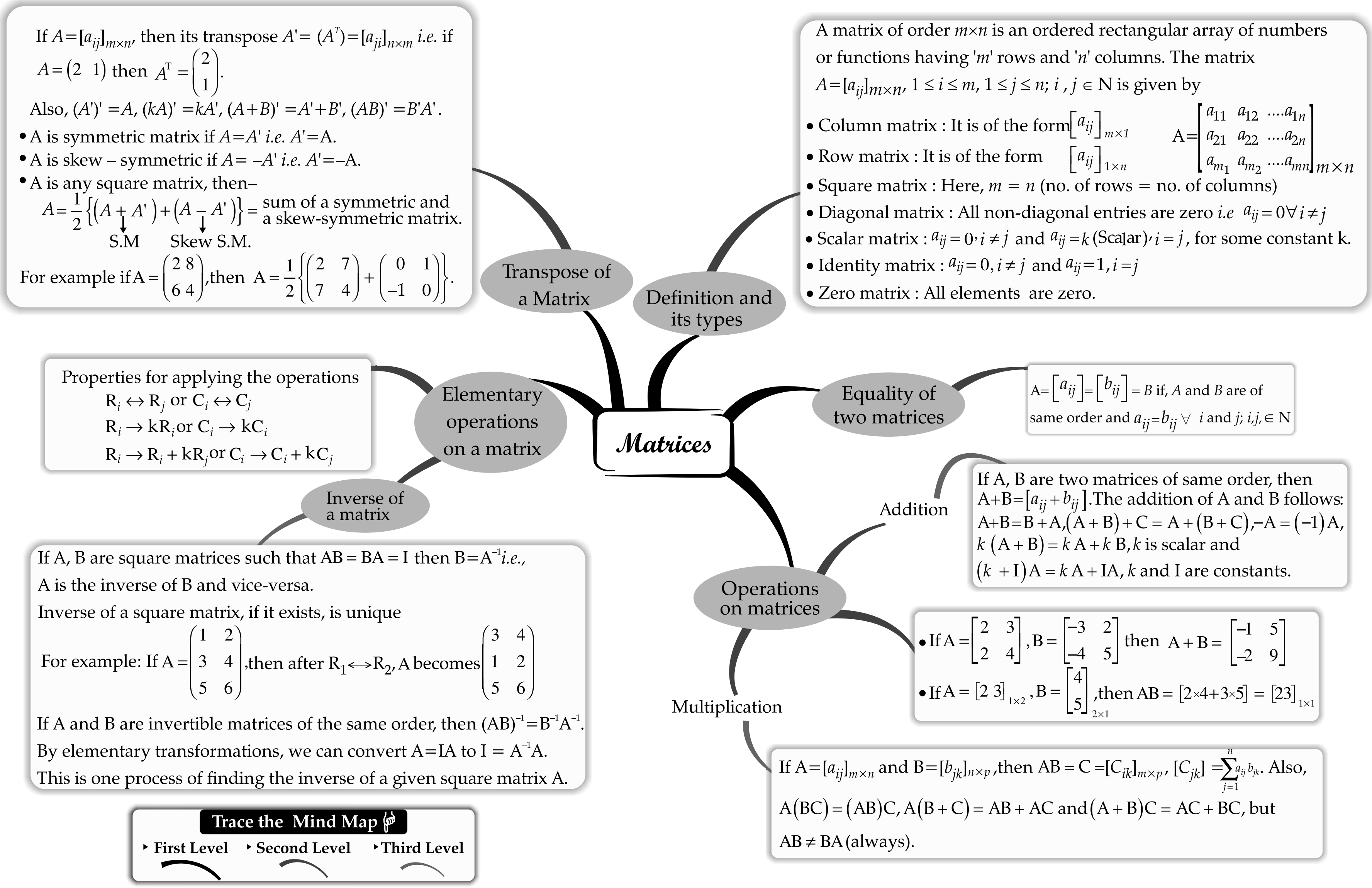


Matrices

Definition and its types

A matrix of order $m\times n$ is an ordered rectangular array of numbers or functions having 'm' rows and 'n' columns. The matrix $A=[a_{ij}]_{m\times n}$, $1 \le i \le m$, $1 \le j \le n$; $i, j \in N$ is given by

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ a_{m_1} & a_{m_2} & \dots a_{mn} \end{bmatrix}_{m\times n}$$

- Column matrix : It is of the form $\left[a_{ij}\right]_{m\times 1}$
- Row matrix : It is of the form $\left[a_{ij}\right]_{1\times n}$
- Square matrix : Here, $m = n$ (no. of rows = no. of columns)
- Diagonal matrix : All non-diagonal entries are zero *i.e* $a_{ij}=0 \forall i \neq j$
- Scalar matrix : $a_{ij}=0, i\neq j$ and $a_{ij}=k$ (Scalar)$, i=j$, for some constant k.
- Identity matrix : $a_{ij}=0, i\neq j$ and $a_{ij}=1, i=j$
- Zero matrix : All elements are zero.

Transpose of a Matrix

If $A=[a_{ij}]_{m\times n}$, then its transpose $A'=(A^T)=[a_{ji}]_{n\times m}$ *i.e.* if $A=\begin{pmatrix}2 & 1\end{pmatrix}$ then $A^T=\begin{pmatrix}2\\1\end{pmatrix}$.

Also, $(A')'=A$, $(kA)'=kA'$, $(A+B)'=A'+B'$, $(AB)'=B'A'$.

- A is symmetric matrix if $A=A'$ *i.e.* $A'=A$.
- A is skew – symmetric if $A=-A'$ *i.e.* $A'=-A$.
- A is any square matrix, then–

$$A=\frac{1}{2}\{\underset{\text{S.M}}{\underset{\downarrow}{(A+A')}}+\underset{\text{Skew S.M.}}{\underset{\downarrow}{(A-A')}}\}=\text{sum of a symmetric and a skew-symmetric matrix.}$$

For example if $A=\begin{pmatrix}2 & 8\\6 & 4\end{pmatrix}$, then $A=\frac{1}{2}\left\{\begin{pmatrix}2 & 7\\7 & 4\end{pmatrix}+\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\right\}$.

Equality of two matrices

$A=\left[a_{ij}\right]=\left[b_{ij}\right]=B$ if, A and B are of same order and $a_{ij}=b_{ij}\ \forall\ i$ and j; $i,j \in N$

Elementary operations on a matrix

Properties for applying the operations

$R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

$R_i \rightarrow kR_i$ or $C_i \rightarrow kC_i$

$R_i \rightarrow R_i + kR_j$ or $C_i \rightarrow C_i + kC_j$

Inverse of a matrix

If A, B are square matrices such that AB = BA = I then $B=A^{-1}$ *i.e.,* A is the inverse of B and vice-versa.

Inverse of a square matrix, if it exists, is unique

For example: If $A=\begin{pmatrix}1 & 2\\3 & 4\\5 & 6\end{pmatrix}$, then after $R_1 \leftrightarrow R_2$, A becomes $\begin{pmatrix}3 & 4\\1 & 2\\5 & 6\end{pmatrix}$

If A and B are invertible matrices of the same order, then $(AB)^{-1}=B^{-1}A^{-1}$.

By elementary transformations, we can convert A=IA to $I = A^{-1}A$.

This is one process of finding the inverse of a given square matrix A.

Operations on matrices

Addition

If A, B are two matrices of same order, then $A+B=\left[a_{ij}+b_{ij}\right]$. The addition of A and B follows: $A+B=B+A$, $(A+B)+C=A+(B+C)$, $-A=(-1)A$, $k(A+B)=kA+kB$, k is scalar and $(k+I)A=kA+IA$, k and I are constants.

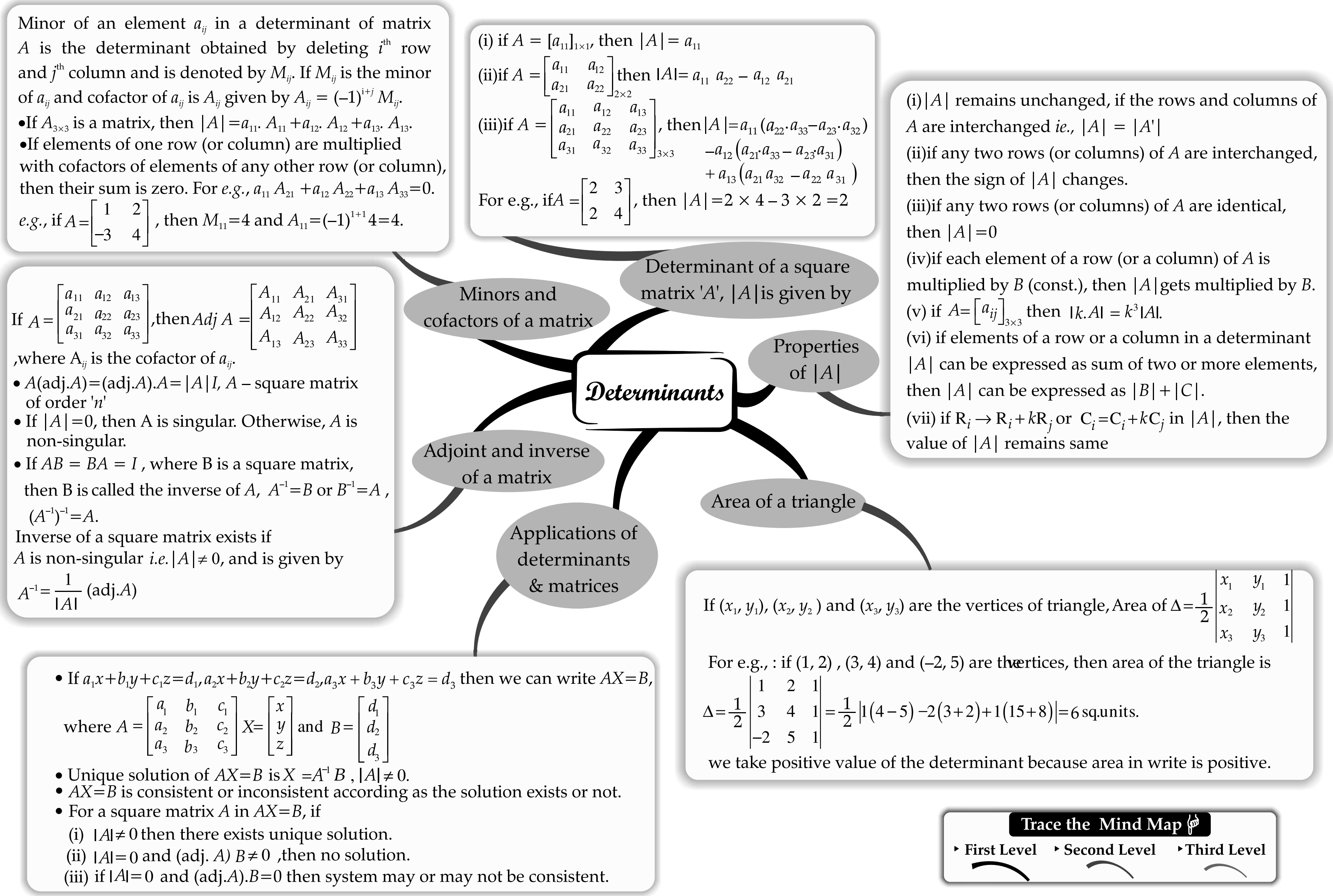
- If $A=\begin{bmatrix}2 & 3\\2 & 4\end{bmatrix}$, $B=\begin{bmatrix}-3 & 2\\-4 & 5\end{bmatrix}$ then $A+B=\begin{bmatrix}-1 & 5\\-2 & 9\end{bmatrix}$
- If $A=\begin{bmatrix}2 & 3\end{bmatrix}_{1\times 2}$, $B=\begin{bmatrix}4\\5\end{bmatrix}_{2\times 1}$, then $AB=\begin{bmatrix}2\times4+3\times5\end{bmatrix}=\begin{bmatrix}23\end{bmatrix}_{1\times 1}$

