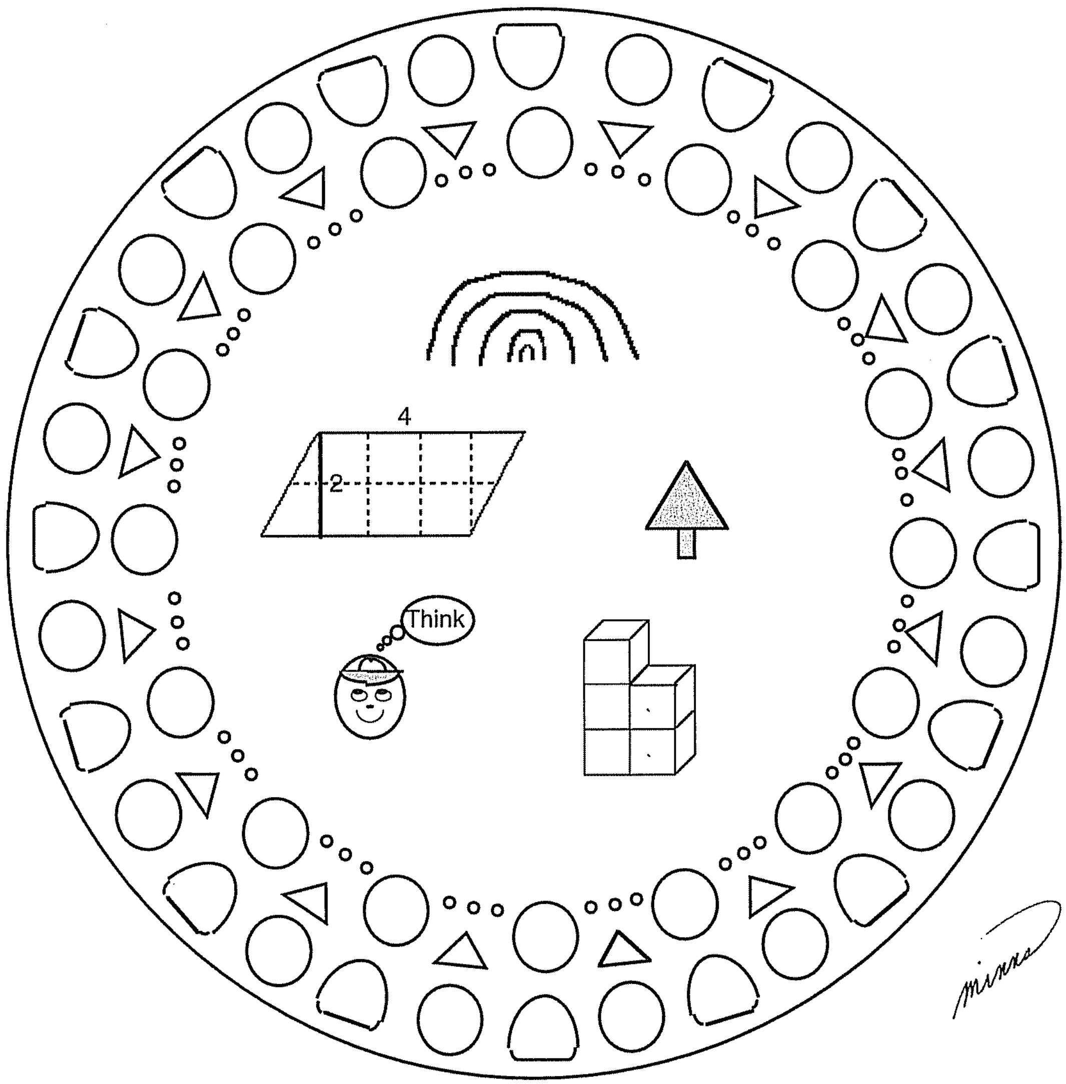
Patterns in Arithmetic

General Math - Booklet 5 PDF

Geometric Formulas, Linear Functions, and Division Relationships

Parent/Teacher Guide



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Based on methods developed by Prof. Michael Butler at the
UCI Farm Elementary School
University of California, Irvine

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Geometric Formulas, Linear Functions, and Division Relationships

Contents

We dedicate this booklet to Suki Glenn, whose tenacity and perseverance made this math series a reality.

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The cover mandala and many delightful illustrations are by Karen Marie Christa Minns. Other illustrations are by Suki Glenn and ClickArt by T/Maker.

To all of the mathematicians, from antiquity to the present, who discovered the principles of mathematics goes our heartfelt appreciation for your dedication.

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Parent/Teacher Guide

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Goldbach's Conjecture and Prime Factors: Review

Purpose

The purpose is to review Prime Numbers and Prime Factors. Goldbach's Conjecture is a fun mathematical idea that reawakens the fun in mathematics.

Prerequisites

Patterns in Arithmetic: Multiplication - Booklet 3 on Factoring and Prime Numbers, or previous instruction on what a prime number is and how to do prime factoring

Materials

Goldbach's Conjecture, page 1
Prime Factors: Review - Worksheets 1 - 6, pages 2 - 7

Warm Up

Fill out the chart on Prime Factors: Review - Worksheets 1 and 2 to remind the student of what a prime number is. 2 x 1 and 1 x 2 are the same factor pair and should be listed only once. Two has only itself and one as factors, as does three. But four has two pairs, 4 x 1 and 2 x 2. Use the Answer Key to be sure the chart is filled out correctly. Then fill out Prime Factors: Review - Worksheet 2 to consolidate the data. If she did Multiplication: Booklet 3, she will recognize this form. Discuss her answers to the last four questions.

Lesson Part 1

Begin the formal lesson with Goldbach's Conjecture. Look up the word 'conjecture.' **"Why do you think Goldbach uses the word 'conjecture' and not just say his pattern is always true?"** Answer: He could not test all numbers, as there are an infinite number. So he could not prove there were no exceptions. Mathematicians are rigorous in their word choice. Allow her to use her list of prime numbers to help her. Have fun filling out the worksheet.

Lesson Part 2

Go over Prime Factors: Review - Worksheet 3. This should be a review of what has been learned in the past.
"Follow the same steps as are shown on the example, except begin with 2 x 6 instead of 3 x 4 when factoring the twelve."

"Will you get the same result at Step 5?" "Yes, because I am finding the prime factors of the same number both times."

"Turn to Prime Factors: Review - Worksheet 4, and prime factor the eighteen two different ways." Watch. She should be able to do this without assistance. Have her finish the page alone and check the answers. If she can not, stay with her and do the rest of the page together.

Practice Worksheets

Prime Factors: Review - Worksheet 5, page 6

Test for Understanding

Prime Factors: Review - Worksheet 6 is an assessment. Reteach any areas where she is not strong.

Rainbow Rectangles and Area and Perimeter Logic Puzzle

Purpose The purpose is to delight a student with a lovely set of patterns and then use these patterns to review the area and perimeter of rectangles.

Prerequisites Basic multiplication tables

Materials Rainbow Rectangles - Worksheets 1 - 6, pages 8 - 13
Crayons or colored pencils
A blank piece of white paper

Warm Up Use a blank piece of white paper. Turn the paper sideways so the long side is on the bottom. On the piece of white paper write this sequence of numbers across the bottom of the page:

7 8 9 10 11 12 13 14

Choose a crayon, for example, blue. Put a blue circle around the seven, and a blue circle around the fourteen. Connect the two circles with a high arching blue line that will be the top of your rainbow. (See the figure on Rainbow Rectangles - Worksheet 1 on the right hand side of the page.) Now choose a green. Circle the eight and the thirteen in green. Draw a connecting arch between these two numbers. This arch will be under the blue one. Repeat with the nine and twelve, and the ten and eleven, choosing a different color for each pair. Now add the numbers in each pair.

"What do you find?" "The sums of each pair are always the same."

"Flip the page over and write another series of consecutive numbers, at least seven, and use an odd number of them this time."

What do you find?" "The sums of each pair are always the same."

"What about the odd number in the middle?" "If you double it, the sum is the same as the others."

"Isn't that cool? We call this a rainbow pattern. All kinds of interesting things happen with this pattern."

Note See a famous use of this pattern in the Extension at the end of the lesson.

Lesson Part 1 Begin with Rainbow Rectangles - Worksheet 1. Have the student trace the upside down rainbow with colored pencils. On the left is the pattern of the sums of the pairs. After he adds the pairs, he will again see the sums are equal.

"Do you think the products of the pairs will be equal also?" Most students will say they do not know.

"Try it." "The products are not equal."

"But what pattern do you see in the numbers as you go down the column?" "The numbers go up."

"Do the numbers go up in a regular pattern?" To answer this question, he must look at the differences between the numbers. There is a second column to the right of the column of products. Take the difference between the first two products twenty and twenty-seven. The difference is seven. Continue down the column taking the difference between the numbers.

Note

See the Answer Key to clarify how the pattern works.

"What pattern do you see?" "The sequence of odd numbers."

"Take the difference between each odd number now and write it to the right in the next column of spaces. What do you see?" "The differences are all two."

Continue on with the worksheet, answering the questions about the patterns. End the session.

Note

This part of the lesson uses the number series 2 - 10 and uses the pairs to form the length and width of rectangles. The lesson not only reviews the calculation of both area and perimeter, but also investigates the relationship between the area and the perimeter. In this lesson, the perimeter is held steady and the area changes in a predictable way. Most students will be surprised to find that the area changes when the perimeters are held steady. The conclusion we want him to see is that the closer the rectangle is to a square shape, where the length and width are nearly the same (the center pair of the rainbow pattern), the larger the area will be.

Lesson Part 2

On Rainbow Rectangles - Worksheet 2, follow the questions on the worksheet. To answer the short essay question at the bottom, draw his attention to the fact that the perimeter numbers match the series of numbers worked with on the top of Rainbow Rectangles - Worksheet 1.

"How is the perimeter of each rectangle related to the sum of the pairs on Rainbow Rectangles - Worksheet 1?" "The perimeter is twice the length of the sum of the pair because each pair is added twice to form the edges of the perimeter."

"At the bottom of the page, use a colored pencil to trace the pair in the rainbow that was used to make the edges of Rectangle C."

Practice Worksheets

Rainbow Rectangles - Worksheets 3 - 5, pages 10 - 12

Test for Understanding

Rainbow Rectangles - Worksheet 6, page 13. See how much of this problem he can do alone. Help him if needed; the questions he asks will tell you what he does and does not understand.

Worksheet Area and Perimeter - Logic Puzzle, page 14

Instructions: The whole, big square is 10 x 10 with an area of 100. The task of the student is to use the information provided on the puzzle and new information discovered while doing the puzzle to figure out the dimensions of each shape. Begin with squares B and D. Then go to square C. The only dimension given is the two, but since it is a square, then the other side must be two as well. To figure out J, use subtraction. You know the length of the whole side is ten and square D takes up five of those units. So the length of rectangle J must be five. Work your way through the puzzle in this fashion.

Extension For Math Lovers Try this: Add these numbers: 1 + 2 + 3 + 4 + 5 + 6 =

Here is a way to use the same pattern as the rainbow pattern to get the answer very fast. Instead of drawing the rainbow, write the sequence of numbers again and then underneath, write the same numbers in reverse order. Now add each pair.

1	2	3	4	5	6
6	5	4	3	2	1

"What pattern do you see?" "All the sums are seven."

"So you have six problems all of which have the sum of seven. If you multiply those two numbers, 6 x 7, you get forty-two. How is that forty-two related to the sum of the first six numbers you added at the beginning?" "The sum was twenty-one; that is half of the forty-two."

"Try it again with a different set of numbers. Try adding the sum of the first ten counting numbers. 1 + 2 + 3......+ 10. What is the sum?" "Fifty-five."

"Now try the reversing pattern you used above. What pattern do you see?" "The sums of each pair is eleven."

1	2	3	4	5	6	7	8	9	10
10	9	8	7	6	5	4	3	2	1

"So you have eleven, ten times. What is the product of those two numbers?" "One hundred ten."

"How is that related to the sum of fifty-five you got the first time?" "The fifty-five is half of the one hundred ten. This is the same as it was on the first one."

"How could you use this pattern to take the sum of any string of numbers?" "You can reverse the string and take the sums. You then multiply the sums by the number of numbers in the string and divide by two."

"How could you shorten that? Can you skip one of those steps?" (Hint: If all the sums are equal, do you need to write out the reversed string and add every number?) "No you could just take the sum of the first and last number, and multiply that by the number of numbers in the string and divide by two."

"Why do you think you have to divide by two to get the actual sum?" This is a hard question. It is because when you reverse the string and add each pair, you are adding each number twice–once at the beginning of the string and once at the end. So when you multiply, the product is two times the actual sum.

"So what would be a really fast way to get the sum of the first hundred counting numbers? 1 + 2 + 3.......+ 99 + 100?" "You would add 100 + 1 to get one hundred one. Then multiply that by one hundred to get 10,100 and then divide by two. That would give you 5,050."

Once upon a time an eight-year-old boy (who grew up to be one of the greatest mathematicians of all time) saw this pattern in his head and came up with the solution to adding the first 100 counting numbers in about thirty seconds. Here is the story.

Born in 1777 in Brunswick, Germany, Carl Friedrich Gauss showed early and unmistakable signs of being an extraordinary youth. As a child of three, he was checking, and occasionally correcting, the books of his father's business, this from a lad who could barely peer over the desktop into the ledger. A famous and charming story is told of Gauss's elementary school training. One of his teachers, apparently eager for a respite from the day's lessons, asked the class to work quietly at their desks and add up the first hundred whole numbers. Surely this would occupy the little tykes for a good long time. Yet the teacher had barely spoken, and the other children had hardly proceeded past "1 + 2 + 3 + 4 + 5 = 15" when Carl walked up and placed the answer on the teacher's desk. One imagines that the teacher registered a combination of incredulity and frustration at this unexpected turn of events, but a quick look at Gauss's answer showed it to be perfectly correct. How did he do it?
First of all, it was not magic, nor was it the ability to add a hundred numbers with lightning speed. Rather, even at this young age, Gauss exhibited the penetrating insight that would remain with him for a lifetime. As the story goes, he simply imagined the sum he sought–which we shall denote by S–being written simultaneously in ascending and in descending order:
Sum = 1 + 2 + 3 + 4 + ... + 98 + 99 + 100 also Sum = 100 + 99 + 98 + 97 + ... + 3 + 2 + 1.
Instead of adding these numbers horizontally across the rows, Gauss added them vertically down the columns. (Reversing the string like we did above.) In so doing, of course, he got 2S = 101 + 101 + 101 + ... + 101 + 101 + 101.

Since the sum of each column is just one hundred one and there are a hundred columns, then 2Sum = 100 × 101 = 10,100, and so the sum of the first hundred whole numbers is just S = 1 + 2 + 3 + 4 + ... + 99 + 100 = 10,100/2 = 5,050.
All this went through Gauss's young head in a flash. It was clear that he was going to make a name for himself.

He just saw the solution in his head, in a flash, at eight years old! Patterns are so wonderful!

Dunham, William. 1990. *Journey through Genius: The Great Theorems of Mathematics*. New York: Wiley, (pages 236 - 237)

Associative Blocks

Purpose

The purpose is to review the calculation of volume while reinforcing the concept of the Associative Property of Multiplication. The physical representation of this property makes it easier for students to understand it.

The Associative Property of Multiplication (and Addition) allows the combination of three or more numbers in any order. 2 x 3 x 4 = 4 x 2 x 3. If all the signs are multiplication signs, changing the order will not change the answer. This property is used in taking volumes of containers, and in factoring, and has many applications in fractions. A 3-D rectangle is called a rectangular solid or a rectangular prism.

Prerequisites

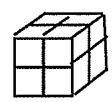
Basic multiplication tables, free exploration with linking cubes, and Commutative Property

Materials

Associative Blocks - Worksheets 1 - 4, pages 15 - 18
Cubes – snap-together type are the best. Linking Cubes work the very best. Base ten unit cubes can be used and taped together to hold the structure together. You could also use sugar cubes and glue them together, or marshmallows and toothpick them together.

Warm Up

After free exploration time: **"Start with one cube. Add cubes to make a larger rectangular solid that has two blocks on each side. Guess how many cubes it will take."**
"How long is your rectangular solid"? "Two cubes."
"How wide is your rectangular solid"? "Two cubes."
"How tall is your rectangular solid"? "Two cubes."



Lesson

Follow the instructions on the worksheets. Have the student build each figure and tape them together. Some students benefit from thinking of these shapes as buildings and each cube as a room.

Note

Do not tell her to just rotate the first structure. As usual, have her discover this. If you have a class, assign different groups to build different structures and then compare them.

