

APPENDIX-A

Conversion Factors

Conversion factors may be read directly from these tables. For example, 1 degree = 2.778×10^{-3} revolutions, so $16.7^\circ = 16.7 \times 2.778 \times 10^{-3}$ rev. The SI units are fully capitalized.

Plane Angle

	$^\circ$	'	"	RADIAN	rev
1 degree = 1	60		3600	1.745×10^{-2}	2.778×10^{-3}
1 minute = 1.667×10^{-2}	1		60	2.909×10^{-4}	4.630×10^{-5}
1 second = 2.778×10^{-4}		1.667×10^{-2}	1	4.848×10^{-6}	7.716×10^{-7}
1 RADIAN = 57.30	3438		2.063×10^5	1	0.1592
1 revolution = 360		2.16×10^4	1.296×10^6	6.283	1

Solid Angle

$$1 \text{ sphere} = 4\pi \text{ steradians} = 12.57 \text{ steradians}$$

Length

cm	m	km	In.	ft	mi
1 centimeter = 1	10^{-2}	10^{-5}	0.3937	3.281×10^{-2}	6.214×10^{-6}
1 meter = 100	1	10^{-3}	39.37 ⁴	3.281	6.214×10^{-4}
1 kilometer = 10^5	1000	1	3.937×10^4	3281	0.6214
1 inch = 2.540	2.540×10^{-2}	2.540×10^{-25}	1	8.333×10^{-2}	1.578×10^{-5}
1 foot = 30.48	0.3048	3.048×10^{-4}	12	1	1.894×10^{-4}
1 mile = 1.609×10^5	1609	1.609	6.336×10^4	5280	1
1 angstrom = 10^{-10} m	1 fermi = 10^{-15} m		1 fathom = 6 ft		1 rod = 16.5 ft
1 nautical mile = 1852 m		1 light-year = 9.460×10^{12} km		1 Bohr radius = 5.292×10^{-11} m	1 mil = 10^{-3} in.
= 1.151 miles = 6076 ft		1 parsec = 3.084×10^{13} km		1 yard = 3 ft	1 nm = 10^{-9} m

Area

m^2	cm^2	ft^2	In.^2
1 SQUARE METER = 1	10^4	10.76	1550
1 square centimetre = 10^{-4}	1	1.076×10^{-3}	0.1550
1 square foot = 9.290×10^{-2}	929.0	1	144
1 square inch = 6.452×10^{-4}	6.452	6.944×10^{-3}	1
1 square mile = 2.788×10^7 ft ² = 640 acres		1 acre = 43 560 ft ²	
1 barn = 10^{-28} m ²		1 hectare = 10^4 m ² = 2.471 acres	

Volume

m³	cm³	L	ft³	in³
1 CUBIC METER = 1	10 ⁶	1000	35.31	6.102 × 10 ⁴
1 cubic centimeter = 10 ⁻⁶	1	1.000 × 10 ⁻³	3.531 × 10 ⁻⁵	6.102 × 10 ⁻²
1 liter = 1.000 × 10 ⁻³	1000	1	3.531 × 10 ⁻²	61.02
1 cubic foot = 2.832 × 10 ⁻²	2.832 × 10 ⁻⁴	28.32	1	1728
1 cubic inch = 1.639 × 10 ⁻⁵	16.39	1.639 × 10 ⁻²	5.787 × 10 ⁻⁴	1

1 U.S. fluid gallon = 4 U.S. fluid quarts = 8 U.S. pints = 128 U.S. fluid ounces = 231 in.³

1 British imperial gallon = 277.4 in.³ = 1.201 U.S. fluid gallons

Mass

Quantities in the colored areas are not mass units but are often used as such. For example, when we write 1 kg “=” 2.205 lb, this means that a kilogram is a *mass* that *weighs* 2.205 pounds at a location where g has the standard value of 9.80665 m/s².

g	Kg	slug	u	oz	lb	ton
1 gram = 1	0.001	6.852 × 10 ⁻⁵	6.022 × 10 ²³	3.527 × 10 ⁻²	2.205 × 10 ⁻³	1.102 × 10 ⁻⁶
1 KILOGRAM = 1000	1	6.852 × 10 ⁻²	6.022 × 10 ²⁶	35.27	2.205	1.102 × 10 ⁻³
1 slug = 1.459 × 10 ⁴	14.59	1	8.786 × 10 ²⁷	514.8	32.17	1.609 × 10 ⁻²
1 atomic mass unit = 1.661 × 10 ⁻²⁴	1.661 × 10 ⁻²⁷	1.138 × 10 ⁻²⁸	1	5.857 × 10 ⁻²⁶	3.662 × 10 ⁻²⁷	1.830 × 10 ⁻³⁰
1 ounce = 28.35	2.835 × 10 ⁻²	1.943 × 10 ⁻³	1.718 × 10 ²⁵	1	6.250 × 10 ⁻²	3.125 × 10 ⁻⁵
1 pound = 453.6	0.4536	3.108 × 10 ⁻²	2.732 × 10 ²⁶	16	1	0.0005
1 ton = 9.072 × 10 ⁵	907.2	62.16	5.463 × 10 ²⁹	3.2 × 10 ⁴	2000	1

1 metric ton = 1000 kg

Density

Quantities in the colored areas are weight densities and, as such, are dimensionally different from mass densities. See the note for the mass table.

slug/ft³	KILOGRAM/METER³	g/cm³	lb/ft³	lb/in.³
1 slug per foot ³ = 1	515.4	0.5154	32.17	1.862 × 10 ⁻²
1 KILOGRAM Per METER ³ = 1.940 × 10 ⁻³	1	0.001	6.243 × 10 ⁻²	3.613 × 10 ⁻⁵
1 gram per centimeter ³ = 1.940	1000	1	62.43	3.613 × 10 ⁻²
1 pound per foot ³ = 3.108 × 10 ⁻²	16.02	16.02 × 10 ⁻²	1	5.787 × 10 ⁻⁴
1 pound per inch ³ = 53.71	2.768 × 10 ⁴	27.68	1728	1

Time

y	d	h	min	s
1 year = 1	365.25	8.766 × 10 ³	5.259 × 10 ⁵	3.156 × 10 ⁷
1 day = 2.738 × 10 ⁻³	1	24	1440	8.640 × 10 ⁴
1 hour = 1.141 × 10 ⁻⁴	4.167 × 10 ⁻²	1	60	3600
1 minute = 1.901 × 10 ⁻⁶	6.944 × 10 ⁻⁴	1.667 × 10 ⁻²	1	60
1 SECOND = 3.169 × 10 ⁻⁸	1.157 × 10 ⁻⁵	2.778 × 10 ⁻⁴	1.667 × 10 ⁻²	1

Speed

ft/s	km/h	m/s	mi/h	cm/s
1 foot per second = 1	1.097	0.3048	0.6818	30.48
1 kilometer per hour = 0.9113	1	0.2278	0.6214	27.78
1 METER per SECOND = 3.281	3.6	1	2.237	100
1 mile per hour = 1.467	1.609	0.4470	1	44.70
1 centimeter per second = 3.281 × 10 ⁻²	3.6 × 10 ⁻²	0.01	2.237 × 10 ⁻²	1

APPENDIX-B

The Greek Alphabet

Alpha	A	α	Iota	I	ι	Rho	P	ρ
Beta	B	β	Kappa	K	κ	Sigma	Σ	σ
Gamma	Γ	γ	Lambda	Λ	λ	Tau	\Tau	τ
Delta	Δ	δ	Mu	M	μ	Upsilon	Y	ν
Epsilon	E	ε	Nu	N	ν	Phi	ϕ	φ
Zeta	Z	ζ	Xi	Ξ	ξ	Chi	X	χ
Eta	H	η	Omicron	O	\o	Psi	Ψ	ψ
Theta	Θ	θ	Pi	Π	π	Omega	Ω	ω

APPENDIX-C

Concept Base Mathematical Formula

ALGEBRA – 1

1.1 Set Identities

Sets : A, B, C , Universal set : U , Complement : A' , Proper subset : $A \subset B$, Empty set : \emptyset , Union of sets : $A \cup B$, Intersection of sets : $A \cap B$, Difference of sets : $A - B$

1. $A \subset U$

2. $A \subset A$

3. $A = B$ if $A \subset B$ and $B \subset A$.

4. Empty set $\emptyset \subset A$

5. Union of Sets

$$C = A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

6. Commutativity $A \cup B = B \cup A$

7. Associativity $A \cup (B \cup C) = (A \cup B) \cup C$

8. Intersection of Sets

$$C = A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

9. Commutativity $A \cap B = B \cap A$

10. Associativity $A \cap (B \cap C) = (A \cap B) \cap C$

11. Distributivity

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

12. Idempotency $A \cap A = A, A \cup A = A$

13. Domination $A \cap \emptyset = \emptyset, A \cup U = U$

14. Identity $A \cup \emptyset = A, A \cap U = A$

15. Complement $A' = \{x \in U \mid x \notin A\}$

16. Complement of intersection and Union

$$A \cup A' = U, A \cap A' = \emptyset$$

17. De - Morgan's Laws

$$(A \cup B)' = A' \cap B', (A \cap B)' = A' \cup B'$$

18. Difference of Sets

$$C = B - A = \{x \mid x \in B \text{ and } x \notin A\}$$

19. $B - A = B - (A \cap B)$

20. $B - A = B \cap A'$

21. $A - A = \emptyset$

22. $A - B = A \text{ iff } A \cap B = \emptyset.$

23. $(A - B) \cap C = (A \cap C) - (B \cap C)$

24. $A' = U - A$

25. Cartesian Product

$$C = A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

1.2 Sets of Numbers

Natural numbers : N , Whole numbers : W , Integers : Z , Positive integers : Z^+ , Negative integers : Z^- , Rational numbers : Q , Real numbers : R , Complex numbers : C

1. Natural Numbers

Counting numbers : $N = \{1, 2, 3, \dots\}$.

2. Whole Numbers

Counting numbers and zero : $W = \{0, 1, 2, 3, \dots\}$.

3. Integers

Whole number and their opposites and zero :

$$\begin{aligned} Z^+ &= N = \{1, 2, 3, \dots\}, \\ Z^- &= \{\dots, -3, -2, -1\}, \\ Z &= Z^- \cup \{0\} \cup Z^+ = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}. \end{aligned}$$

4. Rational numbers

Repeating or terminating decimals:

$$Q = \left\{ x \mid x = \frac{a}{b} \text{ and } a \in Z \text{ and } b \in Z \text{ and } b \neq 0 \right\}.$$

5. Irrational Numbers

Nonrepeating and nonterminating decimals.

6. Real Numbers

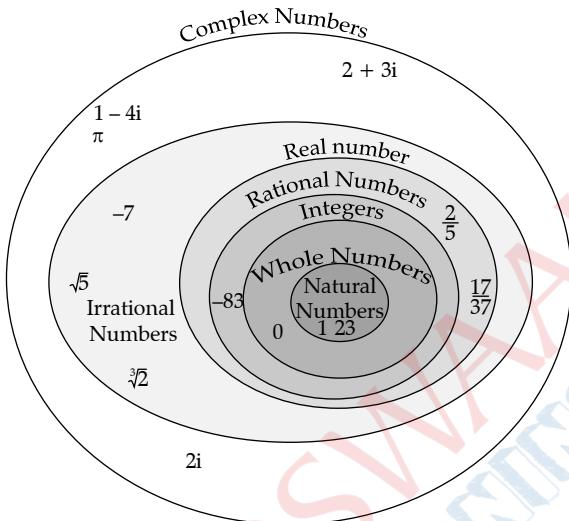
Union of rational and irrational numbers : R .

7. Complex Numbers

$$C = \{x + iy \mid x \in R \text{ and } y \in R\},$$

where i is the imaginary unit.

$$8. N \subset W \subset Z \subset Q \subset R \subset C$$



1.3 Basic Identities

Real numbers: a, b, c

1. **Additive Identity** : $a + 0 = a$
2. **Additive Inverse** : $a + (-a) = 0$
3. **Commutative of Addition** : $a + b = b + a$
4. **Associative of Addition** : $(a + b) + c = a + (b + c)$
5. **Definition of Subtraction** : $a - b = a + (-b)$
6. **Multiplicative Identity** : $a \cdot 1 = a$
7. **Multiplicative Inverse** : $a \cdot \frac{1}{a} = 1, a \neq 0$
8. **Multiplication times 0** : $a \cdot 0 = 0$
9. **Commutative of Multiplication** : $a \cdot b = b \cdot a$
10. **Associative of Multiplication** : $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
11. **Distributive Law** : $a(b + c) = ab + ac$
12. **Definition of Division** : $\frac{a}{b} = a \cdot \frac{1}{b}$

1.4 Complex Numbers

Natural number : n , Imaginary unit : i , Complex number : z , Real part : a, c , Imaginary part : bi, di , Modulus of a complex number : r, r_1, r_2 , Argument of a complex number: $\varphi, \varphi_1, \varphi_2$

1.	$i^1 = i$	$i^5 = i$	$i^{4n+1} = i$
	$i^2 = -1$	$i^6 = -1$	$i^{4n+2} = -1$
	$i^3 = -i$	$i^7 = -i$	$i^{4n+3} = -i$
	$i^4 = 1$	$i^8 = 1$	$i^{4n} = 1$

2. $z = a + bi$
3. **Complex Plane**
4. $(a + bi) + (c + di) = (a + c) + (b + d)i$
5. $(a + bi) - (c + di) = (a - c) + (b - d)i$
6. $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$
7. $\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} \cdot i$
8. **Conjugate Complex Numbers** : $\overline{a + bi} = a - bi$
9. $a = r \cos \varphi, b = r \sin \varphi$
10. **Polar Presentation of Complex Numbers**
 $a + bi = r(\cos \varphi + i \sin \varphi)$
11. **Modulus and Argument of a Complex Number**
If $a + bi$ is a complex number, then
 $r = \sqrt{a^2 + b^2}$ (modulus), and $\varphi = \arctan \frac{b}{a}$ (argument).
12. **Product in Polar Representation**
 $z_1 \cdot z_2 = r_1 (\cos \varphi_1 + i \sin \varphi_1) \cdot r_2 (\cos \varphi_2 + i \sin \varphi_2)$
 $= r_1 r_2 [\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)]$
13. **Conjugate Numbers in Polar Representation**
 $\overline{r(\cos \varphi + i \sin \varphi)} = r[\cos(-\varphi) + i \sin(-\varphi)]$
14. **Inverse of a Complex Number in Polar Representation**
 $\frac{1}{r(\cos \varphi + i \sin \varphi)} = \frac{1}{r} [\cos(-\varphi) + i \sin(-\varphi)]$
15. **Quotient in Polar Representation**
$$\frac{z_1}{z_2} = \frac{r_1 (\cos \varphi_1 + i \sin \varphi_1)}{r_2 (\cos \varphi_2 + i \sin \varphi_2)}$$

 $= \frac{r_1}{r_2} [\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2)]$
16. **Power of a Complex Number**
$$z^n = [r(\cos \varphi + i \sin \varphi)]^n = r^n [\cos(n\varphi) + i \sin(n\varphi)]$$
17. **Formula "De Moivre"**
$$(\cos \varphi + i \sin \varphi)^n = \cos(n\varphi) + i \sin(n\varphi)$$
18. **n^{th} Root of a Complex Number**
$$\sqrt[n]{z} = \sqrt[n]{r(\cos \varphi + i \sin \varphi)}$$

 $= \sqrt[n]{r} \left(\cos \frac{\varphi + 2\pi k}{n} + i \sin \frac{\varphi + 2\pi k}{n} \right),$
where $k = 0, 1, 2, \dots, n - 1$.
19. **Euler's Formula** $e^{ix} = \cos x + i \sin x$

1.5 Basic Algebra

Real numbers: a, b, c , Natural number: n

$$1. \quad a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$2. \quad a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$3. \quad a^5 - b^5 = (a - b)(a^2 + a^3b + a^2b^2 + ab^3 + b^4)$$

$$4. \quad a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$$

5. If n is odd, then

$$a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \dots - ab^{n-2} + b^{n-1}).$$

6. If n is even, then

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + ab^{n-2} + b^{n-1}),$$

$$a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \dots - ab^{n-2} - b^{n-1}).$$

7. Binomial Formula

$$(a + b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \dots + {}^nC_{n-1} a b^{n-1} + {}^nC_n b^n$$

where ${}^nC_k = \frac{n!}{k!(n-k)!}$ are the binomial coefficients.

$$8. \quad (a + b + c + \dots + u + v)^2 = a^2 + b^2 + c^2 + \dots + u^2 + v^2 + 2(ab + ac + \dots + au + av + bc + \dots + bu + bv + \dots + uv)$$

1.6 Properties of Powers and Roots

Base (positive real numbers): a, b , Powers (rational numbers): n, m

Powers

$$1. \quad a^m a^n = a^{m+n}$$

$$6. \quad a^0 = 1, a \neq 0$$

$$2. \quad \frac{a^m}{a^n} = a^{m-n}$$

$$7. \quad a^1 = a$$

$$3. \quad (ab)^m = a^m b^m$$

$$8. \quad a^{-m} = \frac{1}{a^m}$$

$$4. \quad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

$$9. \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

$$5. \quad (a^m)^n = a^{mn}$$

Roots

Bases: a, b , Powers (rational numbers): $n, m, a, b \geq 0$ for even roots ($n = 2k, k \in \mathbb{N}$)

$$1. \quad \sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b}$$

$$6. \quad \left(\sqrt[n]{a}\right)^n = a$$

$$2. \quad \sqrt[n]{a^m \sqrt[n]{b}} = \sqrt[n]{a^m b^n}$$

$$7. \quad \sqrt[n]{a^m} = \sqrt[n]{a^m}$$

$$3. \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}, b \neq 0$$

$$8. \quad \sqrt[n]{a^m} = a^{\frac{m}{n}}$$

$$4. \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a^m}}{\sqrt[n]{b^n}} = \sqrt[n]{\frac{a^m}{b^n}}, b \neq 0$$

$$9. \quad \sqrt[n]{\sqrt[n]{a}} = \sqrt[n]{a}$$

$$5. \quad \left(\sqrt[n]{a^m}\right)^p = \sqrt[n]{a^{mp}}$$

$$10. \quad \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

$$11. \quad \frac{1}{\sqrt[n]{a}} = \frac{\sqrt[n]{a^{n-1}}}{a}, a \neq 0.$$

$$12. \quad \sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}$$

$$13. \quad \frac{1}{\sqrt{a \pm \sqrt{b}}} = \frac{\sqrt{a} \mp \sqrt{b}}{a - b}$$

1.7 Concept of Logarithms

Positive real numbers: x, y, a, c, k , Natural number: n

1. Definition of Logarithm

$y = \log_a x$ if and only if $x = a^y, a > 0, a \neq 1$.

