UNIT – I : RELATIONS AND FUNCTIONS

CHAPTER-1

RELATIONS AND FUNCTIONS

Topic-1 Relations

Concepts covered: Empty, Universal, Identify, Reflexive , Symmetric, Transitive, Equivalence and Inverse Relations, Equivalence Classes, Domain & Range, Co-domain

Revision Notes

- **Definition:**

  A relation R, from a non-empty set A to another non-empty set B is mathematically defined as an arbitrary subset of $A \times B$. Equivalently, any subset of $A \times$ B is a relation from A to B.

  Thus, R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$

  $$\Leftrightarrow R \subseteq \{(a, b) : a \in A, b \in B\}$$

  Illustrations:

  (a) Let $A = \{1, 2, 4\}$, $B = \{4, 6\}$ and let $R = \{(1, 4), (1, 6), (2, 4), (2, 6), (4, 4), (4, 6)\}$, Here, $R \subseteq A \times B$ and therefore R is a relation from A to B.

  (b) Let $A = \{1, 2, 3\}$, $B = \{2, 3, 5, 7\}$ and let $R = \{(2, 3), (3, 5), (5, 7)\}$ Here, $R \not\subseteq A \times B$ and therefore R is not a relation from A to B. Since, $(5, 7) \in R$ but $(5, 7) \notin A \times B$.

  (c) Let $A = \{-1, 1, 2\}$, $B = \{1, 4, 9, 10\}$; let $a \in A$ and $b \in B$ and $a\ R\ b$ means $a^2 = b$ then, $R = \{(-1, 1), (1, 1), (2, 4)\}$.

- **Types of Relations :**

  (a) Empty relation : A relation R from A to B is called an empty relation or a void relation from A to B if $R = \phi$.

  For example, let $A = \{2, 4, 6\}$, $B = \{7, 11\}$

  Let $R = \{(a, b): a \in A, b \in B$ and $a - b$ is even$\}$.

  Here, R is an empty relation.

  (b) Universal relation : A relation R from A to B is said to be a universal relation, if $R = A \times B$.

  For example, let $A = \{1, 2\}$, $B = \{1, 3\}$

  Let $R = \{(1, 1), (1, 3), (2, 1), (2, 3)\}$

  Here, $R = A \times B$,

  So, relation R is a universal relation.

  (c) Identity relation : A relation R defined on a set A is said to be the identity relation on A, if

  $$R = \{(a, b) : a \in A, b \in A \text{ and } a = b\}$$

  Thus, identity relation $R = \{(a, a) : \forall\ a \in A\}$

  The identity relation on set A is also denoted by I_A.

  For example, let $A = \{1, 2, 3, 4\}$

  Then, $I_A = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$.

  But the relation given by $R = \{(1, 1), (2, 2), (1, 3), (4, 4)\}$

  is not an identity relation because element 1 is related to elements 1 and 3.

  (d) Reflexive relation : A relation R defined on a set A is said to be reflexive, if aRa, $\forall\ a \in A$,

  i.e., $(a, a) \in R$, $\forall\ a \in A$.

  For example, let $A = \{1, 2, 3\}$ and R_1, R_2, R_3 be the relation given as :

  $R_1 = \{(1, 1), (2, 2), (3, 3)\}$,

  $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3)\}$

  $R_3 = \{(2, 2), (2, 3), (3, 2), (1, 1)\}$

Important Facts

- A relation from A to B is also called a relation from A into B.
- $(a, b) \in R$ is also written as aRb (read as a is related to b).
- Let A and B be two non-empty finite sets having p and q elements respectively. Then $n(A \times B) = n(A) \cdot n(B) = pq$. Then total number of subsets of $A \times B = 2^{pq}$. Since, each subset of $A \times B$ is a relation from A to B, therefore total number of relations from A to B is given as 2^{pq}.

Important Facts

- The void relation, *i.e.*, ϕ and universal relation, *i.e.*, $A \times A$ on A are respectively the smallest and largest relations defined on the set A. Also these are sometimes called **Trivial Relations.** Any other relation is called a **non-trivial relation.**
- The relation $R = \phi$ and $R = A \times A$ are two extreme relations.

Important Fact

- In an identity relation on A every element of A should be related to itself only.

Here, R_1 and R_2 are reflexive relations on A but R_3 is not reflexive as $3 \in A$ but $(3, 3) \notin R_3$.

(e) Symmetric relation : A relation R defined on a set A is symmetric, if

$(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$, *i.e.*, aRb

$\Rightarrow bRa$ (*i.e.*, whenever aRb then bRa).

For example, let $A = \{1, 2, 3\}$,

$R_1 = \{(1, 2), (2, 1)\}$

$R_2 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$

$R_3 = \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$

$R_4 = \{(1, 3), (3, 1), (2, 3)\}$

Here, R_1, R_2 and R_3 are symmetric relations on A. But R_4 is not symmetric because $(2, 3) \in R_4$ but $(3, 2) \notin R_4$.

Important Facts

- The universal relation on non-void set A is reflexive.
- The identity relation is always a reflexive relation but the converse may or may not be true. As shown in the example, R_1 is both identity as well as reflexive relation on A but R_2 is only reflexive relation on A.

(f) Transitive relation : A relation R on a set A is transitive, if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R, \forall a, b, c \in A$ *i.e.*, aRb and $bRc \Rightarrow aRc$.

For example, let $A = \{1, 2, 3\}$,

$R_1 = \{(1, 2), (2, 3), (1, 3), (3, 2)\}$

and $R_2 = \{(1, 3), (3, 2), (1, 2)\}$,

Here, R_2 is transitive relation whereas R_1 is not transitive because $(2, 3) \in R_1$ and $(3, 2) \in R_1$ but $(2, 2) \notin R_1$.

(g) Equivalence relation : Let A be a non-empty set, then a relation R on A is said to be an equivalence relation, if

(i) R is reflexive, *i.e.*, $(a, a) \in R, \forall a \in A$, *i.e.*, aRa.

(ii) R is symmetric, *i.e.*, $(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$, *i.e.*, $aRb \Rightarrow bRa$.

(iii) R is transitive, *i.e.*, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R, \forall a, b, c \in A$, *i.e.*, aRb and $bRc \Rightarrow aRc$.

For example, let $A = \{1, 2, 3\}$

$R = \{(1, 2), (1, 1), (2, 1), (2, 2), (3, 3)\}$

Here, R is reflexive, symmetric and transitive. So, R is an equivalence relation on A.

Important Facts

- An important property of an equivalence relation is that it divides the set into pair-wise disjoint subsets called equivalence classes whose collection is called a partition of the set.
- Note that the union of all equivalence classes give the whole set.

e.g., Let R denotes the equivalence relation in the set Z of integers given by $R = \{(a, b) : 2 \text{ divides } a - b\}$. Then the equivalence class [0] is $[0] = [0, \pm 2, \pm 4, \pm 6, ...]$.

➢ **Equivalence classes :** Let R be an equivalence relation in a set A and let $a \in A$. Then, the set of all those elements of A which are related to a, is called equivalence class determined by a and it is denoted by $[a]$. Thus, $[a] = \{b \in A : (a, b) \in R\}$

➢ DOMAIN & RANGE OF A RELATION

(a) Domain of a relation : Let R be a relation from A to B. The domain of relation R is the set of all those elements $a \in A$ such that $(a, b) \in R, \forall b \in B$.

Thus, $\text{Dom.}(R) = \{a \in A : (a, b) \in R, \forall b \in B\}$.

That is, the domain of R is the set of first elements of all the ordered pairs which belong to R.

(b) Range of a relation : Let R be a relation from A to B. The range of relation R is the set of all those elements $b \in B$ such that $(a, b) \in R, \forall a \in A$.

Thus, Range of $R = \{b \in B : (a, b) \in R, \forall a \in A\}$.

That is, the range of R is the set of second elements of all the ordered pairs which belong to R.

(c) Co-domain of a relation : Let R be a relation from A to B. Then B is called the co-domain of the relation R. So we can observe that co-domain of a relation R from A into B is the set B as a whole.

Illustrations : Let $a \in A$ and $b \in B$ and

(a) Let $A = \{1, 2, 3, 7\}$, $B = \{3, 6\}$. If aRb means $a < b$.

Then we have $R = \{(1, 3), (1, 6), (2, 3), (2, 6), (3, 6)\}$.

Here, $\text{Dom}(R) = \{1, 2, 3\}$, Range of $R = \{3, 6\}$, Co-domain of $R = B = \{3, 6\}$

(b) Let $A = \{1, 2, 3\}$, $B = \{2, 4, 6, 8\}$.

If $R_1 = \{(1, 2), (2, 4), (3, 6)\}$, and $R_2 = \{(2, 4\}, (2, 6), (3, 8), (1, 6)\}$

Then both R_1 and R_2 are related from A to B because

$R_1 \subseteq A \times B, R_2 \subseteq A \times B$

Here, $\text{Dom}(R_1) = \{1, 2, 3\}$, Range of $R_1 = \{2, 4, 6\}$;

$\text{Dom}(R_2) = \{2, 3, 1\}$, Range of $R_2 = \{4, 6, 8\}$

- **INVERSE RELATION**

  Let $R \subseteq A \times B$ be a relation from A to B. Then, the inverse relation of R, to be denoted by R^{-1}, is a relation from B to A defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$

  Thus $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}, \forall\, a \in A, b \in B$.
  Clearly, Dom (R^{-1}) = Range of R, Range of R^{-1} = Dom (R)
  Also, $(R^{-1})^{-1} = R$.

  For example, Let $A = \{1, 2, 4\}$, $B = \{3, 0\}$ and let $R = \{(1, 3), (4, 0), (2, 3)\}$ be a relation from A to B, then $R^{-1} = \{(3, 1), (0, 4), (3, 2)\}$.

Key Terms

- **(a)** A relation R from A to B is an empty relation or void relation if $R = \phi$.
  (b) A relation R on a set A is an empty relation or void if $R = \phi$.
- **(a)** A relation R from A to B is a universal relation if $R = A \times B$.
  (b) A relation R on a set A is a universal relation if $R = A \times A$.
- A relation R on a set A is reflexive if aRa, $\forall\, a \in A$.
- A relation R on a set A is symmetric if whenever aRb then bRa, for all $a, b \in A$.
- A relation R on a set A is transitive if whenever aRb and bRc, then aRc, for all $a, b, c \in A$.
- A relation R on A is identity relation if $R = \{(a, a), \forall\, a \in A\}$, *i.e.*, R contains only elements of the type (a, a), $\forall\, a \in A$ and it contains no other element.
- A relation R on a non-empty set A is an equivalence relation if it includes reflexive, symmetric and transitive relations.

Mnemonics

Concept: Types of Relations
Mnemonics: Rustee
Interpretation:

R	U	S	T	E	E
↓	↓	↓	↓	↓	↓
REFLEXIVE Relation	UNIVERSAL Relation	SYMMETRIC Relation	TRANSITIVE Relation	EMPTY Relation	EQUIVALENCE Relation

Reflexive Relation: A relation R on set A is said to be a reflexive if $(a, a) \in R$, for every $a \in A$
Symmetric Relation: A relation R on set A is said to be symmetric if $(a, b) \in R \Rightarrow (b, a) \in R$, $\forall\, a, b \in A$
Transitive Relations: A Relation R on set A is said to be transitive if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$, $\forall\, a, b, c \in A$

Concept: Equivalence Relations
Mnemonics:

Ram	Sells	Tickets
↓	↓	↓
Reflexive	Symmetric	Transitive

Topic-2

Functions

Concept covered: Domain Range, Real Function, One-one Function (Injective), Onto Function (Surjective Function), One one Onto (Bijective Function), Equal Function, Invertible Functions

Revision Notes

- **Function :**

  Consider A and B be two non-empty sets, then rule f which associates each element of A with a unique element of B is called a function or the mapping from A to B or f maps A to B. If f is mapping from A to B, then we write $f : A \to B$ which is read as is mapping from A to B or f is a function from A to B.
  If f associates $a \in A$ to $b \in B$, then we say that b is the image of the element a under the function f or b is the f- image

of a or the value of f at 'a' is denoted by $f(a)$ and we write $b = f(a)$. The element a is called the pre-image or inverse-image of b.

Remember the following points :

(a) Negative number should not occur under the square root (even root), *i.e.*, Expression under the square root sign must be always ≥ 0.

(b) Denominator should never be zero.

(c) For $\log_b a$ to be defined $a > 0, b > 0$ and $b \neq 1$. Also, $\log_b 1$ is equal to zero, *i.e.*, 0.

➢ **Difference between relation and function :** A relation from a set A to another set B is any subset of $A \times B$; while a function f from A to B is a subset of $A \times B$ satisfying following conditions:

(a) For every $x \in A$, there exists $y \in B$ such that $(x, y) \in f$.

(b) If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

S. No.	Function	Relation
(i)	Each elements of A is related to some element of B.	There may be some elements of A which are not related to any element of B.
(ii)	An element of A is not be related to more than one element of B.	An element of A may be related to more than one element of B.

➢ **Domain of a function :** If a function is expressed in the form $y = f(x)$, then domain of f means set of all those real values of x for which y is real (*i.e.*, y is well-defined).

Remember the following points for domain :

(a) First find the domain of the given function.

(b) If the domain does not contain an interval, then find the values of y putting these values of x from the domain. The set of all these values of y obtained will be the range.

(c) If domain is the set of all real numbers R or set of all real numbers except a few points, then express x in terms of y and from this find the real values of y for which x is real and belongs to the domain.

➢ **Range of the function :** If a function is expressed in the form $y = f(x)$, then range of f means set of all possible real values of y corresponding to every value of x in its domain.

Important Facts

- It is necessary that every f- image is in B; but there may be some elements in B which are not the f- images of any element of A *i.e.*, whose pre-image under f is not in A.
- Two or more elements of A may have same image in B.
- $f : x \to y$ means that under the function f from A to B, an element x of A has image y in B.
- Usually we denote the function f by writing $y = f(x)$ and read it as y is a function of x.

➢ **Real function :** If the domain and range of a function f are subsets of R (the set of real numbers), then f is said to be a real function.

➢ **Some important real functions and their Domain and Range :**

S. No.	Function	Representation	Domain	Range
(i)	Identity function	$I(x) = x, \forall\, x \in \mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
(ii)	Modulus function or Absolute value function	$f(x) = \lvert x \rvert = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$	$\mathbb{R}$	$[0, \infty)$
(iii)	Greatest integer function or integral function or step function.	$f(x) = [x], \forall\, x \in \mathbb{R}$	$\mathbb{R}$	$\mathbb{Z}$
(iv)	**Signum function**	$f(x) = \begin{cases} \frac{\lvert x \rvert}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ (*i.e.*,) $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$	$\mathbb{R}$	$\{-1, 0, 1\}$
(v)	**Exponential function**	$f(x) = a^x, \forall\, a \neq 1, a > 0$	$\mathbb{R}$	$(0, \infty)$
(vi)	**Logarithmic function**	$f(x) = \log_a x, \forall\, a \neq 1, a > 0$ and $x > 0$	$(0, \infty)$	$\mathbb{R}$
(vii)	**Irrational function**	$f(x) = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
(viii)	**Rational function**	$f(x) = \frac{1}{x}$	$\mathbb{R} - \{0\}$	$\mathbb{R} - \{0\}$

- **Types of Functions :**

(a) One-one function (Injective function or Injection) : A function $f : A \to B$ is one-one function or injective function if distinct elements of A have distinct images in B.

Thus, $f : A \to B$ is one-one $\Leftrightarrow f(a) = f(b) \Rightarrow a = b, \ \forall\ a, b \in A$

$\Leftrightarrow a \neq b \Rightarrow f(a) \neq f(b), \ \forall\ a, b \in A.$

- **If A and B are two sets having m and n elements respectively such that $m \leq n$, then total number of one-one functions from set A to set B is ${}^nC_m \times m\,!$ *i.e.*, ${}^nP_m$.**
- **If $n(A) = m$, then the number if injective functions defined from A onto itself is $m!$.**

Algorithm to check the Injectivity of a Function :

Step 1 : Take any two arbitrary elements a, b in the domain of f.

Step 2 : Put $f(a) = f(b)$

Step 3 : Solve $f(a) = f(b)$. If it gives $a = b$ only, then f is one-one function.

(b) Many-one function : If any two or more elements of set A are connected with a single element of set B, then we call this function as many-one function.

Thus, $f : A \to B$ is many-one $\Leftrightarrow f(a) = f(b) \Rightarrow a \neq b, \ \forall\ a, b \in A$

(c) Onto function (Surjective function or Surjection) : A function $f : A \to B$ is onto function or a surjective function if every element of B is the f-image of some element of A. That implies $f(A) = B$ or range of f is the co-domain of f.

Thus, $f : A \to B$ is onto $\Leftrightarrow f(A) = B$ *i.e.*, range of f = co-domain of f.

Algorithm to check the Surjectivity of a Function :

Step 1: Take an element $b \in B$, where B is the co-domain of the function.

Step 2: Put $f(x) = b$.

Step 3: Solve the equation $f(x) = b$ for x and obtain x in terms of b. Let $x = g(b)$.

Step 4: If for all values of $b \in B$, the values of x obtained from $x = g(b)$ are in A, then f is onto. If there are some $b \in B$ for which values of x, given by $x = g(b)$, is not in A. Then, f is not onto.

(d) Into function : Function 'f' from set A to set B is into function, if set B has atleast an element which is not connected with any of the element of set A.

(e) One-one onto function (Bijective function or Bijection) : A function $f : A \to B$ is said to be one-one onto or bijective if it is both one-one and onto i.e., if the distinct elements of A have distinct images in B and each element of B is the image of some elements of A.

Also, note that a bijective function is also called a one-to-one function or one-to-one correspondence.

(f) Equal function : Two functions f and g having the same domain D are said to be equal if $f(x) = g(x)$, for all $x \in D$.

- For an ordinary finite set A, a one-one function $f : A \to A$ is necessarily onto and an onto function $f : A \to A$ is necessarily one-one for every finite set A.

Thus, for a bijective function from A to B,

(a) A and B should be non-empty.

(b) Each element of A should have image in B.

(c) No element of A should have more than one image in B.

(d) If A and B have respectively m and n number of elements, then the number of functions defined from A to B is n^m.

