MATHEMATICS

MIND MAPS

&

MNEMONICS

Sets and Representations-I

Subset & Power Set

Let A and B be two sets. If every element of A is an element of B, then A is called a subset of B and written as $A \subset B$ or $B \supset A$ (read as 'A' is contained in 'B' or 'B contains A'). B is called superset of A.

Note:

1. Every set is a subset and superset of itself.
2. If A is not a subset of B, we write $A \not\subset B$.
3. The empty set is the subset of every set.
4. Power Set: If A is a set with n (A) = m, then no. of subset in power set $n\,[P(A)] = 2^m$

e.g. Let A = {3, 4}, then subsets of A are ϕ, {3}, {4}, {3, 4}. Here, $n(A) = 2$ and number of subsets of $A = 2^2 = 4$.

Cardinal Number

The number of elements in a finite set is represented by n (A), known as cardinal number.

Eg.: A = {a, b, c, d, e} Then, n (A) = 5

Introduction

A set is a collection of well-defined distinct objects. The sets are usually denoted by capital letters A, B, C etc., and the members or elements of the set are denoted by lower case letters a, b, c etc... If x is a member of the set A, we write $x \in A$ (read as 'x' belongs to A) and if x is not a member of set A, we write $x \notin A$ (read as 'x' doesn't belongs to A). If x and y both belong to A, we write $x, y \in A$.

Some examples of sets are: A: odd numbers less than 10
N: the set of all natural numbers
V : the vowels in the English alphabates
R: the set of all real numbers.

Representation of Sets

Set builder form or Rule Method

In this form, we write a variable (say x) representing any member of the set followed by property satisfied by each member of the set.

e.g.: The set A of all prime number less than 10 in set builder form is written as

A = {x | x is a prime number less than10}

The symbol "|" stands for the word "such that". Sometimes, we use symbol ":" in place of symbol "|"

Roaster or Tabular form

In this form, we first list all the members of the set within braces (curly brackets) and separate these by commas.

e.g.: The set of all odd natural number less than 10 in this form is written as: A = {1, 3, 5, 7, 9}

In roaster form, every element of the set is listed only once.

The order in which the elements are listed is immaterial.

e.g. Each of the following sets denotes the same set {1, 2, 3}, {3, 2, 1}, {1, 3, 2}

Types of Sets

Empty set or Null set

A set which has no element is called null set. It is denoted by symbol ϕ or {}.

e.g: Set of all real numbers whose square is −1.

In set-builder form: {x : x is a real number whose square is −1}

In roaster form: { } or ϕ

Finite and Infinite set

A set which has finite number of elements is called a finite set. Otherwise, it is called an infinite set.

e .g.: The set of all days in a week is a finite set whereas the set of all integers, denoted by {...... −2, −1, 0, 1, 2,......} or {x | x is integers} is an infinite set.

An empty set ϕ which has no element is a finite set is called empty or void or null set.

Equal set

Two sets A and B are set to be equal, written as A=B, if every element of A is in B and every element of B is in A.

e.g.: (i) A = {1, 2, 3,4} and B = {3, 1, 4, 2}, then A = B

(ii) A = {x : x−5 =0} and B = {x : x is an integral positive root of the equation $x^2 - 2x - 15 = 0$}

Then A= B

Singleton set

A set having one element is called singleton set.

e.g.: (i) {0} is a singleton set, whose only member is 0.

(ii) A = {x : $1 < x < 3$, x is a natural number} is a singleton set which has only one member which is 2.

Equivalent set

Two finite sets A and B are said to be equivalent, if $n(A) = n(B)$. Note: equal set are equivalent but equivalent sets need not to be equal.

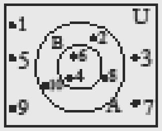
e.g.: The sets A = {4, 5, 3, 2} and B = {1, 6, 8, 9} are equivalent, but are not equal.

Sets, Relations and Functions Part-II

Venn Diagram

A Venn diagram is an illustration of the relationships between and among sets, groups of objects that share something common. These diagrams consist of rectangle and closed curves usually circles

e.g.:



In the given venn diagram U={1,2,3,.......10} universe set of which A={2,4,6,8,10} and B={4,6} are subsets and also B⊂A

Universal Set

The set containing all objects of element and of which all other sets are subsets is known as universal sets and denoted by U.

e.g : For the set of all intergers, the universal set can be the set of rational numbers or the set R of real numbers

Algebra of sets

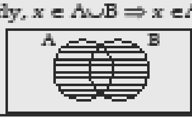
1. For any set A, we have
   (a) $A\cup A=A$, (b) $A\cap A=A$, (c) $A\cup\phi=A$, (d) $A\cap\phi=\phi$, (e) $A\cup U=U$ (f) $A\cap U=A$, (g) $A-\phi=A$, (h) $A-A=\phi$
2. For any two sets A and B we have
   (a) $A\cup B = B\cup A$, (b) $A\cap B= B\cap A$, (c) $A-B\subseteq A$, (d) $B-A\subseteq B$
3. For any three sets A,B and C, we have
   (a) $A\cup(B\cup C)= (A\cup B)\cup C$, (b) $A\cap(B\cap C)= (A\cap B)\cap C$
   (c) $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$, (d) $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$
   (e) $A-(B\cup C)=(A-B)\cap(A-C)$, (f) $A-(B\cap C)=(A-B)\cup(A-C)$

Operations on Sets

Union

The union of two sets A and B, written as $A\cup B$ (read as A union B) is the set of all elements which are either in A or in B or in both. Thus, $A\cup B = \{x : x \in A \text{ or } x \in B\}$

clearly, $x \in A\cup B \Rightarrow x \in A$ or $x \in B$ and $x \notin A\cup B \Rightarrow x \notin A$ and $x \notin B$



eg: If A = {a, b, c, d} and B = {c, d, e, f } then $A\cup B$ = {a, b, c, d, e, f}

Intersection

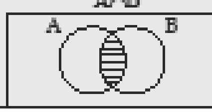
The intersection of two sets A and B, written as $A\cap B$ (read as 'A' intersection 'B') is the set consisting of all the common elements of A and B.

Thus, $A\cap B = \{x : x\in A \text{ and } x\in B\}$

Clearly, $x\in A\cap B \Rightarrow \{x\in A \text{ and } x\in B\}$ and $x\notin A\cap B \Rightarrow \{x \notin A \text{ or } x\notin B\}$.

e.g: If A={a,b,c,d} and B={c,d,e,f}

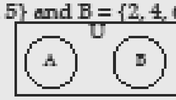
Then $A\cap B$ = {c,d}



Disjoint sets

Two sets A and B are said to be disjoint, if $A\cap B =\phi$ i.e, A and B have no common element. e.g: if A = {1, 3, 5} and B = {2, 4, 6}

Then, $A\cap B =\phi$, so A and B are disjoint.



Symmetric Difference

The symmetric difference of two sets A and B, denoted by A Δ B, in defined as (A Δ B)=(A–B) ∪ (B – A)

e.g. If A = {1, 2,3,4,5}

and B = {1,3,5,7,9} then

(A Δ B)= (A–B) ∪ (B – A)

= {2, 4,} ∪ {7, 9}

= {2, 4, 7, 9}

Difference of two sets

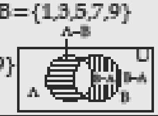
If A and B are two sets, then their difference A-B is defined as:

$A-B = \{x : x \in A \text{ and } x \notin B\}$

Similarly, $B-A = \{x : x \in B \text{ and } x \notin A\}$

e.g. If A={1,2,3,4,5,} and B={1,3,5,7,9} then

A–B={2,4} and B–A={7,9}



Complement set

If U is a universal set and A is a subset of U, then complement of A is the set which contains those elements of U, which are not present in A and is denoted by A′ or A^C. Thus,

$A^C = \{x : x\in U \text{ and } x\notin A\}$

e.g.: If U = {1, 2, 3, 4, ...}

and A = {2, 4, 6, 8, ...}

then A^C = {1, 3, 5, 7, ...}

Properties of complement

- Complement law:
  (i) $A\cup A' = U$ (ii) $A\cap A'= \phi$
- De morgan's Law:
  (i) $(A\cup B)' = A'\cap B'$ (ii) $(A\cap B)' = A'\cup B'$
- Double Complement law:
  $(A')' = A$
- Law of empty set and universal set
  $\phi'=U$ and $U'= \phi$

Subsets of a set of real numbers 'R'

- The set of natural numbers
  N={1, 2, 3, 4, 5, – – –}
- The set of integers
  Z={. . . –3, –2, –1, 0, 1, 2, 3,– – –}
- The set of irrational numbers,
  $T = \{x: x\in R \text{ and } x\notin Q\}$
- The set of rational number
  $Q=\{x : x =\frac{p}{q}, p, q \in Z \text{ and } q \neq 0\}$

Relation among these subsets are

$N \subset Z \subset Q$, $Q \subset R$, $T \subset R$, $N \not\subset T$

Interval Notation

Let a and b be real numbers with $a < b$

Set of Real Numbers	Interval Notation	Region on the real number line
$\{x\|a< x < b\}$	(a, b)	
$\{x\|a \le x < b\}$	[a, b)	
$\{x\|a< x \le b\}$	(a, b]	
$\{x\|a \le x \le b\}$	[a, b]	
$\{x\|x < b\}$	(–∞, b)	
$\{x\|x \le b\}$	(–∞, b]	
$\{x\|x >a\}$	(a, ∞)	
$\{x\|x \ge a\}$	[a, ∞)	
R	(–∞, ∞)	

Sets, Relations and Functions Part-III

Composition of functions and Invertible functions

Composition of functions

The composition of functions $f : A \to B$ and $g : B \to C$ is denoted by gof and is defined as $gof : A \to C$ given by $gof(x) = g(f(x))\ \forall x \in A$. e.g. let $A = N$ and $f, g : N \to N$ such that $f(x) = x^2$ and $g(x) = x^3\ \forall x \in N$. Then $gof(2) = g(f(2)) = g(2^2) = 4^3 = 64$.

Invertible functions

A function $f : X \to Y$ is invertible, if $\exists$ a function $g : Y \to X$ such that $gof = I_X$ and $fog = I_Y$. Then, g is the inverse of f. If f is invertible, then it is both one-one and onto and vice-versa. e.g. If $f(x) = x$ and $f : N \to N$, then f is invertible.

Theorem 1 : If $f : X \to Y$, $g : Y \to Z$ and $H : Z \to S$ are functions, then $ho(gof) = (hog)of$.

Theorem 2 : Let $f : X \to Y$ and $g : Y \to Z$ be two invertible functions, then gof is invertible with $(gof)^{-1} = f^{-1} o g^{-1}$.

Types of Relations

Reflexive relation

A relation $R : A \to A$ is reflexive if $aRa\ \forall a \in A$.

Symmetric relation

A relation $R : A \to A$ is symmetric if $aRb \Leftrightarrow bRa\ \forall a, b \in A$.

Transitive relation

A relation $R : A \to A$ is transitive if aRb and $bRc \Rightarrow aRc\ \forall a, b, c \in A$.

Equivalence relation

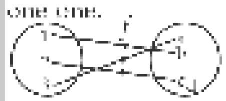
(If a relation has reflexive, symmetric and transitive relations) e.g., Let T = the set of all triangles in a plane and $R : T \to T$ defined by $R = \{(T_1, T_2) : T_1$ is congruent to $T_2\}$. Then, R is equivalence.

Types of Functions

One-one (injective)

$f : X \to Y$ is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2\ \forall x_1, x_2 \in X$. Otherwise, f is many-one.

f is one-one.



Onto (surjective)

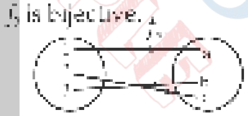
$f : X \to Y$ is onto if every $y \in Y$, $\exists\ x \in X$ such that $f(x) = y$. Then f is surjective.

f_1 is onto.

Bijective

$f : X \to Y$ is both one-one and onto, then f is bijective.

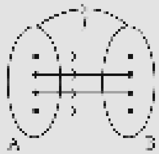
f_1 is bijective.



Sets, Relations and Functions Part-IV

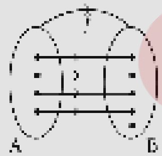
Kinds of Functions

One-One Onto function



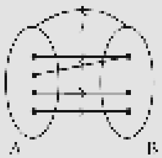
Range = Codomain

One-One Into function



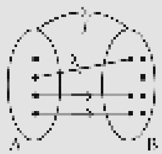
Range ⊂ Codomain

Many One Onto function



Range = Codomain

Many One Into function

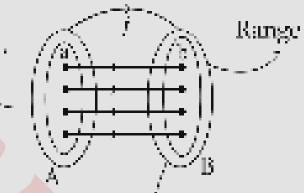


Range ⊂ Codomain

Functions

Definition: A relation 'f' from a set A to set B is said to be a function if every element of set A has one and only one image in set B.

Notations: Domain(Input) ⟶ $\boxed{f}$ ⟶ $y=f(x)$ Range(Output)



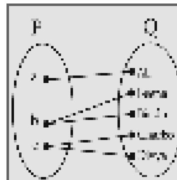
Cartesian product of sets

Given two non empty sets A&B. The cartesian product A×B is the set of all ordered pairs of elements from A&B i.e., $A \times B = \{(a,b) : a \in A ; b \in B\}$. If $n(A)=p$ and $n(B)=q$, then $n(A \times B)=pq$

Relation

Let A&B be two non-empty sets. Then any subset 'R' of A × B is a relation from A to B. If $(a,b) \in R$, then we write $a\,R\,b$, which is read as 'a is related to b' by a relation R. 'b' is also called image of 'a' under R. The total number of relations that can be defined from a set A to a set B is the number of possible subsets of A × B. If $n(A)=p$ and $n(B)=q$, then $n(A \times B)=pq$ and total number of relations is 2^{pq}.

Pictorial representation of a relation



Given, R = {(x,y) : x is the first letter of the name y, x ∈ P, y ∈ Q}. Then, R = {(a, Ali), (b, Beena), (c, charu)} This is a visual or pictorial representation of relation R (called an arrow diagram) is shown in figure.

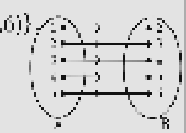
Domain & range of a Relation

If R is a relation from A to B, then the set of first elements in R is called domain & the set of second elements in R is called range of R. Symbolically, Domain of R = {x:(x,y) ∈ R}; Range of R = {y:(x,y) ∈ R}. The set B is called co-domain of relation R.

