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Bachelor Thesis

How can teachers use recreational mathematics to spark students' interest in mathematics and contribute to the educational process?

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Abstract

Have you ever played tic-tac-toe? Well, then you have already engaged in recreational mathematics. This thesis is devoted to the study of recreational mathematics, with a particular emphasis on its application for educational purposes. As there is no universal definition of recreational mathematics, I start by offering an introduction to the world of recreational mathematics and then set out on a quest to find a working definition considering the involvement of amateur and professional mathematicians. There would be no recreational mathematics without its popularizer Martin Gardner, so I introduce the reader to him and his unique life story. Then, I analyze the various benefits of recreational mathematics from a psychological perspective and emphasize its importance for overriding biases and developing critical thinking. Following that, I review a research paper dedicated to recreational mathematics and its implementation in the educational policies of China, England, Finland, India, Japan, Singapore, Sweden, and the U.S. After that, I closely examine educational policies in Austria and Serbia and evaluate whether recreational mathematics is an integral part of their mathematics curricula. Finally, I present examples of recreational mathematics categories such as video games and board games/puzzles that have already been successfully implemented in school curricula.

1 What is recreational mathematics?

The Cambridge Dictionary online edition defines recreation as "a way of enjoying yourself when you are not working." Mathematics, on the other hand, is described as "the study of numbers, shapes, and space using reason and usually a special system of symbols and rules for organizing them." At first glance, recreation and mathematics may not feel well-matched to most people. For example, mathematics is not a traditionally popular subject among students. However, the idea of using math for recreational purposes is not new. As Petković (2013) [36] has noted, papyrus findings dating to around 1650 BC indicate that the early Egyptians had already used puzzles to explore mathematical problems. Petković (2013) has argued that, as no other way of utilizing such activities in everyday life existed, their primary purpose may have been intellectual recreation in the form of play. As every form of play is tremendously satisfying, recreational mathematics is undoubtedly no exception (Trigg, 1978) [45]. However, the question remains: where does one draw a line between serious and recreational mathematics?

What is considered recreational depends on whom you ask. Trigg (1978) [45] asked numerous mathematicians about how they defined the recreational aspect of the subject and concluded at the end of his article "What is Recreational Mathematics?" that it is impossible to agree on one universal definition. For example, many professional mathematicians stated that they regarded their work as highly recreational. Perhaps this is a common description of what happens when people love their job and enjoy what they do. However, Trigg touched on some exciting aspects of what makes mathematics recreational, allowing me to formulate requirements and to sharpen the blurred line between serious and recreational mathematics.

Firstly, Trigg (1978) [45] looked from an academic perspective at why people engage in mathematical problems and observed that, when the primary goal is to publish an article or win an award, then the mathematical activities are not considered recreational. This does not mean that recreational mathematics cannot serve as a starting point for advanced mathematical theories, but rather that the motivation for engaging in recreational mathematics must be intrinsic and not monetary. Secondly, the author (p. 20) pointed out how some experts insisted that an element of play or a game factor must be present. In the literature on recreational mathematics, several keywords involving some affective dimension of learning, such as amusement, joy, fun, pleasure, and satisfaction from solving a problem, appear repeatedly. Playfully performing activities encourages people to put in more effort and get the best out of their mental recreation, and this is the core of recreational mathematics: making the subject fun and approachable (Trigg, 1978). Aside from the fun aspect, Trigg explained that solving a complex mathematical exercise also fulfills the very human need for feeling victorious. Finally, the author pointed out that the simplicity and accessibility of recreational mathematical tasks also play a crucial role. Accessibility refers to the tools that one requires: a simple pen and a piece of paper suffice, making engaging in recreational mathematics accessible to those who cannot afford more (Trigg, 1978). In terms of simplicity, according to Trigg (p. 21), recreational refers to "a piece of mathematics that is both subtle enough to interest the professional mathematician and simple enough to be accessible to the man-in-the-street" - meaning that the problem must be formulated in a particular way so that it becomes memorable and pleasurable but at the same time contains educational value.