2. $\log_a 1 = 0$

3. $\log_a a = 1$

$$4. \quad \log_a 0 = \begin{cases} -\infty & \text{if } a > 1 \\ +\infty & \text{if } a < 1 \end{cases}$$

$$5. \quad \log_a(xy) = \log_a x + \log_a y$$

$$6. \quad \log_a \frac{x}{y} = \log_a x - \log_a y$$

$$7. \quad \log_a(x^n) = n \log_a x$$

$$8. \quad \log_a \sqrt[n]{x} = \frac{1}{n} \log_a x$$

$$9. \quad \log_a x = \frac{\log_c x}{\log_c a} = \log_c x \cdot \log_a c, c > 0, c \neq 1, a \neq 1$$

$$10. \quad \log_a c = \frac{1}{\log_c a}$$

$$11. \quad x = a^{\log_a x}$$

12. Logarithm to Base 10 $\log_{10} x = \log x$

13. Natural Logarithm

$$\log_e x = \ln x, \text{ where } e = \lim_{k \rightarrow \infty} \left(1 + \frac{1}{k}\right)^k = 2.718281828\dots$$

$$14. \quad \log x = \frac{1}{\ln 10} \ln x = 0.434294 \ln x$$

$$15. \quad \ln x = \frac{1}{\log e} \log x = 2.302585 \log x$$

1.8 Concept of Equations

Real numbers: a, b, c, p, q, u, v , Solutions: x_1, x_2, y_1, y_2, y_3

1. Linear Equation in One Variable

$$ax + b = 0, x = -\frac{b}{a}.$$

2. Quadratic Equation

$$ax^2 + bx + c = 0, x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

3. Discriminant $D = b^2 - 4ac$

4. Viete's Formulas

If $x^2 + px + q = 0$, then

$$\begin{cases} x_1 + x_2 = -p \\ x_1 x_2 = q \end{cases}$$

$$5. \quad ax^2 + bx = 0, x_1 = 0, x_2 = -\frac{b}{a}.$$

6. $ax^2 + c = 0, x_{1,2} = \pm \sqrt{-\frac{c}{a}}$.

7. **Cubic Equation. Cardano's Formula.**

$$y^3 + py + q = 0,$$

$$y_1 = u + v, y_{2,3} = -\frac{1}{2}(u+v) \pm \frac{\sqrt{3}}{2}(u+v)i,$$

where

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^2}}, v = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^2}}.$$

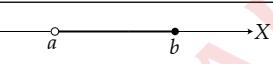
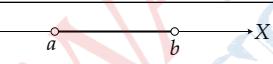
1.9 Inequalities

Variables: x, y, z

Real numbers : $\begin{cases} a, b, c, d \\ a_1, a_2, a_3, \dots, a_n \end{cases}, m, n$

Determinants : D, D_x, D_y, D_z

1. Inequalities, Interval Notations and Graphs

Inequality	Interval Notation	Graph
$a \leq x \leq b$	$[a, b]$	
$a < x \leq b$	$(a, b]$	
$a \leq x < b$	$[a, b)$	
$a < x < b$	(a, b)	
$-\infty < x \leq b, x \leq b$	$(-\infty, b]$	
$-\infty < x < b, x < b$	$(-\infty, b)$	
$a \leq x < \infty, x \geq a$	$[a, \infty)$	
$a < x < \infty, x > a$	(a, ∞)	

2. If $a > b$ and $m > 0$, then $ma > mb$.

3. If $a > b$ and $m > 0$, then $\frac{a}{m} > \frac{b}{m}$.

4. If $a > b$ and $m < 0$, then $ma < mb$.

5. If $a > b$ and $m < 0$, then $\frac{a}{m} < \frac{b}{m}$.

6. If $0 < a < b$ and $n > 0$, then $a^n < b^n$.

7. If $0 < a < b$ and $n < 0$, then $a^n > b^n$.

8. If $0 < a < b$, then $\sqrt[n]{a} < \sqrt[n]{b}$.

9. $\sqrt{ab} \leq \frac{a+b}{2}$, where $a > 0, b > 0$; an equality is valid only if $a = b$.

10. $a + \frac{1}{a} \geq 2$, where $a > 0$; an equality takes place only at $a = 1$.

11. $\sqrt[n]{a_1 a_2 \dots a_n} \leq \frac{a_1 + a_2 + \dots + a_n}{n}$, where $a_1, a_2, \dots, a_n > 0$.

12. If $ax + b > 0$ and $a > 0$, then $x > -\frac{b}{a}$.

13. If $ax + b > 0$ and $a < 0$, then $x < -\frac{b}{a}$.

14. If $x^2 < a$, then $|x| < \sqrt{a}$, where $a > 0$.

15. If $x^2 > a$, then $|x| > \sqrt{a}$, where $a > 0$.

16. If $\frac{f(x)}{g(x)} > 0$, then $\begin{cases} f(x).g(x) > 0 \\ g(x) \neq 0 \end{cases}$

17. If $\frac{f(x)}{g(x)} < 0$, then $\begin{cases} f(x).g(x) < 0 \\ g(x) \neq 0 \end{cases}$

SERIES – 2

2.1 Arithmetic Series

Initial term: a_1 , n^{th} term : a_n , Difference between successive terms : d , Number of terms in the series : n , Sum of the first n terms : S_n

$$1. a_n = a_{n-1} + d = a_{n-2} + 2d = \dots = a_1 + (n-1)d$$

$$2. a_1 + a_n = a_2 + a_{n-1} = \dots = a_i + a_{n+1-i}$$

$$3. a_i = \frac{a_{i-1} + a_{i+1}}{2}$$

$$4. S_n = \frac{a_1 + a_n}{2}.n = \frac{(2a_1 + (n-1)d)}{2}.n$$

2.2 Geometric Series

Initial term: a_1 , n^{th} term : a_n , Common ratio: r , Number of terms in the series: n , Sum of the first n terms: S_n , Sum to infinity: S

$$1. a_n = r a_{n-1} = a_1 r^{n-1}$$

$$2. a_1 \cdot a_n = a_2 \cdot a_{n-1} = \dots = a_i \cdot a_{n+1-i}$$

$$3. a_i = \sqrt{a_{i-1} \cdot a_{i+1}}$$

$$4. S_n = \frac{a_n r - a_1}{r - 1} = \frac{a_1 (r^n - 1)}{r - 1}$$

$$5. S = \lim_{n \rightarrow \infty} S_n = \frac{a_1}{1-r}$$

For $|r| < 1$, the sum S converges as $n \rightarrow \infty$.

2.3 Some Finite Series

Number of terms in the series : n

$$1. 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$2. 2 + 4 + 6 + \dots + 2n = n(n+1)$$

3. $1 + 3 + 5 + \dots + (2n-1) = n^2$
4. $k + (k+1) + (k+2) + \dots + (k+n-1) = \frac{n(2k+n-1)}{2}$
5. $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$
6. $1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$
7. $1^2 + 2^2 + 3^2 + \dots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$
8. $1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = n^2(2n^2-1)$
9. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots = 2$
10. $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} + \dots = \frac{n}{(n+1)}$
11. $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{(n-1)!} + \dots = e$

2.4 Infinite Series

Sequence : $\{a_n\}$, First term : a_1 , N^{th} term : a_N

1. Infinite Series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots + a_n + \dots$$

2. n^{th} Partial Sum

$$S_n = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots + a_n$$

3. Convergence of Infinite Series

$$\sum_{n=1}^{\infty} a_n = L, \text{ if } \lim_{n \rightarrow \infty} S_n = L$$

4. n^{th} Term Test

- If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$.
- If $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series is divergent.

2.5 Properties of Convergent Series

Convergent Series : $\sum_{n=1}^{\infty} a_n = A, \sum_{n=1}^{\infty} b_n = B$, Real number : c

1. $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = A + B$
2. $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n = cA$.

2.6 Convergence Tests

1. The Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series such that $0 < a_n \leq b_n$ for all n .

- If $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is also convergent.
- If $\sum_{n=1}^{\infty} a_n$ is divergent then $\sum_{n=1}^{\infty} b_n$ is also divergent.

2. The Limit Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series such that a_n and b_n are positive for all n .

- If $0 < \lim_{n \rightarrow \infty} \frac{a_n}{b_n} < \infty$ then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are either both convergent or both divergent.
- If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ then $\sum_{n=1}^{\infty} b_n$ converges implies that $\sum_{n=1}^{\infty} a_n$ is also convergent.
- If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ then $\sum_{n=1}^{\infty} b_n$ diverges implies that $\sum_{n=1}^{\infty} a_n$ is also divergent.

3. p -series

p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $0 < p \leq 1$.

4. The Integral Test

Let $f(x)$ be a function which is continuous, positive, and decreasing for all $x \geq 1$. The series

$$\sum_{n=1}^{\infty} f(n) = f(1) + f(2) + f(3) + \dots + f(n) + \dots$$

Converges if $\int_1^{\infty} f(x) dx$ converges, and diverges if

$$\int_1^{\infty} f(x) dx \rightarrow \infty \text{ as } n \rightarrow \infty.$$

5. The Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms.

- If $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$ then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.
- $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$ then $\sum_{n=1}^{\infty} a_n$ may converge or diverge and the ratio test is inconclusive; some other test must be used.

6. The Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms.

- If $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} < 1$ then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

- If $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = 1$ then $\sum_{n=1}^{\infty} a_n$ may converge or diverge, but no conclusion can be drawn from this test.

2.7 Power Series

Real numbers: x, x_0 , Power series : $\sum_{n=0}^{\infty} a_n x^n, \sum_{n=0}^{\infty} a_n (x - x_0)^n$, Whole number : n , Radius of Convergence : R .

1. Power Series in x

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

2. Power Series in $(x - x_0)$

$$\begin{aligned} \sum_{n=0}^{\infty} a_n (x - x_0)^n &= a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \dots \\ &\quad + a_n (x - x_0)^n + \dots \end{aligned}$$

3. Interval of Convergence

The set of those value of x for which the function $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ is convergent is called the interval of convergence.

4. Radius of Convergence

If the interval of convergence is $(x_0 - R, x_0 + R)$ for some $R \geq 0$, the R is called the radius of convergence. It is given as

$$R = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{a_n}} \text{ or } R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

2.8 Power Series Expansions for Some Functions

Whole number: n , Real number : x

$$1. e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

$$2. a^x = 1 + \frac{x \ln a}{1!} + \frac{(x \ln a)^2}{2!} + \frac{(x \ln a)^3}{3!} + \dots + \frac{(x \ln a)^n}{n!} + \dots$$

$$3. \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^n x^{n+1}}{n+1} \pm \dots, -1 < x \leq 1.$$

$$4. \ln \frac{1+x}{1-x} = 2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots \right), |x| < 1.$$

$$5. \ln x = 2 \left[\frac{x-1}{x+1} + \frac{1}{3} \left(\frac{x-1}{x+1} \right)^3 + \frac{1}{5} \left(\frac{x-1}{x+1} \right)^5 \dots \right], x > 0.$$

$$6. \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!} \pm \dots$$

$$7. \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} \pm \dots$$

$$8. \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \dots, |x| < \frac{\pi}{2}.$$

$$9. \cot x = \frac{1}{x} - \left(\frac{x}{3} + \frac{x^3}{45} + \frac{2x^5}{945} + \frac{2x^7}{4725} + \dots \right), 0 < |x| < \pi.$$

$$10. \sin^{-1} x = x + \frac{x^3}{2 \cdot 3} + \frac{1.3x^5}{2 \cdot 4 \cdot 5} + \dots + \frac{1.3.5\dots(2n-1)x^{2n+1}}{2.4.6\dots(2n)(2n+1)} + \dots, |x| < 1.$$

$$11. \cos^{-1} x = \frac{\pi}{2} - \left(x + \frac{x^3}{2 \cdot 3} + \frac{1.3x^5}{2 \cdot 4 \cdot 5} + \dots + \frac{1.3.5\dots(2n-1)x^{2n+1}}{2.4.6\dots(2n)(2n+1)} + \dots \right), |x| < 1$$

$$12. \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + \frac{(-1)^n x^{2n+1}}{2n+1}, |x| < 1.$$

$$13. \cos hx = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots + \frac{x^{2n}}{(2n)!} + \dots$$

$$14. \sin hx = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + \dots$$

2.9 Binomial Series

Whole numbers: n, m , Real number : x Combinations : ${}^n C_m$

$$1. (1+x)^n = 1 + {}^n C_1 x + {}^n C_2 x^2 + \dots + {}^n C_m x^m + \dots + x^n$$

$$2. {}^n C_m = \frac{n(n-1)\dots[n-(m-1)]}{m!}, |x| < 1.$$

$$3. \frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots, |x| < 1.$$

$$4. \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots, |x| < 1.$$

$$5. \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{2 \cdot 4} + \frac{1 \cdot 3x^3}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 3 \cdot 5x^4}{2 \cdot 4 \cdot 6 \cdot 8} + \dots, |x| \leq 1.$$

$$6. \sqrt[3]{1+x} = 1 + \frac{x}{3} - \frac{1.2x^2}{3 \cdot 6} + \frac{1.2 \cdot 5x^3}{3 \cdot 6 \cdot 9} - \frac{1.2 \cdot 5 \cdot 8x^4}{3 \cdot 6 \cdot 9 \cdot 12} + \dots, |x| \leq 1.$$

TRIGONOMETRY – 3

3.1 Periodicity of Trigonometric Functions

$$1. \sin(\alpha \pm 2\pi n) = \sin \alpha, \text{ period } 2\pi \text{ or } 360^\circ.$$

$$2. \cos(\alpha \pm 2\pi n) = \cos \alpha, \text{ period } 2\pi \text{ or } 360^\circ.$$

$$3. \tan(\alpha \pm \pi n) = \tan \alpha, \text{ period } \pi \text{ or } 180^\circ.$$

$$4. \cot(\alpha \pm \pi n) = \cot \alpha, \text{ period } \pi \text{ or } 180^\circ.$$

3.2 Relations between Trigonometric Functions

$$1. \sin \alpha = \pm \sqrt{1 - \cos^2 \alpha} = \pm \sqrt{\frac{1}{2}(1 - \cos 2\alpha)}$$

$$= 2 \cos^2 \left(\frac{\alpha}{2} - \frac{\pi}{4} \right) - 1 = \frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}$$

$$2. \cos \alpha = \pm \sqrt{1 - \sin^2 \alpha} = \pm \sqrt{\frac{1}{2}(1 + \cos 2\alpha)}$$

$$= 2\cos^2 \frac{\alpha}{2} - 1 = \frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}$$

$$3. \tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \pm \sqrt{\sec^2 \alpha - 1} = \frac{\sin 2\alpha}{1 + \cos 2\alpha} = \frac{1 - \cos 2\alpha}{\sin 2\alpha}$$

$$= \pm \sqrt{\frac{1 - \cos 2\alpha}{1 + \cos 2\alpha}} = \frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}$$

$$4. \cot \alpha = \frac{\cos \alpha}{\sin \alpha} = \pm \sqrt{\csc^2 \alpha - 1} = \frac{1 + \cos 2\alpha}{\sin 2\alpha} = \frac{\sin 2\alpha}{1 - \cos 2\alpha}$$

$$= \pm \sqrt{\frac{1 + \cos 2\alpha}{1 - \cos 2\alpha}} = \frac{1 + \tan^2 \frac{\alpha}{2}}{2 \tan \frac{\alpha}{2}}$$

$$5. \sec \alpha = \frac{1}{\cos \alpha} = \pm \sqrt{1 + \tan^2 \alpha} = \frac{1 + \tan^2 \frac{\alpha}{2}}{1 - \tan^2 \frac{\alpha}{2}}$$

$$6. \operatorname{cosec} \alpha = \frac{1}{\sin \alpha} = \pm \sqrt{1 + \cot^2 \alpha} = \frac{1 + \tan^2 \frac{\alpha}{2}}{2 \tan^2 \frac{\alpha}{2}}$$