Practice Worksheets

Associative Blocks - Worksheets 2 and 3, pages 16 and 17

Test for Understanding

1. Show You Know: Associative Blocks - Worksheet 4, page 18

2. Essay Question:
Why is this lesson titled Associative Blocks? What is the connection between the structures you built and the Associative Property of Multiplication?
Possible answer:
"The structures I built had different lengths, widths and heights, but they always

had the same three numbers. The numbers were just put in different orders. This is why the total number of blocks never changed. The lesson is called Associative Blocks because the Associative Property of Multiplication shows that the order you multiply the numbers in will not change the answer. The total number of blocks never changed because the sides were using the same numbers over and over. The way the structure looked was different, but the number of blocks remained the same."

Extension

Greatest Volume Problem
"Fold an 8 x 10 inch piece of paper into a box with no top that will have the greatest possible volume."

Note

Use centimeter grid paper to make calculations easier if you have a group of advanced math students. Try a 16 x 20 centimeter piece of paper.

The student also uses a graph to discover that the problem can be best solved by using a guess and check strategy using a calculator or a spreadsheet.

First, decide how tall you want the box to be. Let's say it will be two inches tall. Measure in two inches on all four sides of the box and draw a heavy scoring line. To fold, you will need to cut the corners in some way so that the edges can fold up. Good geometry problem to solve this part. Tape the corners so that the box can maintain its shape. Fill it with rice, or Rice Krispies, or popped corn, and measure the volume. If you want to get totally accurate, line the box with plastic wrap and use water, (Use heavy paper or poster board if you think you might want to use water–it is heavy.)

If you are in a class, compare the volumes and see what happens. If you are home-schooling, have members of the family make different size boxes.

You can also calculate the volumes. When you fold up the sides, you will see that you lose perimeter off the original paper. You have to take that into account when you calculate.

Example: A two inch side will take a two by two inch square off each of the corners of the paper. If the original paper is 8 x 10, the lengths of the sides would be six inches long and four inches wide. So the box would be six inches by four inches by two inches, or forty-eight cubic inches of volume.

Is it better to have short or tall sides? Where is the maximum volume? Where is the minimum volume? If you want to get really fancy, you can go for fractions of inches.

Finding Composite Factors from Prime Factors

Purpose

The purpose is to reinforce the concept of composite factors, and the relationship between a number and its prime factors. The lesson also uses the Associative Property of Multiplication again and introduces the mathematical grammar used to group factors. The ability to recognize and find prime and composite factors in a number is an important skill used when working with equivalence and simplification of fractions.

Prerequisites

Understanding the Associative Property and prime factoring

Materials

Finding Composite Factors from Prime Factors - Worksheets 1 - 5, pages 19 - 25
Eight copies for each student of Finding Composite Factors from Prime Factors: Using the Associative Property - Practice, page 23

Warm Up

Re-examine Prime Factoring: Review - Worksheet 2, page 3. Let the student refresh her understanding of the terms 'prime numbers' and 'composite numbers.'

Lesson

Finding Composite Factors from Prime Factors - Mystery Numbers: Worksheet 1 begins by reviewing the relationship of the given prime factors 3, 2, 2, 2 and the Mystery Number.
"How do you find out what number has 3, 2, 2, 2 as prime factors?" "Multiply the prime factors together to get twenty-four."

"List all the factor pairs of twenty-four that you can think of." "2 x 12, 3 x 8, 4 x 6 and 1 x 24."

"Are any of those factors prime numbers?" "Yes, two and three."

"In your list of prime factors for twenty-four, there are three 2s. Your list shows only one two. Where do the other two 2s come from?" Many students will be unable to answer this question. Have her prime factor twenty-four with a factor tree. She will see that the other two 2s come from the factors of twenty-four that were composite numbers.

"Which of the factors you listed are composite factors of twenty-four?" "12, 8, 6, and 24."

We are going to use the Associative Property of Multiplication to learn how to find all the composite factors of a number. Grouping signs and parentheses are used to do this. Study Finding Composite Factors from Prime Factors: Using the Associative Property, page 22, for a series of examples.

Complete Finding Composite Factors from Prime Factors: Mystery Numbers - Worksheet 1.
Begin Finding Composite Factors from Prime Factors: Mystery Numbers - Work-

sheet 2. Do this worksheet together. This work can get tedious, so stop when her energy begins to fall. Finish the rest of the worksheets, one or two problems a day for the next week or so.

Note

Notice on the factors of sixty, if the numbers are listed in order instead of just in pairs, you can see that in the middle of the string between ten and twelve, there are no other factors of sixty. This is an indication that you've got them all.

Practice Worksheets

Finding Composite Factors from Prime Factors - Mystery Numbers - Worksheets 2 - 5

Note

Assign only one problem each day.
Use multiple copies of Finding Composite Factors from Prime Factors: Using the Associative Property - Practice, page 23, to solve the problems on Worksheets 4 and 5, pages 24 and 25.

Test for Understanding

Give her the number two hundred forty and have her find all the factors, both prime and composite.
Prime Factors: 5, 3, 2, 2, 2, 2

Composite Factors: 4, 6, 8, 10, 12, 15, 16, 20, 24, 30, 40, 60, 80, 120, 240
If she misses 12 x 20 or 15 x 16 she may not have used the Associative tool but used division or expanded tables.

Prime Factors Meet the Associative Property

Purpose The purpose is to connect prime numbers, prime factoring, and using the Associative Property of Multiplication to solve area and perimeter problems. Most students find these problems challenging.

Prerequisites Previous lessons

Materials Prime Factors Meet the Associative Property - Worksheets 1 - 6, pages 26 - 31
Fourth inch grid paper
Rat Cage Problem Solver - Worksheets 1 and 2, pages 32 and 33 Make five copies of each of these two pages.
More Rat Cage Problems, page 34

Warm Up Prime Factors Meet the Associative Property - Worksheet 1 will serve as a warm up for this lesson. Check the Answer Key to be sure the worksheet is filled out correctly.

Lesson Part 1 What the student is going to do in this lesson is to use the prime and composite factors of a number to find a specific length and width of a rat cage with a given area. He will have to find all the factor pairs that give the area requested. Then he will have to find which pair will give the perimeter requested. This requires many arithmetic and some logic skills.

The Prime Factors Meet the Associative Property - Worksheets 2 - 4 walk him through the first problem. The Answer Key will help you understand what to do on these worksheets.

"What are the prime factors of forty-eight?" "Two, two, two, two, and three."

"Let's put a two and a three in the first parentheses, and all the other twos in the second parentheses. What composite factors would be the product of this grouping?" "Six and eight."

"With the grid paper, cut a rectangle that is six by eight squares. What is the area of this rectangle?" "Forty-eight squares."

"What is a perimeter?" "The distance around the edge."

"So what is the perimeter of this rectangle?" "6 + 6 + 8 + 8 = 28 units."

"Is that what the zoo has ordered?" "No."

"So how can you change the arrangement?" "I would have to put the three in the second parentheses. So I would get (2 x 2) (3 x 2 x 2) = 4 x 12."

"What would the perimeter of this rectangle be?" "Thirty-two units, so this is not the right one either."

"Go to Prime Factors Meet the Associative Property - Worksheet 4. Try grouping four factors together in the parentheses and leave out only one two." (See the Answer Key.) "I get two and twenty-four. This gives me a perimeter of fifty-two, which is again not what the zoo wants."

"What is left to try?" "Three on the outside and all the twos grouped together. That gives me three and sixteen. This perimeter is thirty-eight, so the carpenters should build the rat cage three feet wide and sixteen feet long."

Practice Worksheets

Have him go to Rat Cage Problem 2 on Prime Factors Meet the Associative Property - Worksheet 5, page 30. This one is more difficult because there are many combinations. The worksheet guides him through the process. Have him do this work independently if possible. Check the Answer Key to be sure the work is done correctly.

Lesson Part 2

Make multiple copies of the Rat Cage Problem Solver - Worksheets 1 and 2, pages 32 and 33. He will need five sets to complete all the problems.

Have him try problem 3 on the More Rat Cage Problems, page 34.
Give him a copy of the Rat Cage Problem Solver to fill in his factorizations and associations to aid him in his search for the correct pair.

Practice

Give only one problem every day or two on More Rat Cage Problems 4 through 6. If you are teaching a class, have groups work on these together.

Test for Understanding

Give him a calculator and a Rat Cage Problem Solver and ask him to find the solution for More Rat Cage Problems, problem 7.

Extension

Take all the perimeters from the problem above and put them in order from greatest to least. What is the pattern in the numbers?

Averaging: Manipulative

Purpose The purpose is to manipulatively teach and review the concept of averaging and to have the student develop the formula used for this procedure based on his experience.

Prerequisites Basic division facts

Materials Blank lined paper
Unifix Cubes

Warm Up Have the student freely explore with the Unifix Cubes for several minutes or so before beginning the lesson.

Lesson Have him pick out of the Unifix Cube box seven black cubes, six white cubes, three yellow cubes, two red cubes, and two blue cubes. Separate them into separate groups by color.

Note If you have a class, make the total number of cubes a multiple of five, such as sixty or one hundred. Maintain the total number of colors at five, but give students different numbers of cubes. Each student gets all blocks of the same color group. Example: Six students get black cubes. There are a total of thirty-five black cubes. Each student in the black cube group should not have the same number of black cubes. The black cube students should be sitting next to each other. Run the lesson the same way, but each student places his cubes in the stack for his color.

Make up a story about the cubes that goes something like this. Vary the numbers to fit your group size.

Once upon a time there was a village of people who kept their food in colored boxes. Each neighborhood had a different color of food boxes. The neighborhood with black boxes had seven boxes; the one with white boxes had six boxes; the one with yellow had only three; and the ones with red or blue had only two each. In each neighborhood the same number of people lived. Some people had lots of food and some were starving. The leaders of the town decided this was not fair. They decided that everyone should come to the town square with their boxes. The leaders told the neighborhoods that every group should have the same number of boxes. There would still be five neighborhoods, but each one would have an equal number of food boxes.

So the people came to the town square–some happy, some grumbling–and stacked their boxes in piles.

Have the student snap together all the black cubes into a stack. Do this with all the other colors as well.

Now, fix the stacks so that each stack is the same. The people who had lots of boxes

thought this process was very *Mean*, but they did as they were asked to do. From then on, whenever the boxes were delivered to the neighborhoods, the people all gathered in the town square and did the process of taking the *Mean Average*.
"What was the *Mean Average* number of food boxes each neighborhood ended up with?" "Five."

The next month the food boxes were delivered to the neighborhoods. The blacks got five boxes, the whites got only one; the yellows got fifteen; the reds got nine; and the blues got none at all.
Note: Make sure the blues get a stack, too.

"What was the *Mean Average* this time?" "Six boxes."

"Who grumbled and said this is *Mean*?" "The yellow group."

The next month, this is what happened. The boxes were not delivered on time. But the townspeople got a list telling them ahead of time how many boxes there were going to be of each color. This is what the list said:

Blacks = 12, Whites = 6, Yellows = 8 Reds = 11 Blues = 8

"Can you figure out ahead of time how many boxes each neighborhood will get after the *Mean Average* takes place in the town square?" Wait. See if he can figure out how to do this without the cubes. It is a rare student who will realize at this point that the easiest way to do this is to put all the boxes in the middle and divide by five. Most students will start moving numbers around. For example, he might subtract two from the blacks, leaving them with ten, and give two to the blues, giving them ten also.

Note

If you have a class, have each neighborhood work on the problem. When they are finished, have them share answers and compare how they solved the problem. In a large group it is likely someone will shout out to the group that they should just put all the blocks into the center and divide them up equally. This is in fact how averaging is done.

Conversation to lead to this conclusion might sound like this:
"Is there a faster way to figure out how many boxes each group gets without doing so much adding and subtracting?"

"What are the neighborhoods trying to do?" "Share all the boxes equally."

"In math when we use the words 'share equally,' what operation are we doing? Addition, subtraction, multiplication, or division?" "Division."

"So what are the neighbors dividing up?" "All the boxes."

"In math when we use the word 'altogether' what operation are we doing: ad-

dition, subtraction, multiplication, or division?" "Addition."

"Can you figure out from that how to give the right number of boxes to each neighborhood with those two operations, division and addition? Which one would you have to do first?" "Addition."

"Why?" "Because you can't divide up the boxes until you have them all in one place and know how many there are."

"So try that and see if you get the same answer as when you move the blocks around."

Practice Worksheets

Averaging: Manipulative

Test for Understanding

Make up a problem and see if he can do it with or without the blocks. How he solves the problem will tell you what he understands. Guide him to take all the blocks and put them into a pile and then divide them up. Then match that to the calculation.

Relationships

Purpose The purpose is to examine the relationship between dividends, divisors and quotients. The activity will help lay the foundation of the important concepts of ratio. The lesson first examines a 'the more the more' relationship of the effect on the quotient of doubling or halving the dividend while holding the divisor constant. This is followed by the examination of the effect on the quotient when the divisor is halved and the dividend held constant. The pattern that is revealed here is an inverse pattern. The concept of inverse relationships is critical to understanding fractions and ratio.
Most students find these activities easy but revealing.

$$\text{Divisor}\overline{\left)\text{Dividend}\right.}^{\text{Quotient}}$$

Prerequisites Basic division concepts

Dividend ÷ Divisor = Quotient

Materials Relationships - Worksheets 1 - 14, pages 36 - 49

Lesson Part 1 Relationships: Halving and Doubling the Dividend - Worksheets 1 and 2, pages 36 and 37. Have the student study the icons in the problem solving process. Predicting the pattern is an important part of the process. Do not have her skip it. Have her verbalize what she sees. For example: In the first problem, 12 ÷ 4 goes to 24 ÷ 4.

"What is happening?" "The dividend is doubled."

"How do you think that will change the answer?" "I know it will double the answer because I already know the answer to 24 ÷ 4."

"What do you predict the answer will be?" "Six."

"Write that on the first prediction line. Leave the second one blank; you will use that line later."

"Do you think that will always happen?" Many students will be unsure.
Have her complete the worksheets.

Note The general pattern is a 'the more, the more' relationship: the more the dividend, the more the quotient. The specific pattern revealed here is that if the size of the dividend is doubled, the quotient will be doubled as well. Help her develop and use this vocabulary.

Have her work though Relationships: Halving and Doubling the Dividend - Worksheets 1 and 2 independently. Require a well explained answer to the question on the bottom of Relationships - Worksheet 2. Learning to write about mathematics is an important skill.

Lesson Part 2 Relationships: Halving and Doubling the Dividend (with Remainders) - Worksheets 3 - 5, pages 38 - 40

The next day, warm up with the top two problems carefully and predict what will happen to the quotient. There is a new twist here. The dividend is halved, not doubled as before.

Be sure she fills in the answer to the relationship question, "When I cut the dividend in half, the quotient _______________." The correct answer is that it will be halved also.

"What if a division problem has a remainder and the dividend is doubled? Will the remainder double too?" Have her make a prediction. You make one too.

In the first two problems, the remainder does in fact double also. But in the third problem a complication arises. *Wait*. Let her work it out on her own if possible. The remainder does in fact double, but that doubling creates a remainder that is larger than the divisor. This causes another group to be made that obscures the doubling pattern.

Now she needs to use the second line in the predict answer box. The second line is for her to regroup that remainder that is larger than the divisor. Use the Answer Key to help you figure all this out. It is a good little mind bender.

Worksheets Relationships: Halving and Doubling the Dividend - Worksheets 4, and 5, pages 39 and 40

Lesson Part 3 Relationships: Halving the Divisor - Worksheets 6 - 9, pages 41 - 44
Do Relationships: Halving the Divisor - Worksheet 6 together.

"What pattern is happening when you look at the horizontal arrows connecting the problems in the left hand column to those in the right hand column?" "The dividend is doubling and the divisor is staying the same."

"What pattern is happening when you look at the vertical arrows connecting the top and lower problems ?" "The divisor is halving but the dividend is staying the same."

"What effect on the quotient do you think you will see as the divisor gets smaller?" Many students will be unsure.
Have her complete all the problems and then explain what patterns she sees.

"What happened to the quotients when the divisors were halved?" "They got larger."

"Larger in what way?" "They doubled."

"What pattern do you see again? Be precise." "When the divisor is halved the quotient is doubled."

Note

Add in at this point the words 'more' and 'less' instead of 'halved' and 'doubled.' **"Repeat that sentence using 'more' and 'less' instead of 'halved' and 'doubled.' "** "When the divisor is less, the quotient is more."

"Why does it make sense that when the divisor is halved the quotient is doubled? Try to explain why that would be." Most students will have difficulty with this.

Help her make a concrete example. Suggest this example:
"What if the forty-eight was the number of kids at a camp? At first they get in groups of eight and there are six groups. What happens when they get into groups of four instead?" "The more the groups, the less the kids in a group. The smaller the size of the group, the more groups there will be."

"Now you make up one." The answers will vary.

Vocabulary

A 'the more the less' pattern is called an 'inverse relationship.'

