(y-0)^2} = a$$

or, $$x^2 + y^2 = a^2$$

is the equation of circle in simplest form.

- **Equation of a circle whose centre and radius are given**

Let $C(h, k)$ be the centre of the circle and $a(> 0)$ be its radius. Let $P(x, y)$ be a point in the plane, then P lies on the circle iff $CP = a$



i.e., $$\sqrt{(x-h)^2+(y-k)^2} = a$$

or, $$(x-h)^2 + (y-k)^2 = a^2 \quad \text{...(i)}$$

which is the equation of circle in central form.

The above equation is called the standard equation of the circle.

For example : Find the equation of the circle with centre at (– 2, 5) and having radius 4.

Sol. The equation of circle whose centre is (– 2, 5) and radius is 4, is given by

$$[x-(-2)]^2 + (y-5)^2 = (4)^2$$

$$\Rightarrow \quad (x+2)^2 + (y-5)^2 = 16$$

$$\Rightarrow \quad (x^2 + 4x + 4) + (y^2 - 10y + 25) = 19$$

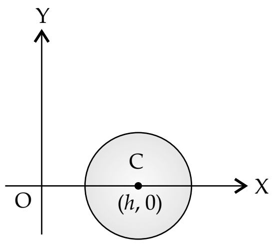
$$\Rightarrow \quad x^2 + y^2 + 4x - 10y + 29 - 16 = 0$$

$$\Rightarrow \quad x^2 + y^2 + 4x - 10y + 13 = 0.$$

- **Equation of circles in particular cases**

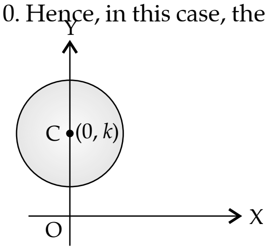
(i) When the centre of the circle be at origin, then clearly $h = 0, k = 0$. Hence, eq. (i) reduces to

$$x^2 + y^2 = a^2$$



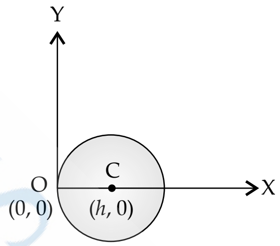
(ii) If the centre of the circle be on the X-axis and the origin is not on the circumference of the circle, then Y-coordinate of the centre will be zero *i.e.*, $k = 0$. Hence, in this case, the equation of the circle is

$$(x-h)^2 + y^2 = a^2$$



(iii) If the centre of the circle is on the Y-axis and the origin does not lie on the circumference of the circle, then the X-coordinate of the centre will be *zero* *i.e.*, $h = 0$. Hence, in this case, the equation of the circle is

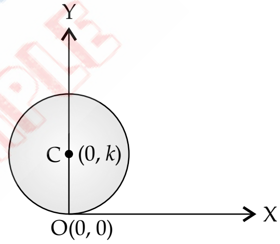
$$x^2 + (y-k)^2 = a^2$$



(iv) If the centre of the circle be on X-axis and the origin be on circumference of the circle, then $k = 0$ and $h = a$. Hence, in this case, the equation of the circle is

$$(x-a)^2 + y^2 = a^2$$

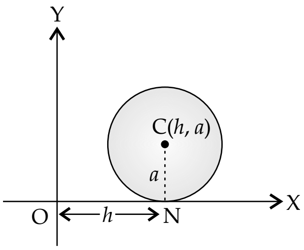
i.e., $$x^2 + y^2 - 2ax = 0$$



(v) If the centre of the circle be on the Y-axis and the origin be on the circumference of the circle, then $h = 0$ and $k = a$. Hence, in this case, the equation of the circle is

$$x^2 + (y-a)^2 = a^2$$

i.e., $$x^2 + y^2 - 2ay = 0$$



(vi) If the circle touches the X-axis, then $k = a$. Hence in this case, the equation of the circle is

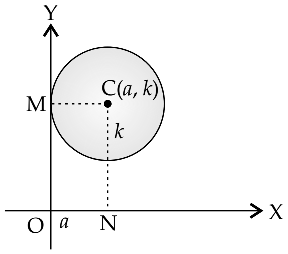
$$(x-h)^2 + (y-a)^2 = a^2$$

i.e., $$x^2 + y^2 - 2hx - 2ay + h^2 = 0$$

(vii) If the circle touches the Y-axis, then $h = a$. Hence, in this case, the equation of the circle is

$$(x-a)^2 + (y-k)^2 = a^2$$

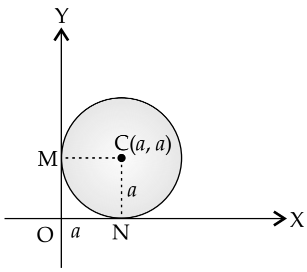
i.e., $$x^2 + y^2 - 2ax - 2ky + k^2 = 0$$