3.3 Addition and Substitution Formulas

$$1. \sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \sin \beta \cos \alpha$$

$$2. \cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$3. \tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}$$

$$4. \cot(\alpha \pm \beta) = \frac{1 \mp \tan \alpha \tan \beta}{\tan \alpha \pm \tan \beta} = \frac{\cot \alpha \cot \beta \mp 1}{\cot \beta \pm \cot \alpha}$$

3.4 Multiple Angle Formulas

$$1. \sin 2\alpha = 2 \sin \alpha \cdot \cos \alpha$$

$$2. \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 1 - 2 \sin^2 \alpha = 2 \cos^2 \alpha - 1$$

$$3. \tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \frac{2}{\cot \alpha - \tan \alpha}$$

$$4. \cot 2\alpha = \frac{\cot^2 \alpha - 1}{2 \cot \alpha} = \frac{\cot \alpha - \tan \alpha}{2}$$

$$5. \sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha = 3 \cos^2 \alpha \cdot \sin \alpha - \sin^3 \alpha$$

$$6. \cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha = \cos^3 \alpha - 3 \cos \alpha \cdot \sin^2 \alpha$$

$$7. \tan 3\alpha = \frac{3 \tan \alpha - \tan^3 \alpha}{1 - 3 \tan^2 \alpha}$$

$$8. \cot 3\alpha = \frac{\cot^3 \alpha - 3 \cot \alpha}{3 \cot^2 \alpha - 1}$$

$$9. \sin 4\alpha = 4 \sin \alpha \cos \alpha (1 - 2 \sin^2 \alpha)$$

$$10. \cos 4\alpha = 8 \cos^4 \alpha - 8 \cos^2 \alpha + 1$$

$$11. \tan 4\alpha = \frac{4 \tan \alpha - 4 \tan^3 \alpha}{1 - 6 \tan^2 \alpha + \tan^4 \alpha}$$

$$12. \cot 4\alpha = \frac{1 - 6 \tan^2 \alpha + \tan^4 \alpha}{4 \tan \alpha - 4 \tan^3 \alpha}$$

$$13. \sin 5\alpha = 5 \sin \alpha - 20 \sin^3 \alpha + 16 \sin^5 \alpha$$

$$14. \cos 5\alpha = 16 \cos^5 \alpha - 20 \cos^3 \alpha + 5 \cos \alpha$$

$$15. \tan 5\alpha = \frac{\tan^5 \alpha - 10 \tan^3 \alpha + 5 \tan \alpha}{1 - 10 \tan^2 \alpha + 5 \tan^4 \alpha}$$

$$16. \cot 5\alpha = \frac{1 - 10 \tan^2 \alpha + 5 \tan^4 \alpha}{\tan^5 \alpha - 10 \tan^3 \alpha + 5 \tan \alpha}$$

3.5 Half Angle Formulas and Identifiers.

$$1. \sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

$$2. \cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

$$3. \tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{\sin \alpha}{1 + \cos \alpha} = \frac{1 - \cos \alpha}{\sin \alpha}$$

$$= \operatorname{cosec} \alpha - \cot \alpha$$

$$4. \cot \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}} = \frac{\sin \alpha}{1 - \cos \alpha} = \frac{1 + \cos \alpha}{\sin \alpha}$$

$$= \operatorname{cosec} \alpha + \cot \alpha$$

$$5. \sin \alpha = \frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}$$

$$6. \cos \alpha = \frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}$$

$$7. \tan \alpha = \frac{2 \tan \frac{\alpha}{2}}{1 - \tan^2 \frac{\alpha}{2}}$$

$$8. \cot \alpha = \frac{1 - \tan^2 \frac{\alpha}{2}}{2 \tan \frac{\alpha}{2}} = \frac{\cot^2 \frac{\alpha}{2} - 1}{2 \cot \frac{\alpha}{2}}$$

3.6 Transforming of Trigonometric Expressions to Product

$$1. \sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$2. \sin \alpha - \sin \beta = 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

$$3. \cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$4. \cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

$$5. \tan \alpha \pm \tan \beta = \frac{\sin(\alpha \pm \beta)}{\cos \alpha \cdot \cos \beta}$$

$$6. \cot \alpha \pm \cot \beta = \frac{\sin(\beta \pm \alpha)}{\sin \alpha \cdot \sin \beta}$$

$$7. \cos \alpha + \sin \alpha = \sqrt{2} \cos\left(\frac{\pi}{4} - \alpha\right) = \sqrt{2} \sin\left(\frac{\pi}{4} + \alpha\right)$$

$$8. \cos \alpha - \sin \alpha = \sqrt{2} \sin\left(\frac{\pi}{4} - \alpha\right) = \sqrt{2} \cos\left(\frac{\pi}{4} + \alpha\right)$$

$$9. \tan \alpha + \cot \beta = \frac{\cos(\alpha - \beta)}{\cos \alpha \cdot \sin \beta}$$

$$10. 1 + \cos \alpha = 2 \cos^2 \frac{\alpha}{2}$$

$$11. 1 - \cos \alpha = 2 \sin^2 \frac{\alpha}{2}$$

$$12. 1 + \sin \alpha = 2 \cos^2\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$$

$$13. 1 - \sin \alpha = 2 \sin^2\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$$

3.7 Transformation of Trigonometric Expression to Sum

$$1. \sin \alpha \cdot \sin \beta = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2}$$

$$2. \cos \alpha \cdot \cos \beta = \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{2}$$

$$3. \sin \alpha \cdot \cos \beta = \frac{\sin(\alpha - \beta) + \sin(\alpha + \beta)}{2}$$

$$4. \tan \alpha \cdot \tan \beta = \frac{\tan \alpha + \tan \beta}{\cot \alpha + \cot \beta}$$

$$5. \cot \alpha \cdot \cot \beta = \frac{\cot \alpha + \cot \beta}{\tan \alpha + \tan \beta}$$

$$6. \tan \alpha \cdot \cot \beta = \frac{\tan \alpha + \cot \beta}{\cot \alpha + \tan \beta}$$

3.8 Powers of Trigonometric Functions

$$1. \sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$$

$$2. \cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}$$

$$3. \sin^3 \alpha = \frac{3 \sin \alpha - \sin 3\alpha}{4}$$

$$4. \cos^3 \alpha = \frac{3 \cos \alpha + \cos 3\alpha}{4}$$

$$5. \sin^4 \alpha = \frac{\cos 4\alpha - 4 \cos 2\alpha + 3}{8}$$

$$6. \cos^4 \alpha = \frac{\cos 4\alpha + 4 \cos 2\alpha + 3}{8}$$

$$7. \sin^5 \alpha = \frac{10 \sin \alpha - 5 \sin 3\alpha + \sin 5\alpha}{16}$$

$$8. \cos^5 \alpha = \frac{10 \cos \alpha + 5 \cos 3\alpha + \cos 5\alpha}{16}$$

$$9. \sin^6 \alpha = \frac{10 - 15 \cos 2\alpha + 6 \cos 4\alpha - \cos 6\alpha}{32}$$

$$10. \cos^6 \alpha = \frac{10 + 15 \cos 2\alpha + 6 \cos 4\alpha + \cos 6\alpha}{32}$$

3.9 Relations between Inverse Trigonometric Functions

$$1. \sin^{-1}(-x) = -\sin^{-1} x$$

$$2. \sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x$$

$$3. \sin^{-1} x = \cos^{-1} \sqrt{1 - x^2}, 0 \leq x \leq 1.$$

$$4. \sin^{-1} x = -\cos^{-1} \sqrt{1 - x^2}, -1 \leq x \leq 0.$$

$$5. \sin^{-1} x = \tan^{-1} \frac{x}{\sqrt{1 - x^2}}, x^2 < 1.$$

$$6. \sin^{-1} x = \cot^{-1} \frac{\sqrt{1 - x^2}}{x}, 0 < x \leq 1.$$

$$7. \sin^{-1} x = \cot^{-1} \frac{\sqrt{1 - x^2}}{x} - \pi, -1 \leq x \leq 0.$$

$$8. \cos^{-1}(-x) = \pi - \cos^{-1} x$$

$$9. \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

$$10. \cos^{-1} x = \sin^{-1} \sqrt{1 - x^2}, 0 \leq x \leq 1.$$

$$11. \cos^{-1} x = \pi - \sin^{-1} \sqrt{1 - x^2}, -1 \leq x \leq 0.$$

$$12. \cos^{-1} x = \tan^{-1} \frac{\sqrt{1 - x^2}}{x}, 0 < x \leq 1.$$

$$13. \cos^{-1} x = \pi + \tan^{-1} \frac{\sqrt{1 - x^2}}{x}, -1 \leq x < 0$$

$$14. \cos^{-1} x = \cot^{-1} x \frac{x}{\sqrt{1 - x^2}}, -1 \leq x < 1.$$

$$15. \tan^{-1}(-x) = -\tan^{-1} x$$

$$16. \tan^{-1} x = \frac{\pi}{2} - \cot^{-1} x$$

$$17. \tan^{-1} x = \sin^{-1} \frac{x}{\sqrt{1 + x^2}}$$

$$18. \tan^{-1} x = \cos^{-1} \frac{1}{\sqrt{1 + x^2}}, x \geq 0.$$

$$19. \tan^{-1} x = \cos^{-1} \frac{1}{\sqrt{1 + x^2}}, x \leq 0.$$

20. $\tan^{-1} x = \frac{\pi}{2} - \tan^{-1} \frac{1}{x}, x > 0.$

21. $\tan^{-1} x = -\frac{\pi}{2} - \tan^{-1} \frac{1}{x}, x < 0.$

22. $\tan^{-1} x = \cot^{-1} \frac{1}{x}, x > 0.$

23. $\tan^{-1} x = \cot^{-1} \frac{1}{x} - \pi, x < 0.$

24. $\cot^{-1}(-x) = \pi - \cot^{-1} x$

25. $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$

26. $\cot^{-1} x = \sin^{-1} \frac{1}{\sqrt{1+x^2}}, x > 0.$

27. $\cot^{-1} x = \pi - \sin^{-1} \frac{1}{\sqrt{1+x^2}}, x < 0.$

28. $\cot^{-1} x = \cos^{-1} \frac{x}{\sqrt{1+x^2}}$

29. $\cot^{-1} x = \tan^{-1} \frac{1}{x}, x > 0.$

30. $\cot^{-1} x = \pi + \tan^{-1} \frac{1}{x}, x < 0.$

3.10 Trigonometric Equations

Whole number : n

1. $\sin x = a, x = (-1)^n \sin^{-1} a + \pi n$

2. $\cos x = a, x = \pm \cos^{-1} a + 2\pi n$

3. $\tan x = a, x = \tan^{-1} a + \pi n$

4. $\cot x = a, x = \cot^{-1} a + \pi n$

3.11 Relations to Hyperbolic Functions

Imaginary unit : i

1. $\sin(ix) = i \sinhx$ 4. $\sec(ix) = \operatorname{sech} x$

2. $\tan(ix) = i \tanh x$ 5. $\operatorname{cosec}(ix) = -i \operatorname{cosech} x$

3. $\cot(ix) = -i \coth x$

MATRICES AND DETERMINANTS – 4

Matrices: A, B, C Transpose of a matrix: A^T, A Inverse of a matrix: A^{-1}

Elements of a matrix : a_{ij}, b_{ij}, c_{ij} Adjoint of a matrix : $\operatorname{adj} A$ Real number: k

Determinants of a matrix : $\det A$ Trace of a matrix : $\operatorname{tr} A$ Natural numbers: m, n

Minor of an element a_{ij} : M_{ij}

Cofactor of an element a_{ij} : C_{ij}

4.1 Determinants

1. 3rd Order Determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

2. n^{th} Order Determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}$$

3. **Minor** : The minor M_{ij} associated with the element a_{ij} of n^{th} order matrix A is the $(n-1)^{\text{th}}$ order determinant derived from the matrix A by deletion of its i^{th} row and j^{th} column.

4. **Cofactor** $C_{ij} = (-1)^{i+j} M_{ij}$

5. **Laplace Expansion of n^{th} Order Determinant**
Laplace expansion by elements of the i^{th} row

$$\det A = \sum_{j=1}^n a_{ij} C_{ij}, i = 1, 2, \dots, n.$$

Laplace expansion by elements of the j^{th} column

$$\det A = \sum_{i=1}^n a_{ij} C_{ij}, j = 1, 2, \dots, n.$$

4.2 Properties of Determinants

1. The value of a determinant remains unchanged if rows are changed to column and columns to rows.

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

2. If two rows (or two columns) are interchanged, the sign of the determinant is changed.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}$$

3. If two rows (or two columns) are identical, the value of the determinant is zero.

$$\begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} = 0$$

4. If the elements of any row (or column) are multiplied by a common factor, the determinant is multiplied by that factor.

$$\begin{vmatrix} ka_1 & kb_1 \\ a_2 & b_2 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

5. If the elements of any row (or column) are increased (or decreased) by equal multiples of the corresponding elements of any other row (or column), the value of the determinant is unchanged.

$$\begin{vmatrix} a_1 + kb_1 & b_1 \\ a_2 + kb_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

4.3 Matrices

1. **Definition :** An $m \times n$ matrix A is a rectangular array of elements (numbers or functions) with m rows and n columns.

$$A = [a_{ij}] = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{vmatrix}$$

2. **Square matrix** is a matrix of order $n \times n$.

A square matrix $[a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$, i.e. it is symmetric about the leading diagonal.

A square matrix $[a_{ij}]$ is skew-symmetric if $a_{ij} = -a_{ji}$.

3. **Diagonal matrix** is a square matrix with all elements zero except those on the leading diagonal.
 4. **Unit matrix** is a diagonal matrix in which the elements on the leading diagonal are all unity. The unit matrix is denoted by I.
 5. **Null matrix** A null matrix is one whose elements are all zero.

4.4 Operations with Matrices

1. Two matrices A and B are equal if, and only if, they are both of the same shape $m \times n$ and corresponding elements are equal.
 2. Two matrices A and B can be added (or subtracted) if, and only if, they have the same shape $m \times n$. If

$$A = [a_{ij}] = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{vmatrix}, \text{ and}$$

$$B = [b_{ij}] = \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{vmatrix},$$

$$\text{then } A + B = \begin{vmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{vmatrix}.$$

3. If k is a scalar, and $A = [a_{ij}]$ is a matrix, then

$$kA = [ka_{ij}] = \begin{vmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{vmatrix}.$$

4. **Multiplication of Two Matrices** : Two matrices can be multiplied together only when the number of columns in the first is equal to the number of rows in the second.

$$\text{If } A = [a_{ij}] = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{vmatrix}, \text{ and}$$

$$B = [b_{ij}] = \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nk} \end{vmatrix},$$

$$\text{Then } AB = C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1k} \\ c_{21} & c_{22} & \dots & c_{2k} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mk} \end{bmatrix},$$

where, $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{\lambda=1}^n a_{i\lambda}b_{\lambda j}$
 $(i=1, 2, \dots, m; j=1, 2, \dots, k)$.

$$\text{Thus if } A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, B = [b_i] = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

$$\text{Then } AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \end{bmatrix}.$$

5. Transpose of a Matrix

If the rows and columns of a matrix are interchanged, then the new matrix is called the transpose of the original matrix. If A is the original matrix, its transpose is denoted A^T or A' .

- The matrix A is orthogonal if $AA^T = I$.
- If the matrix product AB is defined, then $(AB)^T = B^T A^T$.
- Adjoint of Matrix** : If A is a square $n \times n$ matrix, its adjoint, denoted by $\text{adj } A$, is the transpose of the matrix of cofactors C_{ij} of A : $\text{adj } A = [C_{ij}]^T$.
- Trace of Matrix** : If A is a square $n \times n$ matrix, its trace, denoted by $\text{tr } A$, is defined to be the sum of the terms on the leading diagonal : $\text{tr } A = a_{11} + a_{22} + \dots + a_{nn}$.
- Inverse of a Matrix** : If A is a square $n \times n$ matrix with a nonsingular determinant $\det A$, then its inverse A^{-1} is given by

$$A^{-1} = \frac{\text{adj } A}{\det A}.$$
- If the matrix product AB is defined, then

$$(AB)^{-1} = B^{-1} A^{-1}$$
.
- If A is square $n \times n$ matrix, the eigen vectors X satisfy the equation $AX = \lambda X$, While the eigen values λ satisfy the characteristic equation $|A - \lambda I| = 0$.