Practice Worksheets

Relationships: Doubling and Halving - Worksheets 7 - 9, pages 42 - 44
Relationships: Descending Divisor - Worksheets 10 and 11, pages 45 and 46

Lesson Part 4

Relationships: Multiplying Dividends - Worksheets 12 and 13, pages 47 and 48

This lesson extends the pattern of doubling the dividend (increasing it by a factor of two) resulting in doubling the quotient if the divisor is held constant to increasing the dividend by any factor and exploring what happens to the quotient. This understanding is a conceptual underpinning for double digit division, for multiplication of decimals where we multiply by factors of ten to remove decimals and then divide to put them back, and division of decimals in which we increase divisors by factors of ten to remove the decimal point from the divisor.

Vocabulary

Increasing by a factor of two is the same as doubling. $4 \div 4 = 1$ and $16 \div 4 = 4$. "The dividend was increased by a *factor* of four, so the quotient will be increased by a *factor* of four also."

Have her work independently and check her answers immediately in the Answer Key.

Test for Understanding

Relationships: Show You Know - Worksheet 14, page 49. This page is an assessment, so have her complete the page without help.

What's My Rule?

Purpose

The purpose is to learn to spot patterns in number series. When you are looking for a pattern that uncovers a hidden operation, you look at what number you started with and what number you ended with and try to figure out what happened to the numbers in between. This is one thing scientists and mathematicians do when they discover new "formulas." A game that exercises this skill is What's My Rule?

Activity

To play, think of a simple operation such as + 2. The student gives you a number and you add two to it and say the new number back. For example, if she says, "one," you say "three." She says, "five" and you say, "seven." The number she gives you is called the independent variable, or the number In. The number you say back is called the dependent variable, or the number Out.
Record the information on a chart like this:

IN	OUT
1	3
5	7
9	?

What's My Rule? +2

When a student thinks she knows the rule, let her predict the number out loud. Finally, when everyone can predict successfully, let someone formulate the rule of plus 2. Students love to make these up for each other using easy addition, subtraction, multiplication, and division operations.

Play this game regularly. It's a good rainy day and in-the-car game.

One teacher called this activity Black Box and made a symbolic black box out of a milk carton decorated with gears and levers with a slide inside that flipped a card upside down. A card would be put in the slot in the top of the box and come out so the number written on the back of the card came out a bottom slot. The students then guessed the rule. When the box appeared, the students' minds focused to discover the relationship between the In and Out number of the day.

Worksheet

What's My Rule? - Blank page to copy is on the following page.

Sample games:

IN	OUT
2	4
6	8
9	___
20	___

What's My Rule?____________

IN	OUT
7	4
3	0
10	7
8	___
20	___

What's My Rule?____________

*Answers: 9 11 8 5
20 22 20 17
Rule: Add 2 Rule: Subtract 3

What's My Rule? Date ______________

IN	OUT
___	___
___	___
___	___
___	___
___	___

What's My Rule?______________

IN	OUT
___	___
___	___
___	___
___	___
___	___

What's My Rule?______________

IN	OUT
___	___
___	___
___	___
___	___
___	___

What's My Rule?______________

IN	OUT
___	___
___	___
___	___
___	___
___	___

What's My Rule?______________

IN	OUT
___	___
___	___
___	___
___	___
___	___

What's My Rule?______________

IN	OUT
___	___
___	___
___	___
___	___
___	___

What's My Rule?______________

Number Patterns: Functions

Purpose The purpose is to connect manipulative, physical structures to T charts used to record the total number of blocks in the structures, using ordered pairs of numbers used to graph the pattern. The numbers are graphed and then the pattern is extended. This is a major tool used by scientists to find formulas to describe relationships in nature. For example: Isaac Newton used data from astronomers concerning the movements of planets to find the relationship between distance between planets, their size, and how they affect each other's movements to find the function for the force of gravity. This skill forms the base for a significant part of what is studied in Algebra I.

Prerequisites The ability to generate a mathematical rule given a pair of numbers that follows the rule and to place a point on a graph given its coordinates. Coordinate graphing, What's My Rule?, Number Patterns: Functions, and T charts presented in *Patterns in Arithmetic:* General Math - Booklets 3 and 4. Or *GEMS–Algebraic Reasoning*, play the Function Machine Game (look this up online). This site has a good one: www.mathplayground.com/functionmachine.html

Note This lesson is not a beginners lesson in functions.
There are several meanings to the symbol 'x': as multiplication or times sign in arithmetic; to identify the 'x' or horizontal axis on a graph, as 'y' is the vertical axis; to express the unknown in algebra, e.g., $x - 3 = 5, x = 8$). In function lessons it is used to identify the independent variable and is graphed on the x axis. Therefore, multiplication problems will now take the form of 3x, e.g., if $x = 2$, then $3x = 6$.

Materials Number Patterns: Functions - Worksheets 1 - 9, pages 50 - 58
Cubes
A colored pencil
A ruler

Warm Up Play What's My Rule?

Lesson Part 1 Begin with Number Patterns: Functions - Worksheet 1, Thrones. Read the story with the student. Have fun role playing the different parts. Ham it up by using different voices for the characters in the story.

Work together on Number Patterns: Functions - Worksheet 2, Thrones. Fill in the chart for Throne 4.

"What did the boy do each time he built a throne?" "First he built a rectangular shape. Then he took away one cube to make a seat."

"Predict how many cubes Throne 5 will have." "It will have nine."

The student fills in the chart, on page 51, up to Throne Number 6 and answers this question about the function, **"How do you calculate the total number of cubes**

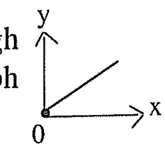
used in a throne?" "Multiply the throne number (x) by two and then take away one." **"How do you write that in an algebraic form?"** "2x - 1."

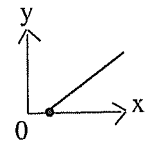
"Predict what the graph will look like and sketch it on the bottom of the page."

Please note that the dialogues in most lessons are idealized, with a student giving all the correct answers. The dialogue you have with your student will be unique. What's most important is to listen to the student and figure out the model of the world she is presenting. From your understanding of what she says, continue to ask probing questions or statements, such as: **"How did you get that?" "Show me what you mean." "Build a model of that." "Tell me more so I can understand what you are saying."**

Note

The Test for Understanding is embedded. If his sketch passes through the origin, he is not yet understanding T charts well enough to graph clearly. The line starts at one, not zero.





Mathematicians have figured out a way to solve this kind of problem. Let's keep exploring.

Now take out Number Patterns: Functions - Worksheet 3.
"Look at the top of Number Patterns: Functions - Worksheet 2 that you just completed. Graph the first pair of numbers." Throne 1 uses one block. So the ordered pair is (1, 1). Go over to the one on the x axis, and up just one little box on the y axis and place a point. Check the Answer Key.

"Put all the points on the graph. What do you notice?" "They make a straight line." **"Connect them."**

"What if the graph had more numbers?" "It would probably be straight forever."

"Where would the seventh throne's total number of blocks show up?" Have him get a colored pencil and put it on the seven on the x axis. Trace that line upwards all the way to the top of the page. Now, take your regular pencil out, and using a ruler, extend the line made of points until it crosses the colored line for the seventh throne and draw a point. Have him place his finger on the point and trace the horizontal line moving left until he reaches the y axis. The number on the scale of the y axis should match the number he would have gotten on the chart for the seventh throne. This is a way to use the graph to tell what the total number of cubes is needed for any throne. See the Answer Key to be sure he is doing the work correctly.

Warm Up

Play What's My Rule? or the Function Machine Game - see Prerequisites.

Lesson Part 2

Number Patterns: Functions - Worksheet 4 - Graph It, page 53. **"Now you are going to figure out a pattern by studying the numbers and then place that pattern**

on the graph on Number Patterns: Functions - Worksheet 5, page 54. Where do you record the x and y columns?" "The x column will be graphed on the horizontal line, the y column on the vertical line."

"So the first point would be placed where?" "Over zero and up eight, so it's placed at (0, 8)."

"Where is the second point placed?" "Over one and up eleven, so it's placed at (1, 11). How to think about what the function might be: When any number is multiplied by zero the resulting product will be zero. On this pattern, when you put in a zero you get back an eight. That tells you there is a + 8 on the end of this function. The pattern is going up by three each time. That tells you there is a multiplication by three going on. So if the x is one, multiply one times three and add eight. That gives the answer of eleven. Try it again with the two; $2 \times 3 + 8 = 14$. So $y = 3x + 8$ is the function for this pattern.

When graphing this function, only two to three points have to be graphed to tell where the line is. All other numbers in the table can be found from the graph like we did on the seventh throne.

Extension

Extension for advanced students (few fifth-graders will get this). You can also see the function in the graph. Notice that the line crosses the y axis at eight. That tells you there is a + 8 in the function. Then notice that as the points go up the graph they always go over one up three. That tells you that there is a multiplication by three in the function. So the function is $y = 3x + 8$.

Lesson Part 3

Number Patterns: Functions - Worksheet 6, Graph It, Table 2, page 55
Use the numbers from Table 2 on Number Patterns: Functions - Worksheet 4, Graph It, page 53.

Possible conversation to figure out the function:
"On the first line the x is a zero and the y is a two. What does that tell you about the function?" "That there is a + 2 on the end."

"So is the function $y = x + 2$?" "No because when I go to one, I can see the y value is not three but seven. That means something else is happening too."

"How can you tell what the something else is?" "The pattern is going up by a skip count of five. That means there is a multiplication by five first."

"So what is the function?" "$y = 5x + 2$."

"Graph it. What pattern do you notice?" "The line crosses the y axis at + 2. The points go over one and up five."

Number Patterns: Functions - Worksheet 7, Graph It, Table 3, page 56
The table is on Number Patterns: Functions - Worksheet 4, Graph It, page 53.

The function for this one is really tricky.

"What pattern do you see in the y side of the chart?" "The eight times table."

"But what is wrong with it?" "It should have the eight across from the one, not the two."

"So what do you have to do to the two to make it act like a one?" "Take away one."

"If you take away one before you multiply by eight, what happens?" "You get the numbers in the table."

"So what is the function?" "You have to take away one from the x and then multiply it by eight. Or $y = (x - 1)$ times eight, which is written $y = 8\ (x - 1)$."

Continuing on Number Patterns: Functions - Worksheet 4, Table 4
This time, there are no y values given at all. The y values must be obtained from the graph on Number Patterns: Functions - Worksheet 8, Graph It, Table 4, page 57. **"What do you notice when you connect the points?"** "It is not a straight line but a very steep curve." **"That tells you there is an exponent attached to the x."**

Practice Worksheets

Number Patterns: Functions - Worksheet 9, page 58
The student completes the tables without graphs. Supply graph paper if the student needs to continue graphing. Copy Grid Paper, page 34

Test for Understanding

Have him predict where the graph of Table 3 on Number Patterns: Functions - Worksheet 9 crosses the y axis, or what the zero value of the table will be.

Make a graph for the function $y = x \div 2$. Use odd numbers also so the points will fall on fractional units. Use large square graph paper. Have him figure out the function from the graph.

Internal Angles of Triangles

Purpose The purpose is to discover a geometric pattern that applies to all triangles. The main point of the lesson is the discovery–not the rule about triangles per se. The pattern can be used to calculate missing angles in a triangle. The pattern is that the sum of the internal angles of all triangles is 180°.

Prerequisites Understand there are 360° in a circle, 90°, and 180° angles.

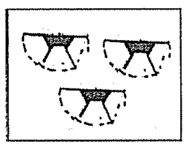
Materials Internal Angles of Triangles - Worksheets 1 and 2, pages 59 and 60
One piece of construction paper (any color)
Several pieces of 8 ½ x 11 paper
Tape–double sided tape makes the work easier

Warm Up Review 90°, 180°, 360° angles and parts of a circle. There are lessons on this available on the Internet.

Lesson Internal Angles of Triangles - Worksheet 1, page 59
Follow the instructions on the worksheet.

On steps 7 and 8, make sure the student does not make the shaded-in area or the part she tears off too small. The shaded corner should be shaded in at least an inch and the torn off corner at least two inches. If the parts are too small, it is difficult to tape the parts down to see the pattern.

Vocabulary Shaded corners, 'vertices,' of a triangle are also called 'internal angles.' A single corner, made by two lines intersecting, is called a 'vertex.' The top of a triangle is called the 'apex.'

The solution to item 12 is this: 

The pattern you are seeing is that no matter what the shape of a triangle is, when you put all the internal angles (corners) together, you will get a straight line along the top. This is also seen as a semicircle, which has 180° in it.
When she gets to item 13, wait before having this conversation to see what she comes up with.

Conversation:
"What pattern do you see when you tape all the shaded corners down?" "They all make a half circle."

"How many degrees are in a half circle?" "One hundred eighty."

"When you taped the internal angles of the triangle together, what arithmetic operation were you doing?" (Clarify if needed and list addition, subtraction, multiplication, or division.) "Putting anything together like this is addition."

"So what pattern occurs when you put the internal angles of a triangle together?" "They always make a 180° angle."

Internal Angles of Triangles - Worksheet 2, item 16 Answer the question: Does this pattern work on all triangles? Record the Rule of Triangles on the line.

"Let's review special types of triangles. What are equilateral and scalene triangles?" "An equilateral triangle has three equal sides and three equal angles. A scalene triangle has no equal sides and no equal angles."

"What is an isosceles triangle?" "A triangle where two of the sides and two of the angles are equal." Have her look it up if she does not know.

"If the sum of the internal angles of a triangle is always 180°, how can you figure out what the missing angle is in problem a?"

Advanced answer: "Since the triangle is an isosceles, two angles must be equal. It can't be 90 because 90 + 90 equals 180, so it must be 45°."

More likely answer: "I need to add 45 + 90 and subtract that from 180."

Practice Worksheets Internal Angles of Triangles: Bottom of Worksheet 2, page 60

Test for Understanding Challenge problem on Internal Angles of Triangles: Bottom of Worksheet 2

Math Journal Add these words and definitions to the Math Journal: equilateral triangle, isosceles triangle, scalene triangle, apex, vertex (plural vertices), sum of the internal angles of any triangle = 180°.

Extension Here is a great website for interactive activities with triangles.

http://www.mathwarehouse.com/geometry/triangles/interactive-triangle.php

Area of Triangles

Purpose

The purpose is to make a mathematical discovery of the pattern that allows the calculation of the area of a triangle.

Prerequisites

Previous lessons

Materials

Area of Triangles - Worksheets 1 - 3, pages 61 - 63
Geoboards and rubber bands
Geoboard grid paper on page 29 of this booklet and available at the following website: www.jamesrahn.com/Geometry/pages/geoboard_activities.htm
Blank scratch paper
Ruler
Scissors
Colored pencils or crayons

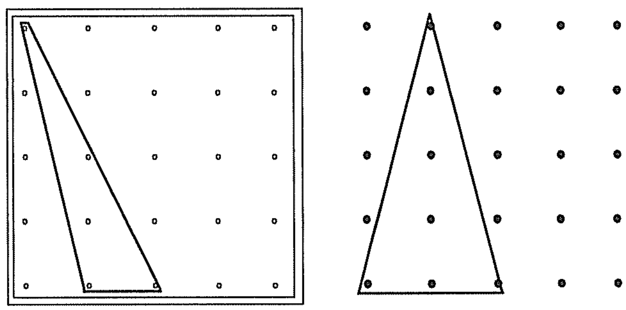
Warm Up

Free exploration time to play with the geoboard and bands to create designs for at least 30 minutes before you attempt to do a focused lesson

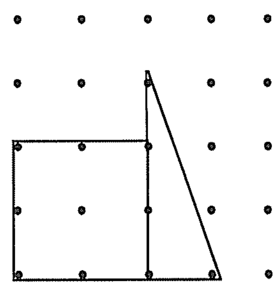
Problem Solving:

1. Make a triangle with a rubber band that touches four and only four pegs and has no right angles.

Two solutions:



2. Make a rectangle and a triangle that share three points of contact with the pegs. The triangle has five points and the rectangle has five points of contact with the pegs.



Lesson Part 1 Exploration

"Create a triangle on your geoboard with one color of rubber band. Then surround it with a rectangle that touches the triangle at all three vertices."

"Is it possible to do this for any triangle of any shape? Examine at least ten different triangles. Record on the grid paper as well as the geoboard."

"What pattern did you find?" "You can surround any triangle with a rectangle that touches all three vertices of the triangle."

Lesson Part 2

Have the student work on Area of Triangles - Worksheet 1, page 61.

Note Do not tell the student how to calculate the area of a triangle, or tell her the triangle is always half of the rectangle. You want her to discover this.

On the first triangle, some students will notice the triangle is half of the rectangle. Most will not be sure about the second one. If this happens, pursue the thread started in Note B below.