- **Inverse of a Function :**

Let $f : A \to B$ be a bijection. Then, a function $g : B \to A$ which associates each element $y \in B$ to a unique element $x \in A$ such that $f(x) = y$ is called the inverse of f *i.e.*, $f(x) = y \Leftrightarrow g(y) = x$.

The inverse of f is generally denoted by f^{-1},

Thus, if $f : A \to B$ is a bijection, then a function $f^{-1} : B \to A$ is such that $f^{-1}(y) = x$.

- **Algorithm to Find the Inverse of a Function :**

Step 1: Put $f(x) = y$ where $y \in B$ and $x \in A$.

Step 2: Solve $f(x) = y$ to obtain x in terms of y.

Step 3: Replace x by $f^{-1}(y)$ in the relation obtained in Step 2.

Step 4: In order to get the required inverse of f *i.e.*, $f^{-1}(x)$, replace y by x in the expression obtained in Step 3 *i.e.*, in the expression $f^{-1}(y)$.

Mnemonics

Concept: Types of functions

Mnemonics: Moving Immense Organs Insert Outdoor Extreme Operations

Interpretation:

Interpretation:

Moving	Immense	Organs	Insert	Outdoor	Extreme	Operations
↓	↓	↓	↓	↓	↓	↓
Many-One Function	Into Function	One-One Function	Identity Function	Onto Function	Equal Function	One-One Onto Function

CHAPTER-2
INVERSE TRIGONOMETRIC FUNCTIONS

Revision Notes

- In mathematics, the inverse trigonometric functions are the inverse function of trigonometric functions. Specifically, they are inverse of the sine, cosine, tangent, cotangent, secant and cosecant functions, and are used to obtain an angle from any of the angle's trigonometric ratios.
- **Domains and Ranges of Inverse Trigonometric Functions :**

Inverse Trigonometric Functions *i.e.*, $f(x)$	Domain/Value of x	Range/Value of $f(x)$
$\sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\text{cosec}^{-1}x$	$\mathbb{R} - (-1, 1)$ $(-\infty, -1] \cup [1, \infty)$	$\left[\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
$\sec^{-1}x$	$\mathbb{R} - (-1, 1)$ $(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$\tan^{-1}x$	$\mathbb{R}$ $(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}x$	$\mathbb{R}$ $(-\infty, \infty)$	$(0, \pi)$

Note

- The symbol $\sin^{-1}x$ is used to denote the smallest angle whether positive or negative, such that the sine of this angle will give us x. Similarly, $\cos^{-1}x$, $\tan^{-1}x$, $\text{cosec}^{-1}x$, $\sec^{-1}x$ and $\cot^{-1}x$ are defined.
- You should note that $\sin^{-1}x$ can be written as arc sin x. Similarly, other inverse trigonometric functions can also be written as arc cos x, arc tan x, arc sec x etc.
- Also, note that $\sin^{-1}x$ (and similarly other inverse trigonometric functions) is entirely different from $(\sin x)^{-1}$. In fact, $\sin^{-1}x$ is measure of an angle in Radians whose sine is x whereas $(\sin x)^{-1}$ is $\frac{1}{\sin x}$ (which is obvious as per the laws of exponents).
- Keep in mind that these inverse trigonometric relations are true only in their domains, *i.e.*, they are valid only for some values of 'x' for which inverse trigonometric functions are well defined.

- **Principal Value :**

Numerically smallest angle is known as the principal value.

For finding the principal value, following algorithm can be followed.

Step 1 : First, draw a trigonometric circle and mark the quadrant in which the angle may be lie.

Step 2 : Select anti-clockwise direction for 1st and 2nd quadrant in which the angle may be lie.

Step 3 : Find the angles in the first rotation.

Step 4 : Select the numerically least (magnitude wise) angle among these two values. The angle thus found will be the principal value.

Step 5 : In case, two angles one with positive sign and the other with the negative sign qualify for the numerically least angle then, it is the convention to select the angle with positive sign as principal value.

Note

A function $f : A \to B$ is said to be invertible if f is bijective (*i.e.*, one-one and onto). The inverse of the function f is denoted by $f : B \to A$ such that $f^{-1}(y) = x$ if $f(x) = y$, $\forall\ x \in A$, $y \in B$. As trigonometric functions are many-one so, their inverse doesn't exist. But they become one-one onto by restricting their domains. Therefore, all the restrictions required so that the inverse of the concerned trigonometric functions do exist. If these restrictions are removed, the terms will represent **Inverse Trigonometric Relations** and not the functions. Note that the inverse trigonometric functions are also called as **Inverse Circular Functions**.

➢ **Elementary Properties of Inverse Trigonometric Functions :**

Property I

(a) $\sin^{-1}(x) = \text{cosec}^{-1}\left(\frac{1}{x}\right), x \in [-1, 1]$

(b) $\text{cosec}^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right), x \in (-\infty, -1] \cup [1, \infty)$

(c) $\cos^{-1}(x) = \sec^{-1}\left(\frac{1}{x}\right), x \in [-1, 1]$

(d) $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right), x \in (-\infty, -1] \cup [1, \infty)$

(e) $\tan^{-1}(x) = \begin{cases} \cot^{-1}\left(\frac{1}{x}\right), & x > 0 \\ -\pi + \cot^{-1}\left(\frac{1}{x}\right), & x < 0 \end{cases}$

(f) $\cot^{-1}(x) = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right), & x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right), & x < 0 \end{cases}$

Property II

(a) $\sin^{-1}(\sin x) = x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(b) $\cos^{-1}(\cos x) = x, 0 \le x \le \pi$

(c) $\tan^{-1}(\tan x) = x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

(d) $\text{cosec}^{-1}(\text{cosec } x) = x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}, x \ne 0$

(e) $\sec^{-1}(\sec x) = x, 0 \le x \le \pi, x \ne \frac{\pi}{2}$

(f) $\cot^{-1}(\cot x) = x, 0 < x < \pi$

Property III

(a) $\sin^{-1}(-x) = -\sin^{-1}x, x \in [-1, 1]$

(b) $\cos^{-1}(-x) = \pi - \cos^{-1}x, x \in [-1, 1]$

(c) $\tan^{-1}(-x) = -\tan^{-1}x, x \in \mathbb{R}$

(d) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x, |x| \ge 1$

(e) $\sec^{-1}(-x) = \pi - \sec^{-1}x, |x| \ge 1$

(f) $\cot^{-1}(-x) = \pi - \cot^{-1}x, x \in \mathbb{R}$

Property IV

(a) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, x \in [-1, 1]$

(b) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, x \in \mathbb{R}$

(c) $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}, x \in (-\infty, -1] \cup [1, \infty)$

Property V

(a) $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$

(b) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right]$

(c) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1$

(d) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right), xy > -1$

(e) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

Property VI

(a) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2}), \text{if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

(b) $3\sin^{-1}x = \sin^{-1}(3x - 4x^3), \text{if } -\frac{1}{2} \le x \le \frac{1}{2}$

Property VII

(a) $2\cos^{-1}x = \cos^{-1}(2x^2 - 1), \text{ if } 0 \le x \le 1$

(b) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x), \text{if } \frac{1}{2} \le x \le 1$

Property VIII

(a) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{if } -1 < x < 1$

(b) $3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), \text{if } \frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Key Formulae

➢ **Trigonometric Formulae :**

- **Relation between trigonometric ratios :**

(a) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ (b) $\tan\theta = \frac{1}{\cot\theta}$ (c) $\tan\theta.\cot\theta = 1$

(d) $\cot\theta = \frac{\cos\theta}{\sin\theta}$ (e) $\text{cosec}\,\theta = \frac{1}{\sin\theta}$ (f) $\sec\theta = \frac{1}{\cos\theta}$

- **Trigonometric Identities :**

(a) $\sin^2\theta + \cos^2\theta = 1$ (b) $\sec^2\theta = 1 + \tan^2\theta$ (c) $\text{cosec}^2\theta = 1 + \cot^2\theta$

- **Addition/subtraction/formulae and some related results :**

(a) $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$ (b) $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

(c) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(d) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(e) $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$ (f) $\cot(A \pm B) = \frac{\cot B\cot A \mp 1}{\cot B \pm \cot A}$

- **Multiple angle formulae involving $2A$ & $3A$:**

(a) $\sin 2A = 2\sin A\cos A$ (b) $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

(c) $\cos 2A = \cos^2 A - \sin^2 A$ (d) $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

(e) $\cos 2A = 2\cos^2 A - 1$ (f) $\cos 2A = 1 - 2\sin^2 A$

(g) $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$ (h) $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

(i) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$ (j) $\sin 3A = 3\sin A - 4\sin^3 A$

(k) $\cos 3A = 4\cos^3 A - 3\cos A$ (l) $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

- **Transformation of sums/differences into products & vice-versa :**

(a) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$ (b) $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(c) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$ (d) $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

(e) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ (f) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(g) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$ (h) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

➢ **Domain and Range of trigonometric functions :**

S. No.	Function	Domain	Range
(i)	sine	R	$[-1, 1]$
(ii)	cosine	R	$[-1, 1]$
(iii)	tangent	$R - \left\{x : x = (2n+1)\frac{\pi}{2};\, n \in Z\right\}$	R
(iv)	cosecant	$R - \{x : x = n\pi,\ n \in Z\}$	$R - (-1, 1)$

(v)	secant	$R-\left\{x: x=(2n+1)\frac{\pi}{2}; n \in Z\right\}$	$R-(-1, 1)$
(vi)	cotangent	$R-\{x: x=n\pi, n \in Z\}$	R

➢ **Relations in different measures of Angle :**

(a) Angle in Radian Measure = (Angle in degree measure) $\times \frac{\pi}{180°}$ radians

(b) Angle in Degree Measure = (Angle in radian measure) $\times \frac{180°}{\pi}$

(c) θ (in radian measure) $= \frac{l}{r} = \frac{\text{arc}}{\text{radius}}$

Also following are of importance as well :

(a) 1 right angle = 90°

(b) $1° = 60', 1' = 60''$

(c) $1° = \frac{\pi}{180°} = 0{\cdot}01745$ radians (Approx.)

(d) 1 radian = $57°17'45''$ or 206265 seconds.

➢ **Relation in Degree & Radian Measures :**

Angles in Degree	0°	30°	45°	60°	90°	180°	270°	360°
Angles in Radian	0 rad	$\left(\frac{\pi}{6}\right)$	$\left(\frac{\pi}{4}\right)$	$\left(\frac{\pi}{3}\right)$	$\left(\frac{\pi}{2}\right)$	(π)	$\left(\frac{3\pi}{2}\right)$	(2π)

➢ **Trigonometric Ratio of Standard Angles :**

Degree	**0°**	**30°**	**45°**	**60°**	**90°**
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
$\cot x$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\operatorname{cosec} x$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec x$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

➢ **Trigonometric Ratios of Allied Angles :**

Angles (→) / **T – Ratios (↓)**	$\frac{\pi}{2}-\theta$	$\frac{\pi}{2}+\theta$	$\pi-\theta$	$\pi+\theta$	$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$	$2\pi-\theta$ or $-\theta$	$2\pi+\theta$
sin	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
cos	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$	$\cos\theta$
tan	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$
cot	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$
sec	$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta$	$-\sec\theta$	$-\sec\theta$	$-\operatorname{cosec}\theta$	$\operatorname{cosec}\theta$	$\sec\theta$	$\sec\theta$
cosec	$\sec\theta$	$\sec\theta$	$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta$	$-\sec\theta$	$-\sec\theta$	$-\operatorname{cosec}\theta$	$\operatorname{cosec}\theta$

UNIT – II : ALGEBRA

CHAPTER-3

MATRICES

Topic-1

Matrices and Operations

Concepts covered: Concept, notation, order, equality, types of matrices, zero and identity matrix, transpose of a matrix, symmetric and skew symmetric matrices.

Operation on Matrices : Addition multiplication and multiplication with a scalar, simple properties of addition, multiplication and scalar multiplication, Non commutativity of multiplication of matrices and existence of non-zero matrices whose product is the zero matrix.

Revision Notes

- **Matrix :**

A matrix is an ordered rectangular array of numbers (real or complex) or functions which are known as elements or the entries of the matrix. It is denoted by the uppercase letters *i.e.*, *A, B, C,* etc.

Consider a matrix A given as,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Here in matrix A depicted above, the horizontal lines of elements are said to constitute rows of the matrix A and vertical lines of elements are said to constitute columns of the matrix. Thus, matrix A has m **rows** and n **columns**. The array is enclosed by square brackets [ ], the parentheses ( ) or the double vertical bars || ||.

- A matrix having m rows and n columns is called a matrix of order $m \times n$ (read as 'm by n' matrix). A matrix 'A' of order $m \times n$ is depicted as $A = [a_{ij}]_{m\times n}; i, j \in N$.
- Also, in general, a_{ij} means an element lying in the i^{th} row and j^{th} column.
- No. of elements in the matrix $A = [a_{ij}]_{m\times n}$ is given as (mn).

- **Types of Matrices :**

(a) Column matrix : A matrix having only one column is called a **column matrix** or **column vector**.

e.g., $A = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}_{3\times 1}$, $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}_{2\times 1}$

General notation : $A = [a_{ij}]_{m\times 1}$

(b) Row matrix : A matrix having only one row is called a **row matrix** or **row vector**.

e.g., $A = [2 \ \ 5 \ -4]_{1\times 3}, B = [\sqrt{2} \ \ 4]_{1\times 2}$

General notation : $A = [a_{ij}]_{1\times n}$

(c) Square matrix : It is a matrix in which the number of rows is equal to the number of columns *i.e.*, an $n\times n$ matrix is said to constitute a square matrix of order $n \times n$ and is known as a **square matrix of order** 'n'.

e.g., $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & -4 \\ 0 & -1 & -2 \end{bmatrix}_{3\times 3}$ is a square matrix of order 3.

General notation : $A = [a_{ij}]_{n\times n}$

(d) Diagonal matrix : A square matrix $A = [a_{ij}]_{m\times n}$ is said to be diagonal matrix if all the elements, except those in the leading diagonal are zero *i.e.,* $a_{ij} = 0$ for all $i \neq j$.

e.g., $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3}$ is a diagonal matrix of order 3.

- Also, there are more notation specifically used for the diagonal matrices. For instance, consider the matrix depicted above, it can also be written as diag (2 5 4) or diag [2, 5, 4]
- Note that the elements $a_{11}, a_{22}, a_{33}, ..., a_{mn}$ of a square matrix $A = [a_{ij}]_{m\times n}$ of order $m \times n$ are said to constitute the principal diagonal or simply **the diagonal of the square matrix** A. These elements are known as **diagonal elements of matrix** A.

(e) Scalar matrix : A diagonal matrix $A = [a_{ij}]_{m\times n}$ is said to be a scalar matrix if its diagonal elements are equal.

i.e., $a_{ij} = \begin{cases} 0, & \text{when } i \neq j \\ k, & \text{when } i = j \text{ for some constant } k \end{cases}$

e.g., $A = \begin{bmatrix} 17 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 17 \end{bmatrix}_{3\times 3}$ is a scalar matrix of order 3.

(f) Unit or Identity matrix : A square matrix $A = [a_{ij}]_{m\times n}$ is said to be an identity matrix if $a_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$.

A unit matrix can also be defined as the scalar matrix if each of its diagonal elements is unity. We denote the identity matrix of order m by I_m or I.

e.g., $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

(g) Zero matrix or Null matrix : A matrix is said to be a **zero matrix** or **null matrix** if each of its elements is 'zero'.

e.g., $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$, $C = [0 \;\; 0]_{1\times 2}$

(h) Triangular matrix :

(i) Lower triangular matrix : A square matrix is called a lower triangular matrix if all the entries above the main diagonal are zero.

e.g., $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 5 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 2 & 3 & 5 \end{bmatrix}$

(ii) Upper triangular matrix : A square matrix is called a **upper triangular matrix** if all the entries below the main diagonal are zero.

e.g., $A = \begin{bmatrix} 1 & -8 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 5 \end{bmatrix}$

➢ **Equality of Matrices :**

Two matrices A and B are said to be equal and written as $A = B$, if they are of the same order and their corresponding elements are identical *i.e.,* $a_{ij} = b_{ij}$ *i.e.,* $a_{11} = b_{11}$, $a_{22} = b_{22}$, $a_{32} = b_{32}$, etc.

- **Addition of Matrix :**

If A and B are two $m \times n$ matrices, then another $m \times n$ matrix obtained by adding the corresponding elements of the matrices A and B is called the sum of the matrices A and B and is denoted by '$A + B$'.

Thus, if $A = [a_{ij}]$, $B = [b_{ij}] \Rightarrow A + B = [a_{ij} + b_{ij}]$.

Properties of matrix addition :

- Commutative property : $A + B = B + A$
- Associative property : $A + (B + C) = (A + B) + C$
- Cancellation law : **(i)** Left cancellation : $A + B = A + C \Rightarrow B = C$

  (ii) Right cancellation : $B + A = C + A \Rightarrow B = C$
- **Existence of additive identity :**

$$A + O = O + A = A$$

where O is the $m \times n$ zero matrix or the additive identity for matrix addition.
- **Existence of additive inverse :**

$$A + (-A) = (-A) + A = 0$$

- **Multiplication of a Matrix by a Scalar :**

If a $m \times n$ matrix A is multiplied by a scalar k (say), then the new kA matrix is obtained by multiplying each element of matrix A by scalar k. Thus, if $A = [a_{ij}]$, and it is multiplied by a scalar k, then $kA = [ka_{ij}]$, *i.e.*. $A = [a_{ij}]$ $\Rightarrow kA = [ka_{ij}]$

e.g.,
$$A = \begin{bmatrix} 2 & -4 \\ 5 & 6 \end{bmatrix} \Rightarrow 3A = \begin{bmatrix} 6 & -12 \\ 15 & 18 \end{bmatrix}$$

- **Multiplication of Two Matrices :**

Let $A = [a_{ij}]$ be a $m \times n$ matrix and $B = [b_{jk}]$ be a $n \times p$ matrix such that the number of columns in A is equal to the number of rows in B, then the $m \times p$ matrix $C = [c_{ik}]$ such that $C_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$ is said to be the product of the matrices A and B in that order and it is denoted by AB, *i.e.*, $C = AB$.