Multiplication

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{jk}]_{n\times p}$, then $AB=C=[C_{ik}]_{m\times p}$, $[C_{jk}]=\sum_{j=1}^{n}a_{ij}b_{jk}$. Also, $A(BC)=(AB)C$, $A(B+C)=AB+AC$ and $(A+B)C=AC+BC$, but $AB\neq BA$ (always).

Trace the Mind Map

‣ First Level ‣ Second Level ‣ Third Level



Determinants

Minors and cofactors of a matrix

Minor of an element a_{ij} in a determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column and is denoted by M_{ij}. If M_{ij} is the minor of a_{ij} and cofactor of a_{ij} is A_{ij} given by $A_{ij} = (-1)^{i+j} M_{ij}$.

- If $A_{3\times3}$ is a matrix, then $|A| = a_{11}. A_{11} + a_{12}. A_{12} + a_{13}. A_{13}$.
- If elements of one row (or column) are multiplied with cofactors of elements of any other row (or column), then their sum is zero. For *e.g.*, $a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{33} = 0$.

e.g., if $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, then $M_{11} = 4$ and $A_{11} = (-1)^{1+1} 4 = 4$.

Determinant of a square matrix 'A', $|A|$ is given by

(i) if $A = [a_{11}]_{1\times1}$, then $|A| = a_{11}$

(ii)if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2}$ then $|A| = a_{11}\ a_{22} - a_{12}\ a_{21}$

(iii)if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$, then $|A| = a_{11}(a_{22}.a_{33} - a_{23}.a_{32}) - a_{12}(a_{21}.a_{33} - a_{23}.a_{31}) + a_{13}(a_{21}\,a_{32} - a_{22}\,a_{31})$

For e.g., if $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$, then $|A| = 2 \times 4 - 3 \times 2 = 2$

Properties of $|A|$

(i) $|A|$ remains unchanged, if the rows and columns of A are interchanged *ie.*, $|A| = |A'|$

(ii)if any two rows (or columns) of A are interchanged, then the sign of $|A|$ changes.

(iii)if any two rows (or columns) of A are identical, then $|A| = 0$

(iv)if each element of a row (or a column) of A is multiplied by B (const.), then $|A|$ gets multiplied by B.

(v) if $A = \left[a_{ij}\right]_{3\times3}$ then $|k.A| = k^3|A|$.

(vi) if elements of a row or a column in a determinant $|A|$ can be expressed as sum of two or more elements, then $|A|$ can be expressed as $|B| + |C|$.

(vii) if $R_i \rightarrow R_i + kR_j$ or $C_i = C_i + kC_j$ in $|A|$, then the value of $|A|$ remains same

Adjoint and inverse of a matrix

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $Adj\ A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

,where A_{ij} is the cofactor of a_{ij}.

- $A(\text{adj}.A) = (\text{adj}.A).A = |A|I$, A – square matrix of order 'n'
- If $|A| = 0$, then A is singular. Otherwise, A is non-singular.
- If $AB = BA = I$, where B is a square matrix, then B is called the inverse of A, $A^{-1} = B$ or $B^{-1} = A$, $(A^{-1})^{-1} = A$.

Inverse of a square matrix exists if A is non-singular *i.e.* $|A| \neq 0$, and is given by

$$A^{-1} = \frac{1}{|A|}(\text{adj}.A)$$

Area of a triangle

If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of triangle, Area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

For e.g., : if (1, 2) , (3, 4) and (–2, 5) are the vertices, then area of the triangle is

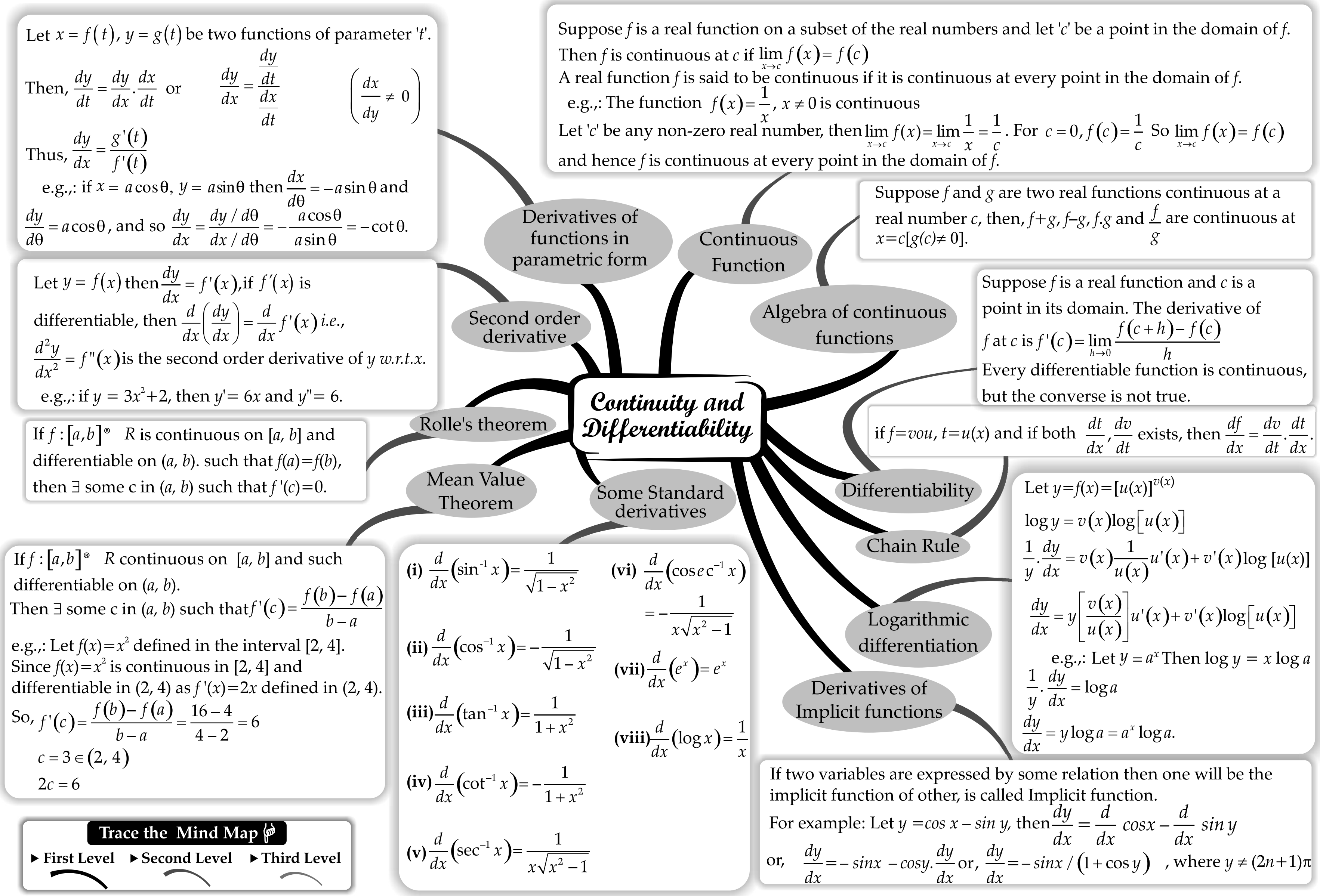
$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ -2 & 5 & 1 \end{vmatrix} = \frac{1}{2}\left|1(4-5) - 2(3+2) + 1(15+8)\right| = 6 \text{ sq.units.}$$

we take positive value of the determinant because area in write is positive.

Applications of determinants & matrices

- If $a_1x + b_1y + c_1z = d_1, a_2x + b_2y + c_2z = d_2, a_3x + b_3y + c_3z = d_3$ then we can write $AX = B$, where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$
- Unique solution of $AX = B$ is $X = A^{-1}B$, $|A| \neq 0$.
- $AX = B$ is consistent or inconsistent according as the solution exists or not.
- For a square matrix A in $AX = B$, if
  - (i) $|A| \neq 0$ then there exists unique solution.
  - (ii) $|A| = 0$ and (adj. A) $B \neq 0$,then no solution.
  - (iii) if $|A| = 0$ and (adj.A).$B = 0$ then system may or may not be consistent.