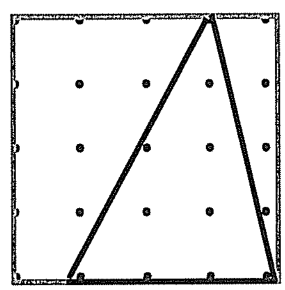
Other students will start counting squares. They will confront the problem of fractional units of squares that are hard to count. If this strategy is the one she uses. pursue the thread started in Note A below.

Note A In the figures at the bottom, A, D, and G are the whole rectangles. B/C, E/F and H/I are those same rectangles cut in half. Have her work on the D–E/F figure first, as it will be the easiest problem to do since the diagonal cuts each square exactly in half. She can see on the E/F problem that each triangle is half of the total D rectangle. When she finishes the worksheet, or if she gets frustrated trying to count fractional squares, proceed to Note B activity below.

Note B If she does bring up the idea that the triangle is half the area of the rectangle on the first one but is not sure on the second one, try this activity:
Take a piece of scratch paper and trim it so that it is square like the geoboard. Have her fold the paper in half and crease it. Open the paper up. Using the ruler, draw a line from the top center point, the apex, down to the lower right hand corner, and again to the lower left hand corner. You have just re-created the triangle shown on the worksheet on the top right. Color in the triangle and then cut it out.

"How could you find out if the colored triangle is half of the original rectangle?"
Two common answers are: "Each little triangle (referring to the ones that were cut off) is the same size as the triangles on each side of the crease, so it must be half of the whole thing."

"If you put the two smaller triangles on top of the colored triangle, they match. The two small ones together add up to the big one. So that means the big triangle has to be half of the original rectangle."



Try it again with a different shaped triangle. The apex of the triangle must be along the top edge of the paper and the sides of the triangle must go from this apex to each corner of the paper for this to work. In other words, the perimeter of the rectangle must touch all three vertices of the triangle as is shown in the figures on Area of Triangles - Worksheet 1.

Yes, the pattern will always work, but she will have to rotate or flip over the pieces to get them to fit. You can demonstrate with cut paper that all triangles can be surrounded by a rectangle and the area of that triangle will always be half of the area of the rectangle.

Lesson Part 2

Area of Triangles - Worksheet 2, page 62

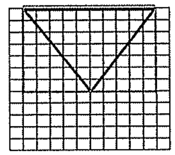
"What pattern did you notice about triangles and rectangles that will help you figure out what the area of any triangle is?" "That the area of the triangle is always half of the area of the rectangle."

"How do you find out the area of a rectangle?" "You multiply the length times the width."

"So how could you calculate the area of a triangle?" "You would put the rectangle around it, get the length and the width and multiply them and then cut that number in half."

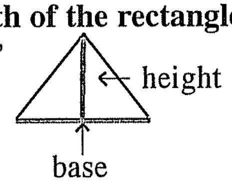
"See if you can use that pattern to calculate the areas of the triangles on Area of Triangles - Worksheet 2."

Have her shade in the triangle with her colored pencil. With a different color have her outline the rectangle. Figure E



Note

On Figures E and F the sides of the rectangle needed to calculate the area of the triangle is outside the triangle. **"What is the length and width of the rectangle around Figure E?"** "The length is ten and the width is seven."



"Show me how you got that." The answer will be unique.

"Draw a line inside the triangle equal to the length and a line inside the triangle equal to the width of the rectangle. Mathematicians call these the base and the height of a triangle. Draw them on Figures F and G."

The algebraic language for this pattern of "Find the length and width of the rectangle, multiply them to get the area of the rectangle, and then cut it in half to get the area of the triangle" is expressed as (L x W) ÷ 2.

Practice Worksheet

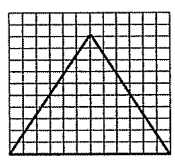
Area of Triangles - Worksheet 3, page 63.

The Internet has more problems and great interactive graphics to help cement the understanding of this concept. Here is one that has some good graphics: www.mathworksheetsgo.com/sheets/geometry/triangles/area-of-triangle-worksheet.php

Test for Understanding

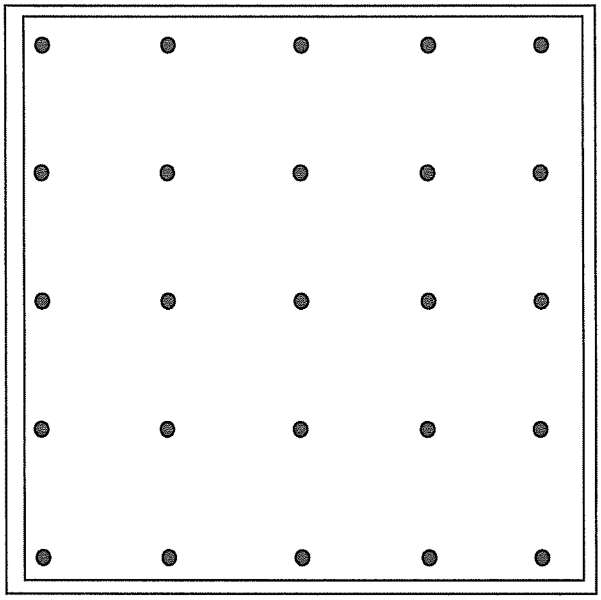
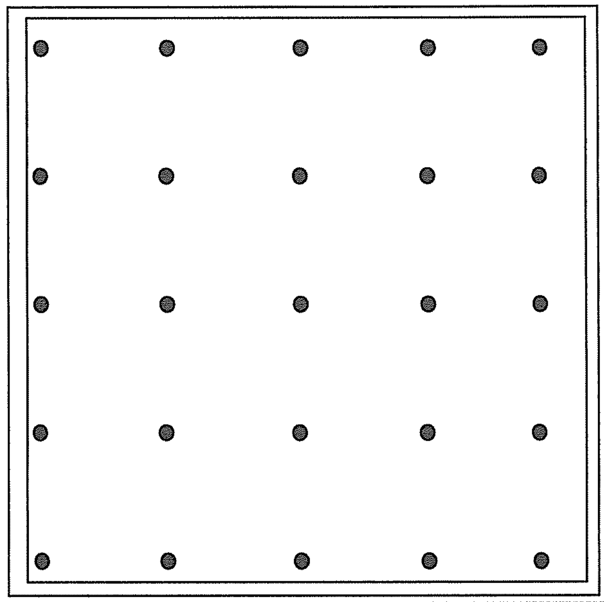
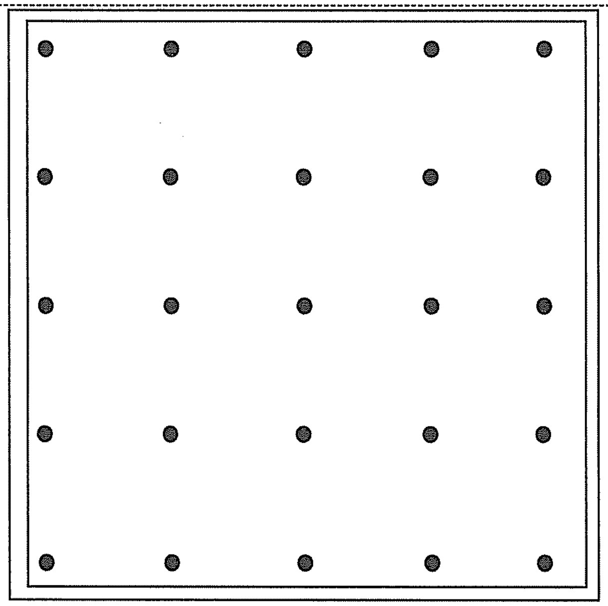
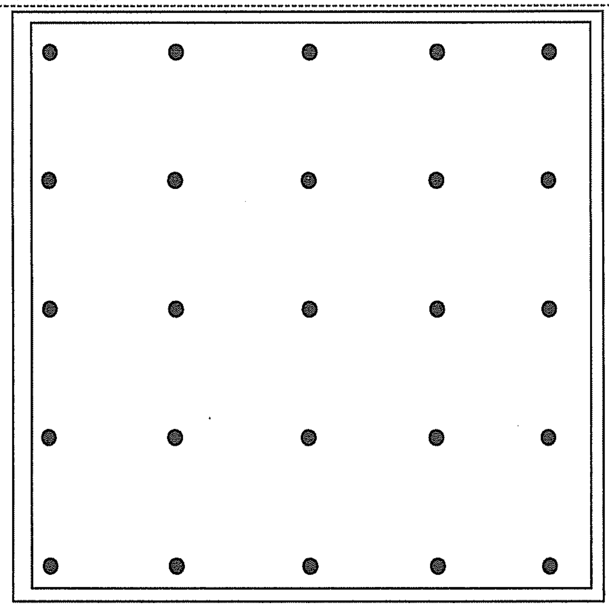
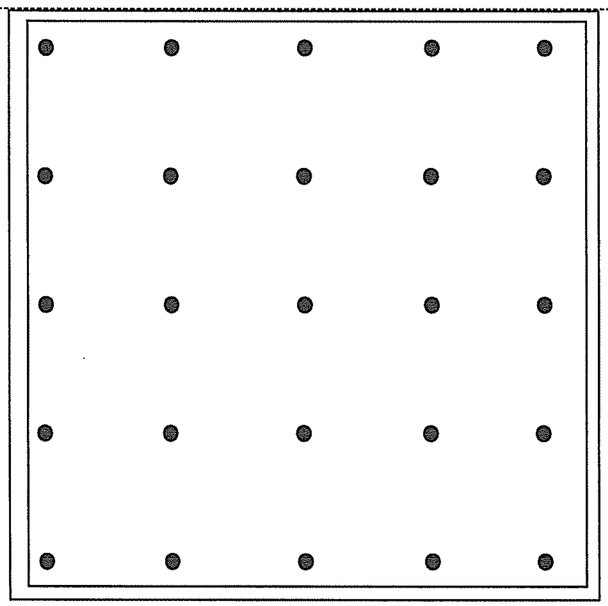
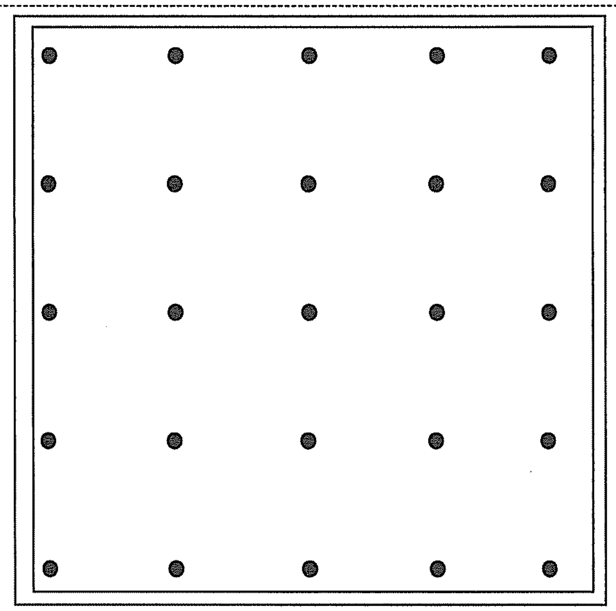
1. Basic - What is the area of a right triangle that is 10 inches tall and 8 inches at the base? Answer: 40 square units

2. Advanced - What is the area of this isosceles triangle? Isosceles triangle that is 12 units at the base and 10 units tall
Answer: 60 square units



Worksheet

Triangle City, page 64, is an independent fun art activity.

Area and Perimeter: Parallelograms

Purpose The purpose is to construct understanding of how to use a rectangle to find the area and perimeter of a parallelogram.

Prerequisites Previous lessons

Materials Area and Perimeter: Parallelograms - Worksheets 1 - 4, pages 65 - 69
Geoboard and rubber bands
Grid paper
Toothpicks
Scissors

Warm Up Draw some rectangles, squares, and triangles on a piece of grid paper. Review calculation of areas of rectangles, squares, and triangles. Then discuss finding the perimeters of those same shapes.

"Why will counting squares not work for finding the perimeter of a triangle?" "The lines cut through the squares at odd angles."

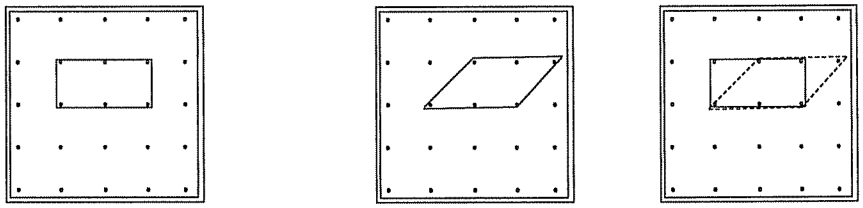
Activity Use toothpicks and make an equilateral triangle on the grid paper. Tape the toothpicks down. Make sure the base of the triangle is sitting exactly on a line with the left hand end of the toothpick at the vertex of a square of grid paper. The other end will probably not be exactly on the edge of one of the squares, but maybe it will. Discuss the way the squares on the grid look against the sides of the triangle. The toothpick will cut through squares on the diagonal. Cut out a strip of grid paper and use it as a ruler to measure the sides of the triangle. They often come out as fractions of squares.

"Is the diagonal of a square equal in length to the side of the square?" "No, the diagonal is longer."

"Prove it." A possible solution is to measure it. Or build it with toothpicks. Toothpicks placed on the diagonal will be too short.

Note This seems like an obvious problem, that you cannot count units to get the perimeter of a triangle or a parallelogram, but it is a source of confusion for many students. It is common for students to get the side of the figure mixed up with the height.

Lesson Part 1 Build a three square long x two square high rectangle on the geoboard with a colored rubber band. Then have the student use his index fingers and thumbs to pinch the top two corners of the rectangle and move them to the right one square. He will have just transformed a rectangle into a parallelogram. Have him transform it back to a rectangle. See the diagrams on the next page.

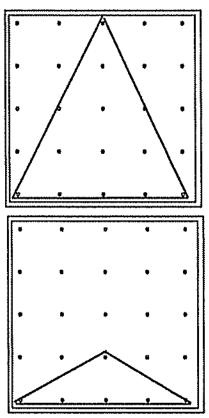


"What is the area of the rectangle?" "Six squares."

"What do you think the area of the parallelogram is?" Whatever he says, such as, "It is also six square units," ask him to prove it and give you a good set of reasons.

Note

The ideal answer is that the area of the parallelogram is six. If you cut off the triangle on the end of the parallelogram and rotate it you can stick it on the other end and re-form the rectangle that has the same height the parallelogram did. This transformation makes it so the height of the parallelogram becomes the side of the rectangle. Some students will be able to visualize this. Others will need to do it with the worksheet.



There is no way to predict what will happen next. Here are some possibilities:

- Challenge his thinking by making a triangle on the geoboard and stretching its apex down within a square of the base. The area shrinks unmistakably. If the area of the triangle shrinks by transforming its shape, why wouldn't the same thing happen with a parallelogram?
- Some students will argue that the triangles on the ends are the same size. Have them prove this to you with examples or a drawing or even cutting paper. Some students will figure out how to do the transformation proof on their own.
- Some students will have no idea.

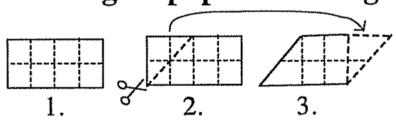
After a discussion: Have him look at the rectangle and parallelogram at the top of Area and Perimeter: Parallelograms - Worksheet 1. Let him read the question above the parallelogram. *Wait.*

Some students will see in their minds that you can transpose the triangle on one end of the parallelogram to the other side to make a rectangle when looking at the dotted lines inside the figure shown on the worksheet. Notice that the vertical dotted lines inside the parallelogram are the same length as the width of the rectangle shown on the left. If you cut off the right triangle on one end of the parallelogram and rotate it, it fits perfectly under the triangle on the other side, resulting in a perfect rectangle. It is easy to find the area of a rectangle. This cut and move strategy proves that the parallelogram has the same area as its related rectangle.

Area and Perimeter: Parallelograms - Worksheet 1 is blank on the back side to allow him to make rectangles on the grid paper and cut them out.

Have him draw a rectangle that does not touch the edges of the grid paper on the worksheet. Cut the rectangle out. Write down its length and width and calculate its area and verify it with a count of the squares inside.

"Transform the rectangle into a parallelogram and trace the new parallelogram on the grid paper to the right." Do not use the bottom grid yet.



Note

This next conversation is important.

"Looking at the two figures at the top of Area and Perimeter: Parallelograms - Worksheet 1, how do you calculate the area of the rectangle?" "You multiply the length, which is four, by the width, which is two, to get eight square units of area."

"What did you discover about the area of the rectangle and the parallelogram in the last activity?" "The area of the parallelogram is equal to that of the rectangle."