4.5 Systems of Linear Equations

Variables: x, y, z, x_1, x_2, \dots , Real numbers: $a_1, a_2, a_3, b_1, a_{11}, a_{12}, \dots$

Determinants: D, D_x, D_y, D_z , Matrices: A, B, X

$$1. \begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

$$2. x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D} \text{ (Cramer's rule),}$$

$$\text{where } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

3. If $D \neq 0$, then the system has a single solution:

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}.$$

If $D = 0$ and $D_x \neq 0$ (or $D_y \neq 0$ or $D_z \neq 0$), then the system has no solution.

If $D = D_x = D_y = D_z = 0$, then the system has infinitely many solutions.

4. Matrix Form of a System of n Linear Equations in n Unknowns. The set of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases}$$

Can be written in matrix form

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix},$$

i.e. $A.X = B$,

$$\text{Where } A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

5. Solution of a Set of Linear Equations $n \times n$
 $X = A^{-1}.B$, where A^{-1} is the inverse of A .

VECTORS – 5

Vectors : $\vec{u}, \vec{v}, \vec{w}, \vec{r}, \vec{AB}, \dots$	Coordinates of vector $\vec{u} : X_1, Y_1, Z_1$
Vector length : $ \vec{u} , \vec{v} , \dots$	
Unit vectors : $\vec{i}, \vec{j}, \vec{k}$	Coordinates of vector $\vec{v} : X_2, Y_2, Z_2$
Null vector : $\vec{0}$	Scalars : λ, μ

Direction cosines: $\cos\alpha, \cos\beta, \cos\gamma$.
Angle between two vectors : θ

5.1 Vector Coordinates

1. Unit Vectors

$$\vec{i} = (1, 0, 0), \quad \vec{k} = (0, 0, 1),$$

$$\vec{j} = (0, 1, 0), \quad |\vec{i}| = |\vec{j}| = |\vec{k}| = 1.$$

$$2. \vec{r} = \vec{AB} = (x_1 - x_0)\vec{i} + (y_1 - y_0)\vec{j} + (z_1 - z_0)\vec{k}$$

$$3. |\vec{r}| = |\vec{AB}| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$$

$$4. \text{ If } \vec{AB} = \vec{r}, \text{ then } \vec{BA} = -\vec{r}.$$

$$5. X = |\vec{r}| \cos\alpha, Y = |\vec{r}| \cos\beta, Z = |\vec{r}| \cos\gamma,$$

$$6. \text{ If } \vec{r} (X, Y, Z) = \vec{r}_1 (X_1, Y_1, Z_1) \text{ then } X = X_1, Y = Y_1, Z = Z_1$$

5.2 Scalar Product

1. Scalar Product of Vectors \vec{u} and \vec{v} $\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta$,

Where θ is the angle between vectors \vec{u} and \vec{v} .

2. Scalar product in Coordinate Form

If $\vec{u} = (X_1, Y_1, Z_1)$, $\vec{v} = (X_2, Y_2, Z_2)$, then

$$\vec{u} \cdot \vec{v} = X_1X_2 + Y_1Y_2 + Z_1Z_2.$$

3. Angle Between Two Vectors

If $\vec{u} = (X_1, Y_1, Z_1)$, $\vec{v} = (X_2, Y_2, Z_2)$, then

$$\cos\theta = \frac{X_1X_2 + Y_1Y_2 + Z_1Z_2}{\sqrt{X_1^2 + Y_1^2 + Z_1^2} \sqrt{X_2^2 + Y_2^2 + Z_2^2}}.$$

4. Commutative Property $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

5. Associative Property $(\lambda\vec{u}) \cdot (\mu\vec{v}) = \lambda\mu\vec{u} \cdot \vec{v}$

6. Distributive Property $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

7. $\vec{u} \cdot \vec{v} = 0$ if \vec{u}, \vec{v} are orthogonal ($\theta = \frac{\pi}{2}$).

8. $\vec{u} \cdot \vec{v} > 0$ if $0 < \theta < \frac{\pi}{2}$ and $\vec{u} \cdot \vec{v} < 0$ if $\frac{\pi}{2} < \theta < \pi$.

9. $|\vec{u} \cdot \vec{v}| \leq |\vec{u}| \cdot |\vec{v}|$ and $\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}|$ if \vec{u}, \vec{v} are parallel ($\theta = 0$).

10. If $\vec{u} = (X_1, Y_1, Z_1)$, then $|\vec{u}| \cdot |\vec{u}| = |\vec{u}|^2 = X_1^2 + Y_1^2 + Z_1^2$.

11. $\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$ and $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$

5.3 Vector Product

1. Vector Product of Vectors \vec{u} and \vec{v} $\vec{u} \times \vec{v} = \vec{w}$, where

$$\bullet \quad |\vec{w}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin\theta, \text{ where } 0 \leq \theta \leq \frac{\pi}{2};$$

$$\bullet \quad \vec{w} \perp \vec{u} \text{ and } \vec{w} \perp \vec{v};$$

$$\bullet \quad \text{Vectors } \vec{u}, \vec{v}, \vec{w} \text{ form a right-handed screw.}$$

$$2. \vec{w} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}$$

$$\vec{w} = \vec{u} \times \vec{v} = \left(\begin{vmatrix} Y_1 & Z_1 \\ Y_2 & Z_2 \end{vmatrix}, - \begin{vmatrix} X_1 & Z_1 \\ X_2 & Z_2 \end{vmatrix}, \begin{vmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{vmatrix} \right)$$

$$3. S = |\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin\theta$$

$$4. \text{ Angle Between Two Vectors } \sin\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|}$$

$$5. \text{ Non commutative Property } \vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$

$$6. \text{ Associative Property } (\lambda\vec{u}) \times (\mu\vec{v}) = \lambda\mu\vec{u} \times \vec{v}$$

$$7. \text{ Distributive Property } \vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$

$$8. \vec{u} \times \vec{v} = \vec{0} \text{ if } \vec{u} \text{ and } \vec{v} \text{ are parallel } (\theta = 0).$$

$$9. \vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0} \text{ and } \vec{i} \times \vec{j} = \vec{k}, \vec{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{i} = \vec{j}$$

5.4 Triple Product

1. Scalar Triple Product

$$[\vec{u} \vec{v} \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{w} \times \vec{u}) = \vec{w} \cdot (\vec{u} \times \vec{v})$$

2. $[\vec{uvw}] = [\vec{wuv}] = [\vec{vuw}] = -[\vec{vuw}] = -[\vec{wvu}] = -[\vec{uwv}]$

3. $\vec{k}\vec{u} \cdot (\vec{v} \times \vec{w}) = k[\vec{uvw}]$

4. **Scalar Triple Product Coordination Form**

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix},$$

Where $\vec{u} = (X_1, Y_1, Z_1)$, $\vec{v} = (X_2, Y_2, Z_2)$, $\vec{w} = (X_3, Y_3, Z_3)$.

5. **Volume of Parallelepiped** $V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$

6. **Volume of Pyramid** $V = \frac{1}{6} |\vec{u} \cdot (\vec{v} \times \vec{w})|$

7. If $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$, then the vectors \vec{u} , \vec{v} , and \vec{w} are **linearly dependent**, so $\vec{w} = \lambda\vec{u} + \mu\vec{v}$ for some scalars λ and μ .

8. If $\vec{u} \cdot (\vec{v} \times \vec{w}) \neq 0$, then the vectors \vec{u} , \vec{v} , and \vec{w} are **linearly independent**.

9. **Vector Triple Product** $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

COORDINATE SYSTEM – 6

6.1 Two – Dimensional Coordinate System

Point coordinates : Positive real number: a, b, c ,

$x_0, x_1, x_2, y_0, y_1, y_2$ Distance between two points : d

Polar coordinates: r, φ Area : A

Real number : λ

1. Distance Between Two Points

$$d = AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

2. Dividing a Line Segment in the Ratio $\lambda : 1$

$$x_0 = \frac{x_1 + \lambda x_2}{1 + \lambda}, \quad y_0 = \frac{y_1 + \lambda y_2}{1 + \lambda}.$$

3. Midpoint of a Line Segment

$$x_0 = \frac{x_1 + x_2}{2}, \quad y_0 = \frac{y_1 + y_2}{2}, \quad \lambda = 1.$$

4. Centroid (Intersection of Medians) of a Triangle

$$x_0 = \frac{x_1 + x_2 + x_3}{3}, \quad y_0 = \frac{y_1 + y_2 + y_3}{3},$$

Where $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ are vertices of the triangle ABC .

5. Incenter (Intersection of Angle Bisectors) of a Triangle

$$x_0 = \frac{ax_1 + bx_2 + cx_3}{a+b+c}, \quad y_0 = \frac{ay_1 + by_2 + cy_3}{a+b+c},$$

where $a = BC$, $b = CA$, $c = AB$ are the sides of $\triangle ABC$

6. Circumcentre (Intersection of the Side Perpendicular Bisectors) of a Triangle

$$x_0 = \begin{vmatrix} x_1^2 + y_1^2 & y_1 & 1 \\ x_2^2 + y_2^2 & y_2 & 1 \\ x_3^2 + y_3^2 & y_3 & 1 \end{vmatrix}, \quad y_0 = \begin{vmatrix} x_1 & x_1^2 + y_1^2 & 1 \\ x_2 & x_2^2 + y_2^2 & 1 \\ x_3 & x_3^2 + y_3^2 & 1 \end{vmatrix}$$

7. Orthocentre (Intersection of Altitudes) of a Triangle

$$x_0 = \begin{vmatrix} y_1 & x_2 x_3 + y_1^2 & 1 \\ y_2 & x_3 x_1 + y_2^2 & 1 \\ y_3 & x_1 x_2 + y_3^2 & 1 \end{vmatrix}, \quad y_0 = \begin{vmatrix} x_1^2 + y_2 y_3 & x_1 & 1 \\ x_2^2 + y_3 y_1 & x_2 & 1 \\ x_3^2 + y_1 y_2 & x_3 & 1 \end{vmatrix}$$

8. Area of a Triangle

$$A = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & 1 \\ x_3 - x_1 & y_3 - y_1 & 1 \end{vmatrix}$$

9. Area of a Quadrilateral

$$A = \frac{1}{2} \left[(x_1 - x_2)(y_1 + y_2) + (x_2 - x_3)(y_2 + y_3) + (x_3 - x_4)(y_3 + y_4) + (x_4 - x_1)(y_4 + y_1) \right]$$

10. Distance Between Two Points in Polar Coordinates

$$d = AB = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\varphi_2 - \varphi_1)}$$

11. Converting Rectangular Coordinates to polar Coordinates $x = r \cos \varphi$, $y = r \sin \varphi$.

12. Converting Polar Coordinates to Rectangular Coordinates $r = \sqrt{x^2 + y^2}$, $\tan \varphi = \frac{y}{x}$.

6.2 Straight Line in Plane

Point coordinates : Angle between two lines : φ

$X, Y, x, x_0, x_1, y_0, y_1, a_1, a_2, \dots$ Normal vector : \vec{n}

Real numbers: Position vectors : $\vec{r}, \vec{a}, \vec{b}$

$k, a, b, p, t, A, B, C, A_1, A_2, \dots$

Angles : α, β

1. General Equation of a Straight Line

$$Ax + By + C = 0$$

2. Normal vector to a Straight Line

The vector $\vec{n}(A, B)$ is normal to the line $Ax + By + C = 0$.

3. Explicit Equation of a Straight Line (Slope – Intercept Form) $y = kx + b$.

4. Gradient of a Line $k = \tan \alpha = \frac{y_2 - y_1}{x_2 - x_1}$

5. Equation of a Line Given a Point and the Gradient $y = y_0 + k(x - x_0)$,

Where k is the gradient, $P(x_0, y_0)$ is a point of the line.

6. Equation of a Line That Passes Through Two Points

$$(x_2 - x_1)(y - y_1) = (y_2 - y_1)(x - x_1) \text{ or } \begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

7. Intercept Form $\frac{x}{a} + \frac{y}{b} = 1$

8. Normal Form $x \cos \beta + y \sin \beta - p = 0$

9. Point Direction Form $\frac{x - x_1}{X} = \frac{y - y_1}{Y},$

where (X, Y) is the direction of the line and $P_1(x_1, y_1)$ lies on the line.

10. Vertical Line $x = a$

11. Horizontal Line $y = b$

12. Vector Equation of a Straight Line $\vec{r} = \vec{a} + t\vec{b},$ where

O is the origin of the coordinates, X is any variable point on the line, \vec{a} is the position vector of a known point A on the line, \vec{b} is a known vector of direction, parallel to the line, t is a parameter, $\vec{r} = \vec{OX}$ is the position vector of any point X on the line.

13. Straight Line in Parametric Form

$$x = a_1 + tb_1 \text{ and } y = a_2 + tb_2$$

where

(x, y) are the coordinates of any unknown point on the line,

(a_1, a_2) are the coordinates of a known point on the line, (b_1, b_2) are the coordinates of a vector parallel to the line, t is a parameter.

14. Distance Form a Point To a Line

The distance from the point $P(a, b)$ to the line

$$Ax + By + C = 0 \text{ is } d = \frac{|Aa + Bb + C|}{\sqrt{A^2 + B^2}}.$$

15. Parallel Lines

Two lines $y = k_1x + b_1$ and $y = k_2x + b_2$ are parallel,

If $k_1 = k_2.$

Two lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$

are parallel if $\frac{A_1}{A_2} = \frac{B_1}{B_2}.$

16. Perpendicular Lines

Two lines $y = k_1x + b_1$ and $y = k_2x + b_2$ are perpendicular if $k_2 = -\frac{1}{k_1}$ or, equivalently, $k_1k_2 = -1.$

Two lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are perpendicular if $A_1A_2 + B_1B_2 = 0.$

17. Angle Between Two Lines

$$\tan \varphi = \frac{k_2 - k_1}{1 + k_1k_2}, \text{ and } \cos \varphi = \frac{A_1A_2 + B_1B_2}{\sqrt{A_1^2 + B_1^2} \cdot \sqrt{A_2^2 + B_2^2}}.$$

18. Intersection of Two Lines

If two lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ intersect, the intersection point has coordinates

$$x_0 = \frac{-C_1B_2 + C_2B_1}{A_1B_2 - A_2B_1}, \quad y_0 = \frac{-A_1C_2 + A_2C_1}{A_1B_2 - A_2B_1}$$

6.3 Circle

Radius : R , Centre of circle: (a, b) , Point coordinates : x, y, x_1, y_1, \dots , Real numbers: $A, B, C, D, E, F, t.$

1. Equation of a circle Centred at the Origin (Standard Form) $x^2 + y^2 = R^2$

2. Equation of a Circle Centred at Any Point (a, b)

$$(x - a)^2 + (y - b)^2 = R^2$$

3. Three Point Form

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

4. Parametric Form

$$\begin{cases} x = R \cos t, & 0 \leq t \leq 2\pi. \\ y = R \sin t \end{cases}$$

5. General Form

$$Ax^2 + Ay^2 + Dx + Ey + F = 0 \text{ (A nonzero, } D^2 + E^2 > 4AF\text{).}$$

The centre of the circle has coordinates (a, b) , where

$$a = -\frac{D}{2A}, \quad b = -\frac{E}{2A}.$$

$$\text{The radius of the circle is } R = \sqrt{\frac{D^2 + E^2 - 4AF}{2|A|}}.$$

6.4 Ellipse

Semimajor axis : a , Semiminor axis: b , Foci : $F_1(-c, 0), F_2(c, 0)$, Distance between the foci : $2c$, Eccentricity : e , Real numbers : A, B, C, D, E, F, t , Perimeter : L , Area : $A.$

1. Equation of an Ellipse (Standard Form) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

2. $r_1 + r_2 = 2a,$

where r_1, r_2 are distances from any point $P(x, y)$ on the ellipse to the two foci.

$$a^2 = b^2 + c^2$$

$$\text{4. Eccentricity } e = \frac{c}{a} < 1$$

$$\text{5. Equations of Directrices } x = \pm \frac{a}{e} = \pm \frac{a^2}{c}$$

6. Parametric Form

$$\begin{cases} x = a \cos t, & 0 \leq t \leq 2\pi. \\ y = b \sin t \end{cases}$$

7. General Form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where $B^2 - 4AC < 0.$

8. General Form with Axes Parallel to the Coordinate Axes

$$Ax^2 + Cy^2 + Dx + Ey + F = 0, \text{ Where } AC > 0.$$

9. Circumference

$$L = 4aE(e),$$

where the function E is the complete elliptic integral of the second kind.

10. Approximate Formulas of the Circumference

$$L = \pi(1.5(a+b) - \sqrt{ab}),$$

$$L = \pi\sqrt{2(a^2 + b^2)}.$$

11. Area of Ellipse $A = \pi ab$

6.5 Hyperbola

Transverse axis : a , Conjugate axis : b , Foci : $F_1(-c, 0), F_2(c, 0)$, Distance between the foci : $2c$, Eccentricity : e ,

Asymptotes : s, t , Real numbers : A, B, C, D, E, F, t, k .