Properties of matrix multiplication :

- Note that the product AB is defined only when the number of columns in matrix A is equal to the number of rows in matrix B.
- If A and B are $m \times n$ and $n \times p$ matrices, respectively, then the matrix AB will be an $m \times p$ matrix, *i.e.*, order of matrix AB will be $m \times p$.
- In the product AB, A is called the **pre-factor** and B is called the **post-factor**.
- If two matrices A and B are such that AB is possible, then it is not necessary that the product BA is also possible.
- If A is an $m \times n$ matrix and both AB as well as BA are defined, then B will be an $n \times m$ matrix.
- If A is an $n \times n$ matrix and I_n be the unit matrix of order n, then $AI_n = I_n A = A$.
- Matrix multiplication is **associative**, *i.e.*, $A(BC) = (AB)C$.
- Matrix multiplication is **distributive** over the **addition**, *i.e.*, $A.(B+C) = AB + AC$.
- Matrix multiplication is not commutative.

- **Existence of non-zero matrices whose product is zero.**

The product of two matrices can be zero without either factor being a zero matrix.

e.g., Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$

Here, $A \neq 0$ and $B \neq 0$.

Also, $AB = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

- **Transpose of a Matrix :**

If $A = [a_{ij}]_{m \times n}$ be a $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of matrix A is said to be a transpose of matrix A. The transpose of A is denoted by A' or A^T, *i.e.*, if $A^T = [a_{ji}]_{n \times m}$.

For example, if $A = \begin{bmatrix} 5 & -4 & 1 \\ 0 & \sqrt{5} & 3 \end{bmatrix}$ then $A^T = \begin{bmatrix} 5 & 0 \\ -4 & \sqrt{5} \\ 1 & 3 \end{bmatrix}$

Properties of Transpose of Matrices :

- $(A+B)^T = A^T+B^T$
- $(A^T)^T = A$
- $(kA)^T = kA^T$, where k is any constant
- $(AB)^T = B^TA^T$
- $(ABC)^T = C^TB^TA^T$

➢ **Symmetric matrix :** A square matrix $A = [a_{ij}]$ is said to be a symmetric matrix if $A^T = A$, *i.e.*, if $A = [a_{ij}]$, then $[a_{ji}] = [a_{ji}]$.

For example, A = $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, $B = \begin{bmatrix} 2+i & 1 & 3 \\ 1 & 2 & 3+2i \\ 3 & 3+2i & 4 \end{bmatrix}$

➢ **Skew Symmetric Matrix :**

A square matrix $A = [a_{ij}]$ is said to be **a skew symmetric matrix** if $A^T = -A$ *i.e.*, if $A = [a_{ij}]$, then $[a_{ji}] = -[a_{ij}]$.

For example, $A = \begin{bmatrix} 0 & 1 & -5 \\ -1 & 0 & 5 \\ 5 & -5 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

Notes

- For any matrices AA^T and A^TA are symmetric matrices
- If A and B are two symmetric matrices of same order, then
  (i) AB is symmetric if and only if $AB = BA$.
  (ii) $A \pm B$, $AB + BA$ are also symmetric matrices.

Notes

- All the diagonal elements in a skew-symmetric matrix are zero.
- If A and B are two symmetric matrices, then $AB - BA$ is a skew symmetric matrix.

Key Formulae

➢ For any square matrix A, the matrix $A + A^T$ is a symmetric and $A - A^T$ is always a skew symmetric matrix.

➢ A square matrix can be expressed as the sum of a symmetric and skew symmetric matrix, *i.e.*, $A = \frac{1}{2}(P)+\frac{1}{2}(Q)$,

where $P = A + A^T$ is a symmetric matrix and $Q = A - A^T$ is a skew symmetric matrix.

Mnemonics

Concept : Types of Matrices

Mnemonics 'Remember Crist Subah Dophar Syam Nite'

Interpretation :

R	C	S	D	S	N	I	T	E
↓	↓	↓	↓	↓	↓	↓	↓	↓
Row Matrix	Column Matrix	Square Matrix	Diagonal Matrix	Scalar Matrix	Null Matrix	Identity Matrix	Triangular Matrix	Equal Matrix

Topic-2

Invertible Matrices and Martin's Rule

Concepts covered : Invertible matrices, Proof of uniqueness of inverse, if it exists, Martin's rule and related problems.

Revision Notes

➢ **Determinant :** A unique number (real or complex) can be associated to every square matrix is known as its determinant. The determinant of matrix A is denoted by det A or $|A|$.

eg. : **(i)** If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2,

then $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

(ii) If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3,

then $|A| = (-1)^{1+1}\, a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

➢ **Singular Matrix & Non-Singular Matrix :**

(a) Singular matrix: A square matrix A is said to be singular if $|A| = 0$, *i.e.*, its determinant is zero.

e.g.,
$$A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 12 \\ 1 & 1 & 3 \end{vmatrix}$$
$$= 1(15-12) - 2(12-12) + 3(4-5)$$
$$= 3 - 0 - 3 = 0$$

$\therefore$ A is singular matrix.

(b) Non-singular Matrix : A square matrix A is said to be non-singular if $|A| \neq 0$.

e.g.,
$$A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$
$$= 0(0-1) - 1(0-1) + 1(1-0)$$
$$= 0 + 1 + 1 = 2 \neq 0$$

$\therefore$ A is a non-singular matrix.

- A square matrix A is **invertible** if and only if A is **non-singular.**

➢ **Minors :** Minors of an element a_{ij} of a determinant (*or a determinant corresponding to matrix A*) is the determinant obtained by deleting its i^{th} row and j^{th} column in which a_{ij} lies. Minor of a_{ij} is denoted by M_{ij}. Hence, we can get 9 minors corresponding to the 9 elements of a third order (*i.e.*, 3×3) determinant.

➢ **Co-factors :** Cofactor of an element a_{ij} denoted by A_{ij} is defined by $A=(-1)^{(i+j)}M_{ij}$, where M_{ij} is minor of a_{ij}. Sometimes C_{ij} is used in place of A_{ij} to denote the cofactor of element a_{ij}.

➢ **Adjoint of a Square Matrix :**

Let $A = [a_{ij}]$ be a square matrix. Also, assume $B = [A_{ij}]$, where A_{ij} is the cofactor of the elements a_{ij} in matrix A. Then the transpose B^T of matrix B is called the **adjoint of matrix** A and it is denoted by ***adj*** **(*A*)**.

To find adjoint of a 2 × 2 matrix: If the adjoint of a square matrix of order 2 can be obtained by interchanging the diagonal elements and changing the signs of off-diagonal elements, *i.e.,* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $adj\ A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

For example, consider a square matrix of order 3 as $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$, then in order to find the adjoint matrix A, we find a matrix B (formed by the co-factors of elements of matrix A as mentioned above in the definition)

i.e., $B = \begin{bmatrix} 15 & -2 & -6 \\ -10 & -1 & 4 \\ -1 & 2 & -1 \end{bmatrix}$. Hence, $adj\, A = B^T = \begin{bmatrix} 15 & -10 & -1 \\ -2 & -1 & 2 \\ -6 & 4 & -1 \end{bmatrix}$

➢ **Algorithm to find A^{-1} by Determinant Method :**

Step 1 : Find $|A|$.

Step 2 : If $|A| = 0$, then, write "A is a singular matrix and hence not invertible". Else write "A is non-singular matrix and hence invertible".

Step 3 : Calculate the co-factors of elements of matrix A.

Step 4 : Write the matrix of co-factors of elements of A and then obtain its transpose to get $adj\ A$ (*i.e.*, adjoint A).

Step 5 : Find the inverse of A by using the relation : $A^{-1}=\frac{1}{|A|}(adj\,A)$

- **Properties associated with various Operations of Matrices & Determinants :**

(a) $AB = I = BA$

(b) $AA^{-1} = I$ or $A^{-1}I = A^{-1}$

(c) $(AB)^{-1} = B^{-1}A^{-1}$

(d) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(e) $(A^{-1})^{-1} = A$

(f) $(A^T)^{-1} = (A^{-1})^T$

(g) $A(adj\,A) = (adj\,A)\,A = |A|\,I$

(h) $adj(AB) = adj\,(B)\,adj\,(A)$

(i) $adj\,(A^T) = (adj\,A)^T$

(j) $(adj\,A)^{-1} = (adj\,A^{-1})$

(k) $|adj\,A| = |A|^{n-1}$, $if\ |A| \neq 0$, where n is of the order of A

(l) $|AB| = |A|\,|B|$

(m) $|A\ adj\,A| = |A|^n$, where n is of the order of A

(n) $|A^{-1}| = \frac{1}{|A|}$, if matrix A is invertible

(o) $|A| = |A^T|$

- $|kA| = k^n|A|$, where n is of the order of square matrix A and k is any scalar.
- If A is non-singular matrix of order n, then $adj\,(adj\,A) = |A|^{n-2}A$.

- **Uniqueness Theorem :** Let A be an invertible square matrix of order n. Suppose B and C are the two inverse of A.

Then $AB = BA = I_n$ (by definition of inverse matrix)

$AC = CA = I_n$

Now, $B = BI_n = B(AC)$ [$\because$ Matrix multiplication is associative]

$= (BA)C$

$= I_nC$

$= C$

$\therefore B = C$, *i.e.*, any two inverse of A are equal matrices.

Hence, the inverse of A is unique.

- **Solving System of Equations by Matrix Method [Martin's Rule]**

Homogeneous and Non-homogeneous system : A system of equations $AX = B$ is said to be a homogeneous system if $B = 0$. Otherwise it is called a non-homogeneous system of equations.

Let given system of equations is :

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

Step 1: Assume $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $B=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ and $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Step 2 : Find $|A|$. Now there may be following situations :

(i) If $|A| \neq 0 \Rightarrow A^{-1}$ exists, then the given system of equations is consistent and therefore, the system has **unique solution.** In that case, write

$$AX = B$$

$$\Rightarrow \quad X = A^{-1}B \quad \left[\text{where } A^{-1}=\frac{1}{|A|}(adj\,A)\right]$$

(ii) If $|A| = 0 \Rightarrow A^{-1}$ does not exist, then the given system of equations is either inconsistent or it has infinitely many solutions. In order to check proceed as follow :

- Compute $(adj\,A)\,B$.
- If $(adj\,A)B \neq 0$, then the given system of equations is inconsistent, *i.e.*, it has no solution.
- If $(adj\,A)B = 0$, then the given system of equations is consistent with infinitely many solutions.
[In order to find these infinitely many solutions, replace one of the variables by some real number and proceed in the same manner in the new two variables system of equations]

CHAPTER-4

DETERMINANTS

Revision Notes

➢ **Determinant :** A unique number (real or complex) can be associated to every square matrix $A = [a_{ij}]$ of order m. This number is called the determinant of the square matrix A, where $a_{ij} = (i, j)^{\text{th}}$ element of A.

The determinant of matrix A is denoted by det A or $|A|$.

(a) Determinant of a square matrix of order 2

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2, then $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{12}\,a_{21}$

It follows that the value of a determinant of order 2 is equal to the product of the elements along the principal diagonal *minus* the product of the off diagonal elements.

e.g., : Let $A = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$, then

$|A| = 5 \times 3 - (-2) \times 4 = 15 + 8 = 23.$

(b) Determinant of a square matrix of order 3

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3, then

$$|A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

or $|A| = a_{11}\,(a_{22}\,a_{33} - a_{23}\,a_{32}) - a_{12}\,(a_{33}\,a_{21} - a_{23}\,a_{31}) + a_{13}\,(a_{21}\,a_{32} - a_{22}\,a_{31})$

It follows that the value of a determinant of order 3 is the sum of the product of elements a_{ij} in first row with $(-1)^{i+j}$ times the determinant of a 2×2 sub matrix obtained by leaving the first row and column passing through elements.

Note

- **Minor :** In a determinant, the minor of an element is a determinant left after deleting the complete row and column in which the element exists and is denoted by corresponding capital letter.
- **Cofactor :** In a determinant, the cofactor of an element is a minor of that element with its respective sign. More briefly co-factors are signed minors.

➢ **Elementary properties of Determinants**

Property I : The value of a determinant is not altered by inter changing its rows into columns and columns into rows, *i.e.,*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & b_3 \end{vmatrix}$$

Property II : If any two adjacent rows or columns of a determinant are interchanged, the sign of the determinant get changed but its numerical value remains unaltered, *i.e.,*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1) \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property III : If any two rows or columns of a determinant are identical, the value of determinant is zero, *i.e.*,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \text{ or, } \begin{vmatrix} a_1 & a_2 & a_1 \\ b_1 & b_2 & b_1 \\ c_1 & c_2 & c_1 \end{vmatrix} = 0$$

Property IV : If every element in a row or a column of a determinant is multiplied by the same non-zero constant k, then the value of the determinant gets multiplied by k, *i.e.*,

$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property V : A determinant can be expressed as the sum of several determinants of the same order, *i.e.*,

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Property VI : The value of the determinant is not affected if the elements of a row or column are increased or diminished by the same multiple of the corresponding elements of any other row or column, *i.e.*,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+pb_1+qc_1 & b_1 & c_1 \\ a_2+pb_2+qc_2 & b_2 & c_2 \\ a_3+pb_3+qc_3 & b_3 & c_3 \end{vmatrix}$$

Property VII : If each elements of a row or column of a determinant is zero, its value is zero, *i.e.*,

$$\begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

➢ **Area of Triangle**

The area of triangle having vertices at (x_1, y_1), (x_2, x_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

➢ **Condition of Collinearity of Three Points**

Let three given points be A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3).

If A, B, C are collinear, then area of triangle ABC is zero.

i.e., A, B, C are collinear ⇔ Area of triangle ABC = 0

or
$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Let the three nonhomogeneous linear equations be

$$a_1x + b_1y + c_1z = c_1z = d_1$$
$$a_2x + b_2y + c_2z = c_2z = d_2$$
$$a_3x + b_3y + c_3z = c_3z = d_3$$

$$x = \frac{D_x}{D}, y = \frac{D_y}{D} \text{ and } z = \frac{D_z}{D}$$

Where,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}, \text{Provided that } D \neq 0$$

➢ **Condition for consistency**

(i) If D ≠ 0, then the given system of equations is consistent and has a unique solution given by

$$x = \frac{D_x}{D}, y = \frac{D_y}{D} \text{ and } z = \frac{D_z}{D}$$

(ii) If $D = 0$ and $D_x = D_y = D_z = 0$, the given system of equations is consistent has infinitely many solutions.

(iii) If D = 0 and atleast one of the determinants D_x, D_y, D_z is non-zero, then the given system of equations is inconsistent.

UNIT – III : CALCULUS

CHAPTER-5

CONTINUITY, DIFFERENTIABILITY & DIFFERENTIATION

Topic-1 Continuity and Differentiability

Concepts covered: Continuity at a point, continuity on an interval, algebra of continuity, discontinuous function, removable discontinuity, differentiability

Revision Notes

- **Limit : For a function $f(x)$, $\lim_{x \to a^-} f(x)$ exists iff**

  $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

- **Continuity at a point**

  A function $f(x)$ is continuous at a point $x = a$, where $a \in$ domain of $f(x)$, if

  $$\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$$

  where, $\lim_{x \to a^-} f(x)$ is **Left Hand Limit** (LHL) of $f(x)$ at $x = a$

  $\lim_{x \to a^+} f(x)$ is **Right Hand Limit** (RHL) of $f(x)$ at $x = a$

  and $f(a)$ is the value of $f(x)$ at $x = a$.

- **Continuity on an interval**
  - **Continuity on an open interval :** A function $f(x)$ is said to be continuous on an open interval (a, b) iff it is continuous at every point on the interval (a, b).
  - **Continuity on a closed interval :** A function $f(x)$ is said to be continuous on closed interval $[a, b]$ iff it is continuous on (a, b) and it is continuous at 'a' from the right side and at 'b' from the left.

    i.e., for $f(x)$ to be continuous on $[a, b]$ iff

    (i) f is continuous on (a, b)

    (ii) $\lim_{x \to a^+} f(x) = f(a)$

    (iii) $\lim_{x \to b^-} f(x) = f(b)$

- **Algebra of continuity**

  Let f and g be two real and continuous function at $x = a$, then

  (i) $f \pm g$ is continuous at $x = a$

  (ii) $f \cdot g$ is continuous at $x = a$

  (iii) $\frac{f}{g}$ is continuous at $x = a$, provided $g(a) \neq 0$

  (iv) $f(x)$ is continuous at $x = a$, where a is a real number.

- **Discontinuous function**

  A point where $f(x)$ is not continuous is called a **point of discontinuity** of $f(x)$ and the function is said to be **discontinuous** at that point. In particular, if either or both of the two limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$.

  do not exist or if both of them exist but are unequal, or even if the two limits are equal but their common value is not equal to $f(a)$, the function $f(x)$ is said to be discontinuous at $x = a$.

- **Removable discontinuity :** In case, $\lim_{x \to a} f(x)$ exist but not equal to $f(a)$, then the function is said to have a removable discontinuity or discontinuity.

- **Differentiability**

  Let $f(x)$ be defined at any point c in the interval (a, b). Then $f(x)$ is said to be differentiable at $x = c$ if the function has a derivative at this point, *i.e.*, if $f'(c)$ exists. Hence, if $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists, then the function $f(x)$ is called differentiable at point $x = c$.

For the existence of this limit it is necessary that when $h \to 0$, the left-hand and right-hand limits both must exist and they must be equal.

The **right-hand derivative** (RHD) of $f(x)$ at the point $x = c$ is defined as

$$Rf'(c) = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h},\ h > 0$$

Similarly, the **left hand derivative** (LHD) of $f(x)$ at $x = c$ is defined as

$$Lf'(c) = \lim_{h\to 0^-} \frac{f(c-h)-f(c)}{h}$$

or

$$Lf'(c) = \lim_{h\to 0} \frac{f(c-h)-f(c)}{-h},\ h > 0$$

Hence, function $f(x)$ is differentiable at $x = c$ iff

$$Rf'(c) = Lf'(c)$$

Know the Facts

- If a function is differentiable at a point, it is continuous at that point as well.
- If a function is not differentiable at a point, it may or may not be continuous at that point.
- If a function is continuous at a point, it may or may not be differentiable at that point.
- If a function is discontinuous at a point, it is not differentiable at that point.