Trace the Mind Map



Continuity and Differentiability

Continuous Function

Suppose f is a real function on a subset of the real numbers and let 'c' be a point in the domain of f. Then f is continuous at c if $\lim_{x\to c} f(x) = f(c)$

A real function f is said to be continuous if it is continuous at every point in the domain of f.

e.g.,: The function $f(x) = \frac{1}{x}$, $x \neq 0$ is continuous

Let 'c' be any non-zero real number, then $\lim_{x\to c} f(x) = \lim_{x\to c} \frac{1}{x} = \frac{1}{c}$. For $c = 0, f(c) = \frac{1}{c}$ So $\lim_{x\to c} f(x) = f(c)$ and hence f is continuous at every point in the domain of f.

Algebra of continuous functions

Suppose f and g are two real functions continuous at a real number c, then, $f+g$, $f-g$, $f.g$ and $\frac{f}{g}$ are continuous at $x=c[g(c)\neq 0]$.

Differentiability

Suppose f is a real function and c is a point in its domain. The derivative of f at c is $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$

Every differentiable function is continuous, but the converse is not true.

Chain Rule

if $f=vou$, $t=u(x)$ and if both $\frac{dt}{dx}, \frac{dv}{dt}$ exists, then $\frac{df}{dx} = \frac{dv}{dt}.\frac{dt}{dx}$.

Logarithmic differentiation

Let $y=f(x)=[u(x)]^{v(x)}$

$\log y = v(x)\log[u(x)]$

$\frac{1}{y}.\frac{dy}{dx} = v(x)\frac{1}{u(x)}u'(x) + v'(x)\log[u(x)]$

$\frac{dy}{dx} = y\left[\frac{v(x)}{u(x)}u'(x) + v'(x)\log[u(x)]\right]$

e.g.,: Let $y = a^x$ Then $\log y = x \log a$

$\frac{1}{y}.\frac{dy}{dx} = \log a$

$\frac{dy}{dx} = y \log a = a^x \log a$.

Derivatives of Implicit functions

If two variables are expressed by some relation then one will be the implicit function of other, is called Implicit function.

For example: Let $y = \cos x - \sin y$, then $\frac{dy}{dx} = \frac{d}{dx}\cos x - \frac{d}{dx}\sin y$

or, $\frac{dy}{dx} = -\sin x - \cos y.\frac{dy}{dx}$ or, $\frac{dy}{dx} = -\sin x/(1+\cos y)$, where $y \neq (2n+1)\pi$

Some Standard derivatives

(i) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

(ii) $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

(iv) $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$

(v) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

(vi) $\frac{d}{dx}(\cos e c^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$

(vii) $\frac{d}{dx}(e^x) = e^x$

(viii) $\frac{d}{dx}(\log x) = \frac{1}{x}$

Mean Value Theorem

If $f:[a,b]$® R continuous on $[a, b]$ and such differentiable on (a, b).

Then ∃ some c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

e.g.,: Let $f(x)=x^2$ defined in the interval [2, 4].

Since $f(x)=x^2$ is continuous in [2, 4] and differentiable in (2, 4) as $f'(x)=2x$ defined in (2, 4).

So, $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{16-4}{4-2} = 6$

$c = 3 \in (2, 4)$

$2c = 6$

Rolle's theorem

If $f:[a,b]$® R is continuous on $[a, b]$ and differentiable on (a, b). such that $f(a)=f(b)$, then ∃ some c in (a, b) such that $f'(c)=0$.

Second order derivative

Let $y = f(x)$ then $\frac{dy}{dx} = f'(x)$, if $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}f'(x)$ *i.e.,*

$\frac{d^2y}{dx^2} = f''(x)$ is the second order derivative of y *w.r.t.x.*

e.g.,: if $y = 3x^2+2$, then $y' = 6x$ and $y'' = 6$.

Derivatives of functions in parametric form

Let $x = f(t)$, $y = g(t)$ be two functions of parameter 't'.

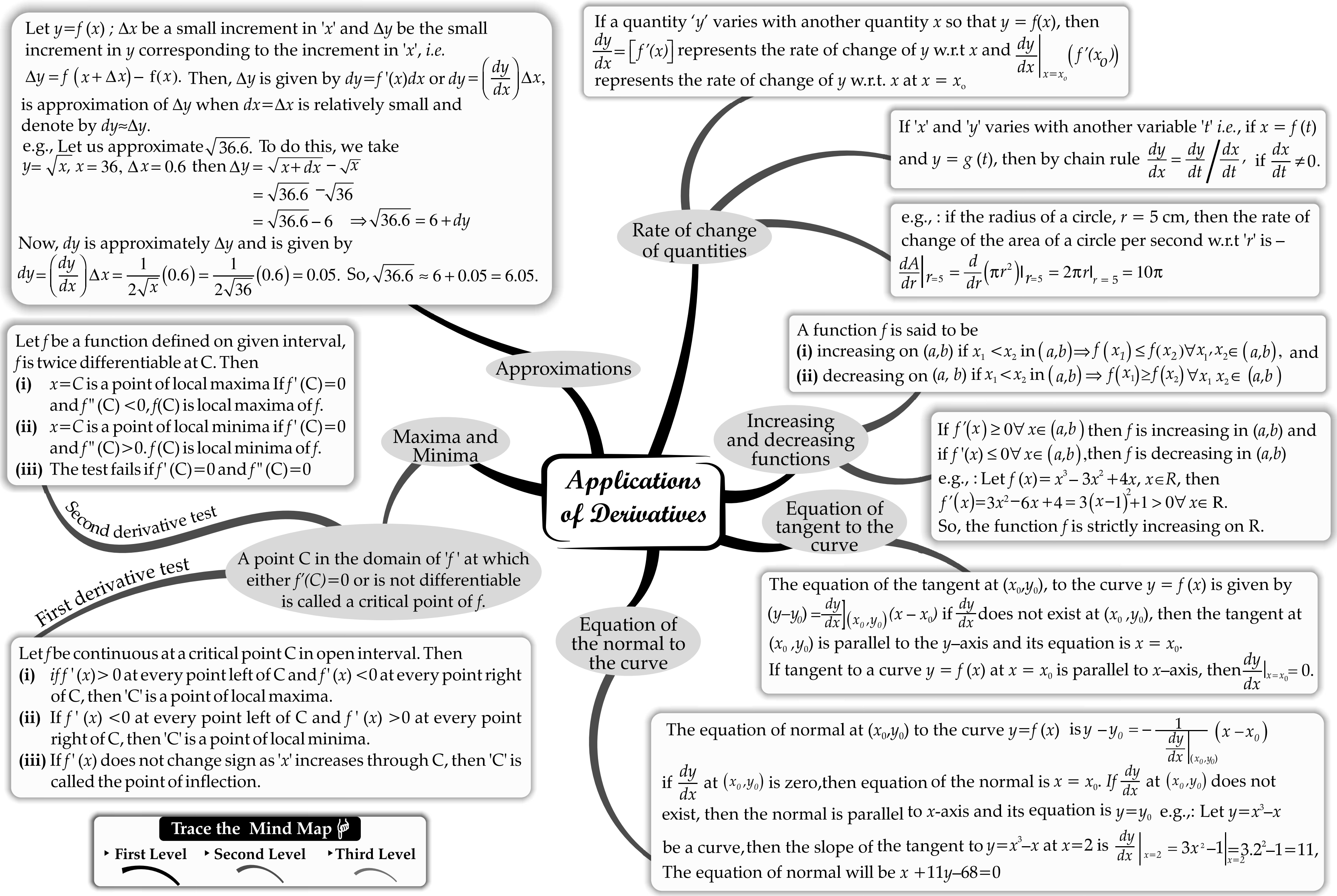
Then, $\frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt}$ or $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $\left(\frac{dx}{dy} \neq 0\right)$

Thus, $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$

e.g.,: if $x = a\cos\theta$, $y = a\sin\theta$ then $\frac{dx}{d\theta} = -a\sin\theta$ and $\frac{dy}{d\theta} = a\cos\theta$, and so $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{a\cos\theta}{a\sin\theta} = -\cot\theta$.

Trace the Mind Map

▸ First Level ▸ Second Level ▸ Third Level



Applications of Derivatives

Rate of change of quantities

If a quantity 'y' varies with another quantity x so that $y = f(x)$, then $\frac{dy}{dx} = [f'(x)]$ represents the rate of change of y w.r.t x and $\left.\frac{dy}{dx}\right|_{x=x_0} (f'(x_0))$ represents the rate of change of y w.r.t. x at $x = x_o$

If 'x' and 'y' varies with another variable 't' *i.e.*, if $x = f(t)$ and $y = g(t)$, then by chain rule $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$, if $\frac{dx}{dt} \neq 0$.

e.g., : if the radius of a circle, $r = 5$ cm, then the rate of change of the area of a circle per second w.r.t 'r' is –

$\left.\frac{dA}{dr}\right|_{r=5} = \frac{d}{dr}(\pi r^2)|_{r=5} = 2\pi r|_{r=5} = 10\pi$

Increasing and decreasing functions

A function f is said to be
(i) increasing on (a,b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) \leq f(x_2) \forall x_1, x_2 \in (a,b)$, and
(ii) decreasing on (a, b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) \geq f(x_2) \forall x_1\ x_2 \in (a,b)$

If $f'(x) \geq 0 \forall\ x \in (a,b)$ then f is increasing in (a,b) and if $f'(x) \leq 0 \forall\ x \in (a,b)$, then f is decreasing in (a,b)
e.g., : Let $f(x) = x^3 - 3x^2 + 4x$, $x \in R$, then
$f'(x) = 3x^2 - 6x + 4 = 3(x-1)^2 + 1 > 0 \forall\ x \in$ R.
So, the function f is strictly increasing on R.