1. Equation of a Hyperbola (Standard Form)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

2. $|r_1 - r_2| = 2a$, where r_1, r_2 are distances from any point $P(x, y)$ on the hyperbola to the two foci.

3. Equations of Asymptotes $y = \pm \frac{b}{a}x$

$$4. c^2 = a^2 + b^2$$

$$5. \text{ Eccentricity } e = \frac{c}{a} > 1$$

$$6. \text{ Equations of Directrices } x = \pm \frac{a}{e} = \pm \frac{a^2}{c}$$

7. Parametric Equations of the Right Branch of a Hyperbola

$$\begin{cases} x = a \cosh t \\ y = b \sinh t \end{cases}, 0 \leq t \leq 2\pi.$$

8. General Form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \text{ where } B^2 - 4AC > 0$$

9. General Form with Axes Parallel to the Coordinate Axes

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

where $AC < 0$.

10. Asymptotic Form

$$xy = \frac{e^2}{4}, \text{ or } y = \frac{k}{x}, \text{ where } k = \frac{e^2}{4}.$$

In this case, the asymptotes have equations $x = 0$ and $y = 0$.

6.6 Parabola

Focal parameter : p , Focus : F , Vertex : $M(x_0, y_0)$,

Real numbers : $A, B, C, D, E, F, p, a, b, c$.

1. Equation of a Parabola (Standard Form)

$$y^2 = 2px$$

- Equation of the directrix $x = -\frac{p}{2}$,

- Coordinates of the focus $F\left(\frac{p}{2}, 0\right)$,

- Coordinates of the vertex $M(0,0)$.

2. General Form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Where $B^2 - 4AC = 0$.

$$3. y = ax^2, p = \frac{1}{2a}.$$

- Equation of the directrix $y = -\frac{p}{2}$

- Coordinates of the focus $F\left(0, \frac{p}{2}\right)$,

- Coordinates of the vertex $M(0,0)$.

4. General Form, Axis Parallel to the y -axis

$$Ax^2 + Dx + Ey + F = 0 (A, E \text{ nonzero}),$$

$$y = ax^2 + bx + c, p = \frac{1}{2a}.$$

- Equation of the directrix $y = y_0 - \frac{p}{2}$,

- Coordinates of the focus $F\left(x_0, y_0 + \frac{p}{2}\right)$

- Coordinates of the vertex

$$x_0 = -\frac{b}{2a}, \quad y_0 = ax_0^2 + bx_0 + c = \frac{4ac - b^2}{4a}.$$

6.7 Three – Dimensional Coordinate System

Point coordinates: $x_0, y_0, z_0, x_1, y_1, z_1, \dots$, Real number : λ , Distance between two points : d , Area : S , Volume: V

1. Distance Between Two Points

$$d = AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

2. Dividing a Line Segment in the Ratio $\lambda : 1$

$$x_0 = \frac{x_1 + \lambda x_2}{1 + \lambda}, \quad y_0 = \frac{y_1 + \lambda y_2}{1 + \lambda}, \quad z_0 = \frac{z_1 + \lambda z_2}{1 + \lambda},$$

3. Midpoint of a Line Segment

$$x_0 = \frac{x_1 + x_2}{2}, \quad y_0 = \frac{y_1 + y_2}{2}, \quad z_0 = \frac{z_1 + z_2}{2}, \quad \lambda = 1$$

4. Area of a Triangle

The area of a triangle with vertices $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$, is given by

$$A = \frac{1}{2} \sqrt{\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2}.$$

5. Volume of a Tetrahedron

The volume of a tetrahedron with vertices $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$, and $P_4(x_4, y_4, z_4)$ is given by

$$V = \pm \frac{1}{6} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}, \text{ or}$$

$$V = \pm \frac{1}{6} \begin{vmatrix} x_1 - x_4 & y_1 - y_4 & z_1 - z_4 \\ x_2 - x_4 & y_2 - y_4 & z_2 - z_4 \\ x_3 - x_4 & y_3 - y_4 & z_3 - z_4 \end{vmatrix}$$

Note : We choose the sign (+) or (-) so that to get a positive answer for volume.

6.8. Plane

Point coordinates: $x, y, z, x_0, y_0, z_0, x_1, y_1, z_1, \dots$

Real numbers: $A, B, C, D, A_1, A_2, a, b, c, a_1, a_2, \lambda, p, t, \dots$

Normal vectors : \vec{n} , \vec{n}_1 , \vec{n}_2 , Direction cosines: $\cos\alpha, \cos\beta, \cos\gamma$, Distance from point to plane: d

1. General Equation of a Plane

$$Ax + By + Cz + D = 0$$

2. Normal Vector to a Plane

The vector \vec{n} (A, B, C) is normal to the plane

$$Ax + By + Cz + D = 0.$$

3. Particular cases of the Equation of a Plane

$$Ax + By + Cz + D = 0$$

- If $A = 0$, the plane is parallel to the x -axis.
- If $B = 0$, the plane is parallel to the y -axis.
- If $C = 0$, the plane is parallel to the z -axis.
- If $D = 0$, the plane lies on the origin.
- If $A = B = 0$, the plane is parallel to the xy -axis.
- If $B = C = 0$, the plane is parallel to the yz -axis.
- If $A = C = 0$, the plane is parallel to the xz -axis.

4. Point Direction Form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

Where the point $P(x_0, y_0, z_0)$ lies in the plane, and the vector (A, B, C) is normal to the plane.

5. Intercept Form

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

6. Three Point Form

$$\begin{vmatrix} x - x_3 & y - y_3 & z - z_3 \\ x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \\ x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \end{vmatrix} = 0, \text{ or, } \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

7. Normal Form

$$x \cos\alpha + y \cos\beta + z \cos\gamma - p = 0,$$

where p is the perpendicular distance from the origin to the plane, and $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of any line normal to the plane.

8. Parametric Form

$$\begin{cases} x = x_1 + a_1 s + a_2 t \\ y = y_1 + b_1 s + b_2 t \\ z = z_1 + c_1 s + c_2 t \end{cases}$$

where (x, y, z) are the coordinates of any unknown point on the line, the point $P(x_1, y_1, z_1)$ lies in the plane, the vectors (a_1, b_1, c_1) and (a_2, b_2, c_2) are parallel to the plane.

9. Dihedral Angle Between Two Planes

If the planes are given by

$A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$, then the dihedral angle between them is

$$\cos\varphi = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \times \sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

10. Parallel Planes

Two planes $A_1x + B_1y + C_1z + D_1 = 0$ and

$A_2x + B_2y + C_2z + D_2 = 0$ are parallel if

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}.$$

11. Perpendicular Planes

Two planes $A_1x + B_1y + C_1z + D_1 = 0$ and

$A_2x + B_2y + C_2z + D_2 = 0$ are perpendicular if

$$A_1A_2 + B_1B_2 + C_1C_2 = 0.$$

12. Equation of a Plane Through $P(x_1, y_1, z_1)$ and Parallel to the Vectors (a_1, b_1, c_1) and (a_2, b_2, c_2)

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

13. Equation of a Plane Through $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, and Parallel to the Vector (a, b, c)

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a & b & c \end{vmatrix} = 0$$

14. Distance From a Point To a Plane

The distance from the point $P_1(x_1, y_1, z_1)$ to the plane $Ax + By + Cz + D = 0$ is

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

15. Intersection of Two Planes

If two planes $A_1x + B_1y + C_1z + D_1 = 0$ and

$A_2x + B_2y + C_2z + D_2 = 0$ intersect, the intersection straight line is given by

$$\begin{cases} x = x_1 + at \\ y = y_1 + bt, \text{ or, } \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}, \\ z = z_1 + ct \end{cases}$$

where

$$a = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, b = \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}, c = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}$$

$$x_1 = \frac{b \begin{vmatrix} D_1 & C_1 \\ D_2 & C_2 \end{vmatrix} - c \begin{vmatrix} D_1 & B_1 \\ D_2 & B_2 \end{vmatrix}}{a^2 + b^2 + c^2},$$

$$y_1 = \frac{c \begin{vmatrix} D_1 & A_1 \\ D_2 & A_2 \end{vmatrix} - a \begin{vmatrix} D_1 & C_1 \\ D_2 & C_2 \end{vmatrix}}{a^2 + b^2 + c^2},$$

$$z_1 = \frac{a \begin{vmatrix} D_1 & B_1 \\ D_2 & B_2 \end{vmatrix} - b \begin{vmatrix} D_1 & A_1 \\ D_2 & A_2 \end{vmatrix}}{a^2 + b^2 + c^2}.$$

6.9 Straight Line in Space

Point coordinates: $x, y, z, x_1, y_1, z_1, \dots$

Direction cosines: $\cos\alpha, \cos\beta, \cos\gamma$

Real numbers : $A, B, C, D, a, b, c, a_1, a_2, t, \dots$

Direction vectors of a line : $\vec{s}, \vec{s}_1, \vec{s}_2$

Normal vector to a plane : \vec{n}

Angle between two lines: φ

1. Point Direction Form of the Equation of a Line

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c},$$

where the point $P_1(x_1, y_1, z_1)$ lies on the line, and (a, b, c) is the direction vector of the line.

2. Two Point Form

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

3. Parametric Form

$$\begin{cases} x = x_1 + t \cot \alpha \\ y = y_1 + t \cos \beta, \\ z = z_1 + t \cos \gamma \end{cases}$$

where the point $P_1(x_1, y_1, z_1)$ lies on the straight line, $\cos \alpha, \cos \beta, \cos \gamma$ are the direction cosines of the direction vector of the line, the parameter t is any real number.

4. Angle Between Two Straight Lines

$$\cos \varphi = \frac{\vec{s}_1 \cdot \vec{s}_2}{|\vec{s}_1| \cdot |\vec{s}_2|} = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

5. Parallel Lines

Two lines are parallel if $\vec{s}_1 \parallel \vec{s}_2$, or, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

6. Perpendicular Lines

Two lines are perpendicular if $\vec{s}_1 \cdot \vec{s}_2 = 0$, or, $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$.

7. Intersection of Two Lines

Two lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and

$$\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2} \text{ intersect if}$$

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

8. Parallel Line and Plane

The straight line $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ and the plane

$Ax + By + Cz + D = 0$ are parallel if $\vec{n} \cdot \vec{s} = 0$, or, $Aa + Bb + Cc = 0$.

9. Perpendicular Line and Plane

The straight line $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ and the plane

$Ax + By + Cz + D = 0$ are perpendicular if $\vec{n} \parallel \vec{s}$,

or $\frac{A}{a} = \frac{B}{b} = \frac{C}{c}$.

DIFFERENTIAL CALCULUS – 7

7.1 Limits of Functions

Function : $f(x), g(x)$, Argument: x , Real constants: a, k

$$1. \lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$$

$$2. \lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

$$3. \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \text{ if } \lim_{x \rightarrow a} g(x) \neq 0.$$

$$4. \lim_{x \rightarrow a} [kf(x)] = k \lim_{x \rightarrow a} f(x)$$

$$5. \lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x))$$

6. $\lim_{x \rightarrow a} f(x) = f(a)$, if the function $f(x)$ is continuous at $x = a$.

$$7. \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad 14. \lim_{x \rightarrow 0} a^x = 1$$

$$8. \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \quad 15. \lim_{x \rightarrow 0} (1+x) = 1$$

$$9. \lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} = 1 \quad 16. \lim_{x \rightarrow 0} e^x = 1$$

$$10. \lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = 1 \quad 17. \lim_{x \rightarrow 0} \frac{e^x - 1}{x}$$

$$11. \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1 \quad 18. \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = l$$

$$12. \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e \quad 19. \lim_{x \rightarrow 0} \frac{x^x - a^n}{x - a} = na$$

$$13. \lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^x = e^k$$

7.2 Definition and Properties of the Derivative

Functions : f, g, y, u, v , Independent variable : x ,

Real constant : k , Angle: α

$$1. y'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

$$2. \frac{dy}{dx} = \tan \alpha$$

$$3. \frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$4. \frac{d(u-v)}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

$$5. \frac{d(ku)}{dx} = k \frac{du}{dx}$$

$$6. \text{Product Rule } \frac{d(u \cdot v)}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$$

$$7. \text{Quotient Rule } \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{\frac{du}{dx} \cdot v - u \cdot \frac{dv}{dx}}{v^2}$$

8. Chain Rule

$$y = f(g(x)), u = g(x),$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

9. Derivative of Inverse Function

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}},$$

Where $x(y)$ is the inverse function of $y(x)$.

10. Reciprocal Rule

$$\frac{d}{dx} \left(\frac{1}{y} \right) = -\frac{dy}{y^2}$$

11. Logarithmic Differentiation $y = f(x)$, $\ln y = \ln f(x)$,

$$\frac{dy}{dx} = f(x) \cdot \frac{d}{dx} [\ln f(x)].$$

7.3 Table of Derivatives

Independent variable : x , Real constants : C, a, b, c , Natural number : n

$$1. \quad \frac{d}{dx}(C) = 0$$

$$2. \quad \frac{d}{dx}(x) = 1$$

$$3. \quad \frac{d}{dx}(ax + b) = a$$

$$4. \quad \frac{d}{dx}(ax^2 + bx + c) = 2ax + b$$

$$5. \quad \frac{d}{dx}(x^n) = nx^{n-1}$$

$$6. \quad \frac{d}{dx}(x^{-n}) = -\frac{n}{x^{n+1}}$$

$$7. \quad \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$8. \quad \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$9. \quad \frac{d}{dx}(\sqrt[n]{x}) = \frac{1}{n\sqrt[n]{x^{n-1}}}$$

$$10. \quad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$11. \quad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}, \quad a > 0, a \neq 1.$$

$$12. \quad \frac{d}{dx}(a^x) = a^x \ln a, \quad a > 0, a \neq 1.$$

$$13. \quad \frac{d}{dx}(e^x) = e^x$$

$$14. \quad \frac{d}{dx}(\sin x) = \cos x$$

$$15. \quad \frac{d}{dx}(\cos x) = -\sin x$$

$$16. \quad \frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x$$

$$17. \quad \frac{d}{dx}(\cot x) = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

$$18. \quad \frac{d}{dx}(\sec x) = \tan x \cdot \sec x$$

$$19. \quad \frac{d}{dx}(\operatorname{cosec} x) = -\cot x \cdot \operatorname{cosec} x$$

$$20. \quad \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$21. \quad \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$22. \quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$23. \quad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$24. \quad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$25. \quad \frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

$$26. \quad \frac{d}{dx}(\sinh x) = \cosh x$$

$$27. \quad \frac{d}{dx}(\cosh x) = \sinh x$$

$$28. \quad \frac{d}{dx}(\tanh x) = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$29. \quad \frac{d}{dx}(\coth x) = -\frac{1}{\sinh^2 x} = -\operatorname{cosech}^2 x$$

$$30. \quad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \cdot \tanh x$$

$$31. \quad \frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \cdot \coth x$$

$$32. \quad \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$$

$$33. \quad \frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$$

$$34. \quad \frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, |x| < 1.$$

$$35. \quad \frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}, 0 < x < 1$$

$$36. \quad \frac{d}{dx}(\operatorname{cosech}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}, x \neq 0$$

$$37. \quad \frac{d}{dx}(\coth^{-1} x) = -\frac{1}{x^2-1}, |x| > 1.$$

$$38. \quad \frac{d}{dx}(u^v) = vu^{v-1} \cdot \frac{du}{dx} + u^v \ln u \cdot \frac{dv}{dx}$$

7.4 Higher Order Derivatives

Functions: f, y, u, v , Independent variable : x , Natural number: n

1. Second derivative

$$f'' = (f')' = \left(\frac{dy}{dx} \right)' = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2 y}{dx^2}$$

2. Higher – Order derivative

$$f^{(n)} = \frac{d^n y}{dx^n} = y^{(n)} = (f^{(n-1)})'$$

$$3. \quad (u \pm v)^{(n)} = u^{(n)} \pm v^{(n)}$$

4. Leibnitz's Formulas

$$(uv)'' = u''v + 2u'v' + uv''$$

$$(uv)''' = u'''v + 3u''v' + 3u'v'' + uv'''$$

$$(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v' + \frac{n(n-1)}{1,2}u^{(n-2)}v'' + \dots + uv^{(n)}$$

$$5. \quad (x^m)^{(n)} = \frac{m!}{(m-n)!} x^{m-n}$$

6. $(x^n)^{(n)} = n!$
7. $(\log_a x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n \ln a}$
8. $(\ln x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}$
9. $(a^x)^{(n)} = a^x \ln^n a$
10. $(e^x)^{(n)} = e^x$
11. $(a^{mx})^{(n)} = m^n a^{mx} \ln^n a$
12. $(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right)$
13. $(\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)$

7.5 Applications of Derivative

Functions : f, g, y , Position of an object : s , Velocity : v , Acceleration : a , Independent variable : x , Time : t , Natural number: n

1. **Velocity and Acceleration** $s=f(t)$ is the position of an object relative to a fixed coordinate system at a time t , $v=s'=f'(t)$ is the instantaneous velocity of the object, $a=v'=s''=f''(t)$ is the instantaneous acceleration of the object.
2. **Tangent Line** $y - y_0 = f'(x_0)(x - x_0)$

3. **Normal Line** $y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

4. Increasing and Decreasing Functions.

If $f'(x_0) > 0$, then $f(x)$ is increasing at x_0 . ($x < x_1, x_2 < x$), If $f'(x_0) < 0$, then $f(x)$ is decreasing at x_0 . ($x_1 < x < x_2$), If $f'(x_0)$ does not exist or is zero, then the test fails.