"So, looking at the parallelogram at the top of Area and Perimeter: Parallelograms - Worksheet 1, where are the numbers 2 and 4 that you would use to calculate the area of the parallelogram?" "The four is along the top. The top is four units long."

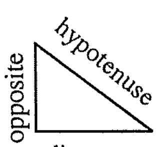
"Where in the parallelogram is the 2?" *Wait.* "It is inside the parallelogram as the length of the vertical dotted lines called the height."

If he can not see it, have him make another rectangle on the bottom left grid and outline the length in one color and the width in another color. Then cut out the rectangle (do not cut off the colors) and transform it into the parallelogram. The colored width will now appear inside the parallelogram. This colored width line from the rectangle is called the height in a parallelogram.

"So, how do you calculate the area of a parallelogram?" "You multiply the base times the height."

Lesson Part 2

Area and Perimeter: Parallelograms - Worksheet 2, page 67 Notice on the top figure, the height of the parallelogram is darkened. Have him write in the area of the rectangle, eight square units, and then the perimeter, which is twelve units (not square units).



"How can you find the perimeter of the parallelogram? "Measure it."

"What is the slanted edge of the parallelogram called?" "The hypotenuse." If he doesn't know, have a discussion about this in regard to a right triangle.

"Is the slanted edge the same length as the width of the rectangle?" "No."

"Check it." He measures and compares the width of the rectangle and the slanted edge (hypotenuse) of the parallelogram.

"So where do we get the length of the side?" He uses a centimeter ruler and measures the edge of the parallelogram and rounds off to the nearest fraction of the unit.

The side of the parallelogram is $2\frac{1}{4}$ units. Two sides + two lengths equals the perimeter; $2\frac{1}{4} + 2\frac{1}{4} + 4 + 4 = 12\frac{1}{2}$. So the perimeter is $12\frac{1}{2}$ units.

Finish Area and Perimeter: Parallelograms - Worksheet 2 together.

Practice Worksheets

Area and Perimeter: Parallelograms - Worksheets 3 and 4, pages 68 and 69

On Area and Perimeter: Parallelograms - Worksheet 3, measure the perimeters with a ruler made from the cutout grid paper on which the parallelograms are drawn.

On Area and Perimeter: Parallelograms - Worksheet 4, he will have to identify and draw in figures within the complex figures in order to calculate areas.

Test for Understanding

Check the clarity of the essay response at the bottom of Area and Perimeter: Parallelograms - Worksheet 2. Insist on complete sentences.

"Write out the procedure for figuring out the area and perimeter of parallelograms in algebraic notation."
A = Base times Height, or A = bh Perimeter is P = 2B + 2S, S being the side of the parallelogram

Math Journal

Add the procedure for figuring out the area and perimeter of parallelograms and record the algebraic notation of the formula.

Grid Paper

General Math - Booklet 5

Goldbach's Conjecture

The German mathematician, Christian Goldbach, made the conjecture that every even number, except two, was the sum of two prime numbers.

a. 6: 2, 4

b. 6: 3, 3

Add the two number in the little triangles to equal the number in the large triangle. Which of these are an example of Goldbach's Conjecture? _______

Write the two prime numbers that equal the even number.

10: 7, 3	8: 3, 5	20: 3, 17	32: 29, 3
48: 5, 43	56: 53, 3	12: 7, 5	28: 23, 5
90: 7, 83	64: 53, 11	102: 5, 97	200: 197, 3

Prime Factors: Review - Worksheet 1

List all the factor pairs for each number from 2 to 25.

2 = 2 x 1,

3 = 3 x 1

4 = 4 x 1, and 2 x 2

5 = 5 x 1

6 = 6 x 1, and 2 x 3

7 = 7 x 1

8 = 8 x 1, and 2 x 4

9 = 9 x 1, and 3 x 3

10 = 10 x 1, and 2 x 5

11 = 11 x 1

12 = 12 x 1, 2 x 6, and 3 x 4

13 = 13 x 1

14 = 14 x 1, and 2 x 7

15 = 15 x 1, and 3 x 5

16 = 16 x 1, 2 x 8, and 4 x 4

17 = 17 x 1

18 = 18 x 1, 2 x 9, and 3 x 6

19 = 19 x 1

20 = 20 x 1, 2 x 10, and 4 x 5

21 = 21 x 1, and 3 x 7

22 = 22 x 1, and 2 x 11

23 = 23 x 1

24 = 24 x 1, 2 x 12, 3 x 8,

and 4 x 6

25 = 25 x 1, and 5 x 5

Prime Factors: Review - Worksheet 2

Go back through your investigation and find all the numbers that had only one factor pair. List the numbers here:

2, 3, 5, 7, 11, 13, 17, 19, 23

These numbers are called PRIME NUMBERS.
A prime number is a number that has only itself and one as a factor.
The only way to make 17 in multiplication is 17 x 1. This means that 17 is a prime number.
The number 1 is not considered to be a prime number.

Go back through your investigation and find all the numbers that had two or more factor pairs. These are numbers that can be made with several different factors.
List those numbers here:

4, 6, 8, 9, 10, 12, 14, 15, 16, 18
20, 21, 22, 24, 25

These numbers are called COMPOSITE (kahm -pahz-it) NUMBERS.
A composite number is any number that has not only itself and one as factors but other numbers too. An example of this is the number 9. Nine can be made in multiplication with 9 x 1 and also 3 x 3.

Examine the list of prime numbers.

How many prime numbers are even? 1

How many prime numbers are odd? 8

Are all odd numbers prime numbers? no

Are any even numbers greater than 2 prime numbers? no

Prime Factors: Review - Worksheet 4

18 → 3 x 6 → 2 x 3 — 3,3,2

18 → 2 x 9 → 3 x 3 — 3,3,2

21 → 3 x 7 — 7,3

22 → 2 x 11 — 11,2

24 → 3 x 8 → 2 x 4 → 2 x 2 — 3,2,2,2

25 → 5 x 5 — 5,5

Prime Factors: Review - Worksheet 5

26 → 2 x 13 — **13, 2**

56 → 2 x 28; 28 → 2 x 14; 14 → 2 x 7 — **7, 2, 2, 2**

44 → 4 x 11; 4 → 2 x 2 — **11, 2, 2**

32 → 8 x 4; 8 → 4 x 2; 4 → 2 x 2; 4 → 2 x 2 — **2, 2, 2, 2, 2**

50 → 5 x 10; 10 → 5 x 2 — **5, 5, 2**

42 → 7 x 6; 6 → 2 x 3 — **7, 3, 2**

58 → 2 x 29 — **29, 2**

98 → 2 x 49; 49 → 7 x 7 — **7, 7, 2**

Prime Factors: Review - Worksheet 6
Show You Know

1. If you need to multiply three numbers together, in what order should you do it? Any order.

2. Explain what a prime number is. A whole number that can only be divided by itself and one.

3. Find the prime factors of 60.

60 → 2 x 30; 30 → 5 x 6; 6 → 2 x 3 — **5, 3, 2, 2**

4. Prime factor 100 two different ways.

100 → 2 x 50; 50 → 2 x 25; 25 → 5 x 5 — **5, 5, 2, 2**

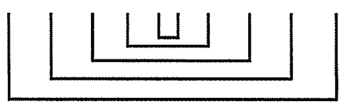
100 → 10 x 10; 10 → 2 x 5; 10 → 2 x 5 — **5, 5, 2, 2**

100 → 4 x 25; 4 → 2 x 2; 25 → 5 x 5

What pattern do you see in the prime factors in the two boxes of problem 4? They are the same numbers. Multiplied together they equal 100.

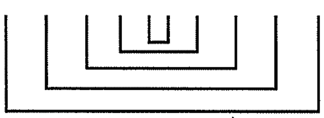
Rainbow Rectangles - Worksheet 1

Here is a series of nine numbers. Multiply the numbers that are connected by the dark lines. Record your results below.



		Differences	
2 + 10 = 12	2 x 10 = 20	7	
3 + 9 = 12	3 x 9 = 27	5	2
4 + 8 = 12	4 x 8 = 32	3	2
5 + 7 = 12	5 x 7 = 35	1	2
6 + 6 = 12	(6 x 6 = 36)		

When you add these pairs the sums are twelve.
When you multiply the pairs the products are not equal.
Circle the pair which gave you the largest product.
Try it again on another sequence. See if the same pattern appears.



Multiply.	Differences	
4 x 12 = 42	7	
5 x 11 = 55	5	2
6 x 10 = 60	3	2
7 x 9 = 63	1	2
8 x 8 = 64		

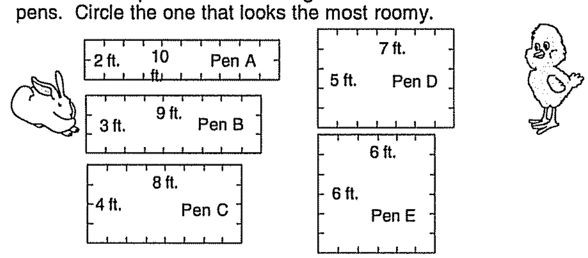
What is the sum of each pair? 16

Circle the pair which had the largest answer this time and write it here. 8X8
Try a few more sequences on another piece of paper.

Make a rule for chosing the multiplication problem with the largest answer.
My rule is The product of the numbers in the middle of the sequence are always the largest.

Rainbow Rectangles - Worksheet 2

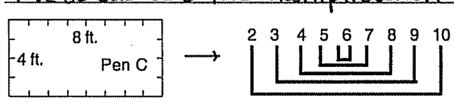
These rainbow patterns help us solve problems dealing with area and perimeter of rectangles. Here are 5 animal pens. Circle the one that looks the most roomy.



Use the numbers or the lines inside each drawing to calculate the area and perimeter of each pen.

Area of Pen A = 20 sq. ft. Perimeter of Pen A = 24 feet.
Area of Pen B = 27 sq. ft. Perimeter of Pen B = 24 feet.
Area of Pen C = 32 sq. ft. Perimeter of Pen C = 24 feet.
Area of Pen D = 35 sq. ft. Perimeter of Pen D = 24 feet.
Area of Pen E = 36 sq. ft. Perimeter of Pen E = 24 feet.

What do you notice about the perimeters? They are the same.
What do you notice about the areas? All are different.
When you compare the length, width, and area of each pen to the number series and multiplication problems that you first worked with, what connection do you see? The perimeters are twice the sum of the rectangle pairs (addition). The areas are the multiplication problems.



Pen C = 32 sq. ft.

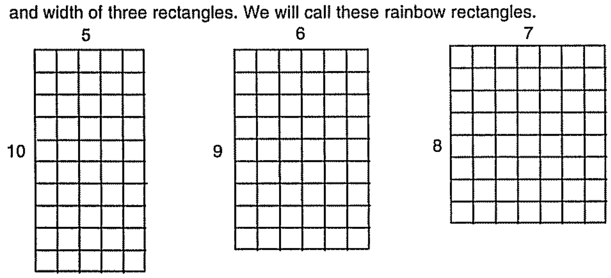
Rainbow Rectangles - Worksheet 4

We will look at the rainbow pattern one more time and then use it to solve problems dealing with animal pens.

Take this series of numbers and do a rainbow pattern.

The sum of each pair is 15.

Here are rectangles which use these number pairs to make the length and width of three rectangles. We will call these rainbow rectangles.



Pen 1	Pen 2	Pen 3
Area 50 sq. units	Area 54 sq. units	Area 56 sq. units
Perimeter 30 units	Perimeter 30 units	Perimeter 30 units

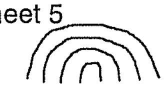
Compare the perimeters of each rectangle to the sum of each pair in the rainbow series. What do you notice? The perimeter is double the sum of each pair.

Go back to the last page. Did this pattern about the perimeters show up in that group of rectangles too? yes

Write a rule about the relationship between the perimeters of rainbow rectangles and the sum of the pairs in rainbows.

The perimeter is double the sum of the addition problems-the pairs in the rainbow rectangle.

Rainbow Rectangles - Worksheet 5



Use what you know about rainbow patterns and rainbow rectangles to solve this problem.

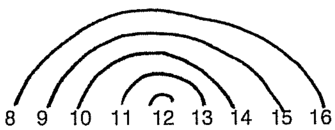
You want to build a dog pen. The pen will be a rectangle. You can afford to buy 48 feet of fence. This includes the gate. So the perimeter of your rectangle will be 48 feet. You want the dog to have the most area possible inside the fence. What should be the length and width of your pen?

Make a rainbow series of numbers which will tell you what size pen will do the job.

HINT 1 : What should the sum of the pairs be in your rainbow series to result in rainbow rectangles with perimeters of 48 feet? 24 1/2 of 48

HINT 2: What shape always gives the largest area in a four sided pen? Solve the problem.

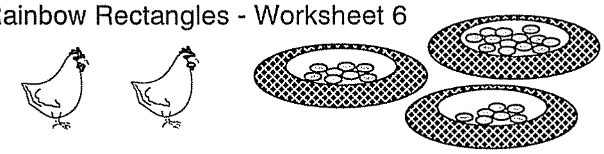
Explain in writing how you solved it. Record how you thought about the problem.



8 + 16 = 24	8 x 16 = 128
9 + 15 = 24	9 x 15 = 135
10 + 14 = 24	10 x 14 = 140
11 + 13 = 24	11 x 13 = 143
12 + 12 = 24	12 x 12 = 144

The perimeter is two times the sum of the pairs of a rainbow rectangle. The pairs add up to 1/2 the perimeter of 48, which is 24. I made a rainbow rectangle for this series. Then I wrote the multiplication and the addition problems. In the rainbow the middle number 12 is multiplied by itself, 12 x 12, which makes a square. The square has the largest area.

Rainbow Rectangles - Worksheet 6



Mary wants to build one rectangular chicken coop for her fifteen hens. Each hen needs six square feet to run around in. She has enough money to buy forty-two feet of fence.

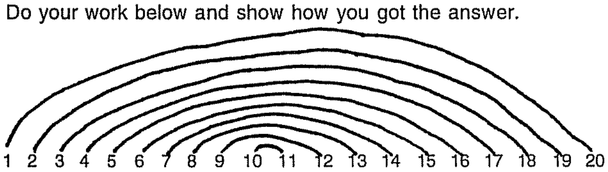
What is the perimeter of the coop? 42

How much area will she need for her hens? 90 square feet

What size coop will she be able to build? 6 X 15

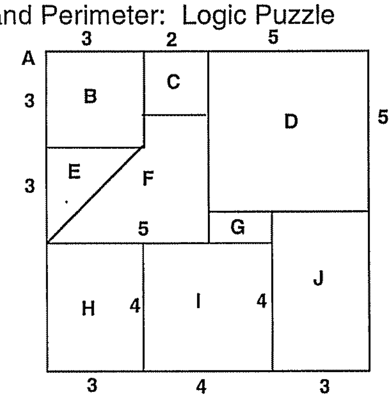
Use what you know about rainbow patterns and rainbow rectangles to solve this problem.

Do your work below and show how you got the answer.



1 + 20 = 21	1 x 20 = 20
2 + 19 = 21	2 x 19 = 38
3 + 18 = 21	3 x 18 = 54
4 + 17 = 21	4 x 17 = 68
5 + 16 = 21	5 x 16 = 80
6 + 15 = 21	6 x 15 = 90
7 + 14 = 21	7 x 14 = 98
8 + 13 = 21	8 x 13 = 104
9 + 12 = 21	9 x 12 = 108
10 + 11 = 21	10 x 11 = 110

Area and Perimeter: Logic Puzzle

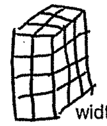


Shape		Dimensions	Area
Large square	A	10 x 10	= 100
Square	B	3 x 3	= 9
Square	C	2 x 2	= 4
Square	D	5 x 5	= 25
Triangle	E	(3 x 3)÷2	= 4 1/2
Combination	F	(2x4)+(3 x 3)÷2	= 12 1/2
Rectangle	G	1 x 2	= 2
Rectangle	H	3 x 4	= 12
Square	I	4 x 4	= 16
Rectangle	J	5 x 3	= 15

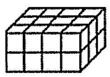
Associative Blocks - Worksheet 1

Take out a set of linking blocks.
Build a figure that is two blocks long, three blocks wide, and four blocks high. It will look something like this:

This figure is called a rectangular solid.
It's a three-dimensional, or 3D rectangle.
In these lessons the size of the sides are always listed in this order: length, width, height.
This solid would be called 2 by 3 by 4.



How many blocks did you use to build this rectangular solid? 24 blocks



Now build this solid.

This one is 4 by 3 by 2.

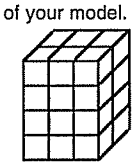
How many blocks will this solid take?

guess _____ actual 24

Compare the number of blocks you used for the first solid to the number of blocks you used in the second solid.
What did you find out? Both used the same amount, 24 blocks.

Arrange the solid another way. Draw a picture of your model.