Equation of tangent to the curve

The equation of the tangent at (x_0, y_0), to the curve $y = f(x)$ is given by $(y - y_0) = \left.\frac{dy}{dx}\right|_{(x_0, y_0)} (x - x_0)$ if $\frac{dy}{dx}$ does not exist at (x_0, y_0), then the tangent at (x_0, y_0) is parallel to the y–axis and its equation is $x = x_0$.
If tangent to a curve $y = f(x)$ at $x = x_0$ is parallel to x–axis, then $\left.\frac{dy}{dx}\right|_{x=x_0} = 0$.

Equation of the normal to the curve

The equation of normal at (x_0, y_0) to the curve $y = f(x)$ is $y - y_0 = -\frac{1}{\left.\frac{dy}{dx}\right|_{(x_0, y_0)}} (x - x_0)$
if $\frac{dy}{dx}$ at (x_0, y_0) is zero, then equation of the normal is $x = x_0$. If $\frac{dy}{dx}$ at (x_0, y_0) does not exist, then the normal is parallel to x-axis and its equation is $y = y_0$ e.g.,: Let $y = x^3 - x$ be a curve, then the slope of the tangent to $y = x^3 - x$ at $x = 2$ is $\left.\frac{dy}{dx}\right|_{x=2} = 3x^2 - 1\Big|_{x=2} = 3.2^2 - 1 = 11$,
The equation of normal will be $x + 11y - 68 = 0$

Maxima and Minima

A point C in the domain of 'f' at which either $f'(C)=0$ or is not differentiable is called a critical point of f.

First derivative test

Let f be continuous at a critical point C in open interval. Then
(i) if $f'(x) > 0$ at every point left of C and $f'(x) < 0$ at every point right of C, then 'C' is a point of local maxima.
(ii) If $f'(x) < 0$ at every point left of C and $f'(x) > 0$ at every point right of C, then 'C' is a point of local minima.
(iii) If $f'(x)$ does not change sign as 'x' increases through C, then 'C' is called the point of inflection.

Second derivative test

Let f be a function defined on given interval, f is twice differentiable at C. Then
(i) $x = C$ is a point of local maxima If $f'(C) = 0$ and $f''(C) < 0$, $f(C)$ is local maxima of f.
(ii) $x = C$ is a point of local minima if $f'(C) = 0$ and $f''(C) > 0$. $f(C)$ is local minima of f.
(iii) The test fails if $f'(C) = 0$ and $f''(C) = 0$

Approximations

Let $y = f(x)$; Δx be a small increment in 'x' and Δy be the small increment in y corresponding to the increment in 'x', *i.e.*
$\Delta y = f(x + \Delta x) - f(x)$. Then, Δy is given by $dy = f'(x)dx$ or $dy = \left(\frac{dy}{dx}\right)\Delta x$, is approximation of Δy when $dx = \Delta x$ is relatively small and denote by $dy \approx \Delta y$.
e.g., Let us approximate $\sqrt{36.6}$. To do this, we take
$y = \sqrt{x}$, $x = 36$, $\Delta x = 0.6$ then $\Delta y = \sqrt{x + dx} - \sqrt{x}$
$= \sqrt{36.6} - \sqrt{36}$
$= \sqrt{36.6} - 6 \quad \Rightarrow \sqrt{36.6} = 6 + dy$
Now, dy is approximately Δy and is given by
$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(0.6) = \frac{1}{2\sqrt{36}}(0.6) = 0.05$. So, $\sqrt{36.6} \approx 6 + 0.05 = 6.05$.

Trace the Mind Map

- First Level
- Second Level
- Third Level

Vectors

Vector

A quantity that has both magnitude and direction is called a vector. The distance between the initial and terminal points of a vector is called its magnitude. Magnitude of vector $\vec{AB}$ is $|\vec{AB}|$.

Position Vector

Position vector of a point *P (x, y, z)* is $x\hat{i}+y\hat{j}+z\hat{k}$ and its magnitude is $OP(r)=\sqrt{x^2+y^2+z^2}$. e.g.,: Position vector of *P (2,3,5)* is $2\hat{i}+3\hat{j}+5\hat{k}$ and its magnitude is $\sqrt{2^2+3^2+5^2}=\sqrt{38}$.

Direction ratios and direction cosines

The scalar components of a vector are its direction ratios, and represent its projections along the respective axes.

The magnitude *(r)* direction ratios *(a,b,c)* and direction cosines *(l,m,n)* of vector $a\hat{i}+b\hat{j}+c\hat{k}$ are related as:

$l=\frac{a}{r}, m=\frac{b}{r}, n=\frac{c}{r}$

e.g., : If $\vec{AB}=\hat{i}+2\hat{j}+3\hat{k}$, then $r=\sqrt{1+4+9}=\sqrt{14}$

Direction ratios are (1,2,3) and direction cosines are $\left(\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}}\right)$

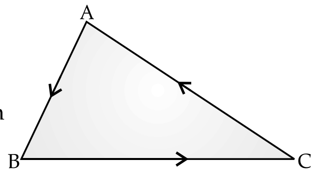
Types of Vectors

(i) Zero vector (initial and terminal points coincide)
(ii) Unit vector (magnitude is unity)
(iii) Coinitial vectors (same initial points)
(iv) Collinear vectors (parallel to the same line)
(v) Equal vectors (same magnitude and direction)
(vi) Negative of a vector (same magnitude, opp. direction)

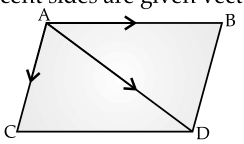
Properties of Vector

The vector sum of the three sides of a triangle taken in order is $\vec{0}$ *.i.e*

if *ABC* is given triangle, then $\vec{AB}+\vec{BC}+\vec{CA}=\vec{0}$.



The vector sum of two coinitials vectors is given by the diagonal of the parallelogram whose adjacent sides are given vectors.



if $\vec{AB}$, $\vec{AC}$ are the given vectors, then $\vec{AB}+\vec{AC}=\vec{AD}$

If we have two vectors $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and λ is any scalar, then-

(i) $\vec{a}\pm\vec{b}=(a_1\pm b_1)\hat{i}+(a_2\pm b_2)\hat{j}+(a_3\pm b_3)\hat{k}$

(ii) $\lambda\vec{a}=(\lambda a_1)\hat{i}+(\lambda a_2)\hat{j}+(\lambda a_3)\hat{k}$

(iii) $\vec{a}.\vec{b}=a_1b_1+a_2b_2+a_3b_3$and

(iv) $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

Cross product of two vectors

$\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, $\hat{n}$ is a unit vector perpendicular to line joining *a,b*.

Scalar product of two vectors

If $\vec{a},\vec{b}$ are the vectors and 'θ', angle between them, then their scalar product $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow \cos\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

Position of vectors

The position vector of a point R dividing a line segment joining *P,Q* whose position vectors are $\vec{a}$, $\vec{b}$ resp., in the ratio $m:n$

(i) internally is $\frac{n\vec{a}+m\vec{b}}{m+n}$, (ii) externally is $\frac{m\vec{b}-n\vec{a}}{m-n}$

Unit vector

For a given vector $\vec{a}$, the vector $\hat{a}=\frac{\vec{a}}{|\vec{a}|}$ gives the unit vector in the direction of $\vec{a}$. e.g., if $\vec{a}=5\hat{i}$, then $\hat{a}=\frac{5\hat{i}}{|5|}=\hat{i}$,which is a unit vector.

Trace the Mind Map

▸ First Level ▸ Second Level ▸ Third Level

Complex Number

Definition of Complex Numbers

A number of the form $a+ib$, where $a,b \in R$ and $i=\sqrt{-1}$ is called a complex number and denoted by 'z'.

$$z = \underset{\text{Real part}}{a} + \overset{\text{Imaginary part}}{ib}$$

Conjugate of a complex number: For a given complex number $z=a+ib$, its conjugate is defined as $\bar{z} = a - ib$

Algebra of Complex Numbers

Let: $z_1 = a+ib$ and $z_1 = c+id$ be two complex numbers, where $a,b,c,d \in R$ and $i=\sqrt{-1}$

1. Addition: $z_1+z_2=(a+ib)+(c+id)=(a+c)+i(b+d)$

2. Subtraction: $z_1-z_2=(a+ib)-(c+id)=(a-c)+(b-d)i$

3. Multiplication: $z_1 \cdot z_2=(a+ib)(c+id)$

$$=a(c+id)+ib(c+id)$$

$$=(ac-bd)+(ad+bc)i$$

$$(\therefore I^2=-1)$$

4. Division: $\frac{z_1}{z_2}=\frac{a+ib}{c+id}=\frac{a+ib}{c+id}.\frac{c-id}{c-id}$

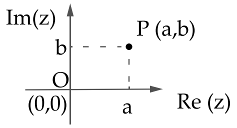
$$=\left(\frac{ac+bd}{c^2+d^2}\right)+\left(\frac{bc-ad}{c^2+d^2}\right)i$$

Note: If $a+ib = c+id$

$\Leftrightarrow a = c$ & $b = d$

Argand Plane

A complex number z=a+ib can be represented by a unique point P(a,b) in the argand plane



z=a+ib is represented by a point P (a,b)

Powers of 'i'

$i=\sqrt{-1}, i^2=-1$

In general, $i^{4k+r}=\begin{cases} 1; r=0 \\ i; r=1 \\ -1; r=2 \\ -i; r=3 \end{cases}$

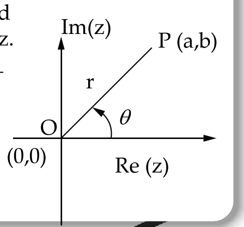
Modulus & Argument of Complex Number

If z= a+ib is a complex number

(i) Distance of z from origin is called as modulus of complex number z.

It is denoted by $r=|z|=\sqrt{a^2+b^2}$

(ii) Angle made by OP with +ve direction of X-axis is called argument of z.



Polar Representation

Let $a = r\cos\theta$

$b = r\sin\theta$

where,

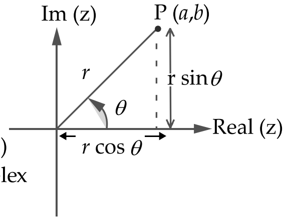
$r=|z|$

and $\theta = \arg(z)$

$\therefore z=a+ib=r(\cos\theta+i\sin\theta)$

The argument 'θ' of complex number $z=a+ib$ is called principal argument of z if

$-\pi<\theta\leq\pi.$



Square root of Complex Number

Let $x+iy=\sqrt{a+ib}$, squaring both sides, we get $(x+iy)^2=a+ib$ i.e. $x^2-y^2=a$, $2xy=b$ solving these equations, we get square root of z.