5. Local extrema

A function $f(x)$ has a local maximum at x_1 if and only if there exists some interval containing x_1 such that $f(x_1) \geq f(x)$ for all x in the interval.

A function $f(x)$ has a local minimum at x_2 if and only if there exists some interval containing x_2 such that $f(x_2) \leq f(x)$ for all x in the interval.

6. Critical Points

A critical point on $f(x)$ occurs at x_0 if and only if either $f'(x_0)$ is zero or the derivative doesn't exist.

7. First Derivative Test for Local Extrema.

If $f(x)$ is increasing ($f'(x) > 0$) for all x in some interval $(a, x_1]$ and $f(x)$ is decreasing ($f'(x) < 0$) for all x in some interval $[x_1, b]$, then $f(x)$ has a local maximum at x_1 .

8. If $f(x)$ is decreasing ($f'(x) < 0$) for all x in some interval $(a, x_2]$ and $f(x)$ is increasing ($f'(x) > 0$) for all x in some interval $[x_2, b]$, then $f(x)$ has a local minimum at x_2 .

9. Second Derivative Test for Local Extrema.

If $f'(x_1) = 0$ and $f''(x_1) < 0$, then $f(x)$ has a local maximum at x_1 .

If $f'(x_2) = 0$ and $f''(x_2) > 0$, then $f(x)$ has a local minimum at x_2 .

10. Concavity.

If $f(x)$ is concave upward at x_0 if and only if $f'(x)$ is increasing at $x_0, x_3 < x$.

If $f(x)$ is concave downward at x_0 if and only if $f'(x)$ is decreasing at $x_0, x < x_3$.

11. Second derivative Test for Concavity.

If $f''(x_0) > 0$, then $f(x)$ is concave upward at x_0 .

If $f''(x_0) < 0$, then $f(x)$ is concave downward at x_0 .

If $f''(x)$ does not exist or is zero, then the test fails.

12. Inflection Points

If $f'(x_3)$ exists and $f''(x)$ changes sign at $x = x_3$, then the point $(x_3, f(x_3))$ is an inflection point of the graph of $f(x)$. If $f''(x_3)$ exists at the inflection point, then $f''(x_3) = 0$.

13. L'Hopital's Rule

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} \text{ if } \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = \begin{cases} 0 \\ \infty \end{cases}.$$

7.6 Differential

Functions: f, u, v , Independent variable: x , Derivative of a function: $y'(x), f'(x)$, Real constant: C , Differential of function $y=f(x)$: dy , Differential of x : dx , Small change in x : Δx , Small change in y : Δy

1. $dy = y' dx$
2. $f(x + \Delta x) = f(x) + f'(x)\Delta x$
3. Small Change in y
 $\Delta y = f(x + \Delta x) - f(x)$
4. $d(u \pm v) = du \pm dv$
5. $d(Cu) = Cdu$
6. $d(uv) = vdu + udv$
7. $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$

INTEGRAL CALCULUS – 8

Functions : f, g, u, v , Independent variables: x, t, ξ

Indefinite integral of a function : $\int f(x)dx, \int g(x)dx, \dots$

Derivative of a function : $y'(x), f'(x), F'(x), \dots$

Real constants: C, a, b, c, d, k , Natural numbers: m, n, i, j

8.1 Indefinite Integral

1. $\int f(ax)dx = \frac{1}{a}F(ax) + C$
2. $\int f(x)f'(x)dx = \frac{1}{2}f^2(x) + C$
3. $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$
4. **Method of Substitution**
 $\int f(x)dx = \int f(u(t))u'(t)dt$ if $x=u(t)$.
5. $\int x^p dx = \frac{x^{p+1}}{p+1} + C, p \neq -1$.
6. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$.
7. $\int \frac{dx}{x} = \ln|x| + C$
8. $\int \frac{dx}{ax+b} = \frac{1}{a} \ln|ax+b| + C$
9. $\int \frac{ax+b}{cx+d} dx = \frac{a}{c}x + \frac{bc-ad}{c^2} \ln|cx+d| + C$

10. $\int \frac{dx}{(x+a)(x+b)} = \frac{1}{a-b} \ln \left| \frac{x+b}{x+a} \right| + C, a \neq b.$

11. $\int \frac{xdx}{a+bx} = \frac{1}{b^2} \left(a + bx - a \ln |a+bx| \right) + C$

12. $\int \frac{x^2 dx}{a+bx} = \frac{1}{b^3} \left[\frac{1}{2} (a+bx)^2 - 2a(a+bx) + a^2 \ln |a+bx| \right] + C$

13. $\int \frac{dx}{x(a+bx)} = -\frac{1}{a} \ln \left| \frac{a+bx}{x} \right| + C$

14. $\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2} \ln \left| \frac{a+bx}{x} \right| + C$

15. $\int \frac{xdx}{(a+bx)^2} = \frac{1}{b^2} \left(\ln |a+bx| + \frac{a}{a+bx} \right) + C$

16. $\int \frac{x^2 dx}{(a+bx)^2} = \frac{1}{b^3} \left(a + bx - 2a \ln |a+bx| - \frac{a^2}{a+bx} \right) + C$

17. $\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} + \frac{1}{a^2} \ln \left| \frac{a+bx}{x} \right| + C$

18. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C$

19. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$

20. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$

21. $\int \frac{xdx}{x^2+a^2} = \frac{1}{2} \ln(x^2+a^2) + C$

22. $\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}} \tan^{-1} \left(x \sqrt{\frac{b}{a}} \right) + C, ab > 0.$

23. $\int \frac{xdx}{a+bx^2} = \frac{1}{2b} \ln \left| x^2 + \frac{a}{b} \right| + C$

24. $\int \frac{dx}{x(a+bx^2)} = \frac{1}{2a} \ln \left| \frac{x^2}{a+bx^2} \right| + C$

25. $\int \frac{dx}{a^2-b^2x^2} = \frac{1}{2ab} \ln \left| \frac{a+bx}{a-bx} \right| + C$

26. $\int \frac{dx}{ax^2+bx+c} = \frac{1}{\sqrt{b^2-4ac}} \ln \left| \frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}} \right| + C, b^2-4ac > 0.$

27. $\int \frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} + C, b^2-4ac < 0.$

8.2 Integral of Irrational Functions

1. $\int \frac{dx}{\sqrt{ax+b}} = \frac{2}{a} \sqrt{ax+b} + C$

2. $\int \sqrt{ax+b} dx = \frac{2}{3a} (ax+b)^{\frac{3}{2}} + C$

3. $\int \frac{xdx}{\sqrt{ax+b}} = \frac{2(ax-2b)}{3a^2} \sqrt{ax+b} + C$

4. $\int x\sqrt{ax+b} dx = \frac{2(3ax-2b)}{15a^2} (ax+b)^{\frac{3}{2}} + C$

5. $\int \frac{dx}{(x+c)\sqrt{ax+b}} = \frac{1}{\sqrt{b-ac}} \ln \left| \frac{\sqrt{ax+b}-\sqrt{b-ac}}{\sqrt{ax+b}+\sqrt{b-ac}} \right| + C, b-ac > 0.$

6. $\int \frac{dx}{(x+c)\sqrt{ax+b}} = \frac{1}{\sqrt{ac-b}} \tan^{-1} \sqrt{\frac{ax+b}{ac-b}} + C, b-ac < 0.$

7. $\int \sqrt{\frac{ax+b}{cx+d}} dx = \frac{1}{c} \sqrt{(ax+b)(cx+d)} - \frac{ad-bc}{c\sqrt{ac}} \ln \left| \sqrt{a(cx+d)} + \sqrt{c(ax+b)} \right| + C, a>0.$

8. $\int \sqrt{\frac{ax+b}{cx+d}} dx = \frac{1}{c} \sqrt{(ax+b)(cx+d)} - \frac{ad-bc}{c\sqrt{ac}} \tan^{-1} \sqrt{\frac{a(cx+d)}{c(ax+b)}} + C, (a<0, c>0).$

9. $\int x^2 \sqrt{ax+b} dx = \frac{2(8a^2-12abx+15b^2x^2)}{105b^3} \sqrt{(a+bx)^3} + C$

10. $\int \frac{x^2 dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3} \sqrt{a+bx} + C$

11. $\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}} \ln \left| \frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}} \right| + C, a>0.$

12. $\int \frac{dx}{x\sqrt{a+bx}} = \frac{2}{\sqrt{-a}} \tan^{-1} \left| \frac{a+bx}{-a} \right| + C, a<0.$

13. $\int \sqrt{\frac{a-x}{b+x}} dx = \sqrt{(a-x)(b+x)} + (a+b) \sin^{-1} \sqrt{\frac{x+b}{a+b}} + C$

14. $\int \sqrt{\frac{a+x}{b-x}} dx = -\sqrt{(a+x)(b-x)} - (a+b) \sin^{-1} \sqrt{\frac{b-x}{a+b}} + C$

15. $\int \sqrt{\frac{1+x}{1-x}} dx = -\sqrt{1-x^2} + \sin^{-1} x + C$

16. $\int \frac{dx}{\sqrt{(x-a)(b-a)}} = 2 \sin^{-1} \sqrt{\frac{x-a}{b-a}} + C$

17. $\int \sqrt{a+bx-cx^2} dx = \frac{2cx-b}{4c} \sqrt{a+bx-cx^2} + \frac{b^2-4ac}{8\sqrt{c^3}} \sin^{-1} \frac{2cx-b}{\sqrt{b^2+4ac}} + C$

18. $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}} \ln \left| 2ax+b+2\sqrt{a(ax^2+bx+c)} \right| + C, a>0.$

19. $\int \frac{dx}{\sqrt{ax^2+bx+c}} = -\frac{1}{\sqrt{a}} \sin^{-1} \frac{2ax+b}{4a} \sqrt{b^2-4ac} + C, a<0.$

20. $\int \sqrt{x^2+a^2} dx = \frac{x}{2} \sqrt{x^2+a^2} + \frac{a^2}{2} \ln \left| x + \sqrt{x^2+a^2} \right| + C$

21. $\int x\sqrt{x^2+a^2} dx = \frac{1}{3} (x^2+a^2)^{\frac{3}{2}} + C$

22. $\int x^2 \sqrt{x^2+a^2} dx = \frac{x}{8} (2x^2+a^2) \sqrt{x^2+a^2} - \frac{a^4}{8} \ln \left| x + \sqrt{x^2+a^2} \right| + C$

23. $\int \frac{\sqrt{x^2+a^2}}{x^2} dx = -\frac{\sqrt{x^2+a^2}}{x} + \ln \left| x + \sqrt{x^2+a^2} \right| + C$

24. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln|x + \sqrt{x^2 + a^2}| + C$

25. $\int \frac{\sqrt{x^2 + a^2}}{x} dx = \sqrt{x^2 + a^2} + a \ln \left| \frac{x}{a + \sqrt{x^2 + a^2}} \right| + C$

26. $\int \frac{x dx}{\sqrt{x^2 + a^2}} = \sqrt{x^2 + a^2} + C$

27. $\int \frac{x^2 dx}{\sqrt{x^2 + a^2}} = \frac{x}{2} \sqrt{x^2 + a^2} - \frac{a^2}{2} \ln|x + \sqrt{x^2 + a^2}| + C$

28. $\int \frac{dx}{x \sqrt{x^2 + a^2}} = \frac{1}{a} \ln \left| \frac{x}{a + \sqrt{x^2 + a^2}} \right| + C$

29. $\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + C$

30. $\int x \sqrt{x^2 - a^2} dx = \frac{1}{3} (x^2 - a^2)^{3/2} + C$

31. $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} + a \sin^{-1} \frac{a}{x} + C$

32. $\int \frac{\sqrt{x^2 - a^2}}{x^2} dx = -\frac{\sqrt{x^2 - a^2}}{x} + \ln|x + \sqrt{x^2 - a^2}| + C$

33. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln|x + \sqrt{x^2 - a^2}| + C$

34. $\int \frac{x dx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2} + C$

35. $\int \frac{x^2 dx}{\sqrt{x^2 - a^2}} = \frac{x}{2} \sqrt{x^2 - a^2} + \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + C$

36. $\int \frac{dx}{x \sqrt{x^2 - a^2}} = -\frac{1}{a} \sin^{-1} \frac{a}{x} + C$

37. $\int \frac{dx}{(x+a)\sqrt{x^2 - a^2}} = \frac{1}{a} \sqrt{\frac{x-a}{x+a}} + C$

38. $\int \frac{dx}{(x-a)\sqrt{x^2 - a^2}} = -\frac{1}{a} \sqrt{\frac{x+a}{x-a}} + C$

39. $\int \frac{dx}{x^2 \sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a^2 x} + C$

40. $\int \frac{dx}{(x^2 - a^2)^{3/2}} = -\frac{x}{a^2 \sqrt{x^2 - a^2}} + C$

41. $\int (x^2 - a^2)^{3/2} dx = -\frac{x}{8} (2x^2 - 5a^2) \sqrt{x^2 - a^2} + \frac{3a^4}{8} \ln|x + \sqrt{x^2 - a^2}| + C$

42. $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$

43. $\int x \sqrt{a^2 - x^2} dx = -\frac{1}{3} (a^2 - x^2)^{3/2} + C$

44. $\int x^2 \sqrt{a^2 - x^2} dx = \frac{x}{8} (2x^2 - a^2) \sqrt{a^2 - x^2} + \frac{a^4}{8} \sin^{-1} \frac{x}{a} + C$

45. $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} + a \ln \left| \frac{x}{a + \sqrt{a^2 - x^2}} \right| + C$

46. $\int \frac{\sqrt{a^2 - x^2}}{x} dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1} \frac{x}{a} + C$

47. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

48. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin \frac{x}{a} + C$

49. $\int \frac{x dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2} + C$

50. $\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$

51. $\int \frac{dx}{(x+a)\sqrt{a^2 - x^2}} = -\frac{1}{2} \sqrt{\frac{a-x}{a+x}} + C$

52. $\int \frac{dx}{(x-a)\sqrt{a^2 - x^2}} = -\frac{1}{2} \sqrt{\frac{a+x}{a-x}} + C$

53. $\int \frac{dx}{(x+b)\sqrt{a^2 - x^2}} = \frac{1}{\sqrt{b^2 - a^2}} \sin^{-1} \frac{bx + a^2}{a(x+b)} + C, b > a.$

54. $\int \frac{dx}{(x+b)\sqrt{a^2 - x^2}} = \frac{1}{\sqrt{a^2 - b^2}} \ln \left| \frac{x+b}{\sqrt{a^2 - b^2} \sqrt{a^2 - x^2} + a^2 + bx} \right| + C, b < a.$

55. $\int \frac{dx}{x^2 \sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x} + C$

56. $\int (a^2 - x^2)^{3/2} dx = \frac{x}{8} (5a^2 - 2x^2) \sqrt{a^2 - x^2} + \frac{3a^4}{8} \sin^{-1} \frac{x}{a} + C$

57. $\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2 \sqrt{a^2 - x^2}} + C$

8.3 Integrals of Trigonometric Functions

1. $\int \sin^2 x dx = \frac{x}{2} - \frac{1}{4} \sin 2x + C$

2. $\int \cos^2 x dx = \frac{x}{2} + \frac{1}{4} \sin 2x + C$

3. $\int \sin^3 x dx = \frac{1}{3} \cos^3 x - \cos x + C = \frac{1}{12} \cos 3x - \frac{3}{4} \cos x + C$

4. $\int \cos^3 x dx = \sin x - \frac{1}{3} \sin^3 x + C = \frac{1}{12} \sin 3x + \frac{3}{4} \sin x + C$

5. $\int \frac{dx}{\sin x} = \int \operatorname{cosec} x dx = \ln \left| \tan \frac{x}{2} \right| + C$

6. $\int \frac{dx}{\cos x} = \int \sec x dx = \ln \left| \tan \left(\frac{x}{2} + \frac{\pi}{2} \right) \right| + C$