It could be 3 by 2 by 4.



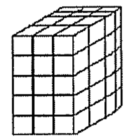
What changes when you change the arrangement of the numbers?
The shape of the solid.

What does not change? The total number of blocks.

Associative Blocks - Worksheet 2

Build another rectangular solid like this: 3 by 4 by 5 solid

Length	Width	Height
3	4	5



How many blocks will it take to build this new solid? guess _____

The number of blocks you use to make a rectangular solid is called its **Volume**.

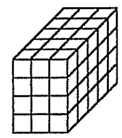
What is the volume of the solid you just built? 60 blocks

Draw it when you are done.

Rotate your solid to make it into a 3 by 5 by 4 solid. Draw it.

Length	Width	Height
3	5	4

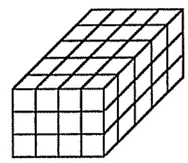
What is the volume of this solid? 60 blocks



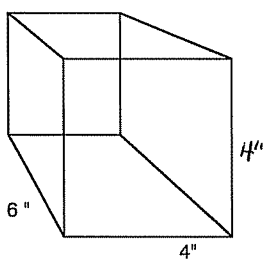
Rotate your solid to make it into a 4 by 5 by 3 solid. Draw it.

Length	Width	Height
4	5	3

What is the volume of this solid? 60 blocks



Associative Blocks - Worksheet 3

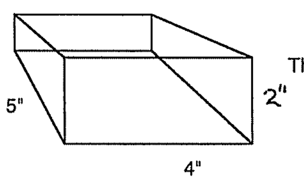


Here is a rectangular solid.
The length, width and height are shown by the little numbers.

What is the volume of this solid?

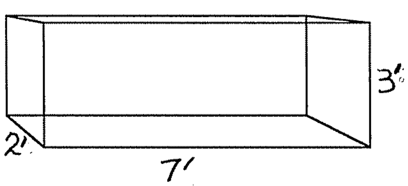
96 cu. "
cu." means cu

Figure out the volumes of each of these rectangular solids.



The volume is 40 cu."

What would the volume of a 2 by 5 by 4 inch solid be? 40 cu."



This is an aquarium. Draw in plants and fish. Color it. If one inch = one foot in this drawing, about what would the volume of this tank be?

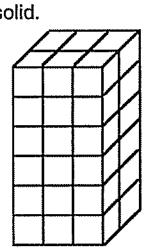
42 cu.

Associative Blocks - Worksheet 4
Show You Know

Build a rectanglular solid.
3 x 2 x 6

Length	Width	Height
3	2	6

Draw it here.

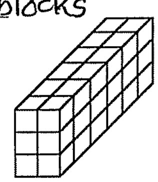


How many blocks will it take to build this solid? Guess _____
36 blocks

What is the volume of the solid you just built? 36 blocks

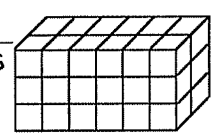
Rotate your solid to make it into a 2 x 6 x 3 solid. Draw it.

What is the volume of this solid? 36 blocks



Rotate your solid to make it into a 6 x 2 x 3 solid. Draw it.

What is the volume of this solid? 36 blocks



Finding Composite Factors from Prime Factors
Mystery Numbers - Worksheet 1

Mystery Number	Prime Factors	Composite Factors List
24	3, 2, 2, 2	6, 4, 8, 12, 2, 3

To find composite factors use theAssociative Principle to make different combinations of the prime numbers.

(3 x 2) x (2 x 2)
6 x 4 =

Six and four are both Composite Factors of twenty-four.Record them on the composite factor list.

Try another combination. 3 x (2 x 2 x 2)
3 x 8 = 24

Eight is another composite factor of twenty-four. Record it on the composite factor list.

A different combination: 2 x (3 x 2 x 2)
2 x 12 = 24

Twelve is another composite factor of twenty-four Record it on the composite factor list.

Here's a different set of prime factors. Solve all the problems and fill in the blanks. List all the composite factors.

Mystery Number	Prime Factors	Composite Factors List
36	3, 3, 2, 2	2, 4, 6, 9, 12, 18, 3

(3 x 3) x (2 x 2)
9 x 4

3 x (3 x 2 x 2)
3 x 12

(3 x 2) x (3 x 2)
6 x 6

2 x (3 x 3 x 2)
2 x 18

Finding Composite Factors from Prime Factors
Mystery Numbers - Worksheet 2

Here's a different set of prime factors. Solve all the problems and fill in the blanks. List all the composite factors.

Mystery Number	Prime Factors	Composite Factors
90	5, 3, 3, 2	2, 3, 6, 9, 10, 18, 15, 30, 45, 5

(5 x 3) x (2 x 3)
15 x 6

2 x (3 x 3 x 5)
2 x 45

(5 x 2) x (3 x 3)
10 x 9

3 x (2 x 3 x 5)
3 x 30

5 x (2 x 3 x 3)
5 x 18

Any more combinations? no

Now, solve a mystery number by yourself.

Mystery Number	Prime Factors	Composite Factors
60	5, 3, 2, 2	2, 4, 6, 10, 12, 15, 20, 30, 3

(5 x 3) x (2 x 2)
15 x 4

2 x (5 x 3 x 2)
2 x 30

(2 x 3) x (5 x 2)
6 x 10

3 x (2 x 2 x 5)
3 x 20

5 x (2 x 2 x 3)
5 x 12

Mystery Number	Prime Factors	Composite Factors
40	5, 2, 2, 2	4, 8, 10, 20

(5 x 2) x (2 x 2)
10 x 4

5 x (2 x 2 x 2)
5 x 8

2 x (5 x 2 x 2)
2 x 20

Finding Composite Factors from Prime Factors
Mystery Numbers - Worksheet 3

You're on your own now.

Mystery Number	Prime Factors	Composite Factors
100	5, 5, 2, 2	4, 10, 20, 50, 25

(5 x 5) x (2 x 2)
25 x 4

5 x (5 x 2 x 2)
5 x 20

(5 x 2) x (5 x 2)
10 x 10

2 x (5 x 2 x 2)
2 x 50

Mystery Number	Prime Factors	Composite Factors
250	5, 5, 5, 2	10, 25, 50, 125

(5 x 5) x (5 x 2)
25 x 10

2 x (5 x 5 x 5)
2 x 125

(5 x 2) x (5 x 5)
10 x 25

5 x (2 x 5 x 5)
5 x 50

Wow! Five numbers.

Mystery Number	Prime Factors	Composite Factors
80	5, 2, 2, 2, 2	4, 8, 10, 16, 20, 40

(5 x 2) x (2 x 2 x 2)
10 x 8

5 x (2 x 2 x 2 x 2)
5 x 16

(2 x 2) x (5 x 2 x 2)
4 x 20

2 x (5 x 2 x 2 x 2)
2 x 40

Finding Composite Factors from Prime Factors
Mystery Numbers - Worksheet 4

Find the mystery number from the prime factors that are given. What are the Composite Factors? Try to do the work in the space given. If you need help, copy the previous page and use one page for each problem.

Mystery Number	Prime Factors	Composite Factors
18	3, 3, 2	6, 9
30	5, 3, 2	6, 10, 15,
12	3, 2, 2	4, 6
54	3, 3, 3, 2	6, 9, 18, 27

Finding Composite Factors from Prime Factors Mystery Numbers - Worksheet 5

Find the mystery number from the prime factors that are given. What are the Composite Factors?

Mystery Number	Prime Factors	Composite Factors
150	5, 5, 3, 2	6, 10, 15, 30, 50, 75, 25
100	5, 5, 2, 2	4, 10, 20, 25, 50
140	7, 5, 2, 2	4, 10, 20, 28, 35, 70

Prime Factors Meet the Associative Property Worksheet 1

Find the Prime Factors of the number 30.

30 → 5 x 6 → 2 x 3 [5, 3, 2]

30 → 2 x 15 → 3 x 5

30 → 3 x 10 → 2 x 5

Now take those three prime factors you found and use the Associative Property on them. Find all the ways to group the three factors.

(5 x 2) x 3 =
10 x 3 = 30

(2 x 3) x 5 =
6 x 5 = 30

(5 x 3) x 2 =
15 x 2 = 30

What happens when you multiply all the prime factors of a number together? You get the original number.

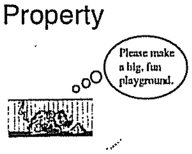
What number has the prime factors 5, 3, 3, 2? 90

Prime Factor the number you found to see if the factors match.

90 → 2 x 45 → 5 x 9 → 3 x 3

Prime Factors Meet the Associative Property Worksheet 2

Find the Prime Factors of the number 48.



48 [3, 2, 2, 2, 2]

When you multiply two prime factors of a number together, you get what is called a composite factor. Sometimes you need to know not the prime factors of a number but the composite factors.

Let's say you were building a cage for a rat playground for a children's zoo. The floor space in the cage can only be 48 square feet in area and it must be a rectangle. You also need to provide 38 feet of perimeter around the edge so that a whole class of kids can look in at the same time. What size should you tell the carpenters to cut the sides?

To do this problem you need to know all the ways two numbers can be multiplied together to make 48 square feet.

A good way to do this is to use the prime factors of 48 and see what composite factors you can find. Forty-eight has 5 prime factors. Let's find all the ways we can group those numbers together to find the composite factors that will give us a cage of the right size.

We will do this on the next page. Below make a sketch of the floor plan of a wonderful rat playground.

Prime Factors Meet the Associative Property Worksheet 3

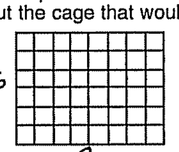
Now, lets find all the composite factors.

List the prime factors of 48 here. 3 2 2 2 2

(3 x 2) x (2 x 2 x 2) =
6 x 8 = 48

These numbers are composite factors of 48. Multiply them together. What number do you get? 48

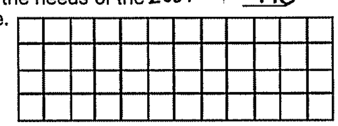
If you made the rat playground with the length and width using these two numbers you would get an area of 48 square feet.
Use some 1/4 inch grid paper and cut out the cage that would be this size. Paste it here.

6

8

What would the perimeter be? 28
Does this number satisfy the needs of the zoo? no
If not, try another arrangement.

(2 x 2) x (3 x 2 x 2) =
4 x 12 = 48

Using these two numbers you would get an area of 48 square feet.
Use the grid paper to show this cage.
What would your perimeter be? 32
Does this number satisfy the needs of the zoo? no
If not, go to the next page.

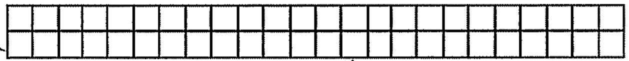
4

12

Prime Factors Meet the Associative Property Worksheet 4

Try it with this arrangement.

2 x (3 x 2 x 2 x 2) =
2 x 24 = 48

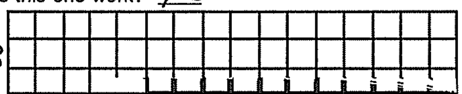
Using these two numbers you would get an area of 48 square feet.
Use the grid paper to show this cage.

2

24

What would your perimeter be? 52
Does this number satisfy the needs of the zoo? yes
If, not try again.

3 x (2 x 2 x 2 x 2) =
3 x 16 = 48

Show this cage.
What is it's perimeter? 38
Does this one work? yes

3

16

What size does this cage have to be? 16 x 3

Prime Factors Meet the Associative Property Worksheet 5

This time the cage must have an area of 60 square feet and a perimeter of 34 feet.

First, Prime factor the 60. Prime Factors are 5, 3, 2, 2

60
2 x 30
5 x 6
2 x 3

Use all four prime factors.
2 x (5 x 3 x 2) = 60
2 x 30 = 60
Draw a picture of a rectangle.
With a width of 2 and a length of 30.

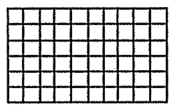
The area of this rectangle is 60.
The perimeter is 64. This is not the rectangle you are looking for.

2

30

Now try these: Put your prime factors in groups of 2.
(2 x 3) x (2 x 5) =
6 x 10 = 60.
Draw this rectangle. Area = 60

Perimeter = 32

6

10

Is this the rectangle you are looking for? no

Prime Factors Meet the Associative Property Worksheet 6

Now try these: Put your prime factors in groups of
(5 x 3) x (2 x 2) =
15 x 4 = 60.

Draw this rectangle. Area = 60

Perimeter = 32

4

15

Is this the rectangle you are looking for? no

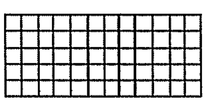
Now try these.
3 x (5 x 2 x 2) = 60 Area = 60

3

20

Draw this rectangle.
Perimeter = 46
Is this the solution? no

5 x (3 x 2 x 2) = 60 Area = 60

5

12

Draw this rectangle.
Perimeter = 34
Is this the solution? yes

More Rat Cage Problems

Hint: Think about perimeter in relation to area. Is the shape you are looking for long skinny or more like a square?

Problem 3
The cage must be a rectangle with an area that must be 72 square feet and a perimeter that must be 44 feet.

What is the length and width of the cage?
18 feet long and 4 feet wide.

Problem 4
The cage must be a rectangle with an area that must be 140 square feet and a perimeter that must be 78

What is the length and width of the cage?
35 feet long and 4 feet wide.

Problem 5
The cage must be a rectangle with an area that must be 180 square feet and a perimeter that must be 54 feet.

What is the length and width of the cage?
15 feet long and 12 feet wide.

Problem 6
The cage must be a rectangle with an area that must be 72 square feet and a perimeter that must be 34 feet.

What is the length and width of the cage?
8 feet long and 9 feet wide.

Problem 7 A doozy! Be patient, 360 has many factors.
The cage must be a rectangle with an area that must be 360 square feet and a perimeter that must be 84 feet.

What is the length and width of the cage?
30 feet long and 12 feet wide.

Averaging: Manipulative

How many are in each stack now? 9

So the average number of baskets you made this week is 9 .

Relationships: Halving and Doubling the Dividend Worksheet 1

quotient
divisor | dividend

Use this process when solving these problems.

1. Solve the first problem.
2. Look at the second problem. Think Think about the pattern.
3. Predict answer ___ ___ Based on your pattern, write your predicted answ You will not need the second line yet.
4. Solve the second problem.
5. Check back. Look again. Look again to see if your prediction was correct. If it is not, figure out why.

Solve this problem.

$4\overline{)12}$ = 3 Look Think Predict answer ___ ___ Solve this problem. $4\overline{)24}$ = 6 Check back. Look again.

$3\overline{)12}$ = 4 Look Think Predict answer ___ ___ $3\overline{)24}$ = 8 Check back. Look again.

Relationships: Halving and Doubling the Dividend Worksheet 2

quotient
divisor | dividend

$5\overline{)10}$ = 2 Look Think Predict answer ___ ___ $5\overline{)20}$ = 4 Check back. Look again.

$8\overline{)32}$ = 4 Look Think Predict answer ___ ___ $8\overline{)64}$ = 8 Check back. Look again.

$9\overline{)27}$ = 3 Look Think Predict answer ___ ___ $9\overline{)54}$ = 6 Check back. Look again.

When I double the dividend the quotient doubles.
Explain why it makes sense that this is so. Answer in complete sentences. Use the words divisior, dividend, and quotient when describing what is going on. Edit.
The divisor is staying the same. The dividend is two times larger, so the quotient is also two times larger. If I have 5 friends and 10 apples, each friend gets 2 apples. If there are 20 apples and still 5 friends, then each friend gets 4 apples, which is double.

Relationships: Halving and Doubling the Dividend Worksheet 3

quotient
divisor | dividend

Solve this problem.

$5\overline{)50}$ = 10 Look Think Predict answer ___ ___ $5\overline{)25}$ = 5 Check back. Look again.

$6\overline{)36}$ = 6 Look Think Predict answer ___ ___ $6\overline{)18}$ = 3 Check back. Look again.

When I cut the dividend in half, the quotient is half .

What happens if you have a remainder?
Does it double too?
What would you predict?

Try it. $3\overline{)16}$ = 5 R1 $3\overline{)32}$ = 10 R2 Again. $4\overline{)13}$ = 3 R1 $4\overline{)26}$ = 6 R2

$4\overline{)19}$ = 4 R3 Look Think Predict answer 8 R6 ___ $4\overline{)38}$ = 8 R6 or 9 R2 Check back. Look again.

Opps! What happened here? The quotient of 19 ÷ 4 is four with a remainder of three. If you double that for 38 ÷ 4, you would get a predicted quotient of eight remainder six.
What is the problem of having a remainder of six when you are dividing by four? There is one group of 4 in 6 with a remainder of 2.
Use the second line in the prediction box to regroup your answer.
8 r6 would regroup to 9 r2.
9 r2 would be your new predicted answer.

Don't worry if this doesn't make sense at first. You'll get it with practice.