Note: the range ⊆ Codomain.

e.g. Given, R = {(1,2),(2,3),(3,4),(4,5),(5,6)} then Domain of R = {1,2,3,4,5}

Range of R = {2,3,4,5,6} and codomain of R = {1,2,3,4,5,6}



Inverse relation

Let A&B be two sets and R be a relation from set A to set B. Then inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b,a) : (a,b) \in R\}$. Note: $(a,b) \in R \Leftrightarrow (b,a) \in R^{-1}$

Also, $Dom(R) = Range(R^{-1})$ and $Range(R) = Dom(R^{-1})$

Even and odd function

Even function $f(-x)=f(x)$, ∀x ∈ Domain

Odd function $f(-x)=-f(x)$, ∀x ∈ Domain

Algebra of functions

Let $f : x \to R$ and $g : x \to R$ be any two real functions where $x \subset R$.

Addition: $(f+g)\,x = f(x)+g(x)$; $\forall x \in R$

Subtraction: $(f-g)\,x = f(x)-g(x)$; $\forall x \in R$

Product: $(fg)\,x = f(x).g(x)$; $\forall x \in R$

Quotient: $(f/g)\,(x) = f(x)/g(x)$; provided $g(x) \neq 0$, $\forall x \in R$

Some standard real functions

Log function: $f(x)=\log_a x$, $a>0$, $a \neq 1$ Domain $= x \in (0,\infty)$ Range $= y \in R$

Identity function: The function f:R → R defined by $y=f(x)=x$ ∀ $x \in R$ is called identity function. Domain = R and Range = R

Constant function: The function f:R → R defined by $y=f(x)=c$, ∀ $x \in R$, where c is a constant, is called constant function. Domain = R and Range = {c}

Modulus function: The function f:R → R defined by $f(x)=\begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$ is called modulus function. It is denoted by $y=f(x)=|x|$. Domain = R and Range = $[0,\infty)$

Signum function: The function f:R → R defined by $f(x)=\begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$ is called signum function. It is usually denoted by $y=f(x)=\text{sgn}(x)$ Domain = R and Range = {−1, 0, 1}

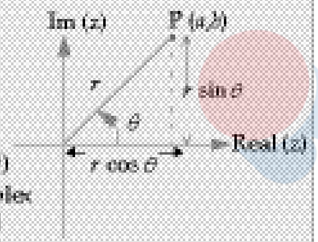
Greatest integer function: The function f:R → R defined by as the greatest integer less than or equal to x. It is usually denoted by $y=f(x)=[x]$. Domain = R and Range = Z (All integers)

Exponential function: $f(x)=a^x$, $a>0$, $a \neq 1$, Domain: $x \in R$; Range: $f(x) \in (0,\infty)$

Complex Numbers & Quadratic Equations

Polar Representation

Let $a = r\cos\theta$
$b = r\sin\theta$
where,
$r = |z|$
and $\theta = \arg(z)$
$\therefore z = a + ib = r(\cos\theta + i\sin\theta)$
The argument 'θ' of complex number $z = a + ib$ is called principal argument of z if
$-\pi < \theta \le \pi$.

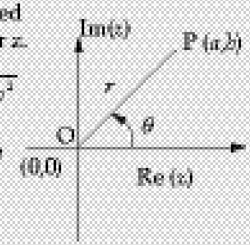


Modulus & Argument of Complex Number

If $z = a + ib$ is a complex number
(i) Distance of z from origin is called as modulus of complex number z.

It is denoted by $r = |z| = \sqrt{a^2 + b^2}$

(ii) Angle θ made by OP with +ve direction of X-axis is called argument of z.



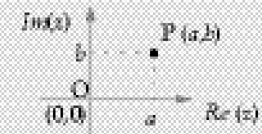
Powers of 'i'

$i = \sqrt{-1}, i^2 = -1$

In general, $i^{4n+r} = \begin{cases} 1; r = 0 \\ i; r = 1 \\ -1; r = 2 \\ -i; r = 3 \end{cases}$

Argand Plane

A complex number $z = a + ib$ can be represented by a unique point $P(a, b)$ in the argand plane



$z = a + ib$ is represented by a point $P(a, b)$

Square root of Complex Number

Let $z = x + iy = \sqrt{a + ib}$, squaring both sides, we get $(x + iy)^2 = a + ib$ i.e. $x^2 - y^2 = a$, $2xy = b$ solving these equations, we get square root of z.

Multiplicative Inverse of complex Number

For a non-zero complex number $z = a + ib$ $(a \neq 0, b \neq 0)$,

there exists a complex number $\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$

denoted by $\frac{1}{z}$ or z^{-1}, called multiplicative inverse of z

Such that: $(a + ib)\left(\frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}\right) = 1 + 0i = 1$

Algebra of Complex Numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers, where $a, b, c, d \in R$ and $i = \sqrt{-1}$

1. **Addition:** $z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d)$
2. **Subtraction:** $z_1 - z_2 = (a + ib) - (c + id) = (a - c) + (b - d)i$
3. **Multiplication:** $z_1 \cdot z_2 = (a + ib)(c + id)$
   $= a(c + id) + ib(c + id)$
   $= (ac - bd) + (ad + bc)i$
   $(\because i^2 = -1)$
4. **Division:** $\frac{z_1}{z_2} = \frac{a + ib}{c + id} = \frac{a + ib}{c + id} \cdot \frac{c - id}{c - id}$
   $= \left(\frac{ac + bd}{c^2 + d^2}\right) + \left(\frac{bc - ad}{c^2 + d^2}\right)i$

Note: If $a + ib = c + id$
$\Leftrightarrow a = c$ & $b = d$

Solution of Quadratic Equation

- General form of quadratic equation in x is $ax^2 + bx + c = 0$,
  Where $a, b, c \in R$ & $a \neq 0$
  The solutions of given quadratic equation are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
  (i) If $b^2 - 4ac > 0$, roots are real and unequal
  (ii) If $b^2 - 4ac = 0$, roots are real and equal
  (iii) If $b^2 - 4ac < 0$, roots are imaginary

Note:
- A monomial equation has one root.
- A polynomial equation of degree n has n roots.

Definition of Complex Numbers

A number of the form $a + ib$, where $a, b \in R$ and $i = \sqrt{-1}$ is called a complex number and denoted by 'z'.

$z = a + ib$ → a: Real part, ib: Imaginary part

Conjugate of a complex number: For a given complex number $z = a + ib$, its conjugate is defined as $\bar{z} = a - ib$

Determinants

Basic Facts

(i) $A^{-1} = \frac{\text{adj } A}{|A|}$ and $|A| \neq 0$

(ii) $|A| = ad - bc$ if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(iii) $AA^{-1} = A^{-1}A = I$

(iv) Area of triangle $ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

(v) If matrix $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ Then $\text{Adj } A = \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}$

(vi) Suppose $AX = B$ be the system of a non-homogeneous linear equations in n variables then

- If $|A| \neq 0$, then the system of equation is consistent and has a unique solution which is obtained as $X = A^{-1}B$
- If $|A| = 0$ and $(adj)B = 0$, then system of equation is consistent and has infinitely many solutions.

(vii) Suppose $AX = B$ be the system of a homogeneous linear equations in n variables then

- If $|A| \neq 0$, then it has only one solution $X = 0$, which is called as trivial solution.
- If $|A| = 0$ then the system has infinitely many solution and is called non-trivial solutions.

(viii) If $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ then $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, where $D \neq 0$

where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$, $D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

(ix) $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$

then, $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, $z = \frac{D_3}{D}$, where $D \neq 0$

where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

(x) $A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ then $|A| = a(ei - hf) - b(di - gf) + c(dh - eg)$

Determinant of a square matrix 'A', |A| is given by

(i) If $A = [a_1]_{1\times 1}$, then $|A| = a_1$

(ii) If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$ then $|A| = a_{11}a_{22} - a_{12}a_{21}$

(iii) If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$, then $|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

For eg: if $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$, then $|A| = 2 \times 4 - 3 \times 2 = 2$

Properties of |A|

(i) $|A|$ remains unchanged, if the rows and columns of A are interchanged i.e., $|A| = |A'|$

(ii) If any two rows (or columns) of A are interchanged, then the sign of $|A|$ changes.

(iii) If any two rows (or columns) of A are identical, then $|A| = 0$

(iv) If each element of a row (or a column) of A is multiplied by k (const.), then $|A|$ gets multiplied by k.

(v) If $A = [a_{ij}]_{n\times n}$, then $|k.A| = k^n|A|$.

(vi) If elements of a row or a column in a determinant $|A|$ can be expressed as sum of two or more elements, then $|A|$ can be expressed as $|B| + |C|$.

(vii) If $R_i \to R_i + kR_j$ or $C_i = C_i + kC_j$ in $|A|$, then the value of $|A|$ remains same

Area of a triangle

If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of triangle, Area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

For eg: if (1, 2), (3, 4) and (–2, 5) are the vertices, then area of the triangle is

$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ -2 & 5 & 1 \end{vmatrix} = \frac{1}{2}\left|1(4-5) - 2(3+2) + 1(15+8)\right| = 6$ sq. units.

we take positive value of the determinant because area is positive.

Matrices

Definition and its types

A matrix of order $m \times n$ is an ordered rectangular array of numbers or functions having 'm' rows and 'n' columns. The matrix $A = [a_{ij}]_{m \times n}$; $1 \le i \le m$, $1 \le j \le n$; $i, j \in N$ is given by

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots a_{2n} \\ a_{m1} & a_{m2} & \cdots a_{mn} \end{vmatrix}_{m \times n}$$

- Column matrix : It is of the form $[a_{ij}]_{n \times 1}$
- Row matrix : It is of the form $[a_{ij}]_{1 \times n}$
- Square matrix : Here, $m = n$ (no. of rows = no. of columns)
- Diagonal matrix : All non-diagonal entries are zero i.e. $a_{ij} = 0 \, \forall i \ne j$
- Scalar matrix : $a_{ij} = 0, i \ne j$ and $a_{ij} = k$(Scalar), $i = j$, for some constant k.
- Identity matrix : $a_{ij} = 0, i \ne j$ and $a_{ij} = 1, i = j$
- Zero matrix : All elements are zero.

Equality of two matrices

$A = [a_{ij}] = [b_{ij}] = B$ if A and B are of same order and $a_{ij} = b_{ij}$ $\forall$ i and j; $i, j \in N$

Operations on matrices

Addition

If A, B are two matrices of same order then $A+B = [a_{ij}+b_{ij}]$. The addition of A and B follows: $A+B = B+A$, $(A+B)+C = A+(B+C)$, $-A + (-1)A$, $k(A+B) = kA + kB$, k is scalar and $(k+l)A = kA + lA$, k and l are constants.

Multiplication

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{jk}]_{n \times p}$, then $AB = C = [c_{ik}]_{m \times p}$, $[c_{ik}] = \sum_{j=1}^{n} a_{ij} b_{jk}$. Also, $A(BC) = (AB)C$, $A(B+C) = AB + AC$ and $(A+B)C = AC + BC$, but $AB \ne BA$ (always).

Transpose of a Matrix

If $A = [a_{ij}]_{m \times n}$ then its transpose $A' = (A^T) = [a_{ji}]_{n \times m}$ i.e. if $A = [2 \;\; 1]$ then $A' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Also, $(A')' = A$, $(kA)' = kA'$, $(A+B)' = A'+B'$, $(AB)' = B'A'$.

- A is symmetric matrix if $A = A'$ i.e. $A' = A$.
- A is skew – symmetric if $A = -A'$ i.e. $A' = -A$.
- A is any square matrix, then $A = \frac{1}{2}\{(A + A')\} + \{A - A'\}$ = sum of a symmetric and a skew-symmetric matrix.

$\underbrace{}_{\text{S.M}}$ $\underbrace{}_{\text{Skew S.M.}}$

Elementary operations on a matrix

Properties for applying the operations

$R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

$R_i \to kR_i$ or $C_i \to kC_i$

$R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

Minors and cofactors of a matrix

Minor of an element a_{ij} in a determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column and is denoted by M_{ij}. If M_{ij} is the minor of a_{ij} and cofactor of a_{ij} is A_{ij} given by $A_{ij} = (-1)^{i+j} M_{ij}$

- If $A_{3 \times 3}$ is a matrix, then $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$
- If elements of one row (or column) are multiplied with cofactors of elements of any other row (or column), then their sum is zero. For e.g., $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$.

e.g., if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $M_{12} = 3$ and $A_{12} = (-1)^{1+2} 3 = -3$.

Adjoint and inverse of a matrix

if $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then adj. $A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

where A_{ij} is the cofactor of a_{ij}

- $A(\text{adj}.A) = (\text{adj}.A).A = |A| I$, A = square matrix of order 'n'
- if $|A| = 0$, then A is singular. Otherwise, A is non-singular.
- if $AB = BA = I$, where B is a square matrix. Then B is called the inverse of A, $A^{-1} = B$ or $B^{-1} = A$, $(A^{-1})^{-1} = A$.

Inverse of a square matrix exists if A is non singular i.e. $|A| \ne 0$, and is given by

$$A^{-1} = \frac{1}{|A|} (\text{adj}.A)$$

Applications of determinants & matrices

- If $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$ then we can write $AX = B$,

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

- Unique solution of $AX = B$ is $X = A^{-1}B$, $|A| \ne 0$.
- $AX = B$ is consistent or inconsistent according as the solution exists or not.
- For a square matrix A in $AX = B$, if
  - (i) $|A| \ne 0$ then there exists unique solution.
  - (ii) $|A| = 0$ and (adj. A) $B \ne 0$, then no solution.
  - (iii) if $|A| = 0$ and (adj.A)$B = 0$ then system may or may not be consistent.

Permutations and Combinations

Fundamental Principle of Counting (FPC)

FPC of Multiplication: If an event can occur in m different ways, following which another event in n different ways, then the total number of occurance of the events in the given order is $m \times n$

FPC of Addition: If there are two events such that they can performed independently in m and n ways respectively, then either of the two events can be performed in $(m+n)$ ways.

Permutation

Each of the different arrangements which can be made by taking some or all of a number of distinct objects is called a permutation.