Key Formulae

- **Formulae For Limits**

(a) $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1 = \lim\limits_{x\to 0} \dfrac{x}{\sin x}$

(b) $\lim\limits_{x\to 0} \dfrac{\tan x}{x} = 1 = \lim\limits_{x\to 0} \dfrac{x}{\tan x}$

(c) $\lim\limits_{x\to 0} \dfrac{\sin^{-1} x}{x} = 1 = \lim\limits_{x\to 0} \dfrac{x}{\sin^{-1} x}$

(d) $\lim\limits_{x\to 0} \dfrac{\tan^{-1} x}{x} = 1 = \lim\limits_{x\to 0} \dfrac{x}{\tan^{-1} x}$

(e) $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \log_e a, a > 0$

(f) $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$

(g) $\lim\limits_{x\to 0} \dfrac{\log_e(1+x)}{x} = 1$

(h) $\lim\limits_{x\to a} \dfrac{x^n - a^n}{x-a} = na^{n-1}$

(i) $\lim\limits_{x\to 0} (1+kx)^{1/x} = e^k$, where k is any constant

(j) $\lim\limits_{x\to \infty} \dfrac{\log_a(1+x)}{x} = \log_a e,\ a > 0,\ a \neq 1$

(k) $\lim\limits_{x\to \infty} \dfrac{\sin x}{x} = \lim\limits_{x\to \infty} \dfrac{\cos x}{x} = 0$

(l) $\lim\limits_{x\to \infty} \cos\left(\dfrac{1}{x}\right) = 1$

Topic-2

Differentiation

Concepts covered: Derivatives of trigonometric functions, exponential functions, logarithmic functions, inverse trigonometric functions, implicit functions, derivative of composite functions, function using chain rule, derivatives of parametric functions.

Differentiation of a function with respect to another function, logarithmic differentiation and successive differentiation.

Revision Notes

- **Derivative :** The derivative of a given function $f(x)$ at $x = 0$ of its domain is defined as :

$f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$, provided the limit exists.

- **Derivatives of Some Standard Functions :**

(a) $\dfrac{d}{dx}(x^n) = nx^{n-1}$

(b) $\dfrac{d}{dx}(k) = 0$, where k is any constant

(c) $\frac{d}{dx}(a^x) = a^x \log_e a, a > 0$

(d) $\frac{d}{dx}(e^x) = e^x$

(e) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a} = \frac{1}{x}\log_a e$

(f) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(g) $\frac{d}{dx}(\sin x) = \cos x$

(h) $\frac{d}{dx}(\cos x) = -\sin x$

(i) $\frac{d}{dx}(\tan x) = \sec^2 x$

(j) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(k) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(l) $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

(m) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, x \in (-1, 1)$

(n) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, x \in (-1, 1)$

(o) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, x \in R$

(p) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, x \in R$

(q) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$, where $x \in (-\infty, -1) \cup (1, \infty)$

(r) $\frac{d}{dx}(\text{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$, where $x \in (-\infty, -1) \cup (1, \infty)$

- **Product rule of derivatives**

$$\frac{d}{dx}(u.v) = u.\frac{d}{dx}(v) + v\frac{d}{dx}(u)$$

- **Quotient rule of derivatives**

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2} = \frac{vu' - uv'}{v^2}$$

- **Chain Rule**

If $y = f(u)$, $u = g(w)$, $w = h(x)$

then, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dw}.\frac{dw}{dx}$ or $\frac{dy}{dx} = f'(u).g'(w).h'(x)$

Important Facts

- **Following derivatives should also be memorized by you for quick use :**
  - $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
  - $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$
  - $\frac{d}{dx}(x^x) = x^x(1 + \log x)$

- **Application of Logarithmic operation :**

If $y = \{f_1(x)\}^{f_2(x)}$ or $y = f_1(x).\, f_2(x)....$ or $y = \frac{f_1(x).f_2(x).f_3(x)....}{g_1(x).g_2(x).g_3(x)....}$

then it is convenient to take the logarithm of the function first and then differentiate.

- **Derivatives of Parametric function**

If $y = f(t)$ and $x = g(t)$, where t is a parameter, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

- **Derivatives of a function w.r.t. to another function**

Let $y = f(x)$; $z = g(x)$, then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$

- **Derivatives of Implicit functions**

If the variables x (independent variable) and y (dependent variable) are connected by a relation of the form $f(x, y) = 0$, then y is said to be an implicit function of x.

To find $\frac{dy}{dx}$ in such cases, we differentiate both sides of given function w.r.t. x, keeping in mind that derivative of

$\phi(y)$ w.r.t. x is $\frac{d\phi}{dy}.\frac{dy}{dx}$.

e.g. $\therefore$ $\frac{dy}{dx}$ of implicit function $xy = 1$ is obtained as following :

$$xy = 1$$

$$\frac{d}{dx}(xy) = \frac{d}{dx}(1)$$

$$\Rightarrow \qquad \frac{d}{dx}(x).y + x\frac{d}{dx}(y) = 0$$

$$\Rightarrow \qquad 1.y + x.\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{-y}{x}$$

CHAPTER-6
APPLICATIONS OF DERIVATIVES

Topic-1 Rate of Change of Bodies

Concept covered: Interpretation of dy/dx as a rate measure

Revision Notes

➢ **Interpretation of $\frac{dy}{dx}$ as a rate measure :**

If two variables x and y are varying with respect to another variables say t, *i.e.*, if $x = f(t)$ and $y = g(t)$, then by the Chain Rule, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0$$

Thus, the rate of change of y with respect to x can be calculated by using the rate of change of y and that of x both with respect to t.

Also, if y is a function of x and they are related as $y = f(x)$ then, $f'(\alpha)$, *i.e.*, represents the rate of change of y with respect to x at the instant when $x = \alpha$.

Topic-2 Tangents and Normals

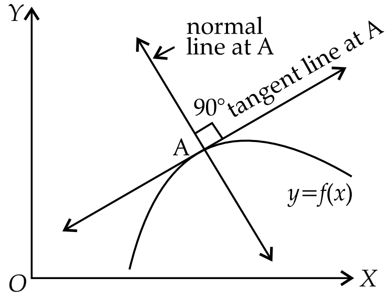
Concepts covered: Slope of a line, equation of tangent, equation of normal, acute angle between two curves

Revision Notes

➢ **Slope or gradient of a line :**

If a line makes an angle θ with the positive direction of X-axis in anti-clockwise direction, then $\tan\theta$ is called the slope or gradient of the line. [Note that θ is taken as positive or negative accordingly as it is measured in anti-clockwise (*i.e.*, from positive direction of X-axis to the positive direction of Y-axis) or clockwise direction respectively.]

➢ **Pictorial representation of tangent and normal :**



➢ **Facts about the slope of a line :**

(i) If a line is parallel to X-axis (or perpendicular to Y-axis), then its slope is 0 (zero) or $\frac{dy}{dx} = 0$.

(ii) If a line is parallel to Y-axis (or perpendicular to X-axis), then its slope is $\frac{1}{0}$, *i.e.*, not defined or $\frac{dx}{dy} = 0$.

(iii) If two lines are perpendicular, then product of their slopes equals – 1, *i.e.*, $m_1 \times m_2 = -1$. Whereas, for two parallel lines, their slopes are equal, *i.e.*, $m_1 = m_2$. (Here in both the cases m_1 and m_2 represent the slopes of respective lines).

➢ **Equation of tangent at (x_1, y_1) :**

$(y - y_1) = m_T(x - x_1)$, where m_T is the slope of tangent such that $m_T = \left[\frac{dy}{dx}\right]_{at(x_1, y_1)}$

➢ **Equation of Normal at (x_1, y_1) :**

$(y - y_1) = m_N(x - x_1)$, where m_N is the slope of such that $m_N = \dfrac{-1}{\left[\frac{dy}{dx}\right]_{at(x_1, y_1)}}$

Note that $m_T \times m_N = -1$, which is obvious because tangent and normal are perpendicular to each other. In other words, the tangent and normal lines are inclined at right angle on each other.

➢ **Acute angle between the two curves whose slopes m_1 and m_2 are known :**

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 \cdot m_2}\right| \quad \Rightarrow \quad \theta = \tan^{-1}\left|\frac{m_2 - m_1}{1 + m_1 \cdot m_2}\right|$$

It is absolutely sufficient to find one angle (generally the acute angle) between the two curves. Other angle between the curves is given by $\pi - \theta$.

Note that if the curves cut orthogonally (*i.e.*, they cut each other at right angles), then it means $m_1 \times m_2 = -1$, where m_1 and m_2 represent the slopes of the tangent of curves at the intersection point.

➢ **Finding the slope of a line $ax + by + c = 0$:**

Step 1 : Express the given line in the standard slope-intercept form $y = mx + c$, *i.e.*, $y = \left(-\frac{a}{b}\right)x - \frac{c}{b}$.

Step 2 : By comparing to the standard form $y = mx + c$, we can conclude $-\frac{a}{b}$ is the slope of given line $ax + by + c = 0$.

Topic-3 Increasing / Decreasing Functions

Concepts covered: Increasing functions, Decreasing functions

Revision Notes

➢ A function $f(x)$ is said to be an increasing function in $[a, b]$, if as x increases, $f(x)$ also increases, *i.e.*, if $\alpha, \beta \in [a, b]$ and $\alpha > \beta \Rightarrow f(\alpha) > f(\beta)$.

If $f'(x) \geq 0$ lies in (a, b), then $f(x)$ is an increasing function in $[a, b]$, provided $f(x)$ is continuous at $x = a$ and $x = b$.

➢ A function $f(x)$ is said to be a decreasing function in $[a, b]$, if as x increase, $f(x)$ decreases, *i.e.*, if $\alpha, \beta \in [a, b]$ and $\alpha > \beta \Rightarrow f(\alpha) < f(\beta)$.

If $f'(x) \leq 0$ lies in (a, b), then $f(x)$ is a decreasing function in $[a, b]$, provided $f(x)$ is continuous at $x = a$ and $x = b$.

(i) A function $f(x)$ is a constant function in $[a, b]$, if $f'(x) = 0$ for each $x \in (a, b)$.

(ii) By monotonic function $f(x)$ in interval I, we mean that f is either only increasing in I or only decreasing in I.

➢ Finding the intervals of increasing and/or decreasing function:

Algorithm

Step 1 : Consider the function $y = f(x)$.

Step 2 : Find $f'(x)$.

Step 3 : Put $f'(x) = 0$ and solve to get the critical point(s).

Step 4 : The value(s) of x for which $f'(x) > 0$, $f(x)$ is increasing and the value(s) of x for which $f'(x) < 0$, $f(x)$ is decreasing.

Mnemonics

Concept: Increasing and Decreasing Function

Mnemonics: Moving Immense Organs Insert Outdoor Extreme Operations

Interpretation:

Interpretation:

Dead	**Z**ombies	**C**onsider	**G**reen	**L**emons
↓	↓	↓	↓	↓
Find **Derivative** of function $f(x)$ *i.e.*, $f'(x)$	Put Derivative equal to **zero** *i.e.*, $f'(x) = 0$	Get **Critical** points *i.e.*, values of x	If $f'(x)$ is **Greater** than zero *i.e.*, $f'(x) > 0$, then $f(x)$ is increasing	If $f'(x)$ is **Less** than zero *i.e.*, $f'(x) < 0$, then $f(x)$ is decreasing

Topic-4

Maxima and Minima

Concepts covered: Stationary points, absolute maxima/minima, local maxima/minima, first derivative test and second derivative test and application problems based on maxima/minima.

Revision Notes

- **Understanding maxima and minima :**

  Consider $y = f(x)$ be a well defined function on an interval I, then

  (a) f is said to have a maximum value in I, if there exists a point c in I such that $f(c) > f(x)$, for all $x \in I$

  The value corresponding to $f(c)$ is called the maximum value of f in I and the point c is called the **point of maximum value of f in I.**

  (b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) < f(x)$, for all $x \in I$.

  The value corresponding to $f(c)$ is called the minimum value of f in I and the point c is called the **point of minimum value of f in I.**

  (c) f is said to have an extreme value in I, if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I.

  The value $f(c)$ in this case, is called an extreme value of f in I and the point c called an **extreme point.**

- Let f be a real valued function and also take a point c from its domain. Then

  (i) c is called a point of **local maxima** if there exists a number $h > 0$ such that $f(c) > f(x)$, for all x in $(c - h, c + h)$.

  The value $f(c)$ is called the **local maximum value of** f.

  (ii) c is called a point of **local minima** if there exists a number $h > 0$ such that $f(c) < f(x)$, for all x in $(c - h, c + h)$.

  The value $f(c)$ is called the **local minimum value** of f.

- **Critical points :**

  It is a point c (say) in the domain of a function $f(x)$ at which either $f'(x)$ vanishes, *i.e.*, $f'(c) = 0$ or f is not differentiable.

- **First Derivative Test :**

  Consider $y = f(x)$ be a well defined function on an open interval I. Now procedure have been mentioned in the following algorithm :

  Step 1 : Find $\frac{dy}{dx}$.

  Step 2 : Find the critical point(s) or stationary point(s) by putting $\frac{dy}{dx} = 0$. Suppose $c \in I$ (where I is the interval) be any critical point and f be continuous at this point c. Then we may have the following situations :

  - $\frac{dy}{dx}$ changes sign from **positive to negative** as x increases through c, then the function attains a **local maximum** at $x = c$.
  - $\frac{dy}{dx}$ changes sign from **negative to positive** as x increases through c, then the function attains a **local minimum** at $x = c$.

- **Second Derivative Test :**

  Consider $y = f(x)$ be a well defined function on an open interval I and twice differentiable at a point c in the interval. Then we observe that :

  - $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$

    The value $f(c)$ is called the local maximum value of f.
  - $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$

    The value $f(c)$ is called the local minimum value of f.

  This test fails if $f'(c) = 0$ and $f''(c) = 0$. In such a case, we use **first derivative test** as discussed in the above.

- **Absolute maxima and absolute minima :**

  If f is a continuous function on a **closed interval** I, then f has the absolute maximum value and f attains it atleast once in I. Also f has the absolute minimum value and the function attains it atleast once in I.

  Algorithm

  Step 1 : Find all the critical points of f in the given interval, *i.e.*, find all the points x where either $f'(x) = 0$ or f is not differentiable.

  Step 2 : Take the end points of the given interval.

  Step 3 : At all these points (*i.e.*, the points found in Step 1 and Step 2) calculate the values of f.

  Step 4 : Identify the maximum and minimum value of f out of the values calculated in Step 3. This maximum value will be the **absolute maximum value** of f and the minimum value will be the **absolute minimum value** of the function f.

  Absolute maximum value is also called as **global maximum value** or **greatest value**. Similarly, absolute minimum value is called as **global minimum value** or the **least value**.

UNIT – III : CALCULUS

CHAPTER-7

INTEGRALS

Topic-1 Indefinite Integral

Concepts covered: Integration as inverse process of differentiation, Integration of a variety of functions by substitution, by partial fraction and by parts, evaluation of simple integral

Revision Notes

- **Meaning of Integral of Function :**

  If differentiation of a function $F(x)$ is $f(x)$ *i.e.*, if $\frac{d}{dx}[F(x)] = f(x)$, then we say that integral or primitive or anti-derivative of $f(x)$ is $F(x)$ and in symbols, we write, $\int f(x)dx = F(x) + C$.

  Therefore, we can say that integration is the inverse process of differentiation.

- **Indefinite Integral :** Let $f(x)$ be a function. Then family of all its Primitives / or anti-derivatives is called the indefinite integral of $f(x)$ and denoted by $\int f(x)dx$.

- **Methods of Integration :**

  (a) Integration by Substitution Method : In this method, we change the integral $\int f(x)dx$, where independent variable is x, to another integral in which independent variable is t (say different from x) such that x and t are related by $x = g(t)$.

  Let $u = \int f(x)dx$ then, $\frac{du}{dx} = f(x)$

  Again as $x = g(t)$ so we have $\frac{dx}{dt} = g'(t)$

  Now $\frac{du}{dt} = \frac{du}{dx}.\frac{dx}{dt} = f(x)\cdot g'(t)$

  On integrating both sides w.r.t. t, we get

  $$\int\left(\frac{du}{dt}\right)dt = \int f(x)g'(t)dt$$

  $$\Rightarrow \quad u = \int f[g(t)]g'(t)dt$$

  i.e., $\int f(x)dx = \int f[g(t)]g'(t)dt$, where $x = g(t)$.

  So, it is clear that substituting $x = g(t)$ in $\int f(x).dx$ will give us the same result as obtained by putting $g(t)$ in place of x and $g'(t)dt$ in place of dx.

  (b) Integration by Partial Fractions : Consider $\frac{f(x)}{g(x)}$ defines a rational polynomial function.