7. $\int \frac{dx}{\sin^2 x} = \int \operatorname{cosec}^2 x dx = -\cot x + C$

8. $\int \frac{dx}{\cos^2 x} = \int \sec^2 x dx = \tan x + C$

$$9. \int \frac{dx}{\sin^3 x} = \int \csc^3 x dx = -\frac{\cos x}{2 \sin^2 x} + \frac{1}{2} \ln \left| \tan \frac{x}{2} \right| + C$$

$$10. \int \frac{dx}{\cos^3 x} = \int \sec^3 x dx = \frac{\sin x}{2 \cos^2 x} + \frac{1}{2} \ln \left| \tan x + \left(\frac{1}{\cos x} \right) \right| + C$$

$$11. \int \sin^2 x \cos^2 x dx = \frac{x}{8} - \frac{1}{32} \sin 4x + C$$

$$12. \int \tan x dx = -\ln |\cos x| + C$$

$$13. \int \frac{\sin x}{\cos^2 x} dx = \frac{1}{\cos x} + C = \sec x + C$$

$$14. \int \frac{\sin^2 x}{\cos x} dx = \ln \left| \tan \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| - \sin x + C$$

$$15. \int \tan^2 x dx = \tan x - x + C$$

$$16. \int \cot x dx = \ln |\sin x| + C$$

$$17. \int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{\sin x} + C = -\csc x + C$$

$$18. \int \frac{\cos^2 x}{\sin x} dx = \ln \left| \tan \frac{x}{2} \right| + \cos x + C$$

$$19. \int \cot^2 x dx = -\cot x - x + C$$

$$20. \int \frac{dx}{\cos x \sin x} = \ln |\tan x| + C$$

$$21. \int \frac{dx}{\sin^2 x \cos x} = -\frac{1}{\sin x} + \ln \left| \tan \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| + C$$

$$22. \int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x} + \ln \left| \tan \frac{x}{2} \right| + C$$

$$23. \int \frac{dx}{\sin^2 x \cos^2 x} = \tan x - \cot x + C$$

$$24. \int \sin mx \sin nx dx = -\frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + C, \\ m^2 \neq n^2.$$

$$25. \int \sin mx \cos nx dx = -\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)} + C, \\ m^2 \neq n^2.$$

$$26. \int \cos mx \cos nx dx = \frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + C, \\ m^2 \neq n^2.$$

$$27. \int \sec x \tan x dx = \sec x + C$$

$$28. \int \cosec x \cot x dx = -\cosec x + C$$

$$29. \int \sin x \cos^n x dx = \frac{\cos^{n+1} x}{n+1} + C$$

$$30. \int \sin^n x \cos x dx = \frac{\sin^{n+1} x}{n+1} + C$$

$$31. \int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1-x^2} + C$$

$$32. \int \cos^{-1} x dx = x \cos^{-1} x - \sqrt{1-x^2} + C$$

$$33. \int \tan^{-1} x dx = x \tan^{-1} x - \frac{1}{2} \ln(x^2 + 1) + C$$

$$34. \int \cot^{-1} x dx = x \cot^{-1} x + \frac{1}{2} \ln(x^2 + 1) + C$$

8.4 Integrals of Hyperbolic Functions

$$1. \int \sinh x dx = \cosh x + C$$

$$2. \int \cosh x dx = \sinh x + C$$

$$3. \int \tanh x dx = \ln |\cosh x| + C$$

$$4. \int \coth x dx = \ln |\sinh x| + C$$

$$5. \int \sech^2 x dx = \tanh x + C$$

$$6. \int \operatorname{cosech}^2 x dx = -\coth x + C$$

$$7. \int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x + C$$

$$8. \int \operatorname{cosech} x \coth x dx = -\operatorname{cosech} x + C$$

8.5 Integrals of Exponential and Logarithmic Functions

$$1. \int e^x dx = e^x + C$$

$$2. \int a^x dx = \frac{a^x}{\ln a} + C$$

$$3. \int e^{ax} dx = \frac{e^{ax}}{a} + C$$

$$4. \int x e^{ax} dx = \frac{e^{ax}}{a^2} (ax - 1) + C$$

$$5. \int \ln x dx = x \ln x - x + C$$

$$6. \int \frac{dx}{x \ln x} = \ln |\ln x| + C$$

$$7. \int x^n \ln x dx = x^{n+1} \left[\frac{\ln x}{n+1} - \frac{1}{(n+1)^2} \right] + C$$

$$8. \int e^{ax} \sin bx dx = \frac{a \sin bx - b \cos bx}{a^2 + b^2} e^{ax} + C$$

$$9. \int e^{ax} \cos bx dx = \frac{a \cos bx + b \sin bx}{a^2 + b^2} e^{ax} + C$$

8.6 Reduction Formulas

$$1. \int x^n e^{mx} dx = \frac{1}{m} x^n e^{mx} - \frac{n}{m} \int x^{n-1} e^{mx} dx$$

$$2. \int \frac{e^{mx}}{x^n} dx = -\frac{e^{mx}}{(n-1)x^{n-1}} + \frac{m}{n-1} \int \frac{e^{mx}}{x^{n-1}} dx, n \neq 1.$$

3. $\int \sin h^n x dx = \frac{1}{n} \sin h^{n-1} x \cosh x - \frac{n-1}{n} \int \sin h^{n-2} x dx$

4. $\int \frac{dx}{\sin h^n x} = -\frac{\cosh x}{(n-1) \sin h^{n-1} x} - \frac{n-2}{n-1} \int \frac{dx}{\sin h^{n-2} x}, n \neq 1.$

5. $\int \cosh^n x dx = \frac{1}{n} \sin h x \cosh^{n-1} x \cosh x + \frac{n-1}{n} \int \cosh^{n-2} x dx$

6. $\int \frac{dx}{\cosh^n x} = -\frac{\sin h x}{(n-1) \cosh^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cosh^{n-2} x}, n \neq 1.$

7. $\int \sin h^n x \cosh^m x dx = \frac{\sin h^{n+1} x \cosh^{m-1} x}{n+m} + \frac{m-1}{n+m} \int \sin h^n x \cosh^{m-2} x dx$

8. $\int \sin h^n x \cosh^m x dx = \frac{\sin h^{n-1} x \cosh^{m+1} x}{n+m} - \frac{n-1}{n+m} \int \sinh^{n-2} x \cosh^m x dx$

9. $\int \tan h^n x dx = -\frac{1}{n-1} \tan h^{n-1} x + \int \tan h^{n-2} x dx, n \neq 1.$

10. $\int \coth^n x dx = -\frac{1}{n-1} \coth^{n-1} x + \int \coth^{n-2} x dx, n \neq 1.$

11. $\int \operatorname{sech}^n x dx = \frac{\operatorname{sech}^{n-2} x \tan h x}{n-1} + \frac{n-2}{n-1} \int \operatorname{sech}^{n-2} x dx, n \neq 1.$

12. $\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx$

13. $\int \frac{dx}{\sin^n x} = -\frac{\cos x}{(n-1) \sin^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\sin^{n-2} x}, n \neq 1.$

14. $\int \cos^n x dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x dx$

15. $\int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1) \cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}, n \neq 1.$

16. $\int \sin^n x \cos^m x dx = \frac{\sin^{n+1} x \cos^{m-1} x}{n+m} + \frac{m-1}{n+m} \int \sin^n x \cos^{m-2} x dx$

17. $\int \sin^n x \cos^m x dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{n+m} + \frac{n-1}{n+m} \int \sin^{n-2} x \cos^m x dx$

18. $\int \tan^n x dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx, n \neq 1.$

19. $\int \cot^n x dx = -\frac{1}{n-1} \cot^{n-1} x - \int \cot^{n-2} x dx, n \neq 1.$

20. $\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx, n \neq 1.$

21. $\int \operatorname{cosec}^n x dx = \frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} \int \operatorname{cosec}^{n-2} x dx, n \neq 1.$

22. $\int x^n \ln^m x dx = \frac{x^{n+1} \ln^m x}{n+1} - \frac{m}{n+1} \int x^n \ln^{m-1} x dx$

23. $\int \frac{\ln^m x}{x^n} dx = -\frac{\ln^m x}{(n-1)x^{n-1}} + \frac{m}{n-1} \int \frac{\ln^{m-1} x}{x^n} dx, n \neq 1.$

24. $\int \ln^n x dx = x \ln^n x - n \int \ln^{n-1} x dx$

25. $\int x^n \sinh x dx = x^n \cosh x - n \int x^{n-1} \cosh x dx$

26. $\int x^n \cosh x dx = x^n \sinh x - n \int x^{n-1} \sinh x dx$

27. $\int x^n \sin x dx = -x^n \cos x + n \int x^{n-1} \cos x dx$

28. $\int x^n \cos x dx = x^n \sin x - n \int x^{n-1} \sin x dx$

29. $\int x^n \sin^{-1} x dx = \frac{x^{n+1}}{n+1} \sin^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{\sqrt{1-x^2}} dx$

30. $\int x^n \cos^{-1} x dx = \frac{x^{n+1}}{n+1} \cos^{-1} x + \frac{1}{n+1} \int \frac{x^{n+1}}{\sqrt{1-x^2}} dx$

31. $\int x^n \tan^{-1} x dx = \frac{x^{n+1}}{n+1} \tan^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{1+x^2} dx$

32. $\int \frac{x^n dx}{ax^n + b} = \frac{x}{a} - \frac{b}{a} \int \frac{dx}{ax^n + b}$

33. $\int \frac{dx}{(ax^2 + bx + c)^n} = \frac{-2ax - b}{(n-1)(b^2 - 4ac)(ax^2 + bx + c)^{n-1}} - \frac{2(2n-3)a}{(n-1)(b^2 - 4ac)} \int \frac{dx}{(ax^2 + bx + c)^{n-1}}, n \neq 1.$

34. $\int \frac{dx}{(x^2 + a^2)^n} = \frac{x}{2(n-1)a^2(x^2 + a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} \int \frac{dx}{(x^2 + a^2)^{n-1}}, n \neq 1.$

35. $\int \frac{dx}{(x^2 - a^2)^n} = -\frac{x}{2(n-1)a^2(x^2 - a^2)^{n-1}} - \frac{2n-3}{2(n-1)a^2} \int \frac{dx}{(x^2 - a^2)^{n-1}}, n \neq 1.$

8.7 Definite Integral – Properties

Definite integral of a function : $\int_a^b f(x) dx, \int_a^b g(x) dx, \dots$

1. $\int_a^a f(x) dx = 0$

2. $\int_a^b f(x) dx = - \int_b^a f(x) dx$

3. $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$ for $a < c < b$.

4. $\int_a^b f(x)dx \geq 0$ if $f(x) \geq 0$ on $[a, b]$.

5. $\int_a^b f(x)dx \leq 0$ if $f(x) \leq 0$ on $[a, b]$.

6. **Fundamental Theorem of Calculus**

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a) \text{ if } F'(x) = f(x).$$

7. **Method of Substitution**

If $x = g(t)$, then $\int_a^b f(x)dx = \int_c^d f(g(t))g'(t)dt$,

Where $c = g^{-1}(a)$, $d = g^{-1}(b)$.

8. **Trapezoidal Rule**

$$\int_a^b f(x)dx = \frac{b-a}{2n} \left[f(x_0) + f(x_n) + 2 \sum_{i=1}^{n-1} f(x_i) \right]$$

9. **Simpson's Rule**

$$\int_a^b f(x)dx = \frac{b-a}{3n} \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 4f(x_{n-1}) + f(x_n) \right]$$

Where $x_i = a + \frac{b-a}{n}i$, $i = 0, 1, 2, \dots, n$.

10. **Area Between Two Curves**

$$A = \int_a^b [f(x) - g(x)]dx = F(b) - G(b) - F(a) + G(a),$$

Where $F'(x) = f(x)$, $G'(x) = g(x)$.

11. $\int xe^{-ax}dx = -\frac{1}{a^2}(ax+1)e^{-ax}$ $\int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}}$

12. $\int x^2 e^{-ax}dx = -\frac{1}{a^3}(a^2x^2 + 2ax + 2)e^{-ax}$

$$\int_0^\infty x^{2n} \cdot e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}}$$

13. $\int_0^\infty x^{2n+1} \cdot e^{-ax^2} dx = \frac{n!}{2a^{n+1}}$

A $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$ ($x^2 < 1$)

B $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

$$\tan \theta = \theta + \frac{\theta^3}{3!} + \frac{2\theta^5}{5!} + \dots$$

DIFFERENTIAL EQUATIONS - 9

9.1 First Order Ordinary Differential Equations

1. **Linear Equations**

$$\frac{dy}{dx} + p(x)y = q(x).$$

The general solution is

$$y = \frac{\int u(x)q(x)dx + C}{u(x)},$$

where $u(x) = \exp\left(\int p(x)dx\right)$.

2. **Separable Equations**

$$\frac{dy}{dx} = f(x, y) = g(x)h(y)$$

The general solution is given by

$$\int \frac{dy}{h(y)} = \int g(x)dx + C,$$

$$H(y) = G(x) + C.$$

3. **Homogeneous Equations**

The differential equation $\frac{dy}{dx} = f(x, y)$ is homogeneous, if the function $f(x, y)$ is homogeneous, that is $f(tx, ty) = f(x, y)$.

The substitution $z = \frac{y}{x}$ (then $y = zx$) leads to the separable equation

$$x \frac{dz}{dx} + z = f(1, z).$$

4. **Bernoulli Equation**

$$\frac{dy}{dx} + p(x)y = q(x)y^n.$$

The substitution $z = y^{1-n}$ leads to the linear equation

$$\frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x).$$

5. **Riccati Equation**

$$\frac{dy}{dx} = p(x) + q(x)y + r(x)y^2$$

If a particular solution y_1 is known, then the general solution can be obtained with the help of substitution

$z = \frac{1}{y - y_1}$, which leads to the first order linear equation

$$\frac{dz}{dx} = -[q(x) + 2y_1r(x)]z - r(x).$$

6. **Exact and Non exact Equations**

The equation $M(x, y)dx + N(x, y)dy = 0$

Is called exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, and non exact otherwise.

The general solution is $\int M(x, y)dx + \int N(x, y)dy = C$.

9.2 Second Order Ordinary Differential Equations

1. Homogeneous Linear Equations with Constant Coefficients $y'' + py' + qy = 0$.

The characteristic equation is $\lambda^2 + p\lambda + q = 0$.

If λ_1 and λ_2 are distinct real roots of the characteristic equation, then the general solution is

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}, \text{ where}$$

C_1 and C_2 are integration constants.

If $\lambda_1 = \lambda_2 = -\frac{p}{2}$, then the general solution is

$$y = (C_1 + C_2 x) e^{-\frac{p}{2}x}.$$

If λ_1 and λ_2 are complex numbers:

$$\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i, \text{ where}$$

$$\alpha = -\frac{p}{2}, \beta = \frac{\sqrt{4q - p^2}}{2},$$

then the general solution is $y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$.

2. Inhomogeneous Linear Equation with Constant Coefficients

$$y'' + py' + qy = f(x).$$

The general solution is given by $y = y_p + y_h$, where y_p is a particular solution of the inhomogeneous equation and y_h is the general solution of the associated homogeneous equation.

If the right side has the form

$f(x) = e^{ax}(P_1(x)\cos \beta x + P_2(x)\sin \beta x)$, then the particular solution y_p is given by

$$y_p = x^k e^{ax} (R_1(x)\cos \beta x + R_2(x)\sin \beta x),$$

Where the polynomials $R_1(x)$ and $R_2(x)$ have to be found by using the method of undetermined coefficients.

- If $\alpha + \beta i$ is not a root of the characteristic equation, then the power $k = 0$,
- If $\alpha + \beta i$ is a simple root, then $k = 1$,
- If $\alpha + \beta i$ is a double root, then $k = 2$,

3. Differential Equations with y missing $y'' = f(x, y')$.

Set $u = y'$. Then the new equation satisfied by v is $u' = f(x, u)$,

Which is a first order differential equation.

4. Differential Equations with x Missing $y'' = f(y, y')$. Set $u = y'$. Since

$$y'' = \frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx} = u \frac{du}{dy},$$

We have $u \frac{du}{dy} = f(y, u)$,

Which is a first order differential equation.

9.3 Some Partial Differential Equations

1. The Laplace Equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Applies to potential energy function $u(x, y)$ for a conservative force field in the xy -plane. Partial differential equations of this type are called elliptic.

2. The Heat Equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$

Applies to the temperature distribution $u(x, y)$ in the xy -plane when heat is allowed to flow from warm areas to cool ones. The equations of this type are called parabolic.

3. The Wave Equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial t^2}$

Applies to the displacement $u(x, y)$ of vibrating membranes and other wave functions. The equations of this type are called hyperbolic.

PROBABILITY – 10

10.1 Permutations and Combinations

Permutations : ${}^n P_m$, Combinations : ${}^n C_m$, Whole numbers : n, m .

1. Factorial

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-2)(n-1)n$$

$$0! = 1$$

2. ${}^n P_n = n!$

$$3. {}^n P_m = \frac{n!}{(n-m)!}$$

$$4. \text{ Binomial Coefficient } {}^n C_m = \binom{n}{m} = \frac{n!}{(n-m)! m!}$$

$$5. {}^n C_m = {}^n C_{n-m}$$

$$6. {}^n C_m + {}^n C_{m+1} = {}^{n+1} C_{m+1}$$

$$7. {}^n C_0 + {}^n C_1 + {}^n C_2 + \dots + {}^n C_n = 2^n$$

8. Pascal's Triangle

Row 0							1
Row 1						1	1
Row 2			1	2		1	
Row 3			1	3	3	1	
Row 4			1	4	6	4	1
Row 5			1	5	10	10	5
Row 6			1	6	15	20	15
						6	1

10.2 Probability Formulas

Events: A, B

Probability: P

Any positive real number : ε

Standard deviation : σ

Variance : σ^2

Density functions : $f(x), f(t)$

Random variable :

X, Y, Z

Values of random variables : x, y, z

Expected value of X : μ

1. Probability of an Event

$$P(A) = \frac{m}{n},$$

where m is the number of possible positive outcomes, n is the total number of possible outcomes.