Multiplicative Inverse of complex Number

For a non-zero complex number $z=a+ib$ $(a\neq 0, b\neq 0)$, there exists a complex number $\frac{a}{a^2+b^2}+\frac{-b}{a^2+b^2}i$ denoted by $\frac{1}{z}$ or z^{-1}, called multiplicative inverse of z such that: $(a+ib)\left(\frac{a}{a^2+b^2}+i\frac{-b}{a^2+b^2}\right)=1+0i=1$

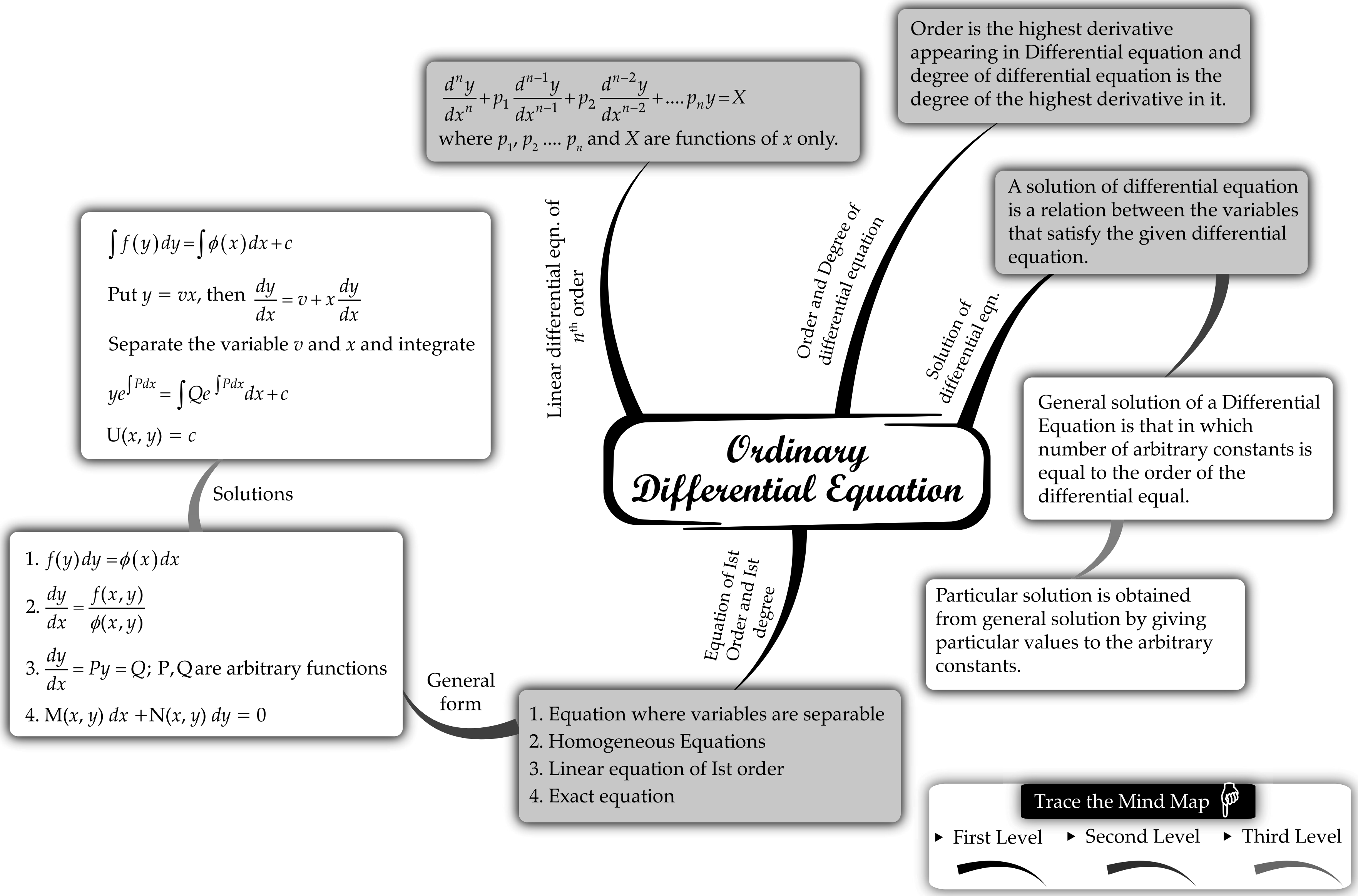
Solution of Quadratic Equation

- General form of quadratic equation in x is $ax^2+bx+c=0$, Where $a, b, c \in R$ & $a\neq 0$
  The solutions of given quadratic equation are given by $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, $a\neq 0$
  $\because b^2-4ac<0$

Note:
- A monomial equation has one root.
- A polynomial equation of degree n has n roots.

Trace the Mind Map

▸ First Level ▸ Second Level ▸ Third Level



Ordinary Differential Equation

Linear differential eqn. of n^{th} order

$$\frac{d^n y}{dx^n}+p_1\frac{d^{n-1}y}{dx^{n-1}}+p_2\frac{d^{n-2}y}{dx^{n-2}}+....p_n y=X$$

where $p_1, p_2 .... p_n$ and X are functions of x only.

Order and Degree of differential equation

Order is the highest derivative appearing in Differential equation and degree of differential equation is the degree of the highest derivative in it.

Solution of differential eqn.

A solution of differential equation is a relation between the variables that satisfy the given differential equation.

General solution of a Differential Equation is that in which number of arbitrary constants is equal to the order of the differential equal.

Particular solution is obtained from general solution by giving particular values to the arbitrary constants.

Equation of Ist Order and Ist degree

1. Equation where variables are separable
2. Homogeneous Equations
3. Linear equation of Ist order
4. Exact equation

General form

1. $f(y)dy=\phi(x)dx$
2. $\frac{dy}{dx}=\frac{f(x,y)}{\phi(x,y)}$
3. $\frac{dy}{dx}=Py=Q$; P, Q are arbitrary functions
4. $M(x, y)\,dx + N(x, y)\,dy = 0$

Solutions

$\int f(y)dy=\int\phi(x)dx+c$

Put $y = vx$, then $\frac{dy}{dx}=v+x\frac{dy}{dx}$

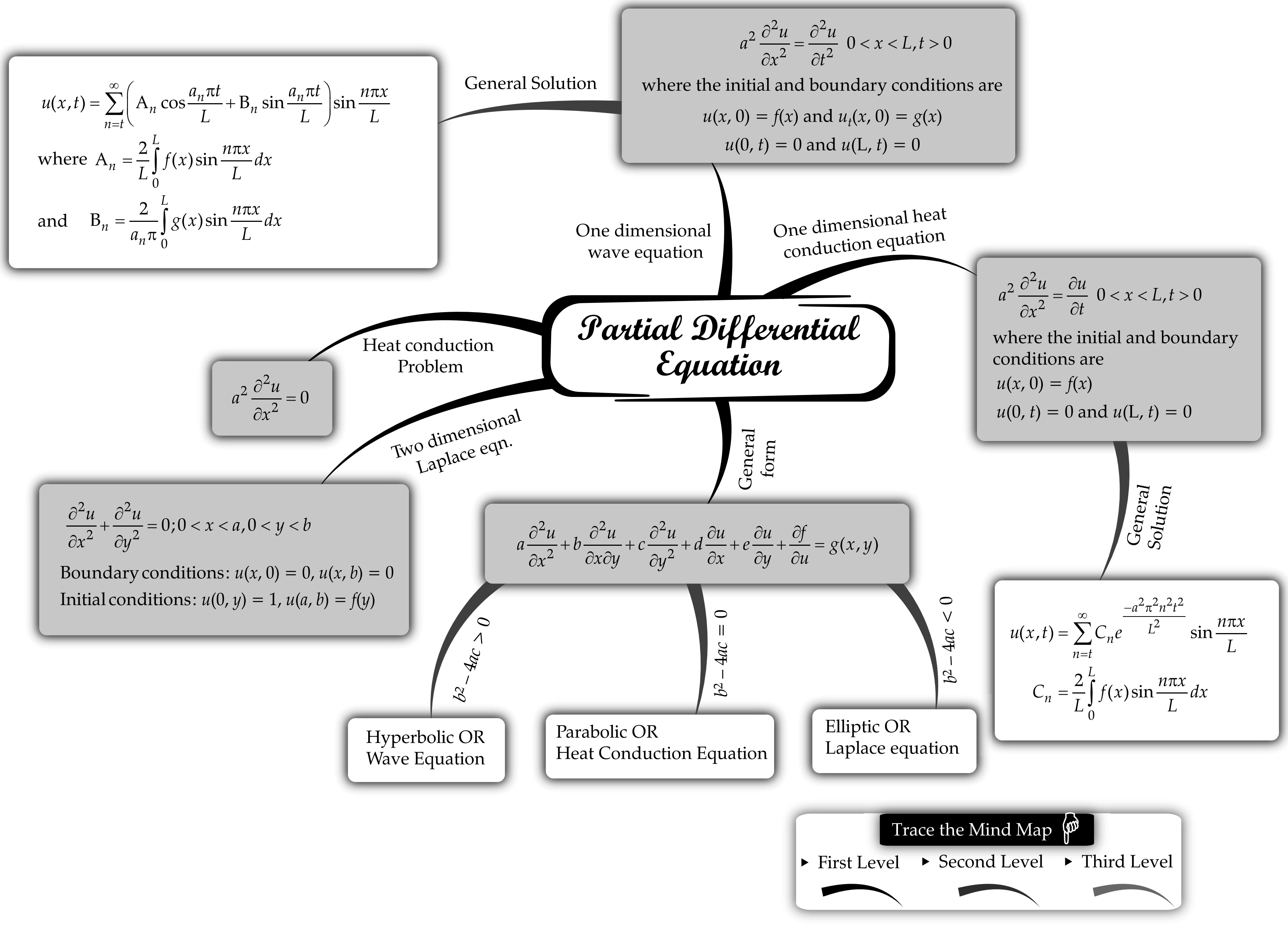
Separate the variable v and x and integrate