  (i) If the degree of numerator *i.e.*, $f(x)$ is **greater than or equal to** the degree of denominator *i.e.*, $g(x)$ then, this type of rational function is called an **improper rational function**. And if degree of $f(x)$ is smaller than the degree of denominator *i.e.*, $g(x)$, then this type of rational function is called a **proper rational function.**

  (ii) ***In rational polynomial function if the degree*** (*i.e.*, highest power of the variable) of numerator (N) is **greater than or equal to** the degree of denominator (D), then (without any doubt) **always perform the division** *i.e.*, divide the **N** by **D** before doing anything and thereafter use the following :

  $$\frac{\text{Numerator}}{\text{Denominator}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Denominator}}$$

Table Demonstrating Partial Fractions of Various Forms

Form of the Rational Functions	Form of the Partial Fractions
$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where x^2+bx+c can't be factorized further.	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

(c) Integration by Parts :

If u and v be two functions of x, then

$$\int \underset{\text{I}}{u}, \underset{\text{II}}{v}\, dx = u\left(\int v\, dx\right) - \int\left\{\frac{du}{dx}\int v\, dx\right\} dx$$

Note

We can choose first function as the one whose initials comes first in the word. 'ILATE', where

I - Inverse Trigonometric function
L - Logarithm function
A - Algebraic function
T - Trigonometric function
E - Exponential function

Key Formulae

(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

(2) $\int \frac{1}{x} dx = \log|x| + C$

(3) $\int e^x dx = e^x + C$

(4) $\int a^x dx = \frac{a^x}{\log_e a} + C$

(5) $\int \sin x\, dx = -\cos x + C$

(6) $\int \cos x\, dx = \sin x + C$

(7) $\int \sec^2 x\, dx = \tan x + C$

(8) $\int \text{cosec}^2 x\, dx = -\cot x + C$

(9) $\int \sec x \tan x\, dx = \sec x + C$

(10) $\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + C$

(11) $\int \tan x\, dx = -\log|\cos x| + C = \log|\sec x| + C$

(12) $\int \cot x\, dx = \log|\sin x| + C$

(13) $\int \sec x\, dx = \log|\sec x + \tan x| + C$

(14) $\int \text{cosec}\, x\, dx = \log|\text{cosec}\, x - \cot x| + C$

(15) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C; |x| < 1$

(16) $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

(17) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x| + C; |x| > 1$

(18) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(19) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(20) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(21) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

(22) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$

(23) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$

(24) $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

(25) $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

(26) $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

➢ **Some Standard Substitutions**

Following are some substitutions useful in evaluating integrals :

Expression	Substitution
$(a^2 + x^2)$, $\sqrt{x^2 + a^2}$, $\frac{1}{\sqrt{x^2 + a^2}}$	$x = a \tan \theta$ or $a \cot \theta$
$(a^2 - x^2)$, $\sqrt{a^2 - x^2}$, $\frac{1}{\sqrt{a^2 - x^2}}$	$x = a \sin \theta$ or $a \cos \theta$
$(x^2 - a^2)$, $\sqrt{x^2 - a^2}$, $\frac{1}{\sqrt{x^2 - a^2}}$	$x = a \sec \theta$ or $a \operatorname{cosec} \theta$
$\sqrt{\frac{a - x}{a + x}}$, $\sqrt{\frac{a + x}{a - x}}$	$x = a \cos 2\theta$
$\sqrt{\frac{x - \alpha}{\beta - x}}$, $\sqrt{(x - \alpha)(\beta - x)}$	$x = \alpha \cos^2 \theta + \beta \sin^2 \theta$
$\sqrt{\frac{x}{a + x}}$, $\sqrt{\frac{a + x}{x}}$, $\sqrt{x(a + x)}$	$x = a \tan^2 \theta$ or $a \cot^2 \theta$
$\sqrt{2ax - x^2}$	$x = a(1 - \cos \theta)$
$\sqrt{\frac{x}{a - x}}$, $\sqrt{\frac{a - x}{x}}$, $\sqrt{x(a - x)}$, $\frac{1}{\sqrt{x(a - x)}}$	$x = a \sin^2 \theta$ or $a \cos^2 \theta$
$\sqrt{\frac{x}{x - a}}$, $\sqrt{\frac{x - a}{x}}$, $\sqrt{x(x - a)}$, $\frac{1}{\sqrt{x(x - a)}}$	$x = a \sec^2 \theta$ or $a \operatorname{cosec}^2 \theta$

➢ **Solving Integrals of following types :**

(a) $\int \frac{dx}{ax^2 + bx + c}$ or $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ or $\int \sqrt{ax^2 + bx + c}\,dx$

Working Rule :

(i) Make the coefficient of x^2 unity by taking the coefficient of x^2 outside the quadratic.

(ii) Express $ax^2 + bx + c$ as sum or difference of two squares

$$i.e.,\ ax^2 + bx + c = \left[\left(x + \frac{b}{2a}\right)^2 \pm \frac{4ac - b^2}{4a^2}\right]$$

(iii) Then integrand is converted into known integrals.

(b) $\int \frac{px + q}{ax^2 + bx + c}dx$ or $\int \frac{px + q}{\sqrt{ax^2 + bx + c}}dx$ or $\int (px + q)\sqrt{ax^2 + bx + c}\,dx$

Working Rule :

(i) Put $px + q = \lambda \frac{d}{dx}(ax^2 + bx + c) + \mu$

(ii) Compare the coefficient of x and constant term on both sides, to obtain values of λ and μ.

(iii) Re-substitute the values of λ and μ and integrate it,

(c) $\int \frac{P(x)}{ax^2+bx+c}dx$, where $P(x)$ is a polynomial in x of degree $n \geq 2$

Working Rule :

(i) Divide $P(x)$ by $ax^2 + bx + c$ and write in the form $\frac{P(x)}{ax^2+bx+c} = Q(x)+\frac{R(x)}{ax^2+bx+c}$

Where $R(x)$ is a linear expression or constant and Q(x) is quotient.

(ii) How, integral reduces to the form discussed earlier.

(d) $\int \frac{dx}{a+b\sin^2 x}, \int \frac{dx}{a+b\cos^2 x}, \int \frac{dx}{a\sin^2 x+b\cos^2 x}, \int \frac{dx}{a\sin^2 x+b\cos^2 x+c}, \int \frac{dx}{(a\sin x+b\cos x)^2}$

(i) Divide numerator and denominator by $\cos^2 x$.

(ii) Replace $\sec^2 x$, if any, in denominator by $1 + \tan^2 x$.

(iii) Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$, then the given integral becomes standard form of integrals in t variables.

(iv) After integration, convert t variables in terms of x variables.

(e) $\int \frac{dx}{a+b\sin x}, \int \frac{dx}{a+b\cos x}, \int \frac{dx}{a\sin x+b\cos x}, \int \frac{dx}{a\sin x+b\cos x+c},$

(i) Put $\sin x = \frac{2\tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}}$ and $\cos x = \frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}$

(ii) Replace $1+\tan^2 \frac{x}{2}$ by $\sec^2 \frac{x}{2}$.

(iii) Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$ *i.e.*, $\sec^2\frac{x}{2}dx = 2\,dt$, then the given integral becomes standard form of integrals in t variables.

(iv) After integration, convert t variables in terms of x variables.

(f) $\int \frac{a\sin x+b\cos x}{c\sin x+d\cos x}dx,$

Working Rule :

(i) Write, numerator = λ (differentiation of denominator) + μ (Denominator, *i.e.*, $a \sin x + b \cos x$)

$= \lambda (c \cos x - d \sin x) + \mu (c \sin x + d \cos x)$

(ii) Obtain the values of λ and μ by equating the coefficients of $\sin x$ and $\cos x$ on both the sides.

(iii) Replace numerator in the integrand and solve it.

(g) $\int \frac{x^2+1}{x^4+\lambda x^2+1}dx, \int \frac{x^2-1}{x^4+\lambda x^2+1}dx, \int \frac{1}{x^4+\lambda x^2+1}dx$

Working Rule :

(i) Divide numerator and denominator by x^2.

(ii) Express the denominator of integrands either of the form of $\left(x+\frac{1}{x}\right)^2 \pm k^2$ or $\left(x-\frac{1}{x}\right)^2 \pm k^2$.

(iii) Introduce $d\left(x+\frac{1}{x}\right)$ or $d\left(x-\frac{1}{x}\right)$ or both in numerator.

(iv) Put $x+\frac{1}{x} = t$ or $x-\frac{1}{x} = t$ as the case may be required.

(v) Integral reduced to the form of $\int \frac{1}{x^2+a^2}dx$ or $\int \frac{1}{x^2-a^2}dx$.

(vi) For the integrals $\int \frac{x^2+a^2}{x^2+kx^2+a^4}dx$ or $\int \frac{x^2-a^2}{x^4+kx^2+a^4}dx$

(vii) After dividing both numerator and denominator by x^2.

(viii) Put $x-\frac{a^2}{x}=t$ or $x+\frac{a^2}{x}=t$, then the given integral becomes standard form of integrals in t variables.

(ix) After integration, convert t variables in terms of x variables.

(h) $\int\sqrt{\tan x}\,dx, \int\sqrt{\cot x}\,dx$

Working Rule :

(i) Put $\tan x = t^2 \Rightarrow \frac{d(\tan x)}{dx}=\frac{d(t^2)}{dt}\Rightarrow \sec^2 x\,dx = 2t\,dt$

(ii) Given integral becomes standard form of integrals in t variables.

➤ **Special form of Integrals :**

(a) $\int e^x\{f(x)+f'(x)\}dx = e^x f(x) + C$

(b) $\int\frac{f'(x)}{f(x)}dx = \log|f(x)| + C$

Topic-2 Definite Integral

Concepts covered: Fundamental theorem of calculus, properties of definite integral and evaluation of definite integrals.

Revision Notes

➤ **Meaning of Definite Integral of Function :**

If $\int f(x)dx = F(x)$, be an integral of $f(x)$, then $F(b) - F(a)$ is called the definite integral of $f(x)$ between the limits a and b and in symbols it is written as $\int_a^b f(x)dx$ or $[F(x)]_a^b = F(b)-F(a)$. Moreover, the definite integral gives a unique and definite value (numeric value) of anti-derivative of the function between the given intervals. It acts as a substitute for evaluating the area analytically.

Key Formulae

➤ **Properties of definite Integral**

(a) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(b) $\int_a^b f(x)dx = \int_a^b f(t)dt$

(c) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, a<c<b$

(d) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

(e) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(f) $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is an even function } i.e., f(x)=f(-x) \\ 0, & \text{if } f(x) \text{ is an odd function } i.e., f(x)=-f(x) \end{cases}$

(g) $\int_{-a}^{a} f(x)dx = \int_0^a \{f(x)+f(-x)\}dx$

(h) $\int_0^{2a} f(x)dx = \int_0^a \{f(x)+f(2a-x)\}dx$

(i) $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x)=f(x) \\ 0, & \text{if } f(2a-x)=-f(x) \end{cases}$

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CHAPTER-8

DIFFERENTIAL EQUATIONS

Topic-1

Basic Concepts and Variable Separable Methods

Concepts covered: Definition of differential equation, degree, order, general and particular solutions of a differential equation and solution of differential equations by method of separation of variables.

Revision Notes

- **Differential Equation :**
An equation consisting of an independent variable, dependent variable and differential coefficients of dependent variable with respect to the independent variable is known as differential equation.

e.g. : **(i)** $\frac{d^2y}{dx^2} = -a^2y$, **(ii)** $\frac{dy}{dx} = \frac{x+y}{x^2}$, **(iii)** $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = p\frac{dy}{dx}$

- **Order of Differential Equation :** The order of a differential equation is the order of the highest derivative appearing in the differential equation.

e.g. : $\left(\frac{d^3y}{dx^3}\right)^2 - 3\left(\frac{dy}{dx}\right)^3 + 2 = 0$ is the differential equation of order 3 because highest order derivative of y w.r.t. x is $\frac{d^3y}{dx^3}$.

- **Degree of Differential Equation :** The degree of the differential equation is the degree (power) of the highest order derivative, when the differential coefficient has been made free from the radicals and fractions.

e.g. : $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{2}{3}} = 3\frac{d^2y}{dx^2}$ is the differential equation of degree 3, because the power of highest order derivative $\frac{d^2y}{dx^2}$ is 3 (after cubing).

- **Formation of Differential Equation :**
If the equation of the family of curves is given then its differential equation is obtained by **eliminating arbitrary constants** occurring in equation with the help of equation of the curve and the equations obtained by differentiating the equation of the curve.
- **Algorithm for the Formation of the Differential Equation :**
Step 1 : Write down the given equation of the curve.
Step 2 : Differentiate the given equation with respect to the independent variable as many times as the number of arbitrary constants.
Step 3 : Eliminate the arbitrary constants by using given equation and the equations obtained by the differentiation in step2.
- **Solution of Differential Equations :**
(a) General solution : The solution which contains as many as arbitrary constants as the order of the differential equations, *e.g.*, $y = \alpha \cos x + \beta \sin x$ is the general solution of $\frac{d^2y}{dx^2} + y = 0$.
Here, the differential equation is of second order and there are two arbitrary constants *i.e.*, α and β in the general solution.
(b) Particular solution: Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution *e.g.*, $y = 3 \cos x + 2 \sin x$ is a particular solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

(c) **Solution of Differential Equation by Variable Separable Method :** A variable separable form of the differential equation is the one which can be expressed in the form of $f(x)\,dx = g(y)\,dy$. The solution is given by

$\int f(x)dx = \int g(y)dy + k$, or $\int g(y)\,dy = \int f(n)\,dn + k$, where k is the constant of integration.

Topic-2 Linear Differential Equations

Concepts covered: Solution of linear differential equation in y and solution of linear differential equation in x.

Revision Notes

- **Linear Differential Equation :**
  A differential equation is said to be linear if dependent variable (say y) and its derivative occurs in the first degree.

(a) **Linear differential equation in y :** It is of the form $\frac{dy}{dx} + P(x)y = Q(x)$, where $P(x)$ and $Q(x)$ are functions of x only.

- **Solution of Linear Differential Equation in y :**

  Step 1 : Write the given differential equation in the form $\frac{dy}{dx} + P(x)y = Q(x)$.

  Step 2 : Find the **integration Factor** (I.F.) $= e^{\int P(x)dx}$.

  Step 3 : The solution is given by, $y(I.F.) = \int Q(x).(I.F.)dx + C$, where C is the constant of integration.

(b) **Linear Differential equation in x :** It is of the form $\frac{dx}{dy} + P(y)x = Q(y)$ where $P(y)$ and $Q(y)$ are functions of x only.

- **Solution of Linear Differential Equation in x :**

  Step 1 : Write the given differential equation in the form $\frac{dx}{dy} + P(y)x = Q(y)$.

  Step 2 : Find the **integration Factor** (I.F.) $= e^{\int P(y)dy}$.

  Step 3 : The **solution** is given by, $x(I.F.) = \int Q(y).(I.F.)dy + \lambda$, where λ is the constant of integration.

Mnemonics

Concept : Linear Differential equation $\frac{dy}{dx} + Py = Q$

Mnemonics : WHY IF KYON IF

Interpretation : Its solution can be remember as :

$$y \times IF = \int (Q \times IF)dx + C$$

WHY IF KYON IF

Topic-3 Homogeneous Differential Equations

Concepts covered: Homogeneous differential equations and their solutions.

Revision Notes

- **Homogeneous Differential Equations and their Solutions**
  - **Identifying a Homogeneous Differential Equation :**

    Step 1 : Write down the given differential equation in the form $\frac{dy}{dx} = F(x, y)$.

    Step 2 : If $f(kx, ky) = k^n f(x, y)$, then the given differential equation is homogeneous of degree 'n'.

- **Solving a homogeneous differential equation :**

Case I: If $\frac{dy}{dx} = f(x, y)$

Put $y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Case II: If $\frac{dx}{dy} = f(x, y)$

Put $x = vy$

$\Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$

Then, we separate the variables to get the required solution.

UNIT – IV : PROBABILITY

CHAPTER-9

PROBABILITY

Topic-1

Conditional Probability and Multiplication Theorem on Probability

Concepts covered: Independent and dependent events, conditional events, law of probability, addition theorem, multiplication theorem, conditional probability.

Revision Notes

➢ **Basic Definition :**

(a) Random Experiment : An experiment, whose all possible outcomes are known in advance.
e.g., Tossing of a coin, Throwing a dice selecting a card from a deck of cards.

(b) Sample-Space : A set of all possible outcomes associated with a random experiment. It is denoted by '*S*'.
e.g., (i) In the experiment of tossing a coin.
Sample Space, S = $\{H, T\}$
(ii) In the experiment of throwing a dice.
Sample, Space, S = $\{1, 2, 3, 4, 5, 6\}$

(c) Event : An event is a subset of a sample space. If an event is a set containing only one element of a sample space is called a 'simple event'. A 'compound event' is one that can be represented as a union of sample points.
e.g., In a throw of a single dice, getting number 4 is a simple event where as getting an even number is a compound event.

(d) Probability :
Let S and E be the sample space and event of an experiment respectively.
Then,

$$\text{Probability} = \frac{\text{Number of favourable events}}{\text{Total number of elementary events}} = \frac{n(E)}{n(S)}$$

$$0 \le n(E) \le n(S) \Rightarrow 0 \le \frac{n(E)}{n(S)} \le 1$$

or $$0 \le P(E) \le 1$$

Hence, if $P(E)$ denotes the probability of occurrence of an event E, then

$0 \le P(E) \le 1$ and $P(\bar{E}) = 1 - P(E)$ such that $P(\bar{E})$ denotes the probability of non-occurrence of the event E.

- Note that $P(\bar{E})$ can also be represented as $P(E')$.

➢ **Mutually Exclusive or Disjoint Events :**

Two events A and B are said to be mutually exclusive, if occurrence of one prevents the occurrence of the other *i.e.*, they can't occur simultaneously.

In this case, sets A and B are disjoint, *i.e.*, $A \cap B = \phi$.

Consider an example of throwing a die. We have the sample space as, $S = \{1, 2, 3, 4, 5, 6\}$

Suppose A = the event of occurrence of a number greater than 4 = $\{5, 6\}$

B = the event of occurrence of an odd number = $\{1, 3, 5\}$

and C = the event of occurrence of an even number = $\{2, 4, 6\}$

In these events, the events B and C are mutually exclusive events but A and B are not mutually exclusive events because they can occur together (when the number 5 comes up). Similarly, A and C are not mutually exclusive events as they can also occur together (when the number 6 comes up).

(i) If A and B are mutually exhaustive events, then we always have :

$$P(A \cap B) = 0 \qquad [\text{As } n(A \cap B) = n(\phi) = 0]$$

$$P(A \cup B) = P(A) + P(B)$$

(ii) If A, B and C are mutually exhaustive events, then we always have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

➢ **Independent Events :**

Two events are independent if the occurrence of one does not affect the occurrence of the other.

Consider an example of drawing two balls one by one with replacement from a bag containing 3 red and 2 black balls.

Suppose A = the event of getting a red ball in first draw.

B = the event of getting of a black ball in the second draw.

Then
$$P(A) = \frac{3}{5}, \qquad P(B) = \frac{2}{5}$$

Here, probability of occurrence of event B is not affected by the occurrence or non-occurrence of the event A. Hence, events A and B are independent events.

➢ **Exhaustive Events :**

Two or more events say A, B and C of an experiment are said to be exhaustive events, if

(i) their union is the total sample space, *i.e.*, $A \cup B \cup C = S$

(ii) the event A, B and C are disjoint in pairs, *i.e.*, $A \cap B = \phi$, $B \cap C = \phi$ and $C \cap A = \phi$.

(iii) $P(A) + P(B) + P(C) = 1$

Consider an example of throwing a die. We have $S = \{1, 2, 3, 4, 5, 6\}$.