(viii) If the circle touches both the axes and the centre C of the circle be in the first quadrant, then $h = k = a$. Hence, in this case, the equation of the circle is

$$(x-a)^2 + (y-a)^2 = a^2$$

i.e., $$x^2 + y^2 - 2ax - 2ay + a^2 = 0$$



Remark : From the particular cases (vi), (vii) and (viii), it is clear that the **distance of centre from any tangent** to the circle is equal to the **radius of the circle.** Thus, if any straight line touches a circle then **the perpendicular distance of the tangent is equal to the radius of the circle.** From this property of the circle, we can find its radius.

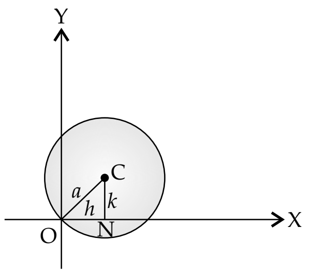
(ix) If the origin be on the circumference of the circle and coordinates of the centre C be (h, k), then

$$h^2 + k^2 = a^2$$

Hence, in this case, the equation of the circle is

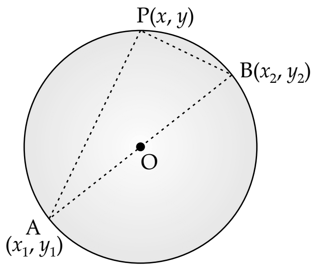
$$(x-h)^2 + (y-k)^2 = h^2 + k^2$$

i.e., $$x^2 + y^2 - 2hx - 2ky = 0$$



➢ **Equation of circle when end points of diameters are given**

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the extremities of a diameter of the circle. $P(x, y)$ be a point which lie on the circle, then



$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$, is the equation of circle in diameter form.

For example : Find the equation of the circle described on the line joining the points (1, 2) and (3, 4) as diameter.

Sol. Let AB be the diameter of the circle where A is the point (x_1, y_1) and B is the point (x_2, y_2).

Therefore, $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (3, 4)$

Hence, the required equation of circle is :

$$(x - 1)(x - 3) + (y - 2)(y - 4) = 0$$
$$\Rightarrow \quad (x^2 - 4x + 3) + (y^2 - 6y + 8) = 0$$
$$\Rightarrow \quad x^2 + y^2 - 4x - 6y + 11 = 0.$$

➢ **Parametric equations of circle**

- **Parametric form of circle $x^2 + y^2 = a^2$**

  The equations $x = a\cos t$ and $y = a\sin t$, $0 \le t \le 2\pi$, represent the circle $x^2 + y^2 = a^2$. These are called parametric equations of the circle and t is called parameter.

  Thus, $(a\cos t, a\sin t)$ is any point on the circle $x^2 + y^2 = a^2$.

- **Parametric form of circle $(x - h)^2 + (y - k)^2 = a^2$**

  The equations $x = h + a\cos t$ and $y = k + a\sin t$, $0 \le t \le 2\pi$, represent the circle $(x - h)^2 + (y - k)^2 = a^2$. These are called parametric equations of the circle and t is called parameter.

  Thus, $(h + a\cos t, k + a\sin t)$ is any point on the circle $(x - h)^2 + (y - k)^2 = a^2$.

  For example : Find the parametric equation of the circle $x^2 + y^2 = 5$.

  Sol. The parametric equation of circle $x^2 + y^2 = 5$ is given as

  $$x = \sqrt{5}\cos t,\ y = \sqrt{5}\sin t, \qquad (\because r^2 = 5 \Rightarrow r = \sqrt{5})$$
  $$0 \le t \le 2\pi.$$

TOPIC-2
General Equation of Circle

Revision Notes

➢ **General equation of circle**

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$, where g, f and c are constants represents the **general equation** of circle.

The coordinates of the **centre** are : $(-g, -f)$ and the radius of the above circle is $\sqrt{g^2 + f^2 - c}$.

Case I : If $g^2 + f^2 - c > 0$, then the radius of the circle is real and hence the circle is also real.