$ye^{\int Pdx}=\int Qe^{\int Pdx}dx+c$

$U(x, y) = c$

Trace the Mind Map

▸ First Level ▸ Second Level ▸ Third Level



Partial Differential Equation

One dimensional wave equation

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad 0 < x < L, t > 0$$

where the initial and boundary conditions are

$u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$

$u(0, t) = 0$ and $u(L, t) = 0$

General Solution

$$u(x,t) = \sum_{n=t}^{\infty}\left(A_n \cos\frac{a_n \pi t}{L} + B_n \sin\frac{a_n \pi t}{L}\right)\sin\frac{n\pi x}{L}$$

where $A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$

and $B_n = \frac{2}{a_n \pi}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$

One dimensional heat conduction equation

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad 0 < x < L, t > 0$$

where the initial and boundary conditions are

$u(x, 0) = f(x)$

$u(0, t) = 0$ and $u(L, t) = 0$

General Solution

$$u(x,t) = \sum_{n=t}^{\infty} C_n e^{\frac{-a^2\pi^2 n^2 t^2}{L^2}} \sin\frac{n\pi x}{L}$$

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

Heat conduction Problem

$$a^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Two dimensional Laplace eqn.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; 0 < x < a, 0 < y < b$$

Boundary conditions: $u(x, 0) = 0$, $u(x, b) = 0$

Initial conditions: $u(0, y) = 1$, $u(a, b) = f(y)$

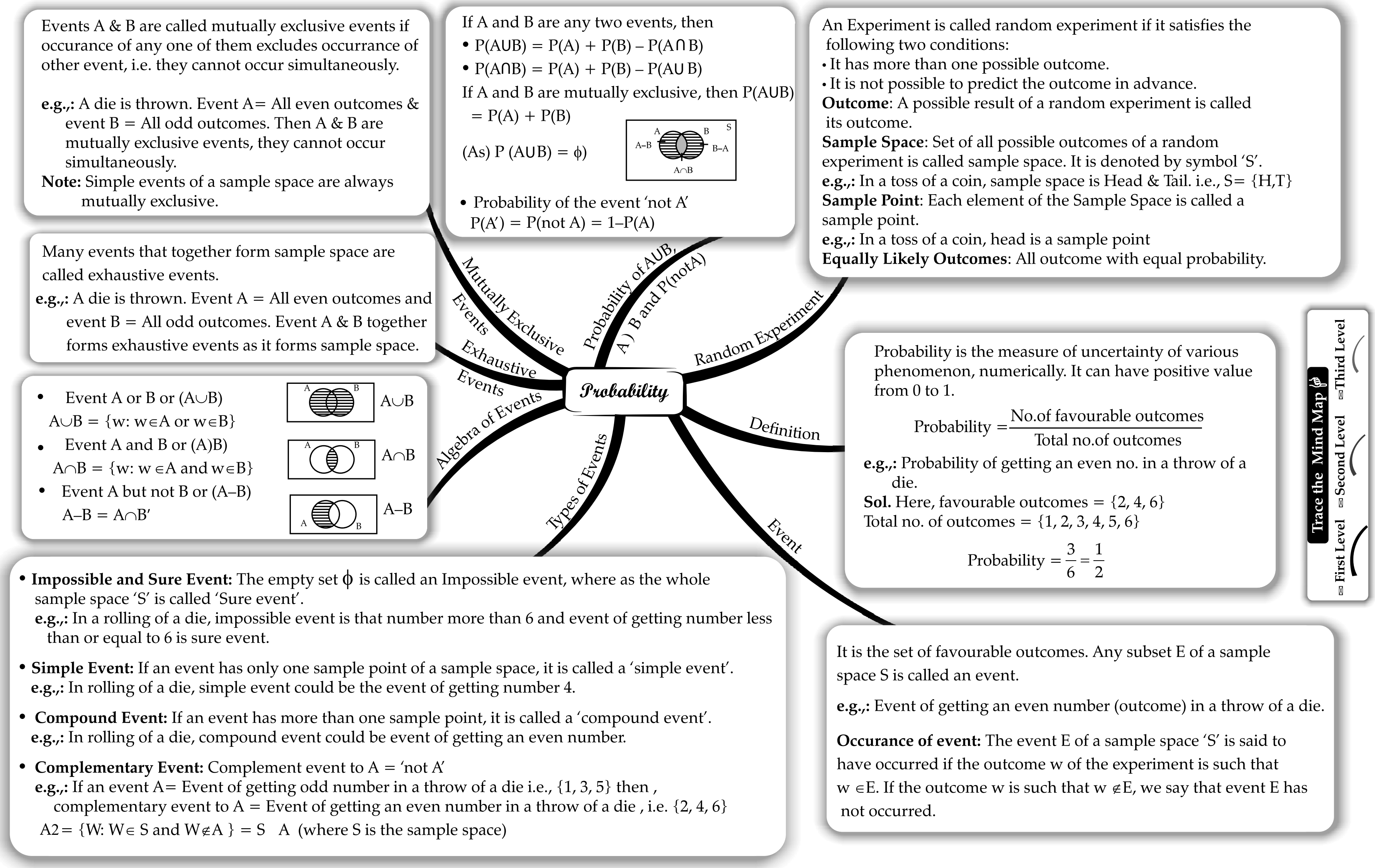
General form

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + \frac{\partial f}{\partial u} = g(x,y)$$

$b^2 - 4ac > 0$ Hyperbolic OR Wave Equation

$b^2 - 4ac = 0$ Parabolic OR Heat Conduction Equation

$b^2 - 4ac < 0$ Elliptic OR Laplace equation



An Experiment is called random experiment if it satisfies the following two conditions:

- It has more than one possible outcome.
- It is not possible to predict the outcome in advance.

Outcome: A possible result of a random experiment is called its outcome.

Sample Space: Set of all possible outcomes of a random experiment is called sample space. It is denoted by symbol 'S'.

e.g.,: In a toss of a coin, sample space is Head & Tail. i.e., S= {H,T}

Sample Point: Each element of the Sample Space is called a sample point.

e.g.,: In a toss of a coin, head is a sample point

Equally Likely Outcomes: All outcome with equal probability.

Probability is the measure of uncertainty of various phenomenon, numerically. It can have positive value from 0 to 1.

$$\text{Probability} = \frac{\text{No.of favourable outcomes}}{\text{Total no.of outcomes}}$$

e.g.,: Probability of getting an even no. in a throw of a die.

Sol. Here, favourable outcomes = {2, 4, 6}

Total no. of outcomes = {1, 2, 3, 4, 5, 6}

$$\text{Probability} = \frac{3}{6} = \frac{1}{2}$$

It is the set of favourable outcomes. Any subset E of a sample space S is called an event.

e.g.,: Event of getting an even number (outcome) in a throw of a die.

Occurance of event: The event E of a sample space 'S' is said to have occurred if the outcome w of the experiment is such that $w \in E$. If the outcome w is such that $w \notin E$, we say that event E has not occurred.

- **Impossible and Sure Event:** The empty set ϕ is called an Impossible event, where as the whole sample space 'S' is called 'Sure event'.
  e.g.,: In a rolling of a die, impossible event is that number more than 6 and event of getting number less than or equal to 6 is sure event.
- **Simple Event:** If an event has only one sample point of a sample space, it is called a 'simple event'.
  e.g.,: In rolling of a die, simple event could be the event of getting number 4.
- **Compound Event:** If an event has more than one sample point, it is called a 'compound event'.
  e.g.,: In rolling of a die, compound event could be event of getting an even number.
- **Complementary Event:** Complement event to A = 'not A'
  e.g.,: If an event A= Event of getting odd number in a throw of a die i.e., {1, 3, 5} then , complementary event to A = Event of getting an even number in a throw of a die , i.e. {2, 4, 6}
  $A2 = \{W: W \in S \text{ and } W \notin A\} = S \quad A$ (where S is the sample space)

- Event A or B or $(A \cup B)$
  $A \cup B = \{w: w \in A \text{ or } w \in B\}$
- Event A and B or (A)B)
  $A \cap B = \{w: w \in A \text{ and } w \in B\}$
- Event A but not B or (A–B)
  $A - B = A \cap B'$

Many events that together form sample space are called exhaustive events.

e.g.,: A die is thrown. Event A = All even outcomes and event B = All odd outcomes. Event A & B together forms exhaustive events as it forms sample space.

Events A & B are called mutually exclusive events if occurance of any one of them excludes occurrance of other event, i.e. they cannot occur simultaneously.

e.g.,: A die is thrown. Event A= All even outcomes & event B = All odd outcomes. Then A & B are mutually exclusive events, they cannot occur simultaneously.

Note: Simple events of a sample space are always mutually exclusive.