The no. of permutations of n different things taken r at a time, where repetition is not allowed, is denoted by ${}^nP_r$ and is given by

$${}^nP_r = \frac{n!}{(n-r)!}$$

Permutations under given Conditions

The no. of all permutations of 'n' different objects taken r at a time:

(i) When a particular object is to be always included in each arrangement is ${}^{n-1}P_{r-1}$

(ii) When a particular object is never taken in each arrangement is ${}^{n-1}P_r$

(iii) The no. of permutation of n different things taken r at a time, where repetition is allowed is $(n)^r$

Permutations of alike Objects

The no. of permutations of n objects taken all at a time, where p_1 objects are of first kind, p_2 objects are of the second kind, ...p_k objects are of the k^{th} kind is $\frac{n!}{p_1!\,p_2!\,p_3!\ldots p_k!}$

Circular Permutations

1. The no. of circular permutations of 'n' distinct objects is $(n-1)!$
2. If anti-clockwise and clockwise order of arrangements are not distinct then the no. of circular permutations of n distinct items is $\frac{1}{2}(n-1)!$

e.g.: arrangements of beads in a necklace, arrangements of flower in a garland etc.

Some Special Results of ${}^nC_r$

Since,

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

- In particular, $r = n$
- ${}^nC_n = \frac{n!}{n!0!} = 1$
- ${}^nC_{n-r} = {}^nC_r$
- ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- ${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = 2^{n-1}$

Relation between ${}^nP_r$ and ${}^nC_r$

${}^nP_r = {}^nC_r \times r!$ $\quad \because 0 < r \leq n$

i.e., Corresponding to each combination of ${}^nC_r$, we have $r!$ permutations, because r objects in every combination can be rearranged in $r!$ ways

Combinations under Certain Conditions

- The number of combinations of n things taken at a time, so that k particular objects are
  (i) Always included: ${}^{n-k}C_{r-k}$
  (ii) Never included: ${}^{n-k}C_r$
- The no. of selections from 'n' different objects, taking at least one $= {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots\ldots {}^nC_n = 2^n - 1$

Combinations

Each of different selections made by taking source or all of a number of distinct objects or item, irrespective of their arrangements or order in which they are placed, is called a combination.

The no. of combinations of n different things taken r at a time, denoted by ${}^nC_r$ is given by

$${}^nC_r = \frac{n!}{r!(n-r)!};\ 0 \leq r \leq n$$

Principle of Mathematical Induction

Induction

Specific Instances to Generalisation
e.g.:Rohit eats food. Vikas eats food. Rohit and Vikas are men. Then all men eat food.
Statement is true for $n=1, n=k$ and $n=k+1$, then, the statement is true for all natural numbers n.

Deduction

Generalisation of Specific Instance
e.g.: Rohit is a man and all men eat food, therefore, Rohit eats food.

Example No.1

Ex: Prove that $2^n > n$ for all positive integer n.
Solution: Step1: Let P(1): $2^n > n$
Step2: When $n=1$, $2^1>1$. Hence P(1) is true
Step3: Assume that P(k) is true for any positive integer k, $2^k > k$
Step4: We shall now prove that P(k+1) is true
Multiplying both sides of step(3) by 2, we get
$2.2^k > 2.k$
$\Rightarrow 2^{k+1} > 2k$
$\Rightarrow 2^{k+1} > k+k$
$\Rightarrow 2^{k+1} > k+1$ (since $k>1$)
Therefore, P(k+1) is true when P(k) is true Hence, by P.M.I., P(n) is true for every positive integer n.

Example No.2

Ex : Prove the following by using the principle of mathematical induction for all $n \in N$: $10^{2n-1}+1$ is divisible by 11.
Answer:
Let the given statement be P(n), i.e.,
P(n): $10^{2n-1}+1$ is divisible by 11.
It can be observed that P(n) is true for $n=1$ since $P(1) = 10^{2.1-1}+1=11$,
which is divisible by 11.
Let P(k) be true for some positive integer k, i.e.,
$10^{2k-1}+1$ is divisible by 11.
$\therefore 10^{2k-1}+1=11m$, where $m \in N$.....(1)
we shall now prove that P(K+1) is true whenever P(k) is true.
Consider $10^{2(k+1)-1}+1$
$=10^{2k+2-1}+1$
$=10^{2k+1}+1$
$=10^2(10^{2k-1}+1-1)+1$
$=10^2(10^{2k-1}+1)-10^2+1$
$=10^2.11m-100+1$
$=100\times 11m-99$
$=11(100m-9)$
$=11r$, where $r=(100m-9)$ is some natural number
Therefore, $10^{2(k+1)-1}+1$ is divisible by 11.
Thus, P(k+1) is true whenever P(k) is true.
Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Steps for Principle of Mathematical Induction Proof

Step1: Let P(n) be a result or statement formulated in terms of n in a given equation
Step2: Prove that P(1) is true.
Step3: Assume that P(k) is true.
Step4: Using step 3, prove that P(k+1) is true.
Step5: Thus, P(1) is true and P(k+1) is true whenever P(k) is true. Hence, by the principle of mathematical induction, P(n) is true for all natural numbers n.

Some important relation

Basic Facts
(i) The sum of square of first n natural number $=\frac{n(n+1)(2n+1)}{6}$
(ii) $1+4+7+10+........+(3n-2)=\frac{n(3n-1)}{2}$
(iii) $1^2+3^2+5^2+........+(2n-1)^2=\frac{n(4n^2-1)}{3}$
(iv) $1^3+2^3+3^3+........+n^3=\left[\frac{n(n+1)}{2}\right]^2$
(v) $2n+7 \le (n+3)^2$, where $n=0,1,2,3,.....$
(vi) $1^2+2^2+3^2+........+n^2 > \frac{n^3}{2}$ where $n \in N$.
(vii) $1+2+3+........+n < \frac{(2n+1)^2}{8}$

Binomial Theorem and its Applications

General & Middle Terms in Expansion of $(a+b)^n$

The general term of an expansion $(a+b)^n$ is

$T_{r+1} = {}^nC_r\, a^{n-r} b^r,\ 0 \le r \le n, r \in N$

Middle Terms:

1. In $(a+b)^n$, if n is even, then the no. of terms in the expansion is odd. Therefore, there is only one middle term and it is $\left(\frac{n+2}{2}\right)^{th}$ term.

2. In $(a+b)^n$, if n is odd then the no. of terms in the expansion is even. Therefore, there are two middle terms and those are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms.

Definition

If $a, b \in R$ and $n \in N$, then

$$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_n a^0 b^n$$

- **Remarks:** If the index of the binomial is n then the expansion contains $n+1$ terms.
- In each term, the sum of indices of a and b is always n.
- Coefficients of the terms in binomial expansion equidistant from both the ends are equal.

$$(a-b)^n = {}^nC_0 a^n b^0 - {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + (-1)^n\, {}^nC_n a^0 b^n$$

Binomial Theorem for any Index

If n is negative integer, then $n!$ is not defined. We state binomial theorem in another form

$$(a+b)^n = a^n + \frac{n}{1!} a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3} b^3 + \dots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} a^{n-r} b^r + \dots + b^n$$

Here, $T_{r+1} = \frac{n(n-1)(n-2)\cdots\cdots(n-r+1)}{r!} a^{n-r} b^r$

Some Special Cases $(a+b)^n$

In the expansion of $(a+b)^n$,

(i) Taking $a = x$ and $b = -y$, we obtain

$(x-y)^n = {}^nC_0 x^n y^0 - {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \dots + (-1)^n\, {}^nC_n y^n$

(ii) Taking $a = 1$, $b = x$, we obtain

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \dots + {}^nC_n x^n$

(iii) Taking $a = 1$, $b = -x$, we obtain

$(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 + \dots + (-1)^n\, {}^nC_n x^n$

(iv) Taking $a = 1$, $b = x$, $n = -n$

$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \dots$

Binomial Coefficients

The coefficient ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$ in the expansion of $(a+b)^n$ are called binomial coefficients and denoted by $C_0, C_1, C_2, \dots, C_n$ respectively

Properties of binomial coefficients:

(i) $C_0 + C_1 + C_2 + \dots + C_n = 2^n$

(ii) $C_0 - C_1 + C_2 - \dots + (-1)^n C_n = 0$

(iii) ${}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 + \dots + {}^nC_n^2 = {}^{2n}C_n$

(iv) ${}^nC_{r_1} = {}^nC_{r_2} \Rightarrow r_1 = r_2$ or $r_1 + r_2 = n$

(v) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(vi) ${}^nC_r = \frac{n}{r}\, {}^{n-1}C_{r-1}$

(vii) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

(viii) ${}^nC_0\,{}^nC_r + {}^nC_1\,{}^nC_{r+1} + {}^nC_2\,{}^nC_{r+2} + \dots + {}^nC_{n-r}\,{}^nC_n = {}^{2n}C_{n+r}$

Sequences and Series

Geometric Progression (GP) · Geometric Mean (GM) · Arithmetico Geometric Series · Properties of AM & GM between two quantities · Relationship between AM, GM and HM · Sum up to n terms of some Special Sequences · Arithmetic Mean (AM) · Arithmetic Progression (A.P.) · Harmonic Mean

Harmonic Progression (HP)

For the solution of HP, we should follow below steps

Make the reciprocal of each terms of HP

Solve by AP method

Make the reciprocal of AP result

eg: if $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}$ ——— in HP, then

$a, (a+d), (a+2d)$ ——— will be in AP

now n^{th} term of AP, $a_n = a+(n-1)d$

So, n^{th} team of HP, $= \frac{1}{a_n} = \frac{1}{a+(n-1)d}$

Some Important result

If H is the harmonic mean (H) between a&b, then $H=\frac{2ab}{a+b}$

If there is n harmonic mean between a&b, then n^{th} $HM(H_n) = \frac{ab(n+1)}{na+b}$

Geometric Progression (GP)

A sequence is said to be a geometric progression or G.P., if the ratio of any term to its preceding term is same throughout. This constant factor is called 'Common ratio'. Usually we denote the first term of a G.P. by 'a' and its common ratio by 'r'. The general or n^{th} term of G.P. is given by $a_n = ar^{n-1}$

The sum S_n of the first n terms of G.P. is given by

$$S_n = \frac{a(r^n-1)}{r-1}, r>1 \text{ or } \frac{a(1-r^n)}{1-r}, r<1 \text{ if } r \neq 1$$

Sum of infinite terms of G.P. is given by

$$S_\infty = \frac{a}{1-r}; |r|<1 \text{ or, } S_\infty = \frac{a}{r-1}; \ r>1$$

Geometric Mean (GM)

- If a, b, c are in G.P., b is the GM between a & c, $b^2 = ac$, therefore ; $a > 0, c > 0$
- If a, b are two given numbers & $a, G_1, G_2, \ldots, G_n, b$ are in G.P., then $G_1, G_2, G_3 \ldots G_n$ are n GMs between a & b.

$$G_1 = a \cdot \left(\frac{b}{a}\right)^{1/n+1}, G_2 = a \cdot \left(\frac{b}{a}\right)^{2/n+1} \ldots G_n = a \cdot \left(\frac{b}{a}\right)^{n/n+1}$$

or $G_1 = ar, G_2 = ar^2, \ldots G_n = ar^n$ where $r = \left(\frac{b}{a}\right)^{1/n+1}$

Arithmetico Geometric Series

In a series, where each term is formed by multiplying the corresponding term of an AP & GP is called Arithmetico-Geometric Series.

Eg: $1+3x+5x^2+7x^3+\ldots$

Here, 1, 3, 5, ... are in AP and $1, x, x^2, \ldots$ are in GP.

Properties of AM & GM between two quantities

- If A.M. : G.M. of two positive numbers a and b is $m : n$, then $a : b = \left(m+\sqrt{m^2-n^2}\right) : \left(m-\sqrt{m^2-n^2}\right)$
- If A and G be the AM & GM between two positive numbers, then the numbers are $A \pm \sqrt{A^2-G^2}$

Relationship between AM, GM and HM

Let A, G and H be the AM, GM and HM of two given positive real numbers a & b, respectively. Then,

$$A = \frac{a+b}{2}, G = \sqrt{ab} \text{ and } H = \frac{2}{a+b}$$

So, $G = \sqrt{AH}$ or, $G^2 = AH$

Arithmetic Progression (A.P.)

An A.P. is a sequence in which terms increase or decrease regularly by the fixed number (same constant). This fixed number is called 'common difference' of the A.P. If 'a' is the first term & 'd' is the common difference and 'l' is the last term of A.P., then general term or the n^{th} term of the A.P. is given by $a_n = a + (n-1)d$ from starting and $a_n = l-(n-1)d$ from the end.

The sum S_n of the first n terms of an A.P. is given by

$$S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[a+l]$$

Sum up to n terms of some Special Sequences

- Sum of first 'n' natural numbers

$$\sum_{k=1}^{n} k = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

- Sum of squares of first n natural numbers

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

- Sum of cubes of first n natural numbers

$$\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\ldots+n^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n} k\right)^2$$

- Sum of first 'n' odd natural numbers

$$\sum_{k=1}^{n} (2k-1) = 1+3+5+\ldots+(2n-1) = n^2$$

Arithmetic Mean (AM)

If three numbers are in A.P., then the middle term is called AM between the other two, so if a, b, c, are in A.P., b is AM of a and c.

AM for any 'n' +ve numbers $a_1, a_2, a_3, \ldots, a_n$ is

$$AM = \frac{a_1+a_2+a_3+\ldots+a_n}{n}$$

n-Arithmetic Mean between Two Numbers: If a, b are any two given numbers & $a, A_1, A_2, \ldots, A_n, b$ are in A.P. then $A_1, A_2, \ldots, A_n$ are n AM's between a & b.

$$A_1 = a+\frac{b-a}{n+1}, A_2 = a+\frac{2(b-a)}{n+1} \ldots A_n = a+\frac{n(b-a)}{n+1}$$

or $A_1 = a+d, A_2 = a+2d, \ldots, A_n = a+nd$,

where $d = \frac{b-a}{n+1}$

Limits and Derivatives

Limit

- We say $\lim_{x \to a^-} f(x)$ is the expected value of f at $x=a$ given that the values of f near x to the left of a. This value is called the left hand limit of f at 'a'
- We say $\lim_{x \to a^+} f(x)$ is the expected value of f at $x=a$ given that the values of f near x to the right of a. This value is called the right hand limit of f 'at a'.
- If the right and left hand limits coincide, we call that common value as the limit of $f(x)$ at $x=a$ and denoted it by $\lim_{x \to a} f(x)$.

eg: Find limit of function $f(x)=(x-1)^2$ at $x=1$.