2. Range of Probability Values $0 \leq P(A) \leq 1$
3. Certain Event $P(A) = 1$
4. Impossible Event $P(A) = 0$
5. Complement $P(\bar{A}) = 1 - P(A)$
6. Independent Events $P(A/B) = P(A), P(B/A) = P(B)$
7. Addition Rule for Independent Events
 $P(A \cup B) = P(A) + P(B)$
8. Multiplication Rule for Independent Events
 $P(A \cap B) = P(A) \cdot P(B)$
9. General Addition Rule
 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$,
 Where $A \cup B$ is the union of events A and B ,
 $A \cap B$ is the intersection of events A and B .
10. Conditional Probability $P(A/B) = \frac{P(A \cap B)}{P(B)}$

11. $P(A \cap B) = P(B) \cdot P(A/B) = P(A) \cdot P(B/A)$

12. Law of Total Probability $P(A) = \sum_{i=1}^n P(B_i)P(A/B_i)$,

Where B_i is a sequence of mutually exclusive events.

13. Bayes' Theorem $P(B/A) = \frac{P(A/B) \cdot P(B)}{P(A)}$

14. Bayes' Formula

$$P(B_i/A) = \frac{P(B_i) \cdot P(A/B_i)}{\sum_{k=1}^m P(B_k)P(A/B_k)},$$

Where B_i is a set of mutually exclusive events (hypotheses),

A is the final event, $P(B_i)$ are the prior probabilities, $P(B_i/A)$ are the posterior probabilities,

15. Law of Large Numbers

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

$$P\left(\left|\frac{S_n}{n} - \mu\right| < \varepsilon\right) \rightarrow 1 \text{ as } n \rightarrow \infty,$$

Where S_n is the sum of random variables, n is the number of possible outcomes.

16. Chebyshev Inequality

$$P(|X - \mu| \geq \varepsilon) \leq \frac{V(X)}{\varepsilon^2},$$

Where $V(X)$ is the variance of X .

17. Normal Density Function

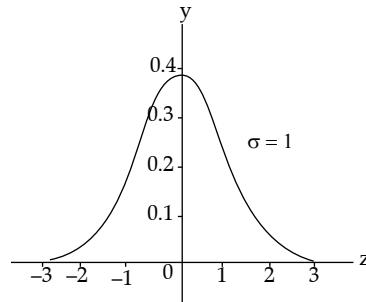
$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

where x is a particular outcome.

18. Standard Normal Density Function

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Average value $\mu=0$, deviation $\sigma=1$.



19. Standard Z Value $Z = \frac{X - \mu}{\sigma}$

20. Cumulative Normal Distribution Function

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt,$$

where x is a particular outcome, t is a variable of integration.

21. $P(\alpha < X < \beta) = F\left(\frac{\beta-\mu}{\sigma}\right) - F\left(\frac{\alpha-\mu}{\sigma}\right)$,

where

X is normally distributed random variable, F is cumulative normal distribution function,

$P(\alpha < X < \beta)$ is interval probability.

22. $P(|X - \mu| < \varepsilon) = 2F\left(\frac{\varepsilon}{\sigma}\right)$

where X is normally distributed random variable, F is cumulative normal distribution function.

23. Cumulative Distribution Function

$$F(x) = P(X < x) = \int_{-\infty}^x f(t)dt,$$

where t is a variable of integration.

24. Bernoulli Trials Process

$$\mu = np, \quad \sigma^2 = npq,$$

where n is a sequence of experiments, p is the probability of success of each experiments, q is the probability of failure, $q = 1 - p$.

25. Binomial Distribution Function

$$b(n, p, q) = {}^n C_k p^k q^{n-k},$$

$$\mu = np, \quad \sigma^2 = npq,$$

$$f(x) = (q + pe^x)^n,$$

where n is the number of trials of selections, p is the probability of success, q is the probability of failure, $q = 1 - p$.

26. Geometric Distribution

$$P(T = j) = q^{j-1} p,$$

$$\mu = \frac{1}{p}, \quad \sigma^2 = \frac{q}{p^2},$$

where T is the first successful event in the series,

j is the event number, p is the probability that any one event is successful, q is the probability of failure, $q = 1 - p$

Poisson Distribution

$$P(X=k) \approx \frac{\lambda^k}{k!} e^{-\lambda}, \lambda = np, \mu = \lambda, \sigma^2 = \lambda,$$

where λ is the rate of occurrence, k is the number of positive outcomes.

27. Expected Value of Discrete Random Variables

$$\mu = E(X) = \sum_{i=1}^n x_i p_i,$$

where x_i is a particular outcome, p_i is its probability.

28. Expected Value of Continuous Random variables

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

29. Properties of Expectations

$$E(X + Y) = E(X) + E(Y),$$

$$E(X - Y) = E(X) - E(Y),$$

$$E(cX) = cE(X),$$

$$E(XY) = E(X) \cdot E(Y),$$

where c is a constant.

$$30. E(X^2) = V(X) + \mu^2,$$

where

$\mu = E(X)$ is the expected value

$V(X)$ is the variance.

31. Markov Inequality

$$\rho(X > k) \leq \frac{E(X)}{k}$$

where k is some constant.

32. Variance of Discrete Random Variables

$$\sigma^2 = V(X) = E[(X - \mu)^2] = \sum_{i=1}^n (x_i - \mu)^2 p_i,$$

where

x_i is a particular outcome,

p_i is its probability.

33. Variance of Continuous Random Variables

$$\sigma^2 = V(X) = E[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

34. Properties of Variance

$$V(X + Y) = V(X) + V(Y),$$

$$V(X - Y) = V(X) - V(Y),$$

$$V(X + c) = V(X),$$

$$V(cX) = c^2 V(X)$$

where c is a constant.

$$35. \text{ Standard Deviation } D(X) = \sqrt{V(X)} = \sqrt{E[(X - \mu)^2]}$$

36. Covariance

$$\text{Cov}(X, Y) = E[(X - \mu(X))(Y - \mu(Y))] = E(XY) - \mu(X)\mu(Y),$$

where X is random variable, $V(X)$ is the variance of X ,

μ is the expected value of X or Y .

$$37. \text{ Correlation } \rho(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{V(X)V(Y)}}$$

Where

$V(X)$ is the variance of X , $V(Y)$ is the variance of Y .

APPENDIX-D

Famous Mathematicians and their contributions

THALES (Greek c. 600 B.C.)

The first Greek known to have used proof and strict logical reasoning to solve mathematical questions

PYTHAGORAS (Greek c. 500 B.C.)

Influential Greek philosopher and religious leader. He taught that numbers and ratios of numbers were the foundation of reality. He discovered many number patterns and the proof that the square root of two is not rational.

EUCLID (Greek c. 300 B.C.)

Organized Greek geometry into a mathematical system based on fundamental definitions, a few postulates and theorems that are logically deduced. This work, known as the Elements, had a profound influence on mathematics for thousands of years.

ARCHIMEDES (Greek c. 250 B.C.)

Discovered many fundamental properties of physics, such as the law of the lever; discovered a way to approximate pi as accurately as desired

APOLLONIUS (Greek c. 600 B.C.)

Discovered the family of curves known as the conic sections. He analyzed their properties using Greek geometry (not,

however, with modern algebra equations or graphing techniques).

PTOLEMY (Greek c. 130 A.D.)

Invented a planetary system that was adopted as truth by the Christian church in Medieval Europe. In this system the Earth does not move and the planets, moon, stars and the Sun revolve around the Earth in circular paths with constant motion. This was described in his book the Almagest.

AI-KHWARIZMI (Hindu – Arabic c. 800 A.D.)

Wrote influential Arabic books on solving algebra problems and the Hindu – Arabic numberation system.

VIETE (Early Modern 1540 - 1603)

Introduced symbols into algebra.

DESCARTES (Early Modern 1596 - 1650)

Developed analytic geometry. He used a sophisticated symbolic algebra to show how algebra can be used to solve geometry problems and how algebra problems can be solved with geometry.

FERMAT (Early Modern 1601 - 1665)

Developed analytic geometry. He showed how a geometric curve, such as a conic section, could be drawn on a coordinate grid from an algebra equation. He also made

important contributions to number theory, including the famous "Fermat's Last Theorem"

KEPLER (Early Modern 1571 - 1630)

Used real astronomical data to show that the planets orbit the sun in elliptical paths at varying rates of speed.

NEWTON (Early Modern 1643 - 1727)

Co-inventor of the calculus, He proved Kepler's laws mathematically in the style of Euclid in his book the Principal

LEIBNIZ (Early Modern 1646 - 1716)

Co-inventor of the calculus. His methods and symbolism is used today.

EULER (Early Modern 1707 - 1783)

A founding father to many branches of mathematics. He lived in the generation that followed Newton and Leibniz. Modern calculus for many modern symbols, such as $f(x)$, e , i , π .

GAUSS (Modern 1777 - 1855)

Discovered non-Euclidean geometry. He was a pioneer in many areas of modern mathematics.

CANTOR (Modern 1845 - 1918)

Invented the theory of infinite sets. He proved that the counting numbers and the real numbers have a different cardinality.

von NEUMANN (Modern 1903 - 1957)

Designed the fundamentals structure of modern computer design, known as the "von Neumann architecture". He also invented a branch of mathematics known as "game theory".

ARYABHATA(476 – 550AD)

1. Aryabhata was born in 476 A.D. Kusumpur, India. He was the first in the line of great mathematicians from the classical age of Indian Mathematics and Astronomy.
2. His famous work are the "Aryabhatiya" and the "Arya-siddhanta". The Mathematical part of the Aryabhatiya covers arithmetic, algebra, plane and spherical trigonometry. The Arya-siddhanta, a lot work on astronomical computation.
3. **Approximation of Pi:** Aryabhata work on approximation for pi (π) and may have come to the conclusion that π is an irrational number. In the 2nd part of Aryabhatiya, he writes the ratio of circumference to diameter is 3.1416.
4. Aryabhata given the formula for area of a triangle. He also discussed the concept of sine in his work by the name of ardha-jya. If we use Aryabhata's table and calculate the value of sin30° which is 1719/3438 = 0.5., the value is correct. His alphabetic code is commonly known as the Aryabhata cipher.
5. He was first person to say that Earth is spherical and it revolves around the sun.
6. He gave the formula $(a + b)^2 = a^2 + b^2 + 2ab$.
7. He taught the method of solving the following problems:

$$1 + 2 + 3 + \dots + n = n(n + 1)/2$$

$$1^2 + 2^2 + 3^2 + \dots + n^2 = n(n + 1)(2n + 1)/6$$

$$1^3 + 2^3 + 3^3 + \dots + n^3 = (n(n + 1)/2)^2$$

BRAHMAGUPT (598 – 668 AD)

1. Brahmagupta was born in 598 A.D. in Bhinmal city in the state of Rajasthan. He was a mathematician and astronomer, who wrote many important works on mathematics and astronomy. His best known work is the "Brahmasphuta-siddhanta", written in 628 AD in Bhinmal.
2. He was the first to use zero as a number. He gave rules to compute with zero.
3. He gave four methods of multiplication.

4. He gave following formulae, used in G.P. series

$$a + ar + ar^2 + ar^3 + \dots + ar^{n-1} = a(r^n - 1)/(r - 1)$$

5. He gave the following formulae (Brahmagupta's formula):

Area of a cyclic quadrilateral with side a , b , c , d
 $= \sqrt{(s-a)(s-b)(s-c)}$, where $2s = a + b + c + d$.

Length of its diagonals

$$= \sqrt{\frac{bc+ad}{ab+cd}}(ac+bd), \sqrt{\frac{ab+cd}{bc+ad}}(ac+bd)$$

BHASKARACHARYA (1114 – 1185 AD)

1. He was born in Bijapur in modern Karnataka. He and his work represent a significant contribution to mathematical and astronomical knowledge in the 12th century.
2. His main work "Siddhanta Shiromani" is divided into four parts called Lilawati, Bijaganit, Grahananita and Goladhyaya. These four sections deal with arithmetic, algebra, mathematics of planets and spheres respectively.
3. He was the first to give that any number divided by zero gives infinity.
4. He was written a lot about zero, surds, permutation and combination.
5. He wrote, "The hundredth part of the circumference of a circle seems to be straight. Our earth is a big sphere and that's why it appear to be flat."
6. He gave the formulae like :

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
7. He calculated derivatives of Trigonometric functions and formulae.
8. He developed spherical trigonometry along with a number of other trigonometric results.
9. He explained solution of quadratic, cubic and quartic indeterminate equations.
10. He developed a proof of Pythagoras Theorem by calculating the same area in two different ways and these cancel out terms to get $a^2 + b^2 = c^2$.
11. He gave first general method for finding the solution of the problem $x^2 - ny^2 = 1$ (so called Pell's equation).
12. He gave solution of Diophantine equations of second order such as $61x^2 + 1 = y^2$.

RAMANUJAN (1887 – 1920)

1. Ramanujan was born on 22nd of December 1887 in Erode, Madras Presidency. He made extraordinary contributions to mathematical analysis, number theory, infinite series, and continued fractions.
2. He demonstrated unusual mathematical skill at school, winning accolades and awards.
3. By 17, he had conducted his own mathematical research on Bernoulli numbers and the Euler-Mascheroni constant.
4. He discovered theorems of his own and rediscovered Euler's identity independently.
5. He sent a set 120 theorems to Professor Hardy of Cambridge. As result he invited Ramanujan to England.
6. He independently compiled nearly 3900 results (mostly identities and equations). Nearly all his claims have claims have now been proved correct.

7. Ramanujan Showed that any big number can be written as sum of not more than four prime numbers.
8. He showed that how to divide the numbers into two or more squares cubes.
9. **Ramanujan's Numbers :** When Mr.G.H. Hardy came to see Ramanujan in taxi number 1729, Ramanujan said that 1729 is the smallest number which can be written in the form of sum of cubes of two numbers in two ways, i.e. $1729 = 9^3 + 10^3 = 1^3 + 12^3$ since than the number 1729 is calld Ramanujan's number.
10. In 1918, Ramanujan and Hardy studied the partition function $P(n)$ extensively and gave a non-convergent asymptotic series that permits exact computation of the number of partition of an integer.
11. He discovered mock theta function in the last year of his life. For many years these functions where a mstry, but they are now known to be the holomorphic parts of harmonic weak mass forms.

SHAKUNTALA DEVI

1. She was born in 1939. She is an indian calculating prodigy.
2. By age 6, She demonstrated her calculation and memorization abilities at university of Mysore. At the age of 8, she had successes at Annamalai University by

doing the same.

3. On June 18, 1980, She demonstrated the multiplication of two 13-digit numbers $7,686,369,774,870 \times 2,465,099,745,779$ picked at random by the Computer Department of Imperial College, London. She answered the question in 28 seconds. However, the time is more likely the time for dictating the answer (a 26-digit number) than the time for mental calculation(the time of 28 seconds was quoted on her website). Her answer was $18,947,668,177,995,426,773,730$. This event is mentioned on page 26 of the 1995 Guinness Book of Records.

4. In Dallas, she competed with a computer to see who give the cube of 188138517 faster, she won. At University of USA she was asked to give the 23rd root of

9167486769200391580986609275853801624831066801443
086224071265164279346570408670965932792057674808
067900227830163549248523803357453169351119035965
7754734007568818688305620821016129132845564895780
158806771.

She answered in 50 seconds. The answer is 546372891. It took a Univac 1108 computer, full one minute (10 seconds more) to confirm that she was right after it was fed with 13000 instructions.

5. Now she is known to be Human Computer.

APPENDIX-E

ROMAN – NUMERALS

(A) Roman Numeral Symbols

Symbol	Number
I	1
V	5
X	10
L	50
C	100
D	500
M	1,000
\bar{V}	5,000
\bar{X}	10,000
\bar{L}	50,000
\bar{C}	100,000
\bar{D}	500,000
\bar{M}	1,000,000

(B) Roman Numerical Table

1	I	14	XIV	27	XXVII	150	CL
2	II	15	XV	28	XXVIII	200	CC
3	III	16	XVI	29	XXIX	300	CCC
4	IV	17	XVII	30	XXX	400	CD
5	V	18	XVIII	31	XXXI	500	D
6	VI	19	XIX	40	XL	600	DC
7	VII	20	XX	50	L	700	DCC
8	VIII	21	XXI	60	LX	800	DCCC
9	IX	22	XXII	70	LXX	900	CM
10	X	23	XXIII	80	LXXX	1000	M
11	XI	24	XXIV	90	XC	1600	MDC
12	XII	25	XXV	100	C	1700	MDCC
13	XIII	26	XXVI	101	CI	1900	MCM