Suppose A = the event of getting of an even number = $\{2, 4, 6\}$

B = the event of getting of an odd number = $\{1, 3, 5\}$

and C = the event of getting a number multiple of 3 = $\{3, 6\}$

In these events, the events A and B are exhaustive events as $A \cup B = S$, but the events A and C or the events B and C are not exhaustive events as $A \cup C \ne S$ and similarly, $B \cup C \ne S$.

➢ **Conditional Probability :**

By the conditional probability, we mean the probability of occurrence of event A when B has already occurred. The conditional probability of occurrence of event A when B has already occurred' is sometimes also called as probability of occurrence of event A w.r.t. B.

- $P(A/B) = \dfrac{P(A \cap B)}{P(B)}, B \ne \phi, i.e., P(B) \ne 0$

- $P(B/A) = \dfrac{P(A \cap B)}{P(A)}, A \ne \phi, i.e., P(A) \ne 0$

- $P(\bar{A}/B) = \dfrac{P(\bar{A} \cap B)}{P(B)}, P(B) \ne 0$

- $P(A/\bar{B}) = \dfrac{P(A \cap \bar{B})}{P(\bar{B})}, P(\bar{B}) \ne 0$

- $P(\bar{A}/\bar{B}) = \dfrac{P(\bar{A}\cap\bar{B})}{P(\bar{B})}, P(\bar{B}) \neq 0$
- $P(A/B) + P(\bar{A}|B) = 1, P(B) \neq \phi$

➢ **Multiplication Theorem of Probability:**

By Multiplication theorem of Probability, if A and B are two events associated with sample space S, then

$P(A \cap B) = P(A)\, P(B/A)$, where $P(A) \neq 0$

Or $P(A \cap B) = P(B)\, P(A/B)$, where $P(B) \neq 0$

Key Formulae

➢ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, *i.e.*, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

➢ $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

➢ $P(\bar{A} \cap B) = P(\text{only } B) = P(B - A) = P(B \text{ but not } A) = P(B) - P(A \cap B)$

➢ $P(A \cap \bar{B}) = P(\text{only } A) = P(A - B) = P(A \text{ but not } B) = P(A) - P(A \cap B)$

➢ $P(\bar{A} \cap \bar{B}) = P(\text{neither } A \text{ nor } B) = 1 - P(A \cup B)$

Events and Symbolic Representations

Description of the event	Equivalent set notation
Event A	A
Not A	$\bar{A}$ or A'
A or B (occurrence of atleast one A or B)	$A \cup B$ or $A + B$
A and B (simultaneous occurrence of both A and B)	$A \cap B$ or AB
A but not B (A occurs but B does not)	$A \cap \bar{B}$ or $A - B$
Neither A nor B	$\bar{A} \cap \bar{B}$
Atleast one A, B or C	$A \cup B \cup C$
All the three A, B and C	$A \cap B \cap C$

Mnemonics

Concept: Independent Events and Mutually Exclusive events

Mnemonics: I is not ME and ME is not I

Interpretations :

I : Independent Events

ME : Mutually Exclusive Events

Topic-2 Random Variable and its Probability Distributions

Concepts covered: Random variable, probability distribution of a random variable, mean and variance of random variable

Revision Notes

➢ **Random Variable :**

A random variable is a real valued function defined over the sample space of an experiment. In other words, a random variable is a real-valued function whose domain is the sample space of a random experiment. A random variable is usually denoted by uppercase letter X, Y, Z etc.

➢ **Probability Distribution of a Random Variable :**

If the values of a random variable together with the corresponding probabilities are given, then this description is called a probability distribution of the random variable.

Key Terms

- **Discrete random variable :** It is a random variable which can take only finite or countable number of values.
- **Continuous random variable :** A variable can take any value between two given limits is called a continuous random variable.

Key Formulae

- Mean of Expectation of a random variable $X = \mu = \sum_{i=1}^{n} x_i P_i$
- Variance $= (\sigma)^2 = \sum_{i=1}^{n} P_i x_i^2 - \mu^2$
- Standard Deviation $= \sigma = \sqrt{\text{Variance}}$

Topic-3 Bayes' Theorem

Concept covered: Bayes' Theorem

Revision Notes

- **Bayes' Theorem :**
  If $E_1, E_2, E_3, ..., E_n$ are n non-empty events constituting a partition of sample space S, *i.e.*, $E_1, E_2, E_3, ..., E_n$ are pair-wise disjoint and $E_1 \cup E_2 \cup E_3 \cup ... \cup E_n = S$ and A is any event of non-zero probability then,

$$P(E/A) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{j=1}^{n} P(E_j) P(A/E_j)}, i = 1, 2, 3, ..., n$$

e.g., $$P(E_1/A) = \frac{P(E_1) \cdot P(A/E_1)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + P(E_3) \cdot P(A/E_3)}$$

- Bayes' theorem is also known as the formula for the probability of causes.
- If $E_1, E_2, E_3, ..., E_n$ form a partition of S and A be any event, then
  $P(A) = P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + ... + P(E_n) \cdot P(A/E_n)$ $[\because \; P(E_i \cap A) = P(E_i) \cdot P(A/E_i)]$
- The probabilities $P(E_1)$, $P(E_2)$, ..., $P(E_n)$ which are known before the experiment takes place are called **prior probabilities** and $P(A/E_n)$ are called **posterior probabilities**.

UNIT – V : VECTORS

CHAPTER-10

VECTORS

Topic-1 Basic Algebra of Vectors and Dot Product

Concepts covered: Vectors and their representation, types of vectors (zero, unit, equal, parallel and collinear, position vector, direction ratio and direction cosines, addition of vectors, multiplication of a vector by a scalar, section formula, scalar (dot) product and projection of vectors.

Revision Notes

- **Vector and their Representation**
  A quantity having magnitude as well as the direction is called a vector. It is denoted by a directed line segment as $\vec{AB}$ or $\vec{a}$. Its magnitude (or modulus) is $|\vec{AB}|$ or $|\vec{a}|$ otherwise, simply AB or a.

A is called the initial point and B is called the terminal point.

- **Types of vectors :**
- **Zero or Null vector :** It is the vector whose initial and terminal points are coincident. It is denoted by $\vec{0}$. Its magnitude is 0 (zero).

  Note : Any non-zero vector is called a **proper vector.**
- **Negative of a vectors :** The vector which has the same magnitude as the $\vec{r}$ but opposite direction. It is denoted by $-\vec{r}$. Hence if, $\vec{AB} = \vec{r} \Rightarrow \vec{BA} = -\vec{r}$ *i.e.,* $\vec{AB} = -\vec{BA}$, $\vec{PQ} = -\vec{QP}$ etc.
- **Unit vector :** It is a vector with the unit magnitude, the unit vector $\vec{r}$ in the direction of vector $\vec{r}$ is given by $\hat{r} = \dfrac{\vec{r}}{|\vec{r}|}$, such that $|\hat{r}| = 1$. So, if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then its unit vector is :

$$\hat{r} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}$$

- **Equal vectors :** Two vectors are said to be equal, if they have the same magnitude as well as direction, regardless of the position of their initial points,

  Thus , $\vec{a} = \vec{b} \Leftrightarrow \begin{cases} |\vec{a}| = |\vec{b}| \\ \vec{a} \text{ and } \vec{b} \text{ have same direction} \end{cases}$

  Also, if $\vec{a} = \vec{b} \Rightarrow a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \Rightarrow a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3$.
- **Collinear or Parallel vectors :** Two vectors $\vec{a}$ and $\vec{b}$ are collinear or parallel, if there exists a non-zero scalar λ such that $\vec{a} = \lambda\vec{b}$
  - It is important to note that the respective coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\vec{a}$ and $\vec{b}$ are proportional provided they are parallel or collinear to each other.
  - The d.r's of parallel vectors are same (or are in proportion).
  - The vectors $\vec{a}$ and $\vec{b}$ will have same or opposite direction as λ is positive or negative respectively.
- **Position Vector :**

  The position vector of a point say $P(x, y, z)$ is $\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and the magnitude is $|\vec{r}| = \left|\sqrt{x^2+y^2+z^2}\right|$.

  The vector $\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is said to be in its component form. Here, x, y, z are called the scalar components or rectangular components of $\vec{r}$ and $x\hat{i}, y\hat{j}, z\hat{k}$ are the vector components of $\vec{r}$ along X, Y, Z-axes, respectively.
  - Also, $\vec{AB} =$ (Position vector of B) – (Position vector of A).

  For example, let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$. Then, $\vec{AB} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$

  Here, $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors along the axes OX, OY and OZ respectively.
- **Direction Ratio and Direction Cosine :**

  If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then coefficient of $\hat{i}, \hat{j}, \hat{k}$ in $\vec{r}$ *i.e.,* x, y, z are called the direction ratio (abbreviated as. d.r's) of vector $\vec{r}$. These are denoted by a, b, c (*i.e.,* $a = x$, $b = y$, $c = z$; in a manner we can say that scalar components of vector $\vec{r}$ and its d.r.'s both are the same).

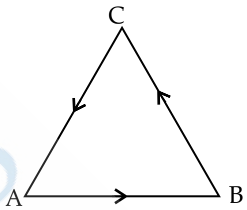
  Also, the coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\hat{r}$ (which is the unit vector of $\vec{r}$) *i.e.,* $\dfrac{x}{\sqrt{x^2+y^2+z^2}}, \dfrac{y}{\sqrt{x^2+y^2+z^2}}, \dfrac{z}{\sqrt{x^2+y^2+z^2}}$ are called direction cosines (which is abbreviated as d.c.'s) of vector $\vec{r}$.

- These direction cosines are denoted by l, m, n such that $l = \cos\alpha, m = \cos\beta, n = \cos\gamma$ and $l^2 + m^2 + n^2 = 1$ $\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.
- It can be easily concluded that $\vec{r} = lr\hat{i} + mr\hat{j} + nr\hat{k} = r(\cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k})$. [Here $r = |\vec{r}|$]

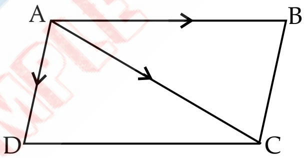
➢ **Addition of vectors :**

(a) Triangular law : If two adjacent sides (say sides *AB* and *BC*) of a triangle *ABC* are represented by $\vec{a}$ and $\vec{b}$ taken in same order, then the third side of the triangle taken in the reverse order gives the sum of vectors $\vec{a}$ and $\vec{b}$ *i.e.*, $\vec{AC} = \vec{AB} + \vec{BC} \Rightarrow \vec{AC} = \vec{a} + \vec{b}$.

Also, since $\vec{AC} = -\vec{CA} \Rightarrow \vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$.



(b) Parallelogram law : If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and the direction by the two adjacent sides (say *AB* and *AD*) of a parallelogram *ABCD*, then their sum is given by that diagonal of parallelogram which is co-initial with $\vec{a}$ and $\vec{b}$ *i.e.*, $\vec{AB} + \vec{AD} = \vec{AC}$.



➢ **Properties of Vector Addition :**

(a) Commutative property : $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Consider $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ be any two given vectors,

Then $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k} = \vec{b} + \vec{a}$

- **Associative property :** $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
- **Additive identity :** $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$
- **Additive inverse :** $(-\vec{a}) + \vec{a} = \vec{a} + (-\vec{a}) = \vec{0}$

➢ **Multiplication of a Vector by a Scalar :** Let $\vec{a}$ be any vector and k be any scalar. Then the product $k\vec{a}$ is defined as a vector whose magnitude is $|k|$ times that of $\vec{a}$ and the direction is

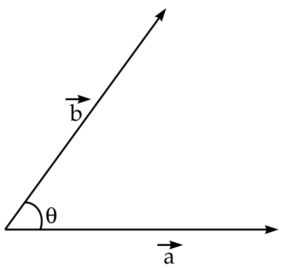
(i) Same as that of $\vec{a}$ if k is positive and

(ii) Opposite that of $\vec{a}$, if k is negative

➢ **Scalar Product or Dot Product :** The dot product of two vectors $\vec{a}$ and $\vec{b}$ is defined by $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$

Consider $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,

then $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.



➢ **Properties of Dot product :**

- $\hat{i}\cdot\hat{i} = |\hat{i}||\hat{i}|\cos 0 = 1 \Rightarrow \hat{i}\cdot\hat{i} = 1 = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k}$
- $\hat{i}\cdot\hat{j} = |\hat{i}||\hat{j}|\cos\frac{\pi}{2} = 0 \Rightarrow \hat{i}\cdot\hat{j} = 0 = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i}$
- $\vec{a}\cdot\vec{b} \in \mathbb{R}$, where $\mathbb{R}$ is real number *i.e.*, any scalar.
- $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (Commutative property of dot product).

- $\vec{a}\cdot\vec{b}=0 \Leftrightarrow \vec{a}\perp\vec{b}$ **or** $|\vec{a}|=0$ or $|\vec{b}|=0$
- If $\theta=0$, then $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|$. Also, $\vec{a}\cdot\vec{a}=|\vec{a}|^2=a^2$ **as** θ **in this case is 0.**
- Moreover if $\theta=\pi$, then $\vec{a}\cdot\vec{b}=-|\vec{a}||\vec{b}|$.
- $\vec{a}\cdot\left(\vec{b}+\vec{c}\right)=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ (Distributive property of dot product).
- $\vec{a}\cdot\left(-\vec{b}\right)=-\left(\vec{a}\cdot\vec{b}\right)=\left(-\vec{a}\right)\cdot\vec{b}$

➢ **Projection of Vectors :**

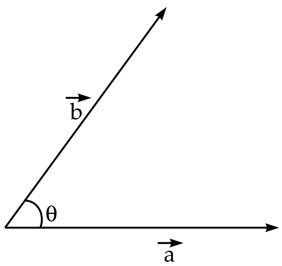
- Projection of a vector $\vec{a}$ on the other vector say $\vec{b}$ is given as $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)$.
- Projection of a vector $\vec{b}$ on the other vector say $\vec{a}$ is given as $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)$.

Key Formulae

➢ **Section Formula :**
The position vector of a point say P dividing a line segment joining the points A and B whose position vectors are $\vec{a}$ and $\vec{b}$ respectively, in the ratio $m:n$.

(a) internally, $\overrightarrow{OP}=\dfrac{m\vec{b}+n\vec{a}}{m+n}$ **(b)** externally, $\overrightarrow{OP}=\dfrac{m\vec{b}-n\vec{a}}{m-n}$

- Also, if point P is the mid-point of line segment AB, then $\overrightarrow{OP}=\dfrac{\vec{a}+\vec{b}}{2}$



➢ **Angle between two vectors $\vec{a}$ and $\vec{b}$ can be found by the expression given below :**

$\cos\theta=\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}\right)$ or $\theta=\cos^{-1}\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$

Topic-2

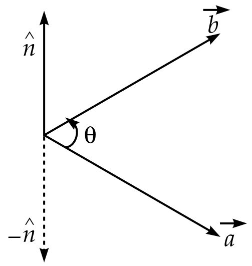
Cross Product

Concepts covered: Cross product of vectors, its properties, triangle inequality, area of triangle and area of parallelogram.

Revision Notes

➢ **The cross product of two vectors $\vec{a}$ and $\vec{b}$ is defined by :**
$\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, $0\le\theta\le\pi$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ (see Figure)
Consider $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$,

then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$



Note : Unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

➢ **Properties of Cross Product :**

- $\hat{i}\times\hat{i} = |\hat{i}||\hat{i}|\sin 0 = \vec{0} \Rightarrow \hat{i}\times\hat{i} = \vec{0} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k}$
- $\hat{i}\times\hat{j} = |\hat{i}||\hat{j}|\sin\dfrac{\pi}{2}\cdot\hat{k} = \hat{k}$. Similarly, $\hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$
- $\vec{a}\times\vec{b}$ is a vector $\vec{c}$ (say), then this vector $\vec{c}$ is perpendicular to both the vectors $\vec{a}$ and $\vec{b}$.
- $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow \vec{a}\,||\,\vec{b}$ or, $\vec{a} = \vec{0}, \vec{b} = \vec{0}$
- $\vec{a}\times\vec{a} = \vec{0}$
- $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ (Commutative property does not hold for cross product).
- $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$ (Left distributive).
- $(\vec{b}+\vec{c})\times\vec{a} = \vec{b}\times\vec{a} + \vec{c}\times\vec{a}$ (Right distributive).

➢ **Triangle Inequality :**

For any two vectors $\vec{a}$ and $\vec{b}$ we always have $|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$

Proof : The given inequality holds trivially when either $\vec{a} = 0$ or $\vec{b} = 0$ *i.e.,* in such a case

$$|\vec{a}+\vec{b}| = 0 = |\vec{a}| + |\vec{b}|$$

So, let us check it for $|\vec{a}| \ne 0 \ne |\vec{b}|$.

Then consider $\quad |\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$

$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta$

For $\cos\theta \le 1$, we have: $\quad 2|\vec{a}||\vec{b}|\cos\theta \le 2|\vec{a}||\vec{b}|$

$\Rightarrow \quad |\vec{a}|^2|\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta \le |\vec{a}|^2|\vec{b}|^2 + 2|\vec{a}||\vec{b}|$

$\Rightarrow \quad |\vec{a}+\vec{b}|^2 \le \left(|\vec{a}| + |\vec{b}|\right)^2$

$\Rightarrow \quad |\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$ **Hence Proved.**

Note : For any two vectors $\vec{a}$ and $\vec{b}$, we always have $|\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|$

Key Formulae

➢ Angle between two vectors $\vec{a}$ and $\vec{b}$ in terms of cross-product can be found by the expression :

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} \quad \textbf{or} \quad \theta = \sin^{-1}\left(\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\right)$$

➢ **Area of Triangle :** If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a Triangle, then the area of triangle can be obtained by evaluating $\dfrac{1}{2}|\vec{a}\times\vec{b}|$.

➢ **Area of Parallelogram**

(i) If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then the area of parallelogram, can be obtained by evaluating $|\vec{a}\times\vec{b}|$.

(ii) The area of the parallelogram with diagonal $\vec{a}$ and $\vec{b}$ is $\dfrac{1}{2}|\vec{a}\times\vec{b}|$.

UNIT – VI : THREE DIMENSIONAL GEOMETRY

CHAPTER-11

THREE-DIMENSIONAL GEOMETRY

Topic-1

Direction Ratios, Direction Cosines and Lines

Concepts covered: Direction cosines and ratios of a line, cartesian and vector equation of a line passing through two given points, angle between two lines, condition for lines to be perpendicular/parallel.