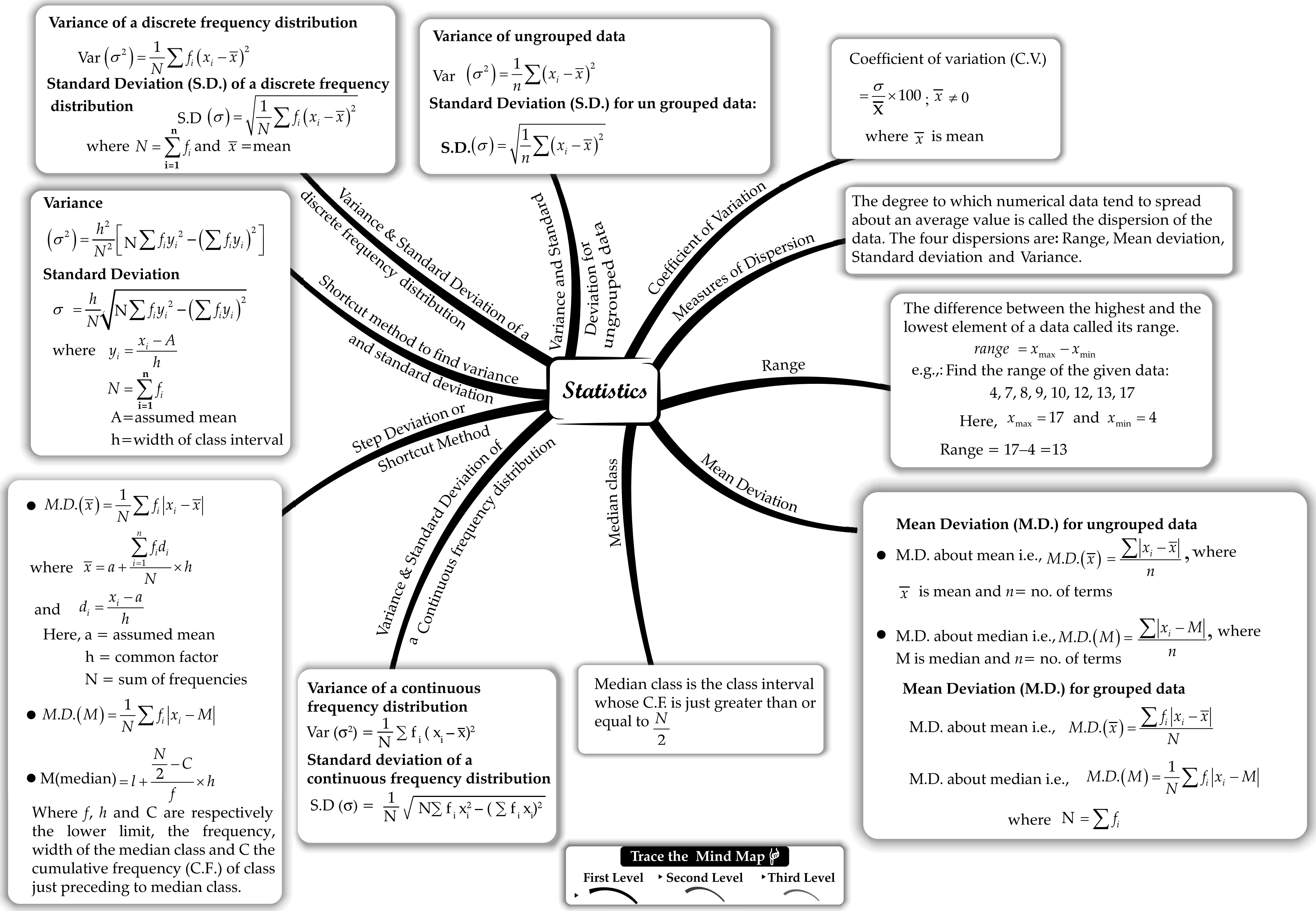
If A and B are any two events, then

- $P(AUB) = P(A) + P(B) - P(A \cap B)$
- $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

If A and B are mutually exclusive, then $P(AUB) = P(A) + P(B)$

(As) $P(AUB) = \phi)$

- Probability of the event 'not A'
  $P(A') = P(\text{not } A) = 1 - P(A)$



Variance of a discrete frequency distribution

$$\text{Var}\left(\sigma^2\right)=\frac{1}{N}\sum f_i\left(x_i-\bar{x}\right)^2$$

Standard Deviation (S.D.) of a discrete frequency distribution

$$\text{S.D}\ (\sigma)=\sqrt{\frac{1}{N}\sum f_i\left(x_i-\bar{x}\right)^2}$$

where $N=\sum_{i=1}^{n} f_i$ and $\bar{x}$ =mean

Variance of ungrouped data

$$\text{Var}\ \left(\sigma^2\right)=\frac{1}{n}\sum\left(x_i-\bar{x}\right)^2$$

Standard Deviation (S.D.) for un grouped data:

$$\textbf{S.D.}(\sigma)=\sqrt{\frac{1}{n}\sum\left(x_i-\bar{x}\right)^2}$$

Coefficient of variation (C.V.)

$$=\frac{\sigma}{\overline{\text{X}}}\times 100\ ;\ \bar{x}\neq 0$$

where $\bar{x}$ is mean

The degree to which numerical data tend to spread about an average value is called the dispersion of the data. The four dispersions are: Range, Mean deviation, Standard deviation and Variance.

The difference between the highest and the lowest element of a data called its range.

$$range\ = x_{\max}-x_{\min}$$

e.g.,: Find the range of the given data:

4, 7, 8, 9, 10, 12, 13, 17

Here, $x_{\max}=17$ and $x_{\min}=4$

Range = 17–4 =13

Variance

$$\left(\sigma^2\right)=\frac{h^2}{N^2}\left[\text{N}\sum f_i y_i^{\,2}-\left(\sum f_i y_i\right)^2\right]$$

Standard Deviation

$$\sigma\ =\frac{h}{N}\sqrt{\text{N}\sum f_i y_i^{\,2}-\left(\sum f_i y_i\right)^2}$$

where $y_i=\dfrac{x_i-A}{h}$

$N=\sum_{i=1}^{n} f_i$

A=assumed mean

h=width of class interval

Mean Deviation (M.D.) for ungrouped data

- M.D. about mean i.e., $M.D.(\bar{x})=\dfrac{\sum\left|x_i-\bar{x}\right|}{n}$, where $\bar{x}$ is mean and n= no. of terms
- M.D. about median i.e., $M.D.(M)=\dfrac{\sum\left|x_i-M\right|}{n}$, where M is median and n= no. of terms

Mean Deviation (M.D.) for grouped data

M.D. about mean i.e., $M.D.(\bar{x})=\dfrac{\sum f_i\left|x_i-\bar{x}\right|}{N}$

M.D. about median i.e., $M.D.(M)=\dfrac{1}{N}\sum f_i\left|x_i-M\right|$

where $\text{N}=\sum f_i$

- $M.D.(\bar{x})=\dfrac{1}{N}\sum f_i\left|x_i-\bar{x}\right|$

where $\bar{x}=a+\dfrac{\sum_{i=1}^{n} f_i d_i}{N}\times h$

and $d_i=\dfrac{x_i-a}{h}$

Here, a = assumed mean

h = common factor

N = sum of frequencies

- $M.D.(M)=\dfrac{1}{N}\sum f_i\left|x_i-M\right|$
- $\text{M(median)}=l+\dfrac{\frac{N}{2}-C}{f}\times h$

Where f, h and C are respectively the lower limit, the frequency, width of the median class and C the cumulative frequency (C.F.) of class just preceding to median class.

Variance of a continuous frequency distribution

$$\text{Var}\ (\sigma^2)=\frac{1}{\text{N}}\sum \text{f}_\text{i}\,(\text{x}_\text{i}-\bar{\text{x}})^2$$

Standard deviation of a continuous frequency distribution

$$\text{S.D}\ (\sigma)=\frac{1}{\text{N}}\sqrt{\text{N}\sum \text{f}_\text{i}\,\text{x}_\text{i}^2-(\sum \text{f}_\text{i}\,\text{x}_\text{i})^2}$$

Median class is the class interval whose C.F. is just greater than or equal to $\dfrac{N}{2}$

Transform Theory

Transform of elementary functions

1. $L\{1\} = \frac{1}{s}$ $\qquad (s > 0)$
2. $L\{t^n\} = \frac{n!}{S^{n+1}}; n = 0, 1, 2..$ $\left[\text{otherwise } \frac{\Gamma(n+1)}{S^{n+1}}\right]$
3. $L\{e^{at}\} = \frac{1}{s-a}$ $\qquad (s > a)$
4. $L\{\sin at\} = \frac{a}{s^2+a^2}$ $\qquad (s < 0)$
5. $L\{\cos at\} = \frac{s}{s^2+a^2}$ $\qquad (s > 0)$
6. $L\{\sinh at\} = \frac{a}{s^2-a^2}$ $\qquad (s > |a|)$
7. $L\{\cosh at\} = \frac{s}{s^2-a^2}$ $\qquad (s > |a|)$

Multiplication by t

If $L\{f(t) = F(s)$, then

$L\{t^n f(t) = (-1)^n \frac{d^n}{ds^n}[F(s)]$

Inverse Laplace Transform

1. $L^{-1}\left[\frac{1}{s}\right] = 1$
2. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
3. $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$ otherwise $\frac{t^n}{\lfloor n}$ $n = 1, 2, 3$
4. $L^{-1}\left[\frac{1}{(s-a)^n}\right] = \frac{e^{at}t^{n-1}}{(n-1)!}$
5. $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$
6. $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$
7. $L^{-1}\left[\frac{a}{s^2-a^2}\right] = \frac{1}{a}\sin hat$
8. $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \cos hat$
9. $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b}e^{at}\sin bt$
10. $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at}\cos bt$
11. $L^{-1}\left[\frac{s}{\left(s^2-a^2\right)^2}\right] = \frac{1}{2a}t\sin at$
12. $L^{-1}\left[\frac{1}{\left(s^2-a^2\right)^2}\right] = \frac{1}{2a^2}(\sin at - at\cos at)$