Sol: For $f(x)=(x-1)^2$

Left hand limit (LHL) (at $x=1$) $= \lim_{x \to 1^-} (x-1)^2 = 0$

and Right hand limit (RHL) (at $x=1$) $= \lim_{x \to 1^+} (x-1)^2 = 0$

$\therefore$ LHL = RHL

$\lim_{x \to 1} (x-1)^2 = 0$

Algebra of Limits

For functions f and g the following holds:

- $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
- $\lim_{x \to a} [f(x).g(x)] = \lim_{x \to a} f(x). \lim_{x \to a} g(x)$
- $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, where $\lim_{x \to a} g(x) \neq 0$

Some Standard Limits

- $\lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}$
- $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$
- $\lim_{x \to a} \frac{\tan^p mx}{\tan^p nx} = \left(\frac{m}{n}\right)^p, x \neq 0$
- $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a, a > 0$
- $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1$
- $\lim_{x \to 0} \frac{e^{\lambda x} - 1}{x} = \lambda, \lambda \neq 0$

Some Standard Derivatives

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
- $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$
- $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$

Algebra of derivatives of functions

For functions u and v the following holds:

- $(u \pm v)' = u' \pm v'$
- $(uv)' = u'v + v'u$
- $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$ provided all are defined and $v \neq 0$

Here, $u' = \frac{du}{dx}$ and $v' = \frac{dv}{dx}$

Derivative of polynomial function

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \text{.........} + a_1 x + a_0$ be a polynomial function, where a_i s are all real numbers and $a_n \neq 0$. Then the derivative function is given by

$$\frac{d f(x)}{dx} = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \text{.........} + 2a_2 x + a_1$$

Derivative of a function at 'a'

The derivative of a function f at a is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

eg: Find derivative of $f(x) = \frac{1}{x}$.

Sol: We have $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{1}{h}\left[\frac{-h}{x(x+h)}\right] = \frac{-1}{x^2}$$

Continuity and Differentiability

Continuous function

Suppose f is a real function on a subset of the real numbers and let c be a point in the domain of f. Then f is continuous at c if $\lim_{x \to c} f(x) = f(c)$

A real function f is said to be continuous if it is continuous at every point in the domain of f.

eg: The function $f(x) = \frac{1}{x}, x \neq 0$ is continuous.

Let c be any non-zero real number, then $\lim_{x \to c} f(x) = \lim_{x \to c} \frac{1}{x} = \frac{1}{c}$. For $c \neq 0$, $f(c) = \frac{1}{c}$. So $\lim_{x \to c} f(x) = f(c)$ and hence f is continuous at every point in the domain of f.

Algebra of continuous functions

Suppose f and g are two real functions continuous at a real number c, then, $f+g$, $f-g$, $f \cdot g$ and $\frac{f}{g}$ are continuous at $x = c\left[g(c) \neq 0\right]$.

Differentiability

Suppose f is a real function and c is a point in its domain. The derivative of f at c is $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

Every differentiable function is continuous, but the converse is not true.

Chain Rule

if $f = v \circ u$, $t = u(x)$ and if both $\frac{dt}{dx}, \frac{dv}{dt}$ exists, then $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$.

Logarithmic differentiation

Let $y = f(x) = \left[u(x)\right]^{v(x)}$

$\log y = v(x) \log\left[u(x)\right]$

$\frac{1}{y}\frac{dy}{dx} = v(x)\frac{1}{u(x)}u'(x) + v'(x)\log\left[u(x)\right]$

$\frac{dy}{dx} = y\left[\frac{v(x)}{u(x)}u'(x) + v'(x)\log\left[u(x)\right]\right]$

e.g.: Let $y = a^x$ Then $\log y = x \log a$

$\frac{1}{y}\frac{dy}{dx} = \log a$

$\frac{dy}{dx} = y \log a = a^x \log a$

Derivatives of implicit functions

If two variables are expressed by some relation then one will be the implicit function of other, is called Implicit function.

For example: Let $y = \cos x - \sin y$, then $\frac{dy}{dx} = \frac{d}{dx}\cos x - \frac{d}{dx}\sin y$

or, $\frac{dy}{dx} = -\sin x - \cos y \cdot \frac{dy}{dx}$ or, $\frac{dy}{dx} = \frac{-\sin x}{1 - \cos y}$, where $y \neq (2n+1)\pi$

Some Standard derivatives

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ (ii) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$ (iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ (iv) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

(v) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$ (vi) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$ (vii) $\frac{d}{dx}(e^x) = e^x$ (viii) $\frac{d}{dx}(\log x) = \frac{1}{x}$

Mean Value Theorem

If $f : [a, b] \to R$ continuous on $[a, b]$ and differentiable on (a, b).

Then $\exists$ some c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

e.g. Let $f(x) = x^2$ defined in the interval $[2, 4]$.

Since $f(x) = x^2$ is continuous in $[2, 4]$ and differentiable in $(2, 4)$ as $f'(x) = 2x$ defined in $(2, 4)$.

So,

$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{16 - 4}{4 - 2} = 6, c \in (2, 4)$.

Rolle's theorem

If $f : [a, b] \to R$ is continuous on $[a, b]$ and differentiable on (a, b). Such that $f(a) = f(b)$, then $\exists$ some c in (a, b) Such that $f'(c) = 0$.

Second order derivative

Let $y = f(x)$ then $\frac{dy}{dx} = f'(x)$, if $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}f'(x)$ i.e.,

$\frac{d^2y}{dx^2} = f''(x)$ is the second order derivative of y w.r.t. x.

For eg: if $y = 3x^2 - 2$, then $y' = 6x$ and $y'' = 6$.

Derivatives of functions in parametric form

Let $x = f(t)$, $y = g(t)$ be two functions of parameter 't'.

Then, $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ or $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $\left(\frac{dx}{dt} \neq 0\right)$

Thus, $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$ (provided $f'(t) \neq 0$)

eg : if $x = a\cos\theta$, $y = a\sin\theta$ then $\frac{dx}{d\theta} = -a\sin\theta$ and $\frac{dy}{d\theta} = a\cos\theta$, and so $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{a\cos\theta}{a\sin\theta} = -\cot\theta$.

Applications of Derivatives

Rate of change of quantities

If a quantity 'y' varies with another quantity x so that $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ represents the rate of change of y w.r.t. x and $\left.\frac{dy}{dx}\right]_{x=x_0}$ $(f'(x_0))$ represents the rate of change of y w.r.t. x at $x = x_0$.

If 'x' and 'y' varies with another variable 't' i.e., if $x = f(t)$ and $y = g(t)$, then by chain rule $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$, if $\frac{dx}{dt} \neq 0$.

e.g. If the radius of a circle, $r = 5$ cm, then the rate of change of the area of a circle per second w.r.t. 'r' is $\left.\frac{dA}{dr}\right|_{r=5} = \frac{d}{dr}(\pi r^2)\Big|_{r=5} = 2\pi r\big|_{r=5} = 10\pi$

Increasing and decreasing functions

A function f is said to be (i) increasing on (a,b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) < f(x_2)\ \forall\ x_1, x_2 \in (a,b)$, and (ii) decreasing on (a,b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) > f(x_2)\ \forall\ x_1, x_2 \in (a,b)$.

If $f'(x) > 0\ \forall\ x \in [a,b]$ then f is increasing in (a,b) and if $f'(x) \leq 0\ \forall\ x \in [a,b]$, then f is decreasing in (a,b).

e.g. Let $f(x) = x^3 - 3x^2 + 4x, x \in R$, then $f'(x) = 3x^2 - 6x + 4 = 3(x-1)^2 + 1 > 0\ \forall\ x \in R$. So, the function f is strictly increasing on R.

Tangents Equation on curve

The equation of the tangent at (x_0, y_0) to the curve $y = f(x)$ is given by $(y - y_0) = \left.\frac{dy}{dx}\right|_{(x_0, y_0)}(x - x_0)$. If $\frac{dy}{dx}$ does not exist at (x_0, y_0), then the tangent at (x_0, y_0) is parallel to the y-axis and its equation is $x = x_0$.

If tangent to a curve $y = f(x)$ at $x = x_0$ is parallel to x-axis, then $\left.\frac{dy}{dx}\right|_{x=x_0} = 0$.

Equation of the normal to the curve

The equation of normal at (x_0, y_0) to the curve $y = f(x)$ is $y - y_0 = \frac{-1}{\left.\frac{dy}{dx}\right|_{(x_0,y_0)}}(x - x_0)$

If $\frac{dy}{dx}$ at (x_0, y_0) is zero, then equation of the normal is $x = x_0$. If $\frac{dy}{dx}$ at (x_0, y_0) does not exist, then the normal is parallel to x-axis and its equation is $y = y_0$. e.g. Let $y = x^3 - x$ be a curve, then the slope of the tangent to $y = x^3 - x$ at $x = 2$ is $\left.\frac{dy}{dx}\right|_{x=2} = 3x^2 - 1\big|_{x=2} = 3.2^2 - 1 = 11$.

Approximations

Let $y = f(x)$; Δx be a small increment in 'x' and Δy be the small increment in y corresponding to the increment in x, i.e., $\Delta y = f(x + \Delta x) - f(x)$. Then, Δy is given by $dy = f'(x)dx$ or $dy = \left(\frac{dy}{dx}\right)\Delta x$, is approximation of Δy when $dx = \Delta x$ is relatively small and denoted by $dy \approx \Delta y$.

e.g. Let us approximate $\sqrt{36.6}$. To do this, we take $y = \sqrt{x}$, $x = 36$, $\Delta x = 0.6$ then $\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{36.6} - \sqrt{36}$

$= \sqrt{36.6} - 6 \Rightarrow \sqrt{36.6} = 6 + dy$

Now, dy is approximately Δy and is given by

$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(0.6) = \frac{1}{2\sqrt{36}}(0.6) = 0.05$. So, $\sqrt{36.6} \approx 6 + 0.05 = 6.05$.

Maxima and Minima

A point C in the domain of 'f' at which either $f'(C) = 0$ or f is not differentiable is called a critical point of f.

First derivative test

Let f be continuous at a critical point C in open interval. Then (i) If $f'(x) > 0$ at every point left of C and $f'(x) < 0$ at every point right of C, then 'C' is a point of local maxima. (ii) If $f'(x) < 0$ at every point left of C and $f'(x) > 0$ at every point right of C, then 'C' is a point of local minima.

(iii) If $f'(x)$ does not change sign as 'x' increases through C, then 'C' is called the point of inflection.

Second derivative test

Let f be a function defined on given interval, f is twice differentiable at C. Then

(i) $x = C$ is a point of local maxima If $f'(C) = 0$ and $f''(C) < 0$, $f(C)$ is local maxima of f.

(ii) $x = C$ is a point of local minima If $f'(C) = 0$ and $f''(C) > 0$, $f(C)$ is local minima of f. (iii) The test fails if $f'(C) = 0$ and $f''(C) = 0$

Integrals Calculus

Integration

It is the inverse of differentiation. Let $\frac{d}{dx}F(x)=f(x)$. Then $\int f(x)dx=F(x)+c$, c = constant of integral. These integrals are called indefinite or general integrals.

Properties of indefinite integrals are

(i) $\int [f(x)\pm g(x)]dx=\int f(x)dx\pm\int g(x)dx$, (ii) $\int kf(x)dx=k\int f(x)dx$,

ex: $\int (3x^2+2x)dx=x^3+x^2+c$, where c is real.

Some standard integrals

(i) $\int x^n dx=\frac{x^{n+1}}{n+1}+c, n\neq -1$ like, $\int dx=x+c$

(ii) $\int \cos x\,dx=\sin x+c$ (iii) $\int \sin x\,dx=-\cos x+c$

(iv) $\int \sec^2 x\,dx=\tan x+c$ (v) $\int \operatorname{cosec}^2 x\,dx=-\cot x+c$

(vi) $\int \sec x\tan x\,dx=\sec x+c$ (vii) $\int \operatorname{cosec} x\cot x\,dx=-\operatorname{cosec} x+c$

(viii) $\int \frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+c$ (ix) $\int \frac{dx}{\sqrt{1-x^2}}=-\cos^{-1}x+c$

(x) $\int \frac{dx}{1+x^2}=\tan^{-1}x+c$ (xi) $\int \frac{dx}{1+x^2}=-\cot^{-1}x+c$

(xii) $\int e^x dx=e^x+c$ (xiii) $\int a^x dx=\frac{a^x}{\log a}+c$

(xiv) $\int \frac{dx}{x\sqrt{x^2-1}}=\sec^{-1}x+c$ (xv) $\int \frac{dx}{x\sqrt{x^2-1}}=-\operatorname{cosec}^{-1}x+c$

(xvi) $\int \frac{1}{x}dx=\log|x|+c$

Integration by partial fractions

A rational function of the form $\frac{P(x)}{Q(x)}, Q(x)\neq 0 = T(x)+\frac{P_1(x)}{Q(x)}$, $P_1(x)$ has degree less than that of $Q(x)$. We can integrate $\frac{P_1(x)}{Q(x)}$ by expressing it in the following forms –

(i) $\frac{px+q}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}, a,b\neq 0.$

(ii) $\frac{px+q}{(x-a)^2}=\frac{A}{x-a}+\frac{B}{(x-a)^2}$ (iii) $\frac{px^2+qx+r}{(x-a)^2(x-b)}=\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$

(iv) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}=\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

(v) $\frac{px+q}{ax^2+bx+c}=\frac{A\frac{d}{dx}(ax^2+bx+c)}{ax^2+bx+c}+\frac{B}{ax^2+bx+c}$

Second fundamental theorem of integral calculus

Let f be a continuous function of x defined on $[a,b]$ and let F be another function such that $\frac{d}{dx}F(x)=f(x)\forall x\in$ domain of f, then $\int_a^b f(x)dx=[F(x)+c]_a^b=F(b)-F(a)$. This is called the definite integral of f over the range $[a,b]$, where a and b are called the limits of integration, a being the lower limit and b be the upper limit.