Case II : If $g^2 + f^2 - c = 0$, *i.e.*, $g^2 + f^2 = c$, then the radius of the circle is **zero** and the circle becomes a point coinciding with the centre $(-g, -f)$. Such a circle is called a **point circle.**

Case III : If $g^2 + f^2 - c < 0$, then the radius of the circle is imaginary and is not possible to draw.

➢ **Special features of general equation**

The general equation $x^2 + y^2 + 2gx + 2fy + c = 0$ of the circle have special features given as follows :

(i) It is quadratic in both x and y.

(ii) Coefficient of x^2 = coefficient of y^2.

(**Note :** In solving problems it is advisable to keep the coefficient of x^2 and y^2 unity.)

(iii) There is no term containing xy *i.e.*, the coefficient of xy is zero.

(iv) It contains three arbitrary constants *i.e.*, g, f and c.

Remarks :

- The equation $ax^2 + ay^2 + 2gx + 2fy + c = 0$, $a \ne 0$ also represents a circle. This equation can also be written as

  $$x^2 + y^2 + \frac{2g}{a}x + \frac{2f}{a}y + \frac{c}{a} = 0$$

The coordinates of the centre are $\left(-\frac{g}{a}, -\frac{f}{a}\right)$ and radius = $\sqrt{\frac{g^2}{a^2}+\frac{f^2}{a^2}-\frac{c}{a}}$.

- On comparing the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$ of a circle with the general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we find that it represents a circle if $a = b$ *i.e.*, coefficient of x^2 = coefficient of y^2 and $h = 0$ *i.e.*, coefficient of $xy = 0$.
- If in the general equation of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ coefficeint of x^2 and y^2 is unity *i.e.*, 1, then

$$X\text{-coordinate of centre} = -\frac{1}{2}(\text{coefficient of } x)$$

$$Y\text{-coordinate of centre} = -\frac{1}{2}(\text{coefficient of } y)$$

and $$\text{radius} = \sqrt{(\frac{1}{2}\text{coefficient of } x)^2 + (\frac{1}{2}\text{coefficient of } y)^2 - (\text{constant term})}$$

For example : Find the centre and radius of the circle $x^2 + y^2 + 4x - 4y - 1 = 0$.

Sol : Here, $$g = \frac{1}{2}(\text{coefficient of } x) = \frac{1}{2}(4) = 2$$

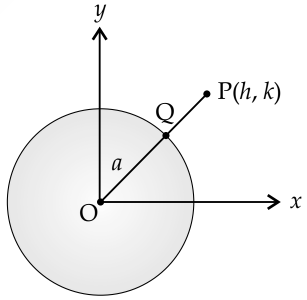
$$f = \frac{1}{2}(\text{coefficient of } y) = \frac{1}{2}(-4) = -2$$

Hence, the centre of the circle is $(-g, -f) = (-2, 2)$

Also, $$\text{radius of the circle} = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{(2)^2 + (-2)^2 - (-1)} = \sqrt{4+4+1} = \sqrt{9} = 3. \text{ unit}$$

➢ Position of a point with respect to a circle



- Position of a point $P(h, k)$ with respect to the circle $x^2 + y^2 = a^2$
  (i) If $h^2 + k^2 > a^2$, then the point $P(h, k)$ will lie outside the circle.
  (ii) If $h^2 + k^2 = a^2$, then the point $P(h, k)$ will lie on the circle.
  (iii) If $h^2 + k^2 < a^2$, then the point $P(h, k)$ will lie inside the circle.
- Position of a point $P(h, k)$ with respect to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$
  (i) If $h^2 + k^2 + 2gh + 2fy + c > 0$, then the point $P(h, k)$ will lie outside the circle.
  (ii) If $h^2 + k^2 + 2gh + 2fy + c = 0$, then the point $P(h, k)$ will lie on the circle.
  (iii) If $h^2 + k^2 + 2gh + 2fy + c < 0$, then the point $P(h, k)$ will lie inside the circle.

For example : Find the position of the point (–4, 2) with respect to the circle $2x^2 + 2y^2 + 16x - 8y - 21 = 0$.

Sol. On putting $x = -4$ and $y = 2$ in the left hand side of the given equation, we get

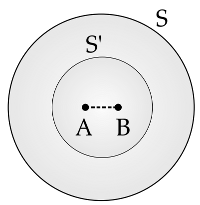
$$\text{L.H.S.} = 2(-4)^2 + 2(2)^2 + 16(-4) - 8(2) - 21$$
$$= 32 + 8 - 64 - 16 - 21$$
$$= -61 < 0$$

Hence, the point (– 4, 2) lies inside the circle.