However, do not be misled by the aspect of simplicity. Recreational mathematics should not be seen as less critical than traditional mathematics, as many recreational problems have helped to develop new mathematical concepts and gain a better understanding of abstract theories such as "number theory, topology, combinatorics, and game theory" (Trigg, 1978 p. 20) [45]. In addition to many amateur mathematicians, numerous professionals in this field also engage in recreational mathematics and have advocated for its use in school curricula. Below are just a few of these individuals.

1.1 Charles L. Dodgson/Lewis Carrol (1832-1898)

Charles L. Dodgson - widely known under his pen name Lewis Carroll, the author of the world-famous *Alice's Adventures in Wonderland* and other children's books - was, believe it or not, a professional mathematician who was fascinated by mental recreation such as logic games, puzzles, and card games (Wilson and Moktefi, 2019) [47]. Wilson and Moktefi (2019) described Dodgson as showing great interest in recreational mathematical problems in the fields of arithmetic, geometry, and probability. He published numerous books on these topics; the best known is titled *Pillow Problems*.

An amusing mathematical rumor tells the story of how Carrol's book *Alice's Adventures in Wonderland* had enchanted Queen Victoria so much that she asked to immediately receive the next book that Mr. Carrol would write (Wilson and Moktefi, 2019) [47]. As it happened, the Queen was indeed puzzled by the author's next book; it was titled *An Elementary Treatise on Determinants: With Their Application to Simultaneous Linear Equations and Algebraical Geometry* (Wilson and Moktefi, 2019, p. 57). Like most rumors, Wilson and Moktefi (2019) explained, this one turned out to be false, and Dodgson himself denied it, too. However, such a story demonstrates Dodgson's dualistic nature as a conservative mathematician and fantastic fiction writer (Wilson and Moktefi, 2019). Even in *Alice's Adventures in Wonderland*, Carrol did not leave behind his affinity for recreational mathematics and introduced into the story numerous logic puzzles, number games, and hidden riddles. Martin Gardner, a famous popularizer of the 20th century's recreational mathematics, published notes on all those hidden gems in his book *Annotated Alice in Wonderland*. I will cover Gardner, "the dean of contemporary recreational mathematics," as Trigg (1978, p. 20) [45] named him; his extraordinary story as a self-taught mathematician; and his immense influence on this scientific field in detail in the next chapter.

But, returning to Dodgson, what was recreational mathematics to him? As Dodgson used mathematical games and puzzles for having fun and entertaining his friends (Wilson and Moktefi, 2019) [47], I conclude that the leitmotif of recreational mathematics to him was amusement. This demonstrates the element of play, previously used to define which part of mathematics is considered recreational.

1.2 Blaise Pascal (1623-1662)

Blaise Pascal once said, "Mathematics is too serious and, therefore, no opportunity should be missed to make it amusing" (Petković, 2013, p. 13) [36]. Pascal was a talented French mathematician who needs no lengthy introduction. He made significant contributions not only to mathematics but also to other fields of science. Together with another mathematician, Pierre de Fermat, Pascal laid the foundation for the concept known today as the theory of probability, which is not only an essential branch of mathematics but also valuable to other scientific disciplines (Petković, 2013). The lesser-known story, according to Petković (2013), is that this discovery originated from a recreational problem. Petković (2013) has explained how a friend of Pascal asked him for help in solving a gambling problem. Pascal contacted Fermat, and together they started working on this brainteaser. As a result of their discussion, they surprisingly developed a new theory (Petković, 2013). This development was of great significance for the probability theory known today, but it was a serendipitous outcome because Pascal was merely looking for a specific practical solution to help a friend.

Can this problem be considered recreational mathematics? For a mathematician like Pascal, all mathematics was probably enjoyable. In this case, however, the famous mathematician did not aim at any academic goal, and his intention was purely intrinsic. As Trigg (1978) [45] stated, this is an important aspect when determining what is considered recreational mathematics. Therefore, the problem Pascal worked on can be seen as recreational, and it simultaneously served as a starting point for an advanced mathematical discovery.