Relationships: Halving and Doubling the Dividend Division on Grids - Worksheet 4

Use Cuisenaire Rods to build these problems.

Examples

$5\overline{)25}$ = 5	Prediction 5 10	$5\overline{)50}$ = 10
$4\overline{)25}$ = 6 R1	Prediction 6 R1 12 R2	$4\overline{)50}$ = 12 R2
$3\overline{)25}$ = 8 R1	Prediction ___ ___	$3\overline{)50}$ = 16 R2
$2\overline{)25}$ = 12 R1	Prediction ___ ___	$2\overline{)50}$ = 25
$1\overline{)25}$ = 25	Prediction ___ ___	$1\overline{)50}$ = 50
$0\overline{)25}$	What is going on here? Can't divide by 0	$0\overline{)50}$

Relationships: Halving and Doubling the Dividend
Worksheet 5

Problem		Predict answer	Problem	
$5\overline{)20}$ = 4	Look, Think	___ ___	$5\overline{)40}$ = 8	Check back. Look again.
$8\overline{)25}$ = 3 R1	Look, Think	___ ___	$8\overline{)50}$ = 6 R2	Check back. Look again.
$9\overline{)26}$ = 2 R8	Look, Think	___ ___	$9\overline{)52}$ = 5 R7	Check back. Look again.
$4\overline{)17}$ = 4 R1	Look, Think	___ ___	$4\overline{)34}$ = 8 R2	Check back. Look again.
$5\overline{)24}$ = 4 R4	Look, Think	___ ___	$5\overline{)48}$ = 9 R3	Check back. Look again.
$6\overline{)35}$ = 5 R5	Look, Think	___ ___	$6\overline{)70}$ = 11 R4	Check back. Look again.

Relationships: Halving the Divisor
Worksheet 6

quotient
divisor | dividend

First, solve only this probem. $8\overline{)48}$ = 6

Then, use the patterns to get all other answers.
Use the pattern going down and across.

$8\overline{)48}$ = 6	→	$8\overline{)96}$ = 12
↓ $4\overline{)48}$ = 12	→	↓ $4\overline{)96}$ = 24
↓ $2\overline{)48}$ = 24	→	↓ $2\overline{)96}$ = 48
↓ $1\overline{)48}$ = 48	→	↓ $1\overline{)96}$ = 96

Check your answers with a calculator.
Did your patterns give you the correct answers? AWV

Relationships: Halving the Divisor
Worksheet 7

quotient
divisor | dividend

Do not do these problems yet. Just examine them.

$24\overline{)24}$ = 1
↓ $12\overline{)24}$ = 2
↓ $6\overline{)24}$ = 4
↓ $3\overline{)24}$ = 8

You can see here that the divisor is being cut in half as you go down the column.

The less the divisor the more the quotient.

Do the problems now.

If you halve the divisor, then you double the quotient.

Explain why this makes sense. If 24 apples are divided among 24 people, each person gets 1 apple. If 24 apples are divided by 12 people, each person gets 2 apples. The amount being divided stayed the same but the amount of people was halved so each person got 2 times more.

We also looked at patterns of doubling the dividend.

$6\overline{)24}$ = 4 → $6\overline{)48}$ = 8

You learned that:

The less the divisor, the more the quotient.

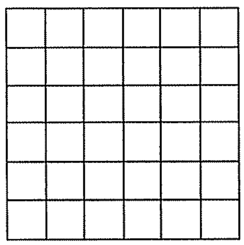
The more the dividend, the more the quotient.

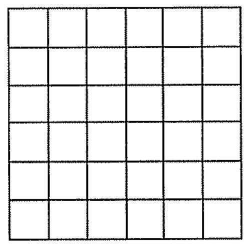
When you half the divisor, you double the quotient.

When you double the dividend, you double the quotient.

On the next pages we will put these two patterns together.

Relationships: Doubling the Dividend - Halving Divisors
Worksheet 8 Division on Grids





Use your patterns and the Cuisenaire Rods to help you find the answers.

$6\overline{)36}$ = 6	→	$6\overline{)72}$ = 12	→	$6\overline{)144}$ = 24
↓ $3\overline{)36}$ = 12		$3\overline{)72}$ = 24		$3\overline{)144}$ = 48

$4\overline{)36}$ = 9	→	$4\overline{)72}$ = 18	→	$4\overline{)144}$ = 36
↓ $2\overline{)36}$ = 18		$2\overline{)72}$ = 36		$2\overline{)144}$ = 72
$1\overline{)36}$ = 36		$1\overline{)72}$ = 72		$1\overline{)144}$ = 144
$0\overline{)36}$		$0\overline{)72}$		$0\overline{)144}$

Relationships: Halving the Dividend
Worksheet 9

divisor | dividend → quotient

1. Solve these problems.

- 4) 32 = 8
- 4) 16 = 4
- 4) 8 = 2
- 4) 4 = 1
- 5) 60 = 12
- 5) 30 = 6
- 5) 15 = 3
- 5) 7 1/2 = 1 1/2

2. What patterns do you see in the problems given? Dividend goes down by half. Quotient goes down by half. Divisor stays the same.

3. What happens to the size of the quotient when the dividend is cut in half? It is also cut in half.

4. At the bottom of each problem set there is a problem with no numbers given. Use the patterns to fill in what problem would come next in the series. Then find the answer. The very last one is a challenge for you to figure out.

5. Explain what happens if the divisor is kept the same and the dividend is cut in half? The quotient goes down by half.

6. Prove it with an example you make up. Choose an even number for a divisor. Choose a dividend that the divisor goes into evenly with no reminders. Hint: Use a times chart to find good numbers.

- 6) 48 = 8
- 6) 24 = 4
- 6) 12
- 6) 6 = 1

7. What did you find out? The pattern works. The quotient goes down by half if the dividend is cut in half.

8. If you halve the dividend it will ____________ the quotient.

Relationships: Descending Divisor - Division on Grids
Worksheet 10

Use Cuisenaire Rods to build these problems

Use your doubling pattern.

7) 49 = 7	→ 7) 98 = 16
6) 49 = 8 R1	→ 6) 98 = 16 R2
5) 49 = 9 R1	→ 5) 98 = 19 R2
4) 49 = 12 R1	→ 4) 98 = 24 R2
3) 49 = 16 R1	→ 3) 98 = 32 R2
2) 49 = 24 R1	→ 2) 98 = 49
1) 49 = 49	→ 1) 98 = 98

What happens to the quotient when the divisor goes down? It gets larger.

Relationships: Descending Divisor Worksheet 11

1. Solve these problems.

- 12) 24 = 2
- 8) 24 = 3
- 6) 24 = 4
- 4) 24 = 6
- 3) 24 = 8
- 2) 24 = 12
- 1) 24 = 24

2. What patterns do you see? Quotients get larger. Dividends are all the same. Divisors are smaller.

3. What happens to the size of the quotient when the divisor goes down? It goes up.

4. So, the less the divisor, the larger the quotient.

5. What will happen to the quotient if you raise the dividend, and keep the divisor the same? It will get larger.

6. Do an experiment to find out. Choose a divisor and keep it the same. Try increasing the dividends and see what happens to the quotient.

Example:

- 5) 5 = 1
- 5) 10 = 2
- 5) 15 = 3
- 5) 20 = 4
- 5) 25 = 5

7. The more the dividend, the more the quotient.

Relationships: Multiplying Dividends
Worksheet 12

divisor | dividend → quotient

1. Solve these problems.

- 4) 4 = 1
- 4) 16 = 4

- 6) 12 = 2
- 6) 48 = 8

Use the pattern to predict the answer.

- 7) 21 = 3
- 7) 84 = 12

2. Study each set of two problems. How are the problems in each pair related? The dividend is multiplied by four. The divisor is the same.

3. What patterns do you notice in the answers in each pair of problems? The answer (quotient) is four times larger.

4. How many fours are in forty? Ten

4) 40 = 10

Now let's make this dividend 4 times larger. 40 x 4 = 160.

4) 160 = 40

5. What happens to the quotient? Multiplied by four.

If needed, use Base Ten Blocks to find the answers.

How can you use this pattern to find the answer to the second problem in each pair? Multiply each quotient by four.

- 5) 80 = 16
- 5) 320 = 64
- 3) 60 = 20
- 3) 240 = 80

You make a pair now. AWV

Relationships: Multiplying Dividends
Worksheet 13

quotient
divisor | dividend

1. Solve these problems.

2 | 2 → 1
2 | 20 → 10

3 | 6 → 2
3 | 60 → 20

Use the pattern to predict the answer.

4 | 8 → 2
4 | 80 → 20

2. Study each set of two problems. How are the problems in each pair related?
The divident is multiplied by ten. The divisor stays the same.

3. What patterns to do notice in the answers in each pair of problems?
The answer is ten time larger.

4. How many fives are in twenty? Four.
5 | 20 → 4

Now lets make this dividend 10 times larger.
20 x 10 = 200.
What happens to the quotient? It's multiplied by ten.
5 | 200 → 40

Use the Base Ten Blocks to find the answers.

How can you use this pattern to find the answer to the second problem in each pair? Multiply each quotient by ten.

5 | 25 → 5
5 | 250 → 50

5 | 15 → 3
5 | 150 → 30

You make a pair now. AWV

Relationships: Show You Know
Worksheet 14

1. Solve these problems.

24 | 48
12 | 48
6 | 48

What's next in the pattern?
3 | 48 → 16

The less the divisor, the more the quotient.

Explain why this makes sense. If the dividend stays the same and the divisor gets smaller, that means the size of the groups being made is getting smaller, so, the number of groups you have to make has to go up.

Make your own that follows the same pattern. AWV

16 | 16 → 1
8 | 16 → 2
4 | 16 → 4
2 | 16 → 8

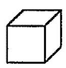
2. Use the pattern in the first set of problems to predict the dividend in the last problem.
Make your own that follows this pattern. Examples:

2 | 8 → 4
2 | 16 → 8
2 | 32 → 16
2 | 64 → 32

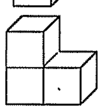
3 | 9 → 3
3 | 18 → 6
3 | 36 → 12
3 | 72 → 24

Number Patterns: Functions - Worksheet 2
Thrones

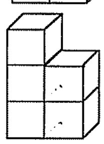
Throne 1 needs 1 cube.



Throne 2 needs 3 cubes.



Throne 3 needs 5 cubes.



X Throne	Y Cube Boxes
1	1
2	3
3	5
4	7
5	9
6	11
Function	Y = 2X-1

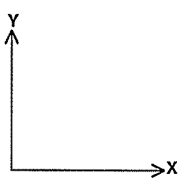
Record the number of cubes Throne 4 needs on the chart.
Write what the boy did as a mathematical function.
First, he built a rectangular shape, four high by two wide.
Then, he took away one cube to make a seat.

X is the Throne number. Y is the total number of Cube Boxes used.

Continue the pattern to Throne 6.

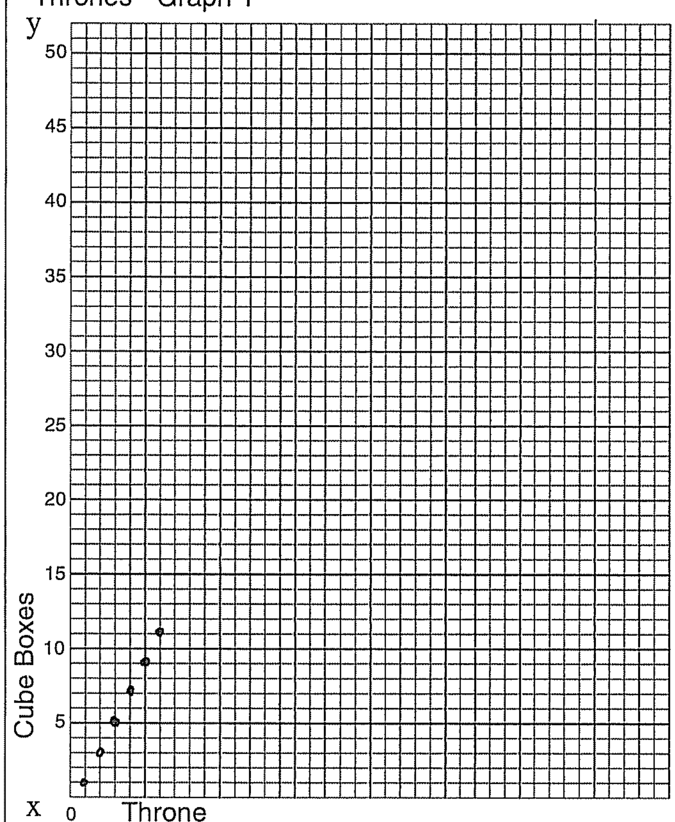
How do you calculate the total number of Cube Boxes used in a Throne? Multiply the throne number by two, then take away one.

What do you think the graph will look like? Sketch it. Do not put in the numbers. Then graph it on the following page. AWV



Note: The line does not pass through the origin.

Number Patterns: Functions - Worksheet 3
Thrones - Graph 1



Number Patterns: Functions - Worksheet 4 - Graph It

Table 1

x	y
0	8
1	11
2	14
3	17
4	20
5	23
6	26
function	y=3x+8
50	158

Table 2

x	y
0	2
1	7
2	12
3	17
4	22
5	27
6	32
function	y=5x+2
50	252

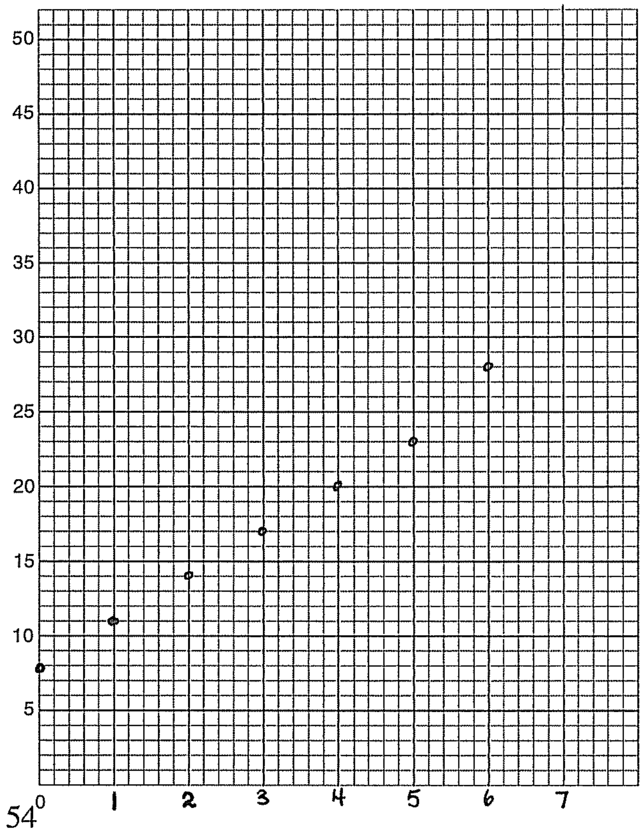
Table 3

x	y
1	0
2	8
3	16
4	24
5	32
6	40
7	48
function	y=(x-1)8
50	392

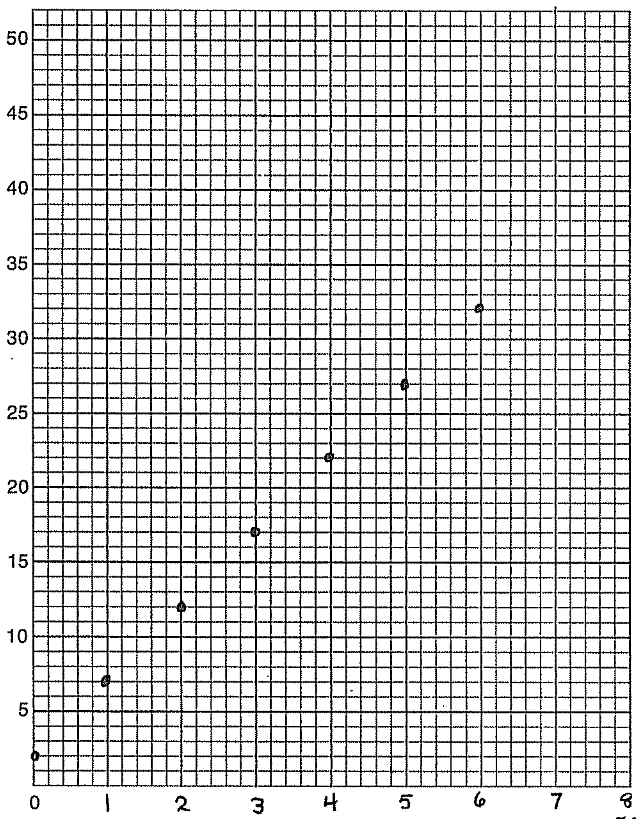
Table 4
Do it backwards.
Get the points off of Table 4.