Revision Notes

➢ **Direction Cosines of a Line :**

If A and B are the two points on a given line L, then direction cosines of vectors $\vec{AB}$ and $\vec{BA}$ are the direction cosines (d.c.´s) of line L. Thus, if α, β, γ are the direction-angles which the line L makes with the positive direction of X, Y, Z-axes respectively, then its d.c.´s are $\cos\alpha$, $\cos\beta$, $\cos\gamma$. If direction of line L is reserved, the direction angles are replaced by their supplements, *i.e.*, $\pi - \alpha$, $\pi - \beta$, $\pi - \gamma$ and so are the d.c.´s, *i.e.*, the direction cosines become $-\cos\alpha$, $-\cos\beta$, $-\cos\gamma$. So, a line in space has two set of d.c.´s *viz.* $\pm\cos\alpha$, $\pm\cos\beta$, $\pm\cos\gamma$.

- The d.c.´s are generally denoted by l, m, n. Also, $l^2 + m^2 + n^2 = 1$ and so we can deduce that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Also, $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$
- The d.c.´s of a line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are $\pm\frac{x_2 - x_1}{AB}, \pm\frac{y_2 - y_1}{AB}, \pm\frac{z_2 - z_1}{AB}$; where AB is the distance between the points A and B, *i.e.*, $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

➢ **Direction Ratios of a Line :**

Any three numbers a, b, c (say) which are proportional to d.c.´s, *i.e.*, l, m, n of a line are called the **direction ratios** (d.r.´s) of the line. Thus, $a = \lambda l$, $b = \lambda m$, $c = \lambda n$ for any $\lambda \in \mathbb{R} - \{0\}$.

Consider, $$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{1}{\lambda} \quad \text{(say)}$$

$$\Rightarrow \quad l = \frac{a}{\lambda}, m = \frac{b}{\lambda}, n = \frac{c}{\lambda}$$

$$\Rightarrow \quad \left(\frac{a}{\lambda}\right)^2 + \left(\frac{b}{\lambda}\right)^2 + \left(\frac{c}{\lambda}\right)^2 = 1 \quad [\text{Using } l^2 + m^2 + n^2 = 1]$$

$$\Rightarrow \quad \lambda = \pm\sqrt{a^2 + b^2 + c^2}$$

Therefore, $$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

- The d.r.´s of a line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are $x_2 - x_1, y_2 - y_1, z_2 - z_1$ or $x_1 - x_2, y_1 - y_2, z_1 - z_2$.
- Direction ratios are sometimes called as **Direction Numbers**.

➢ **Equation of a Line passing through two given points :**

Consider the two given points as $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ with position vectors $\vec{a}$ and $\vec{b}$ respectively. Also, assume $\vec{r}$ as the position vector of any arbitrary point $P(x, y, z)$ on the line L passing through A and B.

Thus, $\vec{OA} = \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\vec{OB} = \vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, $\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

(a) Vector equation of a line : Since the points A, B and P all lie on the same line which means they are all collinear points.

Further it means, $\vec{AP} = \vec{r} - \vec{a}$ and $\vec{AB} = \vec{b} - \vec{a}$ are collinear vectors,

i.e., $\vec{AP} = \lambda \vec{AB}$

$\Rightarrow$ $\vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a})$

$\Rightarrow$ $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, where $\lambda \in \mathbb{R}$.

This is the vector equation of the line.

(b) Cartesian equation of a line : By using the vector equation of the line $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, we get

$$x\hat{i} + y\hat{j} + z\hat{k} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + \lambda[(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}]$$

On equating the coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$, we get

$$x = x_1 + \lambda(x_2 - x_1),\ y = y_1 + \lambda(y_2 - y_1),\ z = z_1 + \lambda(z_2 - z_1)$$

On eliminating λ, we have

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

➢ **Angle between two lines :**

(a) When d.r.´s or d.c.´s of the two lines are given :

Consider two lines L_1 and L_2 with d.r.´s in proportion to a_1, b_1, c_1 and a_2, b_2, c_2 respectively; d.c.´s as l_1, m_1, n_1 and l_2, m_2, n_2. Consider $\vec{b_1} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b_2} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$. These, vectors $\vec{b_1}$ and $\vec{b_2}$ are parallel to the given lines L_1 and L_2. So, in order to find the angle between the lines L_1 and L_2, we need to get the angle between the vectors $\vec{b_1}$ and $\vec{b_2}$.

So the acute angle θ between the vectors $\vec{b_1}$ and $\vec{b_2}$ (and hence lines L_1 and L_2) can be obtained as,

$$\vec{b_1} \cdot \vec{b_2} = |\vec{b_1}||\vec{b_2}|\cos\theta$$

$\therefore$
$$\cos\theta = \frac{\vec{b_1}.\vec{b_2}}{|\vec{b_1}|.|\vec{b_2}|}$$

Thus,
$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

- Also, in terms of d.c.´s : $\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$.
- Sine of angle is given as : $\sin\theta = \left|\frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$

(b) When vector equations of two lines are given :

Consider vector equations of lines L_1 and L_2 as $\vec{r_1} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r_2} = \vec{a_2} + \mu\vec{b_2}$ respectively.

Then, the acute angle θ between the two lines is given by the relation

$$\cos\theta = \left|\frac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}||\vec{b_2}|}\right|$$

(c) When Cartesian equation of two lines are given :

Consider the lines L_1 and L_2 in Cartesian form as,

$$L_1 = \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

Important Facts

- **For two perpendicular lines :** $a_1a_2 + b_1b_2 + c_1c_2 = 0$, $l_1l_2 + m_1m_2 + n_1n_2 = 0$.
- **For two parallel lines :** $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}; \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

$$L_2 = \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

Then the acute angle θ between the Lines L_1 and L_2 can be obtained by,

$$\cos\theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

➢ **Co-planarity of Two lines:**

Assume that the given lines L_1 : $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ such that L_1 passes through $A(x_1, y_1, z_1)$ with position vector $\vec{a_1}$ and is parallel to d.r. s $a_1.b_1.c_1$. Also L_2 passes through B (x_2, y_2, z_2) with position vector $\vec{a_2}$ and is parallel to $\vec{b_2}$ with the d.r. s $a_2b_2c_2$

(a) Vector form of co-plan lines :

We know that $\vec{AB} = \vec{a_2} - \vec{a_1}$. Now the lines L_1 and L_2 are coplanar if $\vec{AB}$ is perpendicular to $\vec{b_1} \times \vec{b_2}$. That implies. $\vec{AB}(\vec{b_1} \times \vec{b_2}) = 0$,

$$(\vec{a_2} - \vec{a_1}).(\vec{b_1} \times \vec{b_2}) = 0$$

(b) Cartesian form of co-planarity of lines :

We know that $\vec{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)k$, $b_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b_2} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$. So, by using $(\vec{a_2} - \vec{a_2}).(\vec{b_2} - \vec{b_2}) = 0$, we get

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Note that only coplanar lines can intersect each other in the plane they exist.

➢ **Skew Lines** : In space, there are lines which are neither intersecting nor parallel. In fact, such pair of lines are non-coplanar and are called the skew lines.

➢ **Shortest Distance between Two Lines :** If two lines are in the same plane *i.e.*, they are coplanar, they will intersect each other if they are non-parallel. Hence, the shortest distance between them is zero. "If the lines are parallel then the shortest distance between them will be the perpendicular distance between the lines i.e., the length of the perpendicular drawn from a point on one line onto the other line".

Key Formulae

➢ **Distance formula :**

The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by the expression

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \text{ units}$$

➢ **Section formula :**

The co-ordinates of a point Q which divides the line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the ratio $m : n$

(a) internally, are

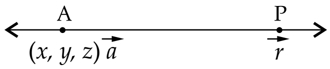
$$\left(\frac{(mx_2 + nx_1)}{m+n}, \frac{(my_2 + ny_1)}{m+n}, \frac{(mz_2 + nz_1)}{m+n} \right)$$

(b) externally, *i.e.*, internally in the ratio $(m) : (-n)$, are $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n} \right)$

Key Term

➢ **Equation of a line in space passing through a given point and parallel to a given vector :**

Consider the line L is passing through the given point $A(x_1, y_1, z_1)$ with the position vector $\vec{a}$, $\vec{b}$ is the given vector with d.r.´s a, b, c and $\vec{r}$ is the position vector of any arbitrary point $P(x, y, z)$ on the line.



Thus,

$$\vec{OA} = \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k},$$
$$\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k},\ \vec{b} = a\hat{i} + b\hat{j} + c\hat{k}.$$

(a) Vector equation of a line : As the line L is parallel to given vector $\vec{b}$ and points A and P are lying on the line so, $\vec{AP}$ is parallel to the $\vec{b}$.

$\Rightarrow \vec{AP} = \lambda\vec{b}$, where $\lambda \in \mathbb{R}$, *i.e.*, set of real numbers

$\Rightarrow \quad \vec{r} - \vec{a} = \lambda\vec{b} \Rightarrow \vec{r} = \vec{a} + \lambda\vec{b}$

This is the vector equation of line.

(b) Parametric equations : If d.r.´s of the line are a, b, c, then by using $\vec{r} = \vec{a} + \lambda\vec{b}$, we get

$$x\hat{i} + y\hat{j} + z\hat{k} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$$

Now, as we equate the coefficients of $\hat{i}, \hat{j}, \hat{k}$, we get the parametric equations of the line given as,

$x = x_1 + \lambda a, y = y_1 + \lambda b, z = z_1 + \lambda c.$

- Co-ordinates of any point on the line considered here are

  $(x_1 + \lambda a, y_1 + \lambda b, z_1 + \lambda c).$

(c) Cartesian equation of a line : If we eliminate the parameter λ from the parametric equations of a line, we get the Cartesian equation of line as

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

- If l, m, n are the d.c.´s of the line, then Cartesian equation of line becomes

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Mnemonics

Direction Cosines

Dance **C**horeographer **P**refer **D**ieting

Direction, Cosines, Positive, Direction

1 glass **L** e **M** o **N** juice

$l^2 + m^2 + n^2 = 1$

Direction Ratios

Directo**r** **R**emo a **P**rofessional **D**ancer

Direction, Ratios, Proportional, Direction

Choreographer created

Cosines

3 **L**ifetime **M**ovies with **N**ew faces **a b c**

l m n

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Topic-2

Plane and Its Equation in Various Forms

Concepts covered: Cartesian and vector equation of a plane, equation of a plane in one point form, normal form, intercept form, angle between a line and a plane, two planes.

Revision Notes

- **Plane and its equation :**

A plane is a surface such that if any two points are taken on it, the line segment joining them lies completely on the surface.

(a) Equation of a plane in Normal form :

Consider a plane at distance d from the origin such that $\vec{ON}$ is the normal from the origin to the plane and the $\hat{n}$ is a unit vector along $\vec{ON}$. Then $\vec{ON} = d\hat{n}$ if $ON = d$. Consider $\vec{r}$ be the position vector of any arbitrary point $P(x, y, z)$ on the plane.

(i) Vector form of the equation of plane : Since P lies on the plane $\vec{NP}$ is perpendicular to the vector $\vec{ON}$.

which implies $\vec{NP} \cdot \vec{ON} = 0$.

$$\Rightarrow \quad (\vec{r} - d\hat{n}) \cdot \hat{n} = 0$$

$$\Rightarrow \quad \vec{r} \cdot \hat{n} - d\hat{n} \cdot \hat{n} = 0$$

$$\Rightarrow \quad \vec{r} \cdot \hat{n} - d(\hat{n} \cdot \hat{n}) = 0$$

$$\Rightarrow \quad \vec{r} \cdot \hat{n} - d|\hat{n}|^2 = 0$$

$$\Rightarrow \quad \vec{r} \cdot \hat{n} - d = 0$$

$$\Rightarrow \quad \vec{r} \cdot \hat{n} = d \qquad [\because |\hat{n}| = 1]$$

This is the vector equation of the plane.

(ii) Cartesian form of the equation of plane : If l, m, n are d.c.´s of the normal $\hat{n}$ to the given plane. Then by using $\vec{r} \cdot \hat{n} = d$

We get, $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (l\hat{i} + m\hat{j} + n\hat{k}) = d$

$$\Rightarrow \quad lx + my + nz = d$$

This is the Cartesian equation of the plane.

- Also, if a, b, c are the d.r.´s of the normal $\hat{n}$ to the plane, then the Cartesian equation of plane becomes $ax + by + cz = d$.

(b) Equation of plane perpendicular to a given vector and passing through a given point :

Assume that the plane passes through a point $A(x_1, y_1, z_1)$ with the position vector $\vec{a}$ and is perpendicular to the vector $\vec{m}$ with d.r.´s as A, B, C ($\therefore \vec{m} = A\hat{i} + B\hat{j} + C\hat{k}$).

Also, consider $P(x, y, z)$ as any arbitrary point on the plane with position vector as $\vec{r}$.

(i) Vector form of the equation of plane : Since P lies on the plane $\vec{NP}$ is perpendicular to the vector $\vec{ON}$.

Vector form of the equation of plane : As $\vec{AP}$ lies in the plane and $\vec{m}$ is perpendicular to the plane. So, $\vec{AP}$ is perpendicular to $\vec{m}$.

$$\Rightarrow \quad \vec{AP} \cdot \vec{m} = 0$$

$$\Rightarrow \quad (\vec{r} - \vec{a}) \cdot \vec{m} = 0$$

This is the vector equation of the plane.

- The above obtained equation of plane can also be expressed as $\vec{r}\cdot\vec{m} = \vec{a}\cdot\vec{m}$.

(ii) Cartesian form of the equation of plane : As $\vec{AP} = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$, and let $\vec{m}$ be $A\hat{i} + B\hat{j} + C\hat{k}$ so by using $(\vec{r} - \vec{a})\cdot\vec{m} = 0$, we get

$$[(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}]\cdot(A\hat{i} + B\hat{j} + C\hat{k}) = 0$$

$$\Rightarrow \quad A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

This is the Cartesian equation of the plane.

(c) Equation of plane passing through three non-collinear points :

- **Cartesian form :**

In order to find the equation of a plane passing through three given points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), the following steps are used :

Step 1 : Write the equation of a plane passing through (x_1, y_1, z_1) as

$$a\,(x - x_1) + b\,(y - y_1) + c\,(z - z_1) = 0 \quad \text{...(i)}$$

Step 2 : If the plane (i) passes through (x_2, y_2, z_2) and (x_3, y_3, z_3), then

$$a\,(x_2 - x_1) + b\,(y_2 - y_1) + c\,(z_2 - z_1) = 0 \quad \text{...(ii)}$$

and

$$a\,(x_3 - x_1) + b\,(y_3 - y_1) + c\,(z_3 - z_1) = 0 \quad \text{...(iii)}$$

Step 3 : Solve equations (ii) and (iii) by cross multiplication method to obtain the values of a, b and c. Substitute the values of a, b and c in eq. (i) to get the equation of the required plane.

Note : On eliminating a, b, c from eqs. (i), (ii) and (iii), we get

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

as the equation of the plane passing through three given points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

Key Formulae

➢ **Angle between a plane and a line :**

- If θ is the angle between line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and the plane $ax + by + cz + d = 0$, then

$$\sin\theta = \left|\frac{al + bm + cn}{\sqrt{a^2 + b^2 + c^2}\,\sqrt{l^2 + m^2 + n^2}}\right|$$

- Vector form : If θ is the angle between a line $\vec{r} = (\vec{a} + \lambda\vec{b})$ and $\vec{r}\,.\,\vec{n} = d$, then

$$\sin\theta = \left|\frac{\vec{b}\,.\,\vec{n}}{|\vec{b}|\,|\vec{n}|}\right|$$

- Condition for perpendicularity : $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$, $\vec{b} \times \vec{n} = 0$

- Condition for parallel : $al + bm + cn = 0$, $\vec{b}\,.\,\vec{n} = 0$

➢ **Angle between two planes :**

- Consider two planes $ax + b + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$. Angle between these plane is the angle between their normals. Since direction ratios of their normals are (a, b, c) and (a', b', c') respectively. Hence, θ the angle between them is given by

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{a^2 + b^2 + c^2}\,\sqrt{a'^2 + b'^2 + c'^2}}$$

Planes are perpendicular, if $aa' + bb' + cc' = 0$ and planes are parallel if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

- The angle θ between the planes $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ is given by

- $\sin\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|.|\vec{n}_2|}$

Planes are perpendicular, is $\vec{n}_1 \cdot \vec{n}_2 = 0$ and planes are parallel if $\vec{n}_1 = \lambda\,\vec{n}_2$.

➢ **Equation the plane in intercept form :**

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Note :
- Equation of XY-plane : $z = 0$
- Equation of YZ-plane : $x = 0$
- Equation of ZX-plane : $y = 0$

UNIT – VI : APPLICATION OF INTEGRALS

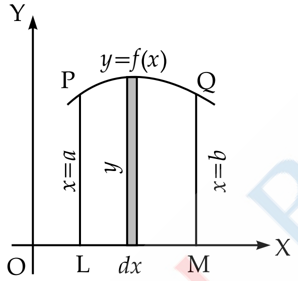
CHAPTER-12

APPLICATION OF INTEGRALS

Revision Notes

➤ **Area under Simple curves :**

(i) Let us find the area bounded by the curve $y = f(x)$, X-axis and the lines $x = a$ and $x = b$. Consider the area under the curve as composed by large number of thin vertical strips.



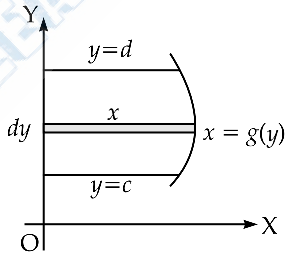
Let there be an arbitrary strip of height y and width dx.