Example

$\int_{-\pi/4}^{\pi/4}\sin^2 x\,dx$

$=2\int_0^{\pi/4}\sin^2 x\,dx$

$=2\int_0^{\pi/4}\left(\frac{1-\cos 2x}{2}\right)dx$

$=\int_0^{\pi/4}(1-\cos 2x)dx$

$=\left[x-\frac{\sin 2x}{2}\right]_0^{\pi/4}$

$=\frac{\pi}{4}-\frac{1}{2}$

Definite integral as the limit of a sum

$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+\dots+f(a+(n-1)h)]$

where $h=\frac{b-a}{n}\to 0$ as $n\to\infty$

First fundamental theorem of integral calculus

Let the area function be defined by $A(x)=\int_a^x f(x)dx\ \forall x\geq a$, where f is continuous on $[a,b]$ then $A'(x)=f(x)\ \forall x\in[a,b]$.

Integration by parts

$\int f_1(x)f_2(x)dx=f_1(x)\int f_2(x)dx-\int\left[\frac{d}{dx}f_1(x)\int f_2(x)dx\right]dx$

Integration of some special functions

(i) $\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$ (ii) $\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$

(iii) $\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$ (iv) $\int\frac{dx}{\sqrt{x^2-a^2}}=\log\left|x+\sqrt{x^2-a^2}\right|+c$

(v) $\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}+c$ (vi) $\int\frac{dx}{\sqrt{x^2+a^2}}=\log\left|x+\sqrt{x^2+a^2}\right|+c$

(vii) $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+c$

(viii) $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+c$

(ix) $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$

Integration by substitution

The method in which we change the variable to some other variable is called the method of substitution. Below problems can be solved by substitution.

$\int\tan x\,dx=\log|\sec x|+c$ $\int\cot x\,dx=\log|\sin x|+c$

$\int\sec x\,dx=\log|\sec x+\tan x|+c$ $\int\operatorname{cosec} x\,dx=\log|\operatorname{cosec} x-\cot x|+c$

Applications of the Integrals

Area bounded by two curves

The area of the region enclosed between two curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$ is given by $A = \int_a^b [f(x) - g(x)]\,dx$, where $f(x) \geq g(x)$ in $[a, b]$

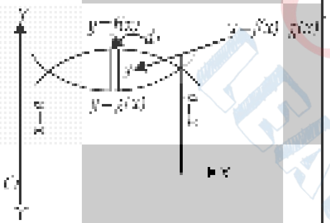
e.g. To find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$, (0,0) and (1,1) are points of intersection of $y = x^2$ and $y^2 = x$ and $y^2 = x \Rightarrow y = \sqrt{x} = f(x)$, and $y = x^2 = g(x)$, where $f(x) \geq g(x)$ in $[0, 1]$.

Area, $A = \int_0^1 [f(x) - g(x)]\,dx$

$= \int_0^1 \left[\sqrt{x} - x^2\right] dx$

$= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1$

$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$ Sq. units.



If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$, $a < c < b$, then the area is

$A = \int_a^c [f(x) - g(x)]\,dx + \int_c^b [g(x) - f(x)]\,dx$

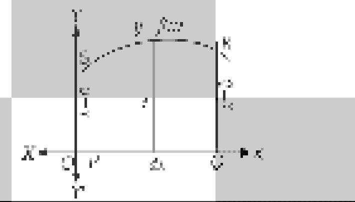
Area under simple curves

The area of the region bounded by the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$ $(b > a)$ is given by

$A = \int_a^b y\,dx$ or $\int_a^b f(x)\,dx$.

eg : the area bounded by $y = x^2$, x-axis in I quadrant and the lines $x = 2$ and $x = 3$ is -

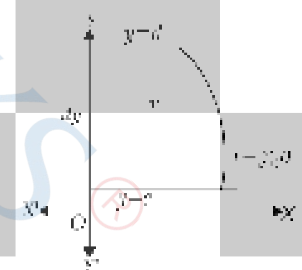
$A = \int_2^3 y\,dx = \int_2^3 x^2\,dx = \left[\frac{x^3}{3}\right]_2^3 = \frac{1}{3}(27 - 8) = \frac{19}{3}$ sq. units.



The area of the region bounded by the curve $x = f(y)$, y-axis and the lines $y = c$ and $y = d$ $(d > c)$ is given by $A = \int_c^d x\,dy$ or $\int_c^d f(y)\,dy$.

eg : the area bounded by $x = y^3$, y-axis in the I quadrant and the lines $y = 1$ and $y = 2$ is

$\int_1^2 x\,dy = \int_1^2 y^3\,dy = \left[\frac{1}{4}y^4\right]_1^2 = \frac{1}{4}[2^4 - 1^4] = \frac{15}{4}$ sq. units



Properties of Definite integrals

Some fundamental properties of definite integral are :

- Value of integration is independent of change of variable $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$
- If the limits of definite integral are interchanged then, its Value changes only by minus sign i.e. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- $\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$
- If $f(t)$ is an odd function, then $\phi(x) = \int_a^x f(t)\,dt$ is an even function.
- If $f(t)$ is an even function, then $\phi(x) = \int_0^x f(t)\,dt$ is an odd function.
- If $f(x)$ is continuous on $[a, \infty]$ then $\int_a^\infty f(x)\,dx$ is called an improper integral and is defined as $\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$
- $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a - x) = f(x) \\ 0, & \text{if } f(2a - x) = -f(x) \end{cases}$
- $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$
- If $f(x)$ is a periodic function with period T, then $\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx$
- If $f(x) \geq 0$ on the interval $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$
- If $f(x) \leq g(x)$ on the interval $[a, b]$, then

Differential Equations

Variable Separation Method

It is used to solve such an equation in which variables can be separated completely. eg : $ydx - xdy$ can be solved as $\frac{dx}{x} - \frac{dy}{y}$: Integrating both sides $\log x = \log y + \log c \Rightarrow \frac{x}{y} = c \Rightarrow x = cy$ is the solution.

Differential equation of family of curves

The order of a differential equations representing a family of curves is same as the number of arbitrary constants present in the equation corresponding to the family of curves. eg: Let the family of curves be $y = mx$, m = constant, then, $y' = m$

$y = y'x \Rightarrow y = \frac{dy}{dx}x \Rightarrow x\frac{dy}{dx} - y = 0.$

Homogeneous Differential Equations

A differential equation which can be expressed in the form $\frac{dy}{dx} = f(x, y)$ or $\frac{dx}{dy} = g(x, y)$, where, $f(x, y)$ and $g(x, y)$ are homogeneous functions of degree zero is called a homogenous differential equation

eg : $(x^2 - xy)dy = (x^2 - y^2)dx$

To solve this, we substitute $y = vx$, and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Linear Differential Equations

Let us take the differential equation is of the form $\frac{dy}{dx} + Py = Q$, where P and Q are constants or the function of 'x' is called a first order linear differential equations. Its solution is given as $ye^{\int Pdx} = \int Q.e^{\int Pdx}\,dx + c.$

eg : $\frac{dy}{dx} + 3y = 2x$ has solution

$ye^{\int 3dx} = \int 2x.e^{\int 3dx}\,dx + c \Rightarrow ye^{3x} = 2\int xe^{3x}dx + c.$

Reducible to Homogeneous Equation

Type 1. Let $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$, here $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Reduce the equation by substituting $x = X + h$ and $y = Y + k$

$\frac{dY}{dX} = \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)}$ ---------(i)

Equate $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$ and find h & k

So, equation (i) can be reduced to $\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$

Now, Solved by homogeneous way

Type 2 : Let $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$, here $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

Put, $a_2x + b_2y = v$ then $a_1x + b_1y = m(a_2x + b_2y) = mv$, here $m = \frac{a_1}{a_2} = \frac{b_1}{b_2}$

So, $\frac{d}{dx}(a_2x + b_2y) = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{b_2}\left(\frac{dv}{dx} - a_2\right)$

Now, substitute above values in given equation and reduce by variable separable type

Order & Degree of a Differential Equation

It is defined if the differential equations is a polynomial equation in its derivatives, and is defined as the highest power (positive integer only) of the highest order derivative.

eg: the degree of $\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} - 0$ is three

Order and degree (if defined) of a differential equation are always positive integers.

Solution of a Differential Equation

A function which satisfies the given differential equation is called its solution. The solution which contains as many arbitrary constants as the order of the differential equation is called a general solution and the solution free from arbitrary constants is called particular solution.

eg: $y = e^x - 1$ is a solution of $y'' - y' = 0$.

Since $y' = e^x$ and $y'' = e^x \Rightarrow y'' - y' = e^x - e^x = 0$.

Formation of Differential Equations

To form a differential equation from a given function, we differentiate the function successively as many times as the no. of arbitrary constants in the given function, and then eliminate the arbitrary constants.

eg: Let the function be $y = ax + b$, then we have to differentiate it two times, since there are 2 arbitrary constants a and b. $\therefore y' = a \Rightarrow y'' = 0$. Thus $y'' = 0$ is the required differential equation.

Straight Lines

Slope Formula

- If θ is the angle at which a straight line is inclined to be +ve direction of x-axis and $0^\circ \le \theta < 180^\circ$, $\theta \ne 90^\circ$, then the slope of the line, denoted by m, is defined by $m = \tan\theta$. If θ is 90°, m doesn't exist, but the line is parallel to y-axis. If $\theta = 0^\circ$, then $m = 0$ and the line is parallel to x-axis.
- If $A(x_1, y_1)$ and $B(x_2, y_2)$, $x_1 \ne x_2$ are points on straight line, then the slope m of the line is given by $m = \frac{(y_1 - y_2)}{(x_1 - x_2)}$

Area of Triangle

Area of triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is $\frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$

Distance Formula

The distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

Section Formula

The $P(x, y)$ divided the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m:n$, then $x = \frac{mx_2 \pm nx_1}{m \pm n}$; $y_1 = \frac{my_2 \pm ny_1}{m \pm n}$

Note: • (+ve) sign tells the division is internal, but (−ve) signs tells, the division is external.

• If $m = n$, then P is the mid-point of the line segment joining A & B.

Equation of Straight Line in various forms

1. POINT-SLOPE FORM: $y - y_1 = m(x - x_1)$ is the equation of a straight line whose slope is 'm' and passes through the point (x_1, y_1).
2. SLOPE INTERCEPT FORM: $y = mx + c$ is the equation of a straight line whose slope is 'm' and makes an intercept 'c' on the y-axis.
3. TWO POINT FORM: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ is the equation of a straight line which passes through (x_1, y_1) & (x_2, y_2).
4. INTERCEPT FORM: $\frac{x}{a} + \frac{y}{b} = 1$ is the equation of a straight line which makes intercepts a & b on x and y axis respectively.
5. NORMAL / PERPENDICULAR FORM: $x\cos\alpha + y\sin\alpha = p$ (where $p > 0, 0 \le \alpha < 2\pi$) is the equation of a straight line where the length of the perpendicular from origin O on the line is p and this perpendicular makes an angle α with +ve x-axis.
6. GENERAL FORM: $ax + by + c = 0$ is the equation of a straight line in general form. In this case, slope of line $= -\frac{a}{b}$

Intersection Point

The point of intersection between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$

Length of the perpendicular from a point on a line

The length of the perpendicular from $P(x_1, y_1)$ on $ax + by + c = 0$ is:

$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

Reflection & foot of perpendicular a point about a line

1. The image of a point (x_1, y_1) about a line $ax + by + c = 0$ is: $[(ar + x_1), (br + y_1)]$ where, $r = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$
2. Perpendicular distance from a point (x_1, y_1) on the line $ax + by + c = 0$ is: $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$
3. The foot of $\perp^{r}$ from a point (x_1, y_1) on the line $ax + by + c = 0$ is: $[(ar + x_1), (br + y_1)]$, where $r = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2}$

Perpendicular lines

1. When two lines of the slope m_1 & m_2 are at right angles, the Product of their slope is -1, i.e., $m_1m_2 = -1$. Thus, any line perpendicular to $y = mx + c$ is of the form, $y = \frac{-1}{m}x + d$ where d is any parameter.
2. Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are perpendicular if $aa' + bb' = 0$. Thus, any line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$, where k is any parameter.

Equations of bisectors

The equation of bisectors of the angles between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{\pm a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

Parallel lines

1. When two lines are parallel their slopes are equal. Thus, any line parallel to $y = mx + c$ is of the type $y = mx + d$, where d is any parameter
2. Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are parallel if $\frac{a}{a'} = \frac{b}{b'} \ne \frac{c}{c'}$
3. The distance between two parallel lines with equations $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $d = \left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$

Angle between two straight lines in terms of their slopes

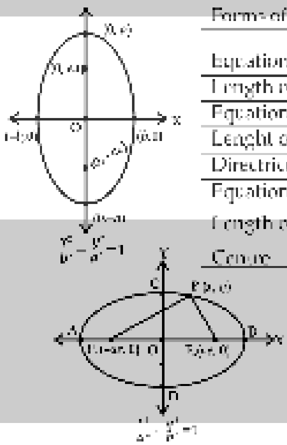
If m_1 and m_2 are the slopes of two intersecting lines ($m_1m_2 \ne -1$) and θ be the acute angle between them then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$

Conic Sections

Ellipse

- An ellipse is the set of all points in a plane, that the sum of their distances from two fixed points in the plane is constant.
- The two fixed points are called the 'foci' of the ellipse.
- The midpoint of line segment joining foci is called the 'centre' of the ellipse.
- The line segment through the foci of the ellipse is called 'major axis'.
- The line segment through centre & perpendicular to major axis is called minor axis.

Forms of ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$
	$a>b$	$a>b$
Equation of major axis	$y=0$	$x=0$
Length of major axis	$2a$	$2a$
Equation of Minor axis	$x=0$	$y=0$
Length of Minor axis	$2b$	$2b$
Directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{a}{e}$
Equation of latus rectum	$x=\pm ae$	$y=\pm ae$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Centre	$(0,0)$	$(0,0)$



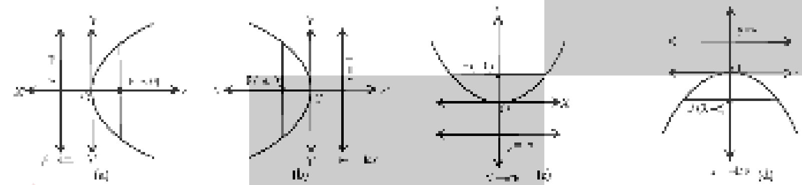
Here, $a>b$ and $b^2=a^2(1-e^2)$, $e<1$

Parabola

- A parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point in the plane. Fixed line is called 'directrix' of parabola. Fixed point F is called the 'focus'. A line through focus & perpendicular to directrix is called 'axis'. Point of intersection of parabola with axis is called 'vertex'.