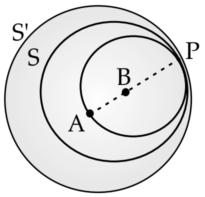
➢ Relative position of Two circles

Let S, S' be two (non-concentric) circles with centre A, B, and radii r_1, r_2 $(r_1 > r_2)$ and d be the distance between their centres, then

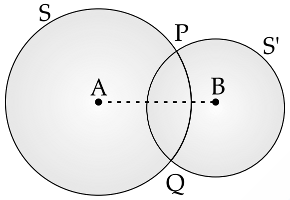
(i) One circle lies completely inside the other iff $d < |r_1 - r_2|$



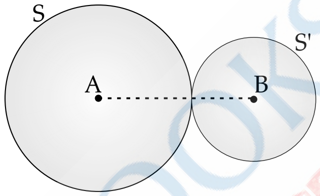
(ii) The two circles touch internally iff $d = |r_1 - r_2|$



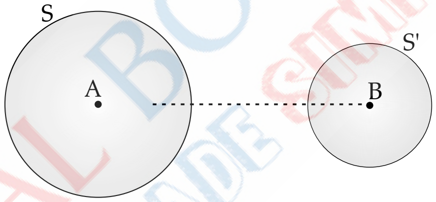
(iii) The two circles intersect in two points iff $d > |r_1 - r_2|$ and $d < r_1 + r_2$



(iv) The two circles touch externally iff $d = r_1 + r_2$



(v) One circle lies completely outside the other circle iff $d > r_1 + r_2$



Remark :

- If two circles intersect, then we can solve the equation of circles simultaneously to find the points of intersection. In particular, the equation of the common chord is given by

$$S - S' = 0$$

- If the two circles touch (internally or externally), then the equation of their common tangent is given by

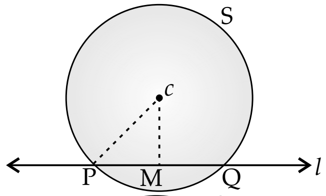
$$S - S' = 0$$

➢ **Line and Circle**

- The condition that the line $y = mx + c$ may intersect the circle $x^2 + y^2 = a^2$ is

$$a^2(1 + m^2) \geq c^2$$

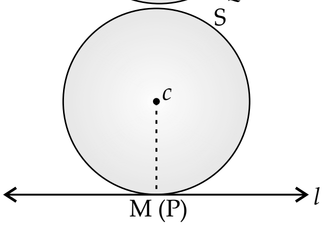
Geometrically, l intersect S in two distinct points iff $d < r$.



- **Condition of the tangency**

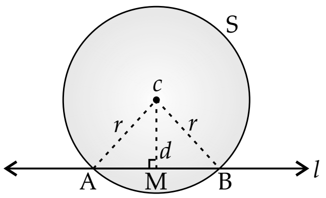
The line $y = mx + c$ will touch the circle $x^2 + y^2 = a^2$ iff $a^2(1 + m^2) = c^2$ or $c = \pm a\sqrt{1+m^2}$

Geometrically, l intersect S in one and only one point iff $d = r$.



Remarks :

- Equations of tangents in slope form $y = mx \pm a\sqrt{1+m^2}$
- Equation of any tangent to the circle $x^2 + y^2 = a^2$ in the slope form is $y = mx + a\sqrt{1+m^2}$.

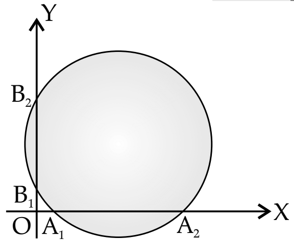


➢ **Length of intercept made by a circle on a line**

Let a line 'l' meet a circle S with centre C and radius r in two distinct points. If d is the distance of C from l, then the length of intercept (AB) is $2\sqrt{r^2 - d^2}$.

- **Length of intercepts on the axes**

  The intercepts made on the axes by the circle whose equation is $x^2 + y^2 + 2gx + 2fy + c = 0$ are given below :

  X-intercept, $A_1A_2 = 2\sqrt{g^2 - c}$

  and Y-intercept, $A_1A_2 = 2\sqrt{f^2 - c}$

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