Area of elementary strip $dA = ydx$, where $y = f(x)$. Total area A of the region between X-axis, lines $x = a$ and $x = b$ and the curve $(y = f(x))$ = sum of areas of elementary thin strips across the region $PQML$.

$$A = \int_a^b y\,dx = \int_a^b f(x)dx$$

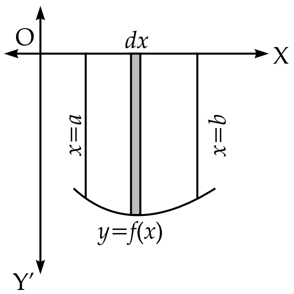
(ii) The area A of the region bounded by the curve $x = g(x)$, Y-axis and the lines $y = c$ and $y = d$ is given by

$$A = \int_c^d x\,dy = \int_c^d g(y)dy$$



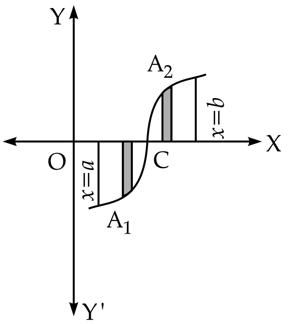
(iii) If the curve under consideration lies below X-axis, then $f(x) < 0$ from $x = a$ to $x = b$, the area bounded by the curve $y = f(x)$ and the lines $x = a$, $x = b$ and X-axis is negative. But, if the numerical value of the area is to be taken into consideration, then

$$\text{Area} = \left|\int_a^b f(x)dx\right|$$



(iv) It may also happen that some portion of the curve is above X-axis and some portion is below X-axis as shown in the figure. Let A_1 be the area below X-axis and A_2 be the area above the X-axis. Therefore, area bounded by the curve $y = f(x)$, X-axis and the **lines** $x = a$ and $x = b$ is given by

$$A = |A_1| + |A_2|$$



➢ **Area between two curves :**

(i) Let the two curves be $y = f(x)$ and $y = g(x)$, as shown in the figure. Suppose these curves intersect at $f(x)$ with width dx.

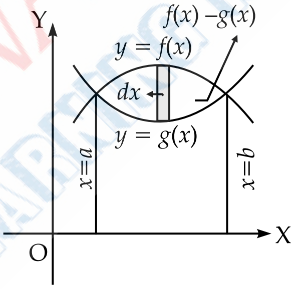
Consider the elementary strip of height y, where $y = f(x)$

$\therefore \quad dA = ydx$

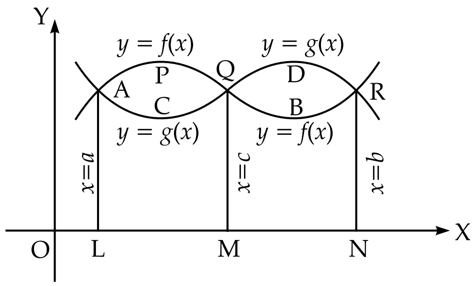
$\Rightarrow \quad A = \int_a^b [f(x) - g(x)]dx$

$= \int_a^b f(x)dx - \int_a^b g(x)dx$

= Area bounded by the curve $\{y = f(x)\}$ – Area bounded by the curve $\{y = g(x)\}$, where $f(x) > g(x)$.



(ii) If the two curves $y = f(x)$ and $y = g(x)$ intersect at $x = a$, $x = c$ and $x = b$, such that $a < c < b$.



If $f(x) > g(x)$ in $[a, c]$ and $g(x) \leq f(x)$ in $[c, b]$, then the area of the regions bounded by the curve :

= Area of region $PACQP$ + Area of region $QDRBQ$

$$= \int_a^c |f(x) - g(x)|\,dx + \int_c^b |g(x) - f(x)|\,dx$$

UNIT – VIII : APPLICATION OF CALCULUS

CHAPTER-13

APPLICATION OF CALCULUS

Revision Notes

- **Cost function :**

If x denotes the quantity produced of a certain commodity at total cost c, then the total cost function is given as $C = c(x)$ (explicit form) and $f(c, x) = 0$ in the implicit form. The total cost is divided into two parts, the fixed cost and the variable cost.

(a) Fixed Cost :

Fixed costs are those which incurred regardless of the level of production - like interest, rent, wages of permanent staff etc.

Thus, Total Fixed Cost (TFC) $= TC$, when $x = 0$

Fixed cost does not change whether there is any increase or decrease in level of production.

(b) Variable Cost :

Variable costs are those which vary with output.

e.g., Raw materials and wages of casual labour etc. Thus,

Total Cost = Total Fixed Cost + Total Variable Cost

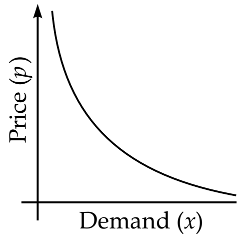
or $TC = TFC + TVC$

(c) Average Cost :

The average cost represents the cost per unit, *i.e.*, $AC = \dfrac{TC}{x} = \dfrac{\text{Total cost}}{\text{number of commodities}}$

(d) Demand Function :

The demand function is the functional relationship between demand and price of a commodity. If p denotes the price per unit and x is the number of units demanded by a consumer at that price, then demand function in explicit form will be written as $x = f(p)$ and in implicit form as $f(x, p) = 0$.



$\therefore$ Price (p) is shown against demand x in the following figure :

(e) Revenue Function :

Revenue is the amount received by a company on selling a certain number of units of a commodity. Let p be the price per unit and x be the number of units sold. Then, total revenue

R or $R(x) = p \times x$

Total revenue = Selling price per unit of the commodity × Quantity sold

(f) Profit Function :

Profit function is the difference of revenue function and cost function, *i.e.*,

$$P(x) = R(x) - C(x)$$

(g) Break-even Point :

The break-even point is that point, where total revenue equals to the total cost incurred, *i.e.*, $P(x) = 0$ or $R(x) = C(x)$. At break-even point, a company begins to earn profit.

(h) Marginal Cost :

The marginal cost is the rate of change of the total cost with respect to x (Output).

$$MC = \frac{d}{dx}(C) = \frac{dC}{dx},\ x > 0$$

(i) Relation between Average Cost (AC) and Marginal Cost (MC) :

If C is the total cost of producing and marketing x units of commodity, then

$$\frac{d}{dx}(AC) = \frac{1}{x}(MC - AC)$$

Here, three cases arise :

Case 1 : For $MC > AC \Rightarrow AC$ increases with x and AC curve is rising.
Case 2 : For $MC = AC \Rightarrow AC$ is constant at all levels of output.
Case 3 : For $MC < AC \Rightarrow AC$ decreases with x and AC curve is falling.

(j) Average Revenue and Marginal Revenue :

(i) Average Revenue is the revenue generated per unit of output sold. It is denoted by AR, *i.e.*,

$$AR = \frac{\text{Total revenue}}{\text{Number of units sold}} = \frac{R}{x} = \frac{p \cdot x}{x} = p$$

(ii) Marginal Revenue is the rate of change of total revenue with respect to quantity sold.

i.e.,
$$MR = \frac{dR}{dx} = \frac{d}{dx}(px)$$

(11) Marginal Average Cost :

Marginal average cost (MAC) is $\frac{d}{dx}$ (AC), *i.e.*, derivative of the average cost function. This is also called as slope of average cost curve.

➢ **Minimisation of AC and MC Functions :**

Using the concept of maxima and minima, we can determine the level of output where per unit cost is minimum corresponding to a given total or average cost function. Following steps are used to find the optimal level of output :

Step 1 : From AC, determine $\frac{d}{dx}(AC)$.

Step 2 : Let $\frac{d}{dx}(AC) = 0$ and solve for x.

Step 3 : Find $\frac{d^2}{dx^2}(AC)$.

Step 4 : The value of x for which $\frac{d^2}{dx^2}(AC) > 0$ is the minimum per unit cost.

Step 5 : Minimum average cost can be determined by substituting this value of x in $AC(x)$. Similarly, we can find the minimum marginal cost or minimum cost

➢ **Maximisation of Total Revenue :**

It is possible to determine the level of output at which the total revenue is maximum, if demand function is given. Total revenue, $R(x)$, is the maximum, when marginal revenue is zero.

To maximise total revenue, following steps are to be followed :

Step 1 : Determine $\frac{dR}{dx}$.

Step 2 : Let $\frac{dR}{dx} = 0$ and solve for x.

Step 3 : Determine $\frac{d^2R}{dx^2}$.

Step 4 : The value for which $\frac{d^2R}{dx^2} < 0$ gives the maximum total revenue.

Step 5 : The maximum revenue can be obtained by putting this value of x in $R(x)$. Similarly, maximum profit can be determined.

Mnemonics

Concept : Types of Function in Application of calculus

Interpretations :

- **Trained** → Total Cost Function
- **Dogs** → Demand Function
- **Pluck** → Profit Function
- **Roadblocks** → Revenue Function

Concept : Types of Cost in Application of calculus

Interpretations :

- **A**ntisocial → Average Cost
- **F**ishes → Fixed Cost
- **M**anufacture → Marginal Cost
- **V**oices → Variable Cost

UNIT – IX : LINEAR REGRESSION

CHAPTER-14

LINEAR REGRESSION

Revision Notes

- **Regression :** Regression is an estimate or prediction of unknown values of one variable from known values of another variable. Regression measures the extent of correlation and the nature of correlation.
- **Linear Regression :** If there are only two variables under consideration, then the regression is called as simple regression. This simple regression is called linear regression.
- **Regression Lines :** If two variables given in a bivariate frequency distribution are correlated, then the dots in the scatter diagram of the distribution cluster around a straight line, called the line of regression.
- **Two methods of drawing regression lines :**

  (i) Curve Fitting Method :

  In this method, if the data $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ is plotted on a scatter diagram and then visualize a smooth curve which is approximating the data is called curve fitting method.

  (ii) Method of least squares :

  In this method, a regression line is fitted through different points in such a way that the sum of squares of the deviations of the observed value from the fitted line shall be least. The line drawn by this method is called **line of best fit**.
- **Regression Equation :** Regression equations are the algebraic formulation of regression lines. They represent regression lines.

  Regression equations can be fixed in following ways :

  (a) Regression Equation of y on x : This equation is used to estimate the probable values of y on the basis of the given values of x.

  i.e., $y = a + bx$

  where, a and b are constants.

  Regression equation of y on x can also be written as

  $$y - \bar{y} = r \cdot \frac{\sigma_y}{\sigma_x}(x - \bar{x}) \qquad \left[\text{where } \bar{x} = \frac{\Sigma x}{n},\ \bar{y} = \frac{\Sigma y}{n}\right]$$

  or $$y - \bar{y} = b_{yx}(x - \bar{x})$$

  where, b_{yx} = Regression coefficient of y on x.

  (b) Regression Equation of x on y : This equation is used to estimate the probable values of x on the basis of the given values of y.

  i.e., $x = a_0 + b_0 y$

  where, a_0 and b_0 are constants.

  Regression equation of x on y can also be written as

  $$x - \bar{x} = r \cdot \frac{\sigma_x}{\sigma_y}(y - \bar{y}) \qquad \left[\text{where, } \bar{x} = \frac{\Sigma x}{n},\ \bar{y} = \frac{\Sigma y}{n}\right]$$

  [$\because$ σ_x = standard deviation for x, σ_y = standard deviation for y]

  or $$x - \bar{x} = b_{xy}(y - \bar{y})$$

  where, b_{xy} = Regression coefficient of x on y.
- **Method to Identify the Regression :**

  Suppose two equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are given and we have to identify the regression line of y on x or x on y, we use the following steps :

  Step 1 : Firstly, we consider any one of the equations say first, *i.e.,* $a_1x + b_1y + c_1 = 0$ as y on x and other equation say second, *i.e.,* $a_2x + b_2y + c_2 = 0$ as x on y.

  Step 2 : Adjust the first equation y on x such that y should be on left side with coefficient 1 and rest of the terms be on right side (*i.e.,* $y = -\frac{a_1}{b_1}x - \frac{c_1}{b_1}$). The coefficient of x will considered as regression coefficient y on x, *i.e.,* $b_{yx} = -\frac{a_1}{b_1}$. Similarly, for another equation x on y, adjust equation like as

$x = -\dfrac{b_2}{a_2} y - \dfrac{c_2}{a_2}$. The coefficient of y will give regression coefficient of x on y, *i.e.*, $b_{xy} = -\dfrac{b_2}{a_2}$.

Step 3 : If b_{xy} and b_{yx} are not of same sign, then we do not determine the regression line. If they have the same sign, then determine coefficient by using the formula.

$$r = \sqrt{b_{yx} \times b_{xy}}$$

Step 4 : If the value of r lies 0 to 1, then our consideration is true, otherwise we have to change the considering from equations.

➢ **Properties of Regression Coefficient and Lines of Regression :**

(i) Coefficient of correlation is the geometric mean between the regression coefficient.

i.e., $$r = \sqrt{b_{xy} \times b_{yx}}$$

(ii) Both the regression coefficients must have the same algebraic signs.

(iii) The coefficient of correlation will have the same sign as that of regression coefficients.

(iv) Both the regression coefficients cannot be greater than unity.

(v) If one regression coefficient is greater than unity, then other regression coefficient must be less than unity.

(vi) If $r = 0$, the variables are uncorrelated, the lines of regression becomes perpendicular to each other.

(vii) If $r = \pm 1$, the two lines of regression either coincide or parallel to each other.

(viii) Arithmetic mean of two regression coefficients is either equal to or greater than the correlation coefficient.

i.e., $$\frac{b_{yx} + b_{xy}}{2} \geq r$$

(ix) The two regression lines coincide if and only if there is perfect linear relation between X and Y.

i.e., $$\rho(X, Y) = \pm 1$$

(x) The intersection points of two regression lines is, $(\bar{x}, \bar{y})$, were $\bar{x}$ is mean of x-series and $\bar{y}$ is mean of y-series.

(xi) The angle between regression lines indicates the degree of dependence between the variables.

Key Formulae

➢ **Regression coefficient of y on x :**

$$b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x} = \frac{\Sigma xy - \dfrac{\Sigma x \cdot \Sigma y}{n}}{\Sigma x^2 - \dfrac{(\Sigma x)^2}{n}}$$

➢ **Regression coefficient of x on y :**

$$b_{xy} = r \cdot \frac{\sigma_x}{\sigma_y} = \frac{\Sigma xy - \dfrac{\Sigma x \cdot \Sigma y}{n}}{\Sigma y^2 - \dfrac{(\Sigma y)^2}{n}}$$

➢ **Regression equation of y on x :**

$$(y - \bar{y}) = b_{yx}(x - \bar{x})$$

or

$$(y - \bar{y}) = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

➢ **Regression equation of x on y :**

$$(x - \bar{x}) = b_{xy}(y - \bar{y})$$

or

$$(x - \bar{x}) = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

➢ **Correlation coefficient :** $$r = \sqrt{b_{yx} \times b_{xy}}$$

$0 \leq r \leq 1$ and r, b_{yx} and b_{xy} have same sign.

Mnemonics

Concept : Correlation coefficient : $r = \sqrt{b_{yx} \times b_{xy}}$ $0 \leq r \leq 1$ and r, b_{yx} and b_{xy} have same sign.

Mnemonic : Rohit Sometimes Builds Building

Rohit		Sometimes	Builds	Building
↓		↓	↓	↓
r	=	*Square rood*	b_{xy}	b_{yx}

UNIT – X : LINEAR PROGRAMMING

CHAPTER-15

LINEAR PROGRAMMING

Revision Notes

- **Linear programming problems :** Problems which minimize or maximize a linear function Z subject to certain conditions determined by a set of linear inequalities with non-negative variables are known as linear programming problems.
- **Objective function :** A linear function $Z = ax + by$, where a and b are constants which has to be maximized or minimized according to a set of given conditions, is called as linear objective function.
- **Decision variables :** In the objective function $Z = ax + by$, the variables x, y are said to be decision variables.
- **Constraints :** The restrictions in the form of inequalities on the variables of a linear programming problem are called constraints. The condition $x \geq 0, y \geq 0$ are known as non-negative restrictions.
- **Feasible region :** The common region determined by all the constraints including non-negative constraints $x, y \geq 0$ of linear programming problem is known as the feasible region.
- **Feasible solution :** Points within and on the boundary of the feasible region represents feasible solutions of constraints. In the feasible region, there are infinitely many points (solutions) which satisfy the given conditions.
- **Theorem 1:** Let R be the feasible region for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where variables x and y are subject to constraints described by linear inequalities, the optimal value must occur at a corner p into (vertex) of the feasible region.
- **Theorem 2 :** Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function R has both maximum and minimum values of R and each of these occurs at a corner point (vertex) of R.

  However, if the feasible region is unbounded, the optimal value obtained may not be maximum or minimum.

  CORNER OR EXTREME POINT METHOD

Step 1 :	Formulate the linear programming problem in x and y with given conditions.
Step 2 :	Convert the inequality constraints into equality constraints and plot each line on the graph paper.
Step 3 :	Find the feasible region and check if the feasible region is bounded or unbounded.
Step 4 :	Evaluate the value of the objective function Z at each corner point. Let M be the greatest and m be the smallest value of the objective function Z.

  (i) When the feasible region is bounded : M and m are the maximum and minimum values of the objective function Z, respectively.

  (ii) When the feasible region is unbounded :

  (a) M is the maximum value of the objective function Z, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region.
  Otherwise the objective function has no maximum value.

  (b) m is the minimum value of the objective function Z, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region.
  Otherwise the objective function has no minimum value.

- **Different types of linear programming problems :** A few important linear programming problems are as follows :

  (i) Manufacturing problem : In such problem, we determine :

  (a) Number of units of different products to be produced and sold.

  (b) Manpower required, machines hours needed, warehouse space available, etc. Objective function is to maximize profit.

  (ii) Diet problem : Here, we determine the amount of different types of constituent or nutrients which should be included in the diet.

Objective function is to minimize the cost of production.

(iii) Blending problem : To determine the optimum amount of several constituents used in producing a set of products while determining the optimum quantity of each product to be produced.

(iv) Investment problem : To determine the amount of investment in fixed income securities to maximize the return on these investment.

- **Limitation of Linear Programming :**

(i) To specify an objective function in mathematical form is not an easy task.

(ii) Even if objective function is determined, it is difficult to determine social, institutional, financial and other constraints.

(iii) It is also possible that the objective function of constraints may not be directly specified by linear inequality equations.

Important Fact

- If two corner points of the feasible region are both optimal solutions of the same type (max or min), then any point on the line segment joining these two points is also an optimal of the same type.

Mnemonics

Concept : LPP Parameter

Mnemonics : NOC

Interpretations :

N → Non-negative Variables

O → Objective Function

C → Constraints