Main facts about the parabola

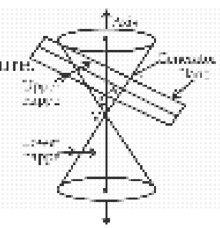
Forms of Parabola	$y^2=4ax$	$y^2=-4ax$	$x^2=4ay$	$x^2=-4ay$
Axis	$y=0$	$y=0$	$x=0$	$x=0$
Directix	$x=-a$	$x=a$	$y=-a$	$y=a$
Vertex	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
Focus	$(a,0)$	$(-a,0)$	$(0,a)$	$(0,-a)$
Lenght of latus rectum	$4a$	$4a$	$4a$	$4a$
Equations of latus rectum	$x=a$	$x=-a$	$y=a$	$y=-a$



Definition

Circles, ellipses, parabolas and hyperbolas are known as conic sections because they can be obtained as intersections of plane with a double napped right circular cone as from the given figure.

(i) Section will represent circle, if $\beta=90°$
(ii) Section will represent an Ellipse, if $\alpha<\beta<\pi/2$
(iii) Section will represent a parabola if if $\alpha=\beta$
(iv) Section will represent a hyperbola if $0\le\beta<\alpha$



Circle

A circle is a set of all points in a plane that are equidistant from a fixed point in the plane. The fixed point is called the 'centre' of the circle and the distance from the centre to a point on the circle is called the 'radius' of the circle.

The equation of a circle with centre (h,k) and the radius r is $(x-h)^2+(y-k)^2=r^2$

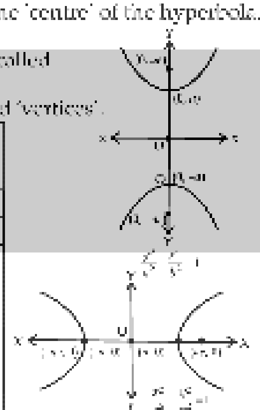
The general equation of circle is $x^2+y^2+2gx+2fy+c=0$

its centre is $(-g,-f)$ and radius $r=\sqrt{g^2+f^2-c}$

Hyperbola

- A hyperbola is the set of all points in a plane, that the difference of whose distances from two fixed points in the plane is a constant.
- The two fixed points are called the 'foci' of the hyperbola.
- The mid-point of the line segment joining the foci is called the 'centre' of the hyperbola.
- The line through the foci is called 'transverse axis'.
- Line through centre and perpendicular to transverse axis is called 'conjugate axis'.
- Points at which hyperbola intersects transverse axis are called 'vertices'.

Forms of the hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Equation of transverse axis	$y=0$	$x=0$
Equation of conjugate axis	$x=0$	$y=0$
Length of transverse axis	$2a$	$2a$
Foci	$(\pm ae,0)$	$(0,\pm ae)$
Equation of latus rectum	$x=\pm ae$	$y=\pm ae$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Centre	$(0,0)$	$(0,0)$



Here, $b^2=a^2(e^2-1)$, $e>1$

Conic Sections Important Facts

Ellipse

- The standard euation of ellipse is given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $b^2=a^2(1-e^2)$
- Eccentricity of ellipse $e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{(minor\ axis)^2}{(major\ axis)^2}}$
- The equation of tangent at a point (x_1, y_1) on the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$
- The point of contact are $\left(\mp\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$
- Equations of normal on the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, at a point (x_1, y_1)

  In point form $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$

  In parametric form $(a\sec\theta)x-(b\csc\theta)y=a^2-b^2$

  In slope form $y=mx-\frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$

Hyperbola

- Eccentricity of Hyperbola $e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{(conjugate\ axis)^2}{(transverse\ axis)^2}}$
- Length of latus-rectum $=2a(e^2-1)$
- The Conjugate hyperbola of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
- A line $y=mx+c$ will be the tangent on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, if $c=\pm\sqrt{a^2m^2-b^2}$
- The equation of tangent at a point (x_1, y_1) is given by $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$
- The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, at $(a\sec\theta, b\tan\theta)$ is $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$
- Equation of normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at a point (x_1, y_1)
- In point form $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$
- In parametric form $ax\cos\theta+by\cot\theta=a^2+b^2$
- In slope form $y=mx\mp\frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$

Parabola

- The equation of the tangent slope m to the parabola $y^2=4ax$ at point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$, $y=mx+\frac{a}{m}$
- The equation of the tangent slope m to the parabola $(y-k)^2=4a(x-h)$ is : $(y-k)=m(x-h)+\frac{a}{m}$
- The coordinate of the point of contact are $\left(h+\frac{a}{m^2}, k+\frac{2a}{m}\right)$
- The equation of normal to the parabola $y^2=4ax$ at a point (x_1, y_1) is : $y-y_1=-\frac{y_1}{2a}(x-x_1)$
- The equation of normal of slope m to the parabola $y^2=4ax$ at a point $(am^2, -2am)$ is : $y=mx-2am-am^3$
- The equation of normal to the parabola $y^2=4ax$ at the point $(at^2, 2at)$ is given by $y+tx=2at+at^3$

Circle

- Standard equation of a circle $(x-h)^2+(y-k)^2=r^2$
- When the coordinates of the end points of diameter are (x_1,y_1) and (x_2,y_2), then the equation of the circle is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$
- General equation of the circle is $x^2+y^2+2gx+2fy+c=0$
- The above equation is called general equation of the circle, whose centre is $(-g, -f)$ and radius is $\sqrt{g^2+f^2-c}$.

Three Dimensional Geometry-I

Coordinates of the Centroid of a Triangle

The coordinates of the centroid of the triangle, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$

eg: The centroid of a triangle ABC is at the point (1, 1, 1). If the coordinates of A and B are (3, –5, 7) and (–1, 7, –6), respectively, find the coordinates of the point C.

Sol: Let the coordinates of C be (x, y, z) and the coordinates of the centroid G be (1, 1, 1). Then $\frac{x+3-1}{3} = 1$, *i.e.*, $x = 1$;

$$\frac{y-5+7}{3} = 1, \text{ i.e., } y = 1;$$

$$\frac{z+7-6}{3} = 1, \text{ i.e., } z = 2. \text{ So, } C\,(x, y, z) = (1, 1, 2)$$

Coordinates of a Mid point

The coordinates of the midpoint of the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

eg: Find the midpoint of the line joining two points P(1, –3, 4) and Q(–4, 1, 2).

Sol: Coordinates of the midpoint of the line joining the points P & Q are

$$\left(\frac{1-4}{2}, \frac{-3+1}{2}, \frac{4+2}{2}\right) \text{ i.e. } \left(\frac{-3}{2}, -1, 3\right)$$

Introduction

- In three dimensions, the coordinate axes of a rectangular cartesian coordinate system are three mutually perpendicular lines. The axes are called x, y and z-axes.
- The three planes determined by the pair of axes are the coordinate planes, called xy, yz and zx-planes.
- The three coordinate planes divide the space into eight parts known as octants.
- The coordinates of a point P in 3D Geometry is always written in the form of triplet like (x, y, z). Here, x, y and z are the distances from yz, zx and yx planes, respectively.

eg:
- Any point on x-axis is : $(x, 0, 0)$
- Any point on y-axis is : $(0, y, 0)$
- Any point on z-axis is : $(0, 0, z)$

Section Formula

The coordinates of the point R which divides the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally and externally in the ratio m : n are given by

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right) \;\&\; \left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}, \frac{mz_2-nz_1}{m-n}\right)$$

respectively.

eg: Find the coordinates of the point which divides the line segment joining the points (1, –2, 3) and (3, 4, –5) in the ratio 2:3 internally.

Sol : Let $P(x, y, z)$ be the point which divides line segment joining A (1, –2, 3) and B (3, 4, –5) internally in the ratio 2:3. Therefore,

$$x = \frac{2(3)+3(1)}{2+3} = \frac{9}{5} \quad y = \frac{2(4)+3(-2)}{2+3} = \frac{2}{5} \quad z = \frac{2(-5)+3(3)}{2+3} = \frac{-1}{5}$$

Thus, the required point is $\left(\frac{9}{5}, \frac{2}{5}, \frac{-1}{5}\right)$.

Distance between Two Points

Distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

eg: Find the distance between the points P(1, –3, 4) and (–4, 1, 2).

Sol: The distance PQ between the points P & Q is given by

$$PQ = \sqrt{(-4-1)^2+(1+3)^2+(2-4)^2}$$

$$= \sqrt{25+16+4} = \sqrt{45} = 3\sqrt{5} \text{ units}$$

Three Dimensional Geometry-II

Direction ratios and direction cosines of a line

D. Cs of a line are the cosines of the angles made by the line with the positive direction of the co-ordinate axes. If l, m, n are the D. Cs of a line, then $l^2 + m^2 + n^2 = 1$. D. Cs of a line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$ where $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

D.Rs of a line are the nos which are proportional to the D.Cs of the line if l, m, n are the D.Cs and a, b, c are D.Rs of a line, then

$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$

Skew lines

These are the lines in space which are neither parallel nor intersecting. They lie in different planes. Angle between skew lines is the angle between two intersecting lines drawn from any point (origin) parallel to each of the skew lines

Angle between the two lines

If l_1, m_1, n_1 & l_2, m_2, n_2 are the D.Cs and $a_1, b_1, c_1, a_2, b_2, c_2$ are the D.Rs of the two lines and θ is the acute angle between them, then

$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2| = \left|\frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$

Line of Vector equations in 3D

Vector equation of a line passing through the given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$

Vector equation of a line which passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

Equation of a line through point (x_1, y_1, z_1) and having D.Cs l, m, n is $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$. Also, equation of a line that passes through two points is $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

Angle between two lines

If θ is the acute angle between $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ & $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$

then, $\cos\theta = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$

if $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$

are the equations of two lines, then acute angle between them is $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$

Characteristics of planes

Two lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$, $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ are coplanar if $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 0$. Equation of a plane that cuts co-ordinate axes at $(a,0,0), (0,b,0), (0,0,c)$ is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

The distance of a point with position vector $\vec{a}$ from the plane $\vec{r} \cdot \hat{n} = d$ is $|d - \vec{a} \cdot \hat{n}|$. The distance from a point (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is $\left|\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right|$

Vector equation of a plane

(i) which contains three non collinear points having position vectors $\vec{a}, \vec{b}, \vec{c}$ is $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$.

(ii) That passes through the intersection of planes $\vec{r} \cdot \vec{n}_1 = d_1$ & $\vec{r} \cdot \vec{n}_2 = d_2$ is $\vec{r} \cdot (\vec{n}_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$, λ = non-zero constant.

Equation of a plane

(i) which is at distance 'd' from origin and D.Cs of the normal to the plane as l, m, n is $lx + my + nz = d$.

(ii) $\perp$ to a given line with D.Rs. A, B, C and passing through (x_1, y_1, z_1) is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

(iii) Passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$.

Shortest distance between

(i) two skew lines is the line segment perpendicular to both the lines

(ii) $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is $\left|\frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$

(iii) $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is

$= \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2 + (a_1 b_2 - a_2 b_1)^2}}$

(iv) Distance between parallel line $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$ is $\left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$

Vector Algebra

Vector

A quantity that has both magnitude and direction is called a vector. The distance between the initial and terminal points of a vector is called its magnitude. Magnitude of vector $\vec{AB}$ is $|\vec{AB}|$.

Position Vector

Position vector of a point $P(x, y, z)$ is $x\hat{i} + y\hat{j} + z\hat{k}$ and its magnitude is $OP(\vec{r}) = \sqrt{x^2 + y^2 + z^2}$. eg. Position vector of $P(2, 3, 5)$ is $2\hat{i} + 3\hat{j} + 5\hat{k}$ and its magnitude is $\sqrt{2^2 + 3^2 + 5^2} = \sqrt{38}$.

Direction ratios and direction cosines

The scalar components of a vector are its direction ratios, and represent its projections along the respective axes.

The magnitude (r), direction ratios (a, b, c) and direction cosines (l, m, n) of vector $a\hat{i} + b\hat{j} + c\hat{k}$ are related as:

$l = \frac{a}{r}, m = \frac{b}{r}, n = \frac{c}{r}$

eg.: If $\vec{AB} = \hat{i} + 2\hat{j} + 3\hat{k}$, then $r = \sqrt{1 + 4 + 9} = \sqrt{14}$

Direction ratios are $(1, 2, 3)$,

and direction cosines are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

Types of Vectors

(i) Zero vector (initial and terminal points coincide)

(ii) Unit vector (magnitude is unity)

(iii) Coinitial vectors (same initial points)

(iv) Collinear vectors (parallel to the same line)

(v) Equal vectors (same magnitude and direction)

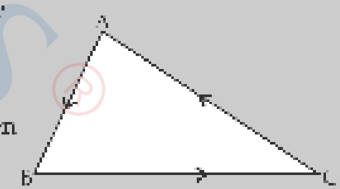
(vi) Negative of a vector (same magnitude, opp. direction)

Properties of Vector

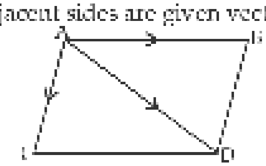
The vector sum of the three sides of a triangle taken in order is $\vec{0}$, i.e.

if $\triangle ABC$ is given triangle, then

$\vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$.



The Vector sum of two coinitial vectors is given by the diagonal of the parallelogram whose adjacent sides are given vectors.



if $\vec{AB}, \vec{AC}$ are the given vectors, then $\vec{AB} + \vec{AC} = \vec{AD}$

If we have two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and λ is any scalar, then-

(i) $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

(ii) $\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$

(iii) $\vec{a}.\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ and

(iv) $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Cross product of two vectors

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, $\hat{n}$ is a unit vector perpendicular to line joining $\vec{a}, \vec{b}$.

Properties

- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $\lambda(\vec{a} \times \vec{b}) = \vec{a} \times (\lambda\vec{b})$
- $\vec{a} \times \vec{b} = \vec{0}$ if $\vec{a} \parallel \vec{b}$

Scalar product of two vectors

If $\vec{a}, \vec{b}$ are the vectors and 'θ' is the angle between them, then their scalar product $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\rightarrow \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

Position of vectors

The Position vector of a point R dividing a line segment joining P, Q whose position vectors are $\vec{a}, \vec{b}$ respectively, in the ratio $m : n$

(i) internally is $\frac{n\vec{a} + m\vec{b}}{m + n}$, (ii) externally is $\frac{m\vec{b} - n\vec{a}}{m - n}$

Unit vector

For a given vector $\vec{a}$, the vector $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$ gives the unit vector in the direction of $\vec{a}$. eg., if $\vec{a} = 5\hat{i}$, then $\hat{a} = \frac{5\hat{i}}{5} = \hat{i}$, which is a unit vector.