x	y
0	3
1	5
2	7
3	9
4	11
5	13
6	15
7	17
8	19
9	21
10	22
function	y=2x+1

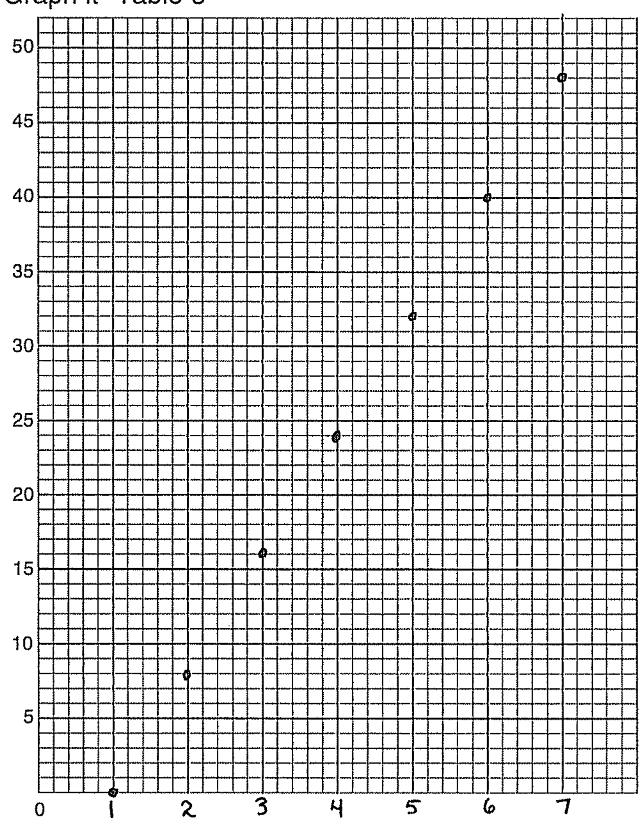
Number Patterns: Functions - Worksheet 5
Graph It -Table 1



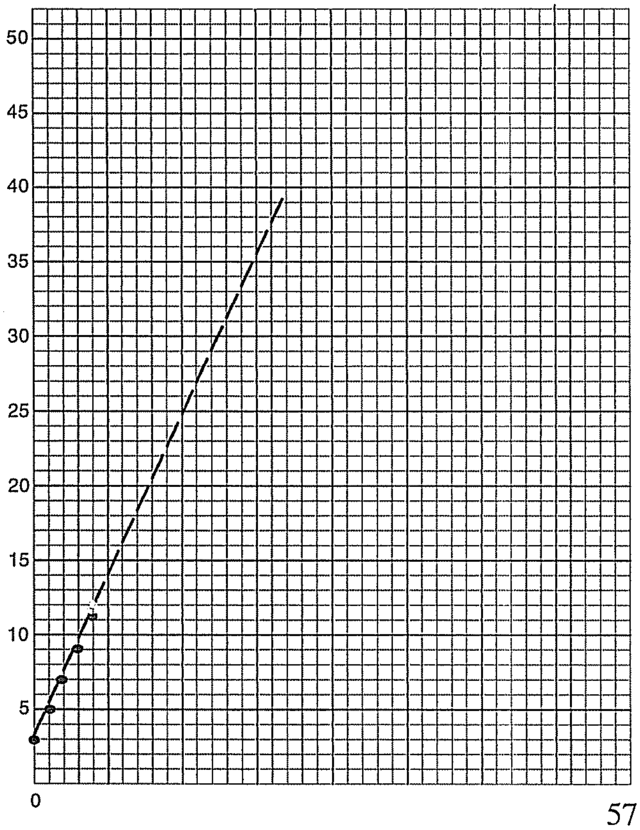
Number Patterns: Functions - Worksheet 6
Graph It -Table 2



Number Patterns: Functions - Worksheet 7
Graph It -Table 3



Number Patterns: Functions - Worksheet 8
Graph It -Table 4



Number Patterns: Functions - Worksheet 9 - Tables
Complete the tables.

Table 1

x	y
0	4
1	9
2	16
3	25
4	36
5	49
6	64
function	y=(x+2)²
50	2,704

Table 2

x	y
1	2
2	4
3	6
4	8
5	10
6	12
7	14
function	y=2x
50	100

Table 3

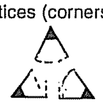
x	y
1	5
2	9
3	13
4	17
5	21
6	25
7	29
function	y=4x+1
50	201

Table 4 AWV
Make your own.

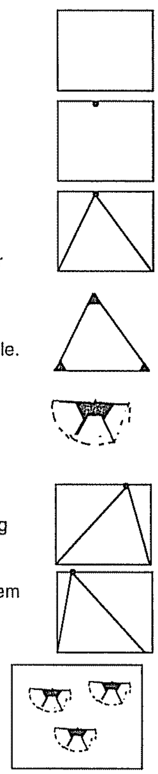
x	y

Internal Angles of Triangles - Worksheet 1
Here is a neat pattern to investigate.

1. Take a regular size piece of paper (8 1/2 x1).
2. Turn it sideways.
3. Draw a dot somewhere along the top edge.
4. Using a ruler draw a line from the dot to the lower left-hand corner.
5. Draw a line from the dot to the lower right-hand corner
6. Cut out the large triangle.
7. Shade in all of the three vertices (corners) of the triangle.
8. Tear off all three vertices.



9. Put the three shaded corners together and tape them to a colored piece of paper
10. Repeat twice but draw the dot in a different place along top edge.
11. Remember to shade in the vertices before you tear them off of the big triangle.
12. Tape each set down on the colored piece of paper
13. What patterns do you notice? They all come out as half circles. They all make a straight line at the top.



Internal Angles of Triangles- Worksheet 2
The shaded vertices you tore off of each triangle are called Internal Angles of a triangle. They are called this because they are the angles on the inside.

14. How many degrees are in a straight line or half of a circle? 180°
15. What was the sum of the Internal Angles of each of the triangles? 180°

Try it with the small triangles you cut off the papers while you were making the big triangles. What did you find out? The pattern is true for all triangles.

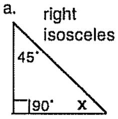
16. Does this pattern work on all triangles? yes

Rule of Triangles

The sum of the Internal Angles of ANY triangle will always be 180°.

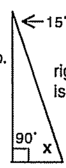
Use this rule to figure out the missing angles. The missing angle is labeled with an **X**.

a. right isosceles



x = 45°

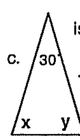
b. right isosceles



90 + 15 = 105; 180 − 105 = 75

x = 75°

c. isosceles

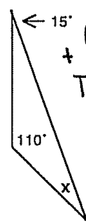


180 − 30 = 150

x = 75°

y = 75°

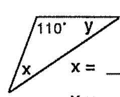
d. scalene



110 + 15 = 125; 180 − 125 = 55

x = 55°

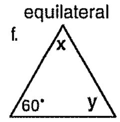
e. Challenge (No right angle at x). isosceles



x = 35°

y = 35°

f. equilateral



x = 60°

y = 60°

The angles must be equal because it is an isosceles triangle, so x and y are each 75°.

Area of Triangles - Worksheet 1

Use a geoboard.
Make rectangles and triangles like this with rubber bands:

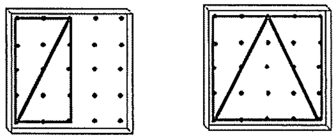
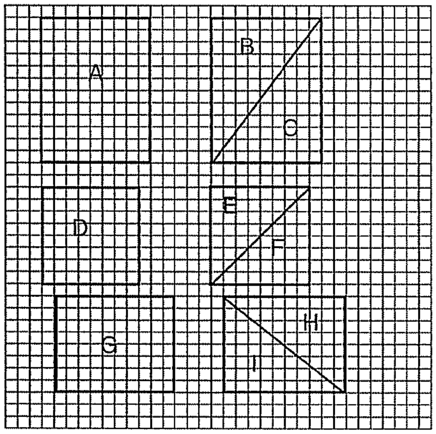


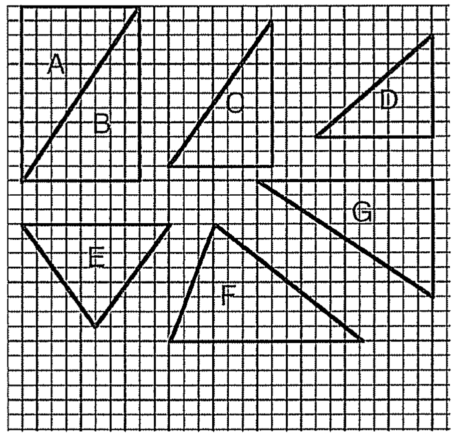
Figure out how to find the area of a triangle.
Describe your method here.
Multiply the base times the height, then divide by 2. You can do this because a rectangle, or parallelogram, can be divided into 2 triangles.



What is the

A 108
B 54
C 54
D 64
E 32
F 32
G 80
H 40
I 40

Area of Triangles - Worksheet 2

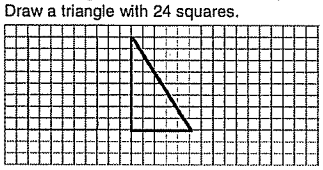


What is the area?
A 48
B 48
C 35
D 28
E 35
F 52
G 48

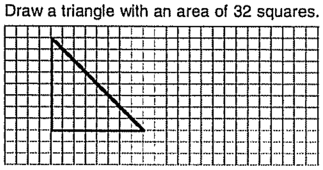
Write how to find the area of a triangle in pattern language
B x H ÷ 2 or 1/2 B x H or L x W ÷ 2 or 1/2 L x W

Area of Triangles - Worksheet 3

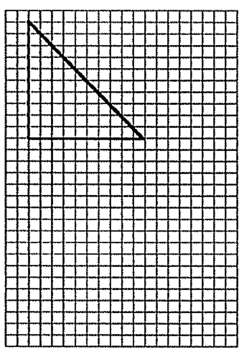
This rectangle has an area of 48 squares.
Draw a triangle with 24 squares.



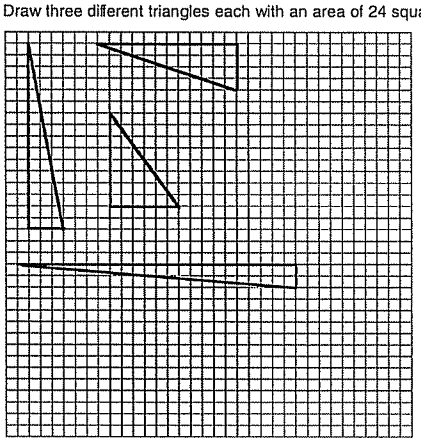
Draw a triangle with an area of 32 squares.



Draw a triangle with an area of 50 squares.



Draw three different triangles each with an area of 24 squares.



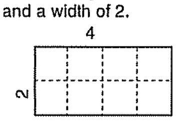
Write all the factors of the number that will help you.

1 x 48
2 x 24
3 x 16
4 x 12
6 x 8
8 x 6
___ x ___
___ x ___

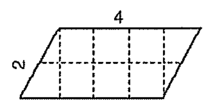
Area and Perimeter: Parallelograms - Worksheet 1

You already know how to figure out the area of a rectangle. Put that knowledge to work to figure out how to find the area of a parallelogram.

This rectangle has a length of 4 and a width of 2.

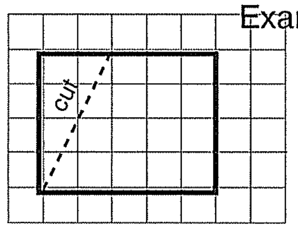


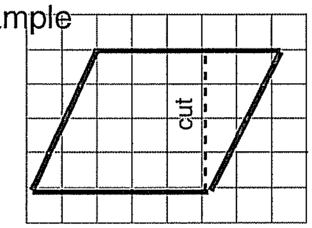
This parallelogram has a length of 4 and a width of 2.



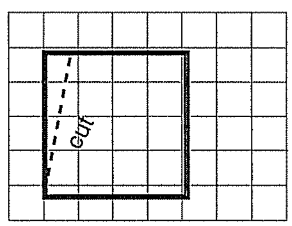
On the grid below draw and then cut out a rectangle. Rearrange and transform the rectangle into a parallelogram. Draw the parallelogram here.

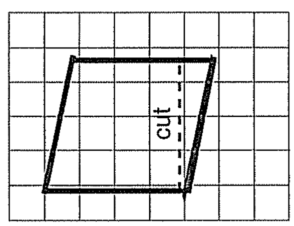
Example



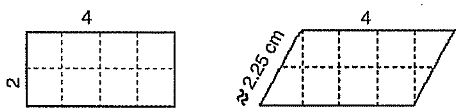


On the grid below draw and then cut out a square. Rearrange and transform the square into a parallelogram. Draw the parallelogram here.



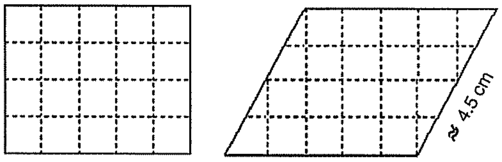


Area and Perimeter: Parallelograms - Worksheet 2



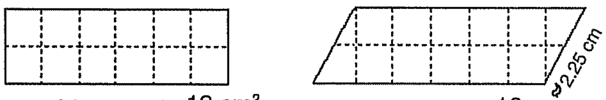
What is the area of the rectangle? 8 cm² What is the perimeter? 12 cm

What is the area of the parallelogram? 8 cm² What is the perimeter? ≈12.5 cm



Area of the rectangle. 20 cm² Perimeter of the rectangle. 18 cm

Area of the parallelogram. 20 cm² Perimeter of the parallelogram. ≈19 cm



Area of the rectangle. 12 cm² Perimeter of the rectangle. 16 cm

Area of the parallelogram. 12 cm² Perimeter of the parallelogram. ≈16.5 cm

What parts of a parallelogram must be measured to give you the two numbers you need to calculate the area?

The length of the bottom line and the distance from the lower corner to the top line, called the height.

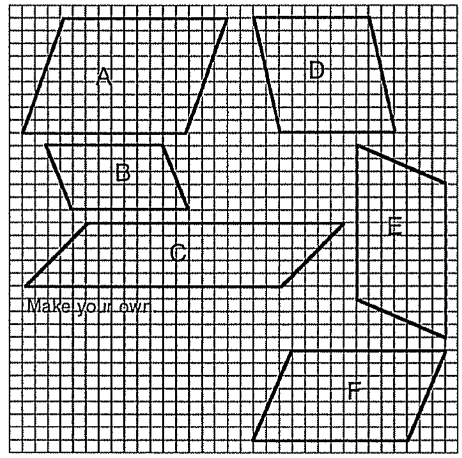
Explain how to find the area of a parallelogram.

Measure the base and the height and multiply them together. Or take the product of the base and the height.

Explain how to find the perimeter of a parallelogram.

Add together the length of each side.

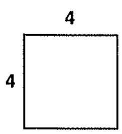
Area and Perimeter: Parallelograms -Worksheet 3



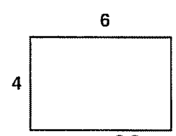
Base	Height	Figure	Area	Hypotenuse		Perimeter
13 u	9 u	A	117 u²	10 u²	A	46 units
9 u	5 u	B	45 u²	5½ u²	B	29 units
20 u	5 u	C	100 u²	7 u²	C	54 units
9 u	9 u	D	81 u²	9½ u²	D	37 units
12 u	7 u	E	84 u²	8 u²	E	40 units
12 u	7 u	F	84 u²	8 u²	F	40 units

Area and Perimeter: Parallelograms Worksheet 4

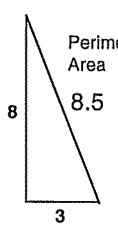
Calculate the area and perimeter of each figure.



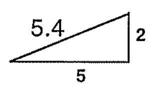
Perimeter 16 u
Area 16 u²



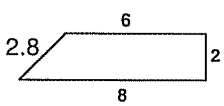
Perimeter 20 u
Area 24 u²



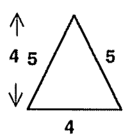
Perimeter 19.5 u
Area 12 u²



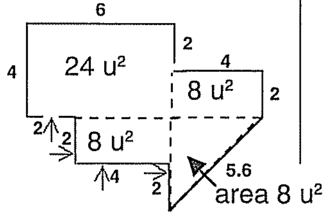
Perimeter 12.4 u
Area 5



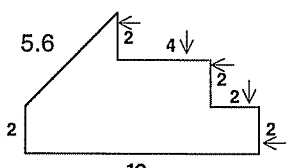
Perimeter 18.8 u
Area 12 + 2 = 14 u²



Perimeter 14 u
Area 8 u²



Perimeter 33.6 u
Area 48 u²



Perimeter 29.6 u
Area 36 u²

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