Division by t

If $L\{f(t)\} = F(s)$,

$L\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty F(s)\,ds$

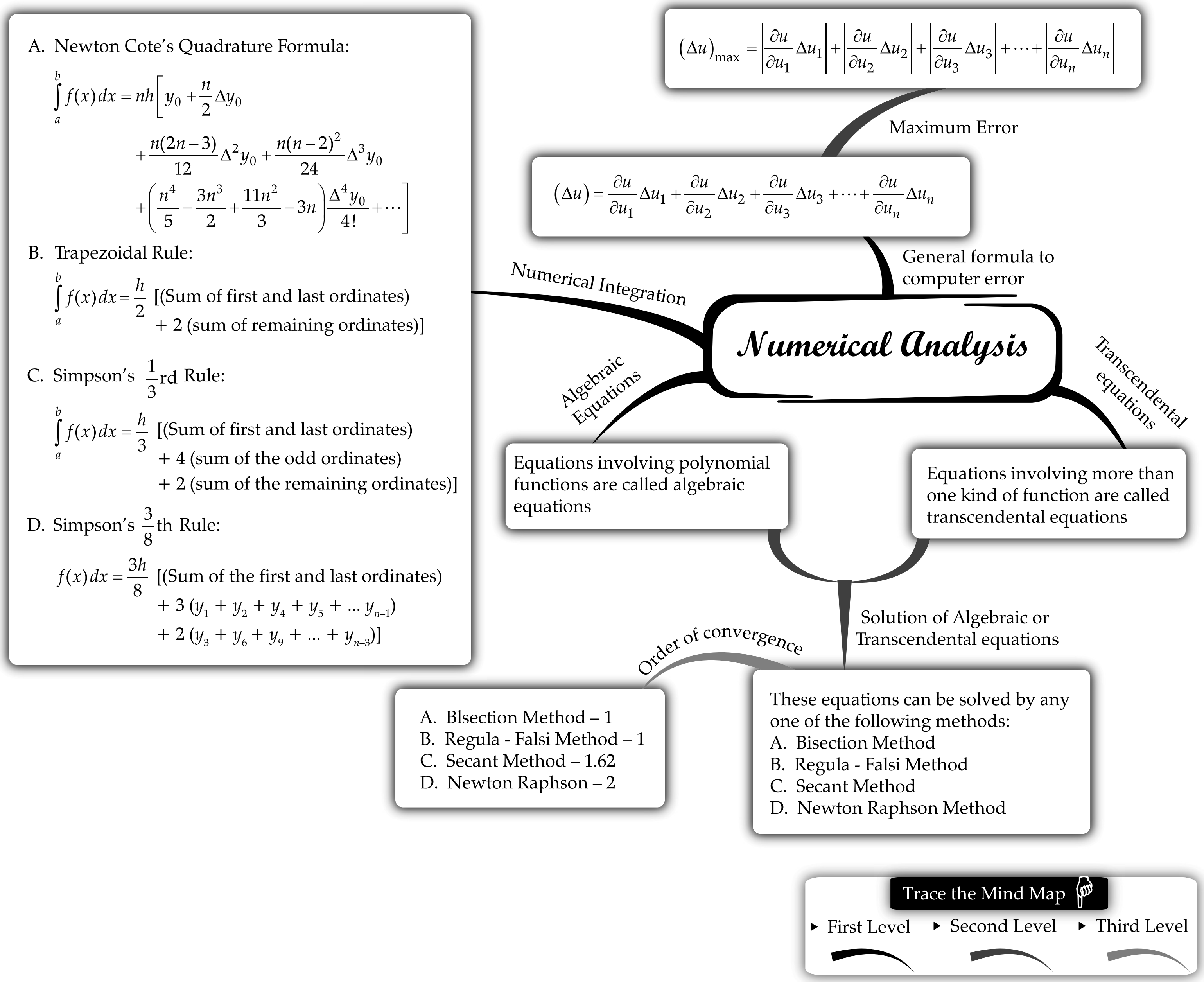
Tansform of integral

If $L\{f(t)\} = F(s)$ then

$L\left\{\int_0^t f(4)\,du\right\} = \frac{1}{s}F(s)$

Trace the Mind Map

- First Level
- Second Level
- Third Level



Numerical Analysis

General formula to computer error

$$(\Delta u) = \frac{\partial u}{\partial u_1}\Delta u_1 + \frac{\partial u}{\partial u_2}\Delta u_2 + \frac{\partial u}{\partial u_3}\Delta u_3 + \cdots + \frac{\partial u}{\partial u_n}\Delta u_n$$

Maximum Error

$$(\Delta u)_{\max} = \left|\frac{\partial u}{\partial u_1}\Delta u_1\right| + \left|\frac{\partial u}{\partial u_2}\Delta u_2\right| + \left|\frac{\partial u}{\partial u_3}\Delta u_3\right| + \cdots + \left|\frac{\partial u}{\partial u_n}\Delta u_n\right|$$

Numerical Integration

A. Newton Cote's Quadrature Formula:

$$\int_a^b f(x)\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!} + \cdots\right]$$

B. Trapezoidal Rule:

$\int_a^b f(x)\,dx = \frac{h}{2}$ [(Sum of first and last ordinates)
+ 2 (sum of remaining ordinates)]

C. Simpson's $\frac{1}{3}$rd Rule:

$\int_a^b f(x)\,dx = \frac{h}{3}$ [(Sum of first and last ordinates)
+ 4 (sum of the odd ordinates)
+ 2 (sum of the remaining ordinates)]

D. Simpson's $\frac{3}{8}$th Rule:

$f(x)\,dx = \frac{3h}{8}$ [(Sum of the first and last ordinates)
+ 3 ($y_1 + y_2 + y_4 + y_5 + \ldots y_{n-1}$)
+ 2 ($y_3 + y_6 + y_9 + \ldots + y_{n-3}$)]

Algebraic Equations

Equations involving polynomial functions are called algebraic equations

Transcendental equations

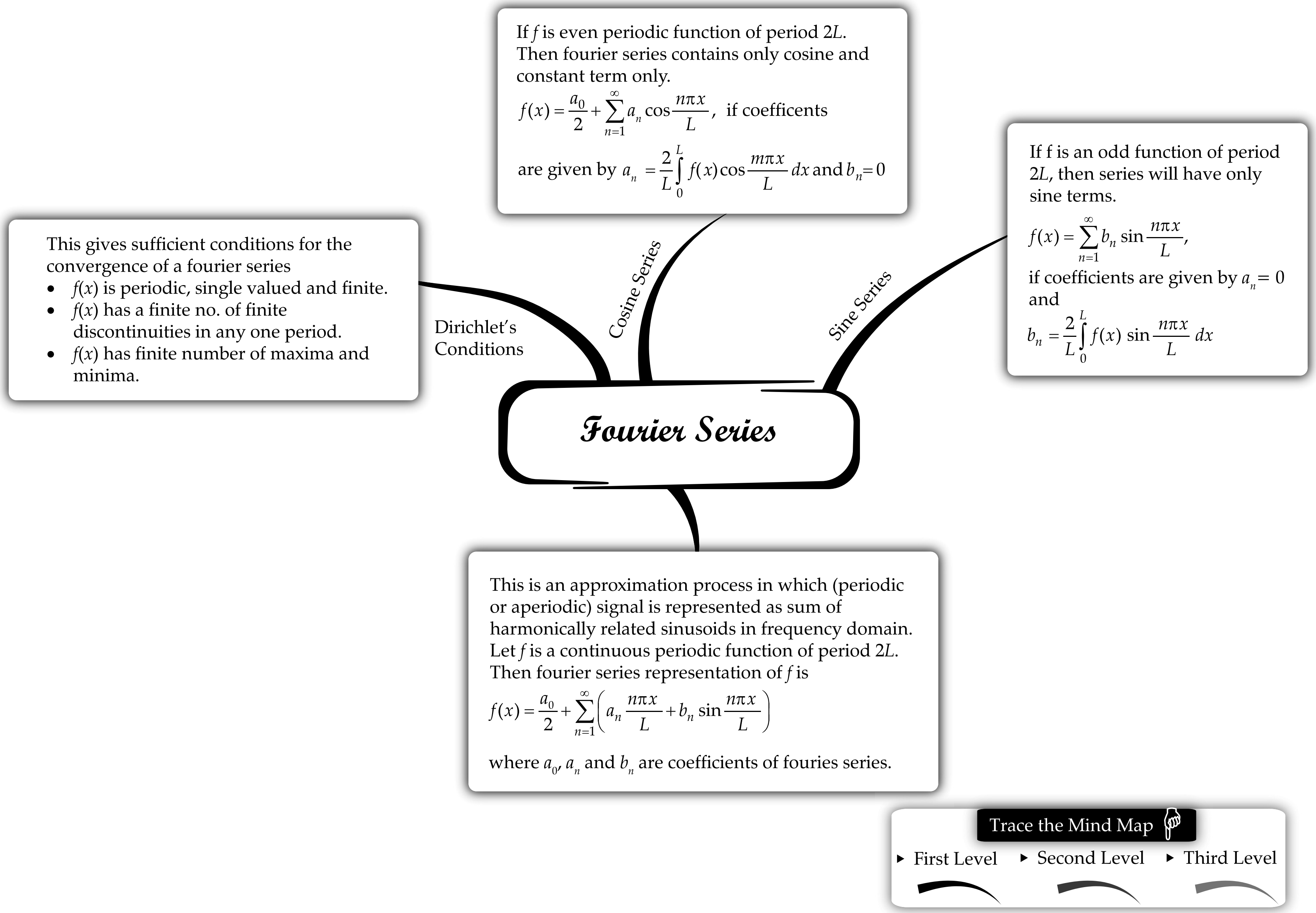
Equations involving more than one kind of function are called transcendental equations

Solution of Algebraic or Transcendental equations

These equations can be solved by any one of the following methods:
A. Bisection Method
B. Regula - Falsi Method
C. Secant Method
D. Newton Raphson Method

Order of convergence

A. BIsection Method – 1
B. Regula - Falsi Method – 1
C. Secant Method – 1.62
D. Newton Raphson – 2



Fourier Series

This is an approximation process in which (periodic or aperiodic) signal is represented as sum of harmonically related sinusoids in frequency domain. Let *f* is a continuous periodic function of period $2L$. Then fourier series representation of *f* is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where a_0, a_n and b_n are coefficients of fouries series.

Dirichlet's Conditions

This gives sufficient conditions for the convergence of a fourier series

- $f(x)$ is periodic, single valued and finite.
- $f(x)$ has a finite no. of finite discontinuities in any one period.
- $f(x)$ has finite number of maxima and minima.

Cosine Series

If *f* is even periodic function of period $2L$. Then fourier series contains only cosine and constant term only.

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$, if coefficents are given by $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx$ and $b_n = 0$

Sine Series

If f is an odd function of period $2L$, then series will have only sine terms.

$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$,

if coefficients are given by $a_n = 0$ and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Trace the Mind Map

- First Level
- Second Level
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