Statistics

Variance of a discrete frequency distribution

$\text{Var}(\sigma^2) = \frac{1}{N}\sum f_i (x_i - \bar{x})^2$

Standard Deviation (S.D.) of a discrete frequency distribution

$\text{S.D}(\sigma) = \sqrt{\frac{1}{N}\sum f_i (x_i - \bar{x})^2}$

where $N = \sum_{i=1}^{n} f_i$ and $\bar{x}$ = mean

Variance of ungrouped data

$\text{Var}\ (\sigma^2) = \frac{1}{n}\sum (x_i - \bar{x})^2$

Standard Deviation (S.D.) for ungrouped data:

$\text{S.D.}(\sigma) = \sqrt{\frac{1}{n}\sum (x_i - \bar{x})^2}$

- Geometric Mean (G) $= \text{antilog}\ \dfrac{\sum_{i=1}^{n} f_i \log x_i}{\sum_{i=1}^{n} f_i}$
- Harmonic Mean (H) $= \dfrac{\sum_{i=1}^{n} f_i}{\sum_{i=1}^{n} \frac{f_i}{x_i}}$

Quartiles : $Q_1 = l + \dfrac{\frac{N}{4} - c}{f} \times h$, $Q_3 = l + \dfrac{\frac{3N}{4} - c}{f} \times h$

Inter quartile range $= Q_3 - Q_1$

Coeff. of quartile deviation $= \dfrac{Q_3 - Q_1}{Q_3 + Q_1}$

Decile: $D_i = \dfrac{i(n+1)}{10}$ team

Percentile $= P_i = 100\dfrac{(i - 0.5)}{n}$

Variance

$(\sigma^2) = \frac{h^2}{N^2}\left[N\sum f_i y_i^2 - \left(\sum f_i y_i\right)^2\right]$

Standard Deviation

$\sigma = \frac{h}{N}\sqrt{N\sum f_i y_i^2 - \left(\sum f_i y_i\right)^2}$

where $y_i = \frac{x_i - A}{h}$, $N = \sum_{i=1}^{n} f_i$

A = assumed mean, h = width of class interval

Karl Pearson's Coeff. of correlation

$r = \dfrac{\text{Cov}(x, y)}{\sqrt{\text{Var}(x)\ \text{Var}(y)}} = \dfrac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\sqrt{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}\sqrt{n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2}}$

Here, $\text{Cov}(x, y) = \frac{1}{n}\sum_{i=1}^{n} x_i y_i - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)$

Regression Equations

Regression equation of x on y

$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y}) = b_{xy}(y - \bar{y})$

Regression equation of y on x.

$(y - \bar{y}) = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}) = b_{yx}(x - \bar{x})$

Angle between two regression lines

$\theta = \tan^{-1}\left(\frac{1 - r^2}{b_{xy} + b_{yx}}\right) = \tan^{-1}\left(\frac{1 - r^2}{r}\right)\left(\frac{\sigma_x \times \sigma_y}{\sigma_x^2 + \sigma_y^2}\right)$

Variance & Standard Deviation of a discrete frequency distribution

Variance and Standard Deviation for ungrouped data

Sequence Mean

Quartiles, Decile and Percentile

Decile

Perecentile

Shortcut method to find variance and standard deviation

Step Deviation or Shortcut Method

Mean, Median, Mode

Mean Deviation

Variance & Standard Deviation of a continuous frequency distribution

- $M.D.(\bar{x}) = \frac{1}{N}\sum f_i |x_i - \bar{x}|$

  where $\bar{x} = a + \dfrac{\sum_{i=1}^{n} f_i d_i}{N} \times h$

  and $d_i = \frac{x_i - a}{h}$

  Here, a = assumed mean

  h = height

  N = sum of frequencies

- $M.D.(M) = \frac{1}{N}\sum f_i |x_i - M|$

  where, M(median) $= l + \dfrac{\frac{N}{2} - C}{f} \times h$

Mode (for grouped data) $M = L + \dfrac{f_1 - f_0}{2f_1 - f_0 - f_2} \times h$

Relation among Mean, Median and Mode

3 Median = 2 Mean + Mode

Mean Deviation (M.D.) for ungrouped data

- M.D. about mean i.e., $M.D.(\bar{x}) = \dfrac{\sum |x_i - \bar{x}|}{N}$, where $\bar{x}$ is mean and N = no. of terms
- M.D. about median i.e., $M.D.(M) = \dfrac{\sum |x_i - M|}{N}$, where M is median and N = no. of terms

Mean Deviation (M.D.) for grouped data

M.D. about mean i.e., $M.D.(\bar{x}) = \dfrac{\sum f_i |x_i - \bar{x}|}{N}$

M.D. about median i.e., $M.D.(M) = \frac{1}{N}\sum f_i |x_i - M|$

Variance of a continuous frequency distribution

$\text{Var}(\sigma^2) = \frac{1}{N}\sum f_i (x_i - \bar{x})^2$

Standard deviation of a continuous frequency distribution

$\text{S.D}(\sigma) = \frac{1}{N}\sqrt{N\sum f_i x_i^2 - \left(\sum f_i x_i\right)^2}$

Probability-I

Random Experiment

An Experiment is called random experiment if it satisfies the following two conditions:

- It has more than one possible outcome.
- It is not possible to predict the outcome in advance.

Outcome: A possible result of a random experiment is called its outcome.

Sample Space: Set of all possible outcomes of a random experiment is called sample space. It is denoted by symbol 'S'.
eg: In a toss of a coin, sample space is Head & Tail. i.e., S= {H,T}

Sample Point: Each element of the Sample Space is called a sample point.
eg: In a toss of a coin, head is a sample point

Equally Likely Outcomes: All outcome with equal probability.

Definition

Probability is the measure of uncertainty of various phenomenon, numerically. It can have positive value from 0 to 1.

$$\text{Probability} = \frac{\text{No. of favourable outcomes}}{\text{Total no. of outcomes}}$$

eg: Probability of getting an even no. in a throw of a die.

Sol. Here, favourable outcomes = {2, 4, 6}
Total no. of outcomes = {1, 2, 3, 4, 5, 6}

$$\text{Probability} = \frac{3}{6} = \frac{1}{2}$$

Event

It is the set of favourable outcomes. Any subset E of a sample space S is called an event.

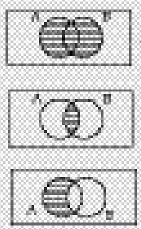
eg: Event of getting an even number (outcome) in a throw of a die.

Occurance of event: The event E of a sample space 'S' is said to have occurred if the outcome w of the experiment is such that $w \in E$. If the outcome w is such that $w \notin E$, we say that event E has not occurred.

Types of Events

- Impossible and Sure Event: The empty set ϕ is called an Impossible event, where as the whole sample space 'S' is called 'Sure event'.
  eg: In a rolling of a die, impossible event is that number more than 6 and event of getting number less than or equal to 6 is sure event.
- Simple Event: If an event has only one sample point of a sample space, it is called a 'simple event'.
  eg: In rolling of a die, simple event could be the event of getting number 4.
- Compound Event: If an event has more than one sample point, it is called a 'compound event'.
  eg: In rolling of a die, compound event could be event of getting an even number.
- Complementary Event: Complement event to A = 'not A'
  eg: If an event A = Event of getting odd number in a throw of a die i.e., {1, 3, 5} then, complementary event to A = Event of getting an even number in a throw of a die, i.e. {2, 4, 6}
  $A' = \{w : w \in S \text{ and } w \notin A\} = S - A$ (where S is the sample space)

Algebra of Events



- Event A or B or $(A \cup B)$
  $A \cup B = \{w : w \in A \text{ or } w \in B\}$
- Event A and B or $(A \cap B)$
  $A \cap B = \{w : w \in A \text{ and } w \in B\}$
- Event A but not B or $(A - B)$
  $A - B = A \cap B'$

Exhaustive Events

Many events that together form sample space are called exhaustive events.
eg: A die is thrown. Event A = All even outcomes and event B = All odd outcomes.
Event A & B together forms exhaustive events as it forms sample space.

Mutually Exclusive Events

Events A & B are called mutually exclusive events if occurance of any one of them excludes occurrance of other event, i.e. they cannot occur simultaneously.
eg: A die is thrown. Event A = All even outcomes & event B = All odd outcomes. then, A & B are mutually exclusive events, they cannot occur simultaneously.
Note: Simple events of a sample space are always mutually exclusive.

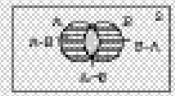
Probability of A∪B, A∩B and P(not A)

If A and B are any two events, then

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

If A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$

(As $P(A \cap B) = \phi$)



- Probability of the event 'not A'
  $P(A') = P(\text{not } A) = 1 - P(A)$

Probability - II

Conditional Probability

The probability of the event E is called the conditional probability of E given that F has already occurred, and is denoted by $P(E/F)$. Also, $P(E/F)=\frac{P(E\cap F)}{P(F)}$, $P(F)\neq 0$.

Properties

(i) $0\le P(E/F)\le 1$, $P(F/F)=1$, $P(E'/F)=1-P(E/F)$

(ii) $P((E\cup F)/G)=P(F/G)+P(F/G)-P((E\cap F)/G)$

(iii) $P(E\cap F)=P(E)P(F/E), P(E)\neq 0$

(iv) $P(E\cap F)=P(F)P(E/F), P(F)\neq 0$

eg: if $P(A)=\frac{7}{13}$, $P(B)=\frac{9}{13}$ and $P(A\cap B)=\frac{4}{13}$, then

$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{4}{13}}{\frac{9}{13}}=\frac{4}{9}$.

Independent Event

If E and F are independent, then $P(E\cap F)=P(E)P(F)$, $P(E/F)=P(E)$, $P(F)\neq 0$ and $P(F/E)=P(F)$, $P(E)\neq 0$.

If A,B,C are mutually independent events then

(i) $P(A\cap B)=P(A).P(B)$

(ii) $P(A\cap C)=P(A).P(C)$

(iii) $P(B\cap C)=P(B).P(C)$

(iv) $P(A\cap B\cap C)=P(A).P(B).P(C)$

Theorem of total probability

Let $\{E_1, E_2, ..., E_n\}$ be a partition of a sample space S and suppose that each of $E_1, E_2, ...E_n$ has non zero probability. Let A be any event associated with S, then

$P(A)=P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+....+P(E_n)P(A/E_n)$

$=\sum_{j=1}^{n}P(E_j)P(A/E_j)$

Bayes' Theorem

If $E_1, E_2, ..., E_n$ are events which constitute a partition of sample space S, i.e., $E_1, E_2, ..., E_n$ are pairwise disjoint and $E_1\cup E_2\cup ....\cup E_n=S$ and A be any event with non-zero probability.

then $P(E_i/A)=\frac{P(E_i)P(A|E_i)}{\sum_{j=1}^{n}P(E_j)P(A/E_j)}$, $i=1,2,3,......,n$

Bernoulli's Trials

Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions :

(i) There should be a finite no. of trials.

(ii) The trials should be independent.

(iii) Each trial has exactly two outcomes: success or failure.

(iv) The probability of success remains the same in each trial.

For Binomial distribution,

$B(n,p)$, $P(X=x)=\left({}^{n}C_x q^{n-x} p^x\right)$, $x=0,1,...,n$ $(q=1-p)$

Variance and standard deviation

Let x be a random variable whose possible values $x_1, x_2, ..., x_n$ occurs with probabilities $p(x_1), p(x_2), ..., p(x_n)$ respectively.

Let, $\mu=E(x)$ be the mean of x. The variance of x, var (x)

or $\sigma^2=\sum_{i=1}^{n}(x_i-\mu)^2 p(x_i)$ or $E(x-\mu)^2$

The non-negative number

$\sigma_x=\sqrt{\operatorname{var}(X)}=\sqrt{\sum_{i=1}^{n}(x_i-\mu)^2 p(x_i)}$ is called the standard deviation of the random variable 'X'. Also,

$\operatorname{var}(X)=E(X^2)-[E(X)]^2$ eg: $E(X)=3$ and $E(X^2)=10$, then var $X=10-9=1$ and $SD=\sqrt{1}=1$.

Mean of a random variable

Let x be a random variable whose possible values $x_1, x_2, ..., x_n$ occur with probabilities $p_1, p_2, ..., p_n$ resp. Then, mean of x, $\mu=\sum_{i=1}^{n}x_i p_i$. It is also called the expectation of x, denoted by $E(x)$.

Probability Distribution

The probability distribution of a random variable x is the system of numbers

x: x_1 x_2 . . . . . x_n

$P(x)$: p_1 p_2 . . . . . p_n

where, $p_i>0$,

$\sum_{i=1}^{n}p_i=1$, $i=1,2,...,n$

Distribution

Normal Distribution

$P(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$

Here, $z=\frac{x-\mu}{\sigma}$ = normal variables for which mean is 0 and standard deviation is 1

and $\sigma=\sqrt{npq}$

Poisson Distribution

$P(r)=\lim_{n\to\infty}{}^{n}C_r p^r q^{n-r}$

$=\frac{\lambda^r e^{-\lambda}}{r!}$

Here, $\lambda=np$ is called Poisson Distribution $\lambda=np$

$r=0, 1, 2, 3,$--------

Trigonometric Functions

Important Trigonometric Ratios values

- $\sin 15°$ or $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75°$ or $\cos\frac{5\pi}{12}$
- $\cos 15°$ or $\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75°$ or $\sin\frac{5\pi}{12}$
- $\tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75°$
- $\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15°$
- $\sin\frac{\pi}{10}$ or $\sin 18° = \frac{\sqrt{5}-1}{4}$
- $\cos\frac{\pi}{5}$ or $\cos 36° = \frac{\sqrt{5}+1}{4}$

Standard General Solutions of Trigonometrical ratios

General solution: The solution consisting of all possible solutions of a trigonometric equation is called its General Solution.

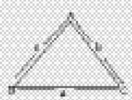
- $\sin\theta = 0 \Leftrightarrow \theta = n\pi$, $\cos\theta = 0 \Leftrightarrow \theta = (2n+1)\frac{\pi}{2}$
- $\tan\theta = 0 \Leftrightarrow \theta = n\pi$, $\sin\theta = \sin\alpha \Leftrightarrow \theta = n\pi + (-1)^n\alpha$, where $\alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
- $\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi \pm \alpha$, where $\alpha \in [0, \pi]$
- $\tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha$, where $\alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
- $\sin^2\theta = \sin^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha$, $\cos^2\theta = \cos^2\alpha \Leftrightarrow n\pi \pm \alpha$
- $\tan^2\theta = \tan^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha$, $\sin\theta = 1 \Leftrightarrow \theta = (4n+1)\frac{\pi}{2}$
- $\cos\theta = 1 \Leftrightarrow \theta = 2n\pi$, $\cos = -1 \Leftrightarrow \theta = (2n+1)\pi$
- $\sin\theta = \sin\alpha \Leftrightarrow \theta = n\pi + (-1)^n\alpha$

Solutions of Triangle

In, ΔABC,

(i) Sines Law

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



(ii) Projection formula

(a) $a\cos C + c\cos A = b$

(b) $b\cos A + a\cos B = c$

(c) $c\cos B + b\cos C = a$

(iii) Cosines Law

(a) $\cos A = \frac{b^2+c^2-a^2}{2bc}$

(b) $\cos B = \frac{a^2+c^2-b^2}{2ac}$

(c) $\cos C = \frac{a^2+b^2-c^2}{2ab}$

Area of Triangle

Area of triangle ABC

$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$

$= \sqrt{s(s-a)(s-b)(s-c)}$; here, $s = \frac{a+b+c}{2}$

Trigonometrical Equations

The equation involving trigonometric functions of unknown angles are known as Trigonometric equations.

e.g $\cos\theta = 0$, $\cos^2\theta - 4\cos\theta = 1$

A solution of trigonometric equation is the value of the unknown angle that satisfies the equation.

e.g. $\sin\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \ldots$

Factorisation of the Sum or Difference of sine and cosine with two variables

- $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
- $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$
- $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
- $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

Transformation of Product into Sum or Difference of Sines or Cosines

- $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
- $2\cos A\sin B = \sin(A+B) - \sin(A-B)$
- $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
- $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Trigonometrical Function sum and Difference of Two Angles

- $\sin(A\pm B) = \sin A\cos B \pm \cos A\sin B$
- $\cos(A\pm B) = \cos A\cos B \mp \sin A\sin B$
- $\tan(A\pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
- $\cot(A\pm B) = \frac{\cot A\cot B \mp 1}{\cot A \pm \cot B}$
- $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B)\sin(A-B)$
- $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B)\cos(A-B)$

Degree & Radian Measures

If in a circle of radius r, an arc of length l subtends an angle of θ radians, then $l = r\theta$.

- Radian Measure $= \frac{\pi}{180} \times$ Degree Measure
- Degree Measure $= \frac{180}{\pi} \times$ Radian Measure

Trigonometric Identities

- $\sin^2\theta + \cos^2\theta = 1$, $-1 \le \sin\theta \le 1$, $-1 \le \cos\theta \le 1$ $\forall\theta\in R$
- $\sec^2\theta - \tan^2\theta = 1$, $|\sec\theta| \ge 1$, $\forall\theta\in R$
- $\mathrm{cosec}^2\theta - \cot^2\theta = 1$, $|\mathrm{cosec}\,\theta| \ge 1$, $\theta\in R$

Multiple and Half Angles

- $\sin 2A = 2\sin A\cos A$
- $\cos 2A = \cos^2 A - \sin^2 A$
  $= 2\cos^2 A - 1$
  $= 1 - 2\sin^2 A$
- $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$
- $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$
- $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$
- $\sin 3A = 3\sin A - 4\sin^3 A$
- $\cos 3A = 4\cos^3 A - 3\cos A$
- $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$
- $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$
- $\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}$
- $\tan\theta = \left(2\tan\frac{\theta}{2}\right)/\left(1-\tan^2\frac{\theta}{2}\right)$

Trigonometrical Ratio of allied Angles

- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = \cos\theta$
- $\tan(-\theta) = -\tan\theta$
- $\cot(-\theta) = -\cot\theta$
- $\sec(-\theta) = \sec\theta$
- $\mathrm{cosec}(-\theta) = -\mathrm{cosec}\theta$

- $\sin\theta\left(\frac{\pi}{2}\mp\theta\right) = \cos\theta$
- $\cos\left(\frac{\pi}{2}\mp\theta\right) = \pm\sin\theta$
- $\tan\left(\frac{\pi}{2}\mp\theta\right) = \pm\cot\theta$
- $\cot\left(\frac{\pi}{2}\mp\theta\right) = \pm\tan\theta$
- $\sec\theta\left(\frac{\pi}{2}\mp\theta\right) = \pm\mathrm{cosec}\theta$
- $\mathrm{cosec}\left(\frac{\pi}{2}\mp\theta\right) = \sec\theta$

- $\cos((\pi\mp\theta)) = -\cos\theta$
- $\sin(\pi\mp\theta) = \pm\sin\theta$
- $\tan(\pi\mp\theta) = \mp\tan\theta$
- $\cot(\pi\mp\theta) = \mp\cot\theta$
- $\sec(\pi\mp\theta) = -\sec\theta$
- $\mathrm{cosec}(\pi\mp\theta) = \pm\mathrm{cosec}\theta$

- $\sin(2\pi\mp\theta) = \mp\sin\theta$
- $\cos(2\pi\mp\theta) = \cos\theta$
- $\tan(2\pi\mp\theta) = \mp\tan\theta$
- $\cot(2\pi\mp\theta) = \mp\cot\theta$
- $\sec(2\pi\mp\theta) = \sec\theta$
- $\mathrm{cosec}(2\pi\mp\theta) = \mp\mathrm{cosec}\theta$

- $\sin\left(\frac{3\pi}{2}\mp\theta\right) = -\cos\theta$
- $\cos\left(\frac{3\pi}{2}\mp\theta\right) = \mp\sin\theta$
- $\tan\left(\frac{3\pi}{2}\mp\theta\right) = \pm\cot\theta$
- $\cot\left(\frac{3\pi}{2}\mp\theta\right) = \pm\tan\theta$
- $\sec\left(\frac{3\pi}{2}\mp\theta\right) = \mp\mathrm{cosec}\theta$
- $\mathrm{cosec}\left(\frac{3\pi}{2}\mp\theta\right) = -\sec\theta$

Inverse Trigonometric Functions

Domain and range of inverse trigonometric functions

(i) $y = \sin^{-1}x$, Domain $= [-1, 1]$, Range $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $y = \cos^{-1}x$, Domain $= [-1, 1]$, Range $= [0, \pi]$

(iii) $y = \operatorname{cosec}^{-1}x$, Domain $= R - (-1, 1)$, Range $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

(iv) $y = \sec^{-1}x$, Domain $= R - (-1, 1)$, Range $= [0, \pi] - \left\{\frac{\pi}{2}\right\}$

(v) $y = \tan^{-1}x$, Domain $= R$, Range $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(vi) $y = \cot^{-1}x$, Domain $= R$, Range $= (0, \pi)$

Trigonometric functions

(i) $\sin : R \to [-1, 1]$

(ii) $\cos : R \to [-1, 1]$

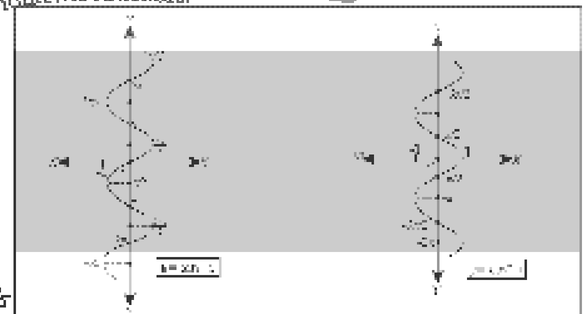
(iii) $\tan : R - \left\{x : x = (2n-1)\frac{\pi}{2}, n \in Z\right\} \to R$

(iv) $\cot : R - \{x : x = n\pi, n \in Z\} \to R$

(v) $\sec : R - \left\{x : x = (2n-1)\frac{\pi}{2}, n \in Z\right\} \to R - (-1, 1)$

(vi) $\operatorname{cosec} : R - \{x : x = n\pi, n \in Z\} \to R - (-1, 1)$

Graphs of trigonometric functions and Inverse trigonometric functions



Some important relations

(i) $y = \sin^{-1}x \Rightarrow x = \sin y$

(ii) $x = \sin y \Rightarrow y = \sin^{-1}x$

(iii) $\sin(\sin^{-1}x) = x$

(iv) $\sin^{-1}(\sin x) = x$

(v) $\sin^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}x$

(vi) $\cos^{-1}(-x) = \pi - \cos^{-1}x$

(vii) $\cos^{-1}\frac{1}{x} = \sec^{-1}x$

(viii) $\cot^{-1}(-x) = \pi - \cot^{-1}x$

(ix) $\tan^{-1}\frac{1}{x} = \cot^{-1}x$

(x) $\sec^{-1}(-x) = \pi - \sec^{-1}x$

(xi) $\sin^{-1}(-x) = -\sin^{-1}x$

(xii) $\tan^{-1}(-x) = -\tan^{-1}x$

(xiii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

(xiv) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$, $|x| \geq 1$

(xv) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$, $xy < 1$

(xvi) $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$, $-1 < x < 1$

(xvii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$, $xy > -1$

(xviii) $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2}$

$\sin^{-1}x \neq (\sin x)^{-1} = \frac{1}{\sin x}$ and same for other trigonometric functions.

Principal value branch and principal value

The range of an inverse trigonometric function is the principal value branch and those values which lies in the principal value branch is called the principal value of that inverse trigonometric functions.

$0 \leq \sin^{-1}x \leq \frac{\pi}{2}$	$-\frac{\pi}{2} \leq \sin^{-1}x < 0$
$0 \leq \cos^{-1}x \leq \frac{\pi}{2}$	$\frac{\pi}{2} < \cos^{-1}x \leq \pi$
$0 \leq \tan^{-1}x < \frac{\pi}{2}$	$-\frac{\pi}{2} < \tan^{-1}x < 0$
$0 < \cot^{-1}x \leq \frac{\pi}{2}$	$\frac{\pi}{2} < \cot^{-1}x < \pi$
$0 \leq \sec^{-1}x < \pi/2$	$\frac{\pi}{2} < \sec^{-1}x \leq \pi$
$0 < \operatorname{cosec}^{-1}x \leq \frac{\pi}{2}$	$-\frac{\pi}{2} \leq \operatorname{cosec}^{-1}x < 0$

Mathematical Reasoning

Algebric of Statements

Idempotent Law
(i) $A \wedge A \Leftrightarrow A$,
(ii) $A \vee A \Leftrightarrow A$

Associative Law
(i) $(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$,
(ii) $(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$

Commutative Law
(i) $A \vee B \Leftrightarrow B \vee A$,
(ii) $A \wedge B \Leftrightarrow B \wedge A$

Distributive Law
(i) $A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$
(ii) $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$

Identity Laws
(i) $A \wedge T \Leftrightarrow A$
(ii) $A \wedge F \Leftrightarrow F$
(iii) $A \vee T \Leftrightarrow T$
(iv) $A \vee F \Leftrightarrow A$

Complement Laws
(i) $A \vee (\sim A) \Leftrightarrow T$
(ii) $A \wedge (\sim A) \Leftrightarrow F$
(iii) $\sim T \Leftrightarrow F$
(iv) $\sim F \Leftrightarrow T$

Absorption Law
(i) $A \vee (A \wedge B) \Leftrightarrow A$
(ii) $A \wedge (A \vee B) \Leftrightarrow A$
(ii) $\sim(A \wedge B) \Leftrightarrow (\sim A) \vee (\sim B)$

Involution Law
$\sim(\sim A) \Leftrightarrow A$

Contrapositive and Converse Statements

- The contrapositive of a statement $p \Rightarrow q$ is the statement $\sim q \Rightarrow \sim p$
- The converse of a statement $p \Rightarrow q$ is the statement $q \Rightarrow p$

e.g. If the physical environment changes, then biological environmental changes.

- Contrapositive : If the biological environment does not change then the physical environment does not change.
- Converse: If the biological environment changes then physical environment changes.

Validity of Statements

The following methods are used to check the validity of statements
(i) Direct Method
(ii) Contrapositive Method
(iii) Method of Contradiction
(iv) Using a Counter example

Implications

- These are statements with word "if then", "only if" and "if and only if".

e.g.

- r: if a number is a multiple of 9, then it is multiple of 3.
- p: a number is a multiple of 9.
- q: a number is a multiple of 3.
- Then, if p then q is the same as following:

→ p implies q is denoted by $p \Rightarrow q$, then symbol $\Rightarrow$ stands for implies,
→ p is a sufficient condition for q, then symbol $\Rightarrow$
→ p only if $q \rightarrow$ q is a necessary condition for $p \rightarrow \sim q$ implies $\sim p$

Quantifiers

In this statement, the two important symbols are used.

- The symbol '$\forall$' stand for "all values of";
- The symbol '$\exists$' stand for "there exists"

e.g. For every prime number P, P is an irrational number.

- This means that if S denotes the set of all prime numbers, then for all the members of P of the set S, P is an irrational number.

Compound Statement

Many mathematical statements are obtained by combining one or more statements using some connecting words like "and", "or" etc. Each statement is called a "Compound Statement."

e.g.

'The sky is blue' and 'the grass is green' is a compound statement where connecting word is "and";
the components of Compound Statement are
p : The sky is blue
q : The grass is green. So, $p \wedge q$

Sentence

A sentence is called a mathematically acceptable statement if it is either true or false but not both.

e.g.

- The sum of two positive numbers is positive.
- All prime numbers are odd numbers.
- In above statements, first is 'true' and second is 'false'.

Negation

A statement which is formed by changing the true value of a given statement by using the word like 'no', 'not' is called negation of given statement.

- If p is a statement, then negation is denoted by $\sim p$

e.g. New Delhi is a city.
The negation of this statement:
→ It is not the case that New Delhi is a city.
→ It is false that New Delhi is a city.
→ New Delhi is not a city.