1.3 Henry Dudeney (1857-1930) and Sam Loyd (1841-1911)

Henry Dudeney, dubbed Britain's greatest puzzlist, was an English author and amateur mathematician since he only obtained a primary education and only learned advanced mathematics in his spare time (Guy and Woodrow, 1994) [22]. Nevertheless, Guy and Woodrow (1994) pointed out, Dudeney's published mathematical problems overshadowed any other puzzlist's work before and after his time in not only quantity but also quality. In the literature, Dudeney often appears alongside Sam Loyd, another leading creator of mathematical problems around the turn of the century. Though they were great rivals, they did not hesitate to borrow inspiration from each other and exchange mathematical ideas (Guy and Woodrow, 1994; Trigg, 1978) [22] [45]. They both published an extraordinary number of puzzles that are still popular today (Petković, 2013) [36]. However, Loyd was a controversial person, often perceived to be dishonest, as can be taken from Costello's book *The Greatest Puzzles Of All Time*, in which the author labeled Loyd as a "fast-talking snake oil salesman" (Costello, 1988, p. 45) [11]. What was incredibly aggravating to Dudeney was that Loyd went so far as to steal Dudeney's puzzles and publish them under his own name (Dudeney and Gardner, 1967) [12].

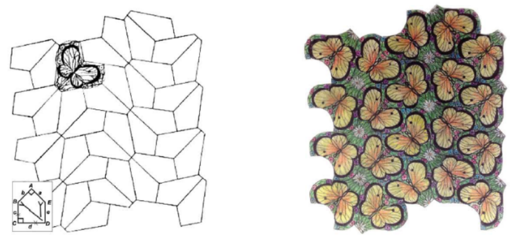
What was recreational mathematics to Dudeney? As Dudeney was passionately writing for multiple journals at the time, his goal was seemingly to popularize recreational mathematics and make it accessible to the general public. This demonstrates the aspect of accessibility that was included in the definition of recreational mathematics.

1.4 Marjorie Rice (1923-2017)

Tessellation, a popular branch of mathematics, means covering a surface or a plane without gaps or overlaps with identical geometric shapes, called tiles (Ardila and Stanley, 2010) [2]. A honeycomb is the best-known tessellated structure in nature. One does not have to be a mathematician to come up with various forms. Many artists have used tessellation in their artwork; ancient mosaics found worldwide, particularly in Islamic architecture, are examples (Schattschneider, 1981) [38].

Marjorie Rice, an ordinary housewife and a surprising amateur mathematician with no formal training in mathematics beyond high school, made significant discoveries in geometry (Schattschneider, 2018) [39]. Schattschneider (2018) remembered Rice as an admirer and frequent reader of Gardner's column "Mathematical Games." In July 1975, Gardner's article "On Tessellating the Plane with Convex Polygon Tiles" sparked Marjorie's curiosity, and she began the pursuit of a new type of tiling polygon (Schattschneider, 1981; Schattschneider, 2018) [38] [39]. Gardner's article on this topic covered polygons, which are tricky and do not always tile a plane (Schattschneider, 1981). The writer's column illustrated eight known types of convex polygons that could tile a plane (Schattschneider, 2018). At the time, it was a complete list - or so mathematicians thought. In February 1976, Rice discovered a new type of tiling polygon and sent her discovery to Gardner, who, in turn, forwarded it for verification to Doris Schattschneider, an expert on this subject (Schattschneider, 2018). Schattschneider (1981) verified Rice's discovery and encouraged her curiosity by providing Rice with various articles on tessellation. Over two years, Rice discovered three other unknown types of pentagon tiles and created 60 different combinations of pentagon tilings (Schattschneider, 1981; Schattschneider, 2018) - a noteworthy accomplishment for a "mere amateur."

What was recreational mathematics to Rice? Rice received neither remuneration nor training in the course of her research; she devoted her free time out of fascination with this topic and for the sake of the beautiful artwork that she created through the combination of geometric shapes (Schattschneider, 1981) [38]. To an amateur such as Rice, recreational mathematics equaled pure satisfaction. Again, this demonstrates one of Trigg's (1978) [45] significant factors that define recreational mathematics: intrinsic motivation. Other than that, what inspired her to pursue a serious scientific subject like mathematics was, in fact, the recreational element. Therefore, making mathematics less abstract and more accessible to an average person is crucial to attracting interest and encouraging further study in this area.



Type 13: A = C = 90°, 2B + D = 360°, 2E + D = 360°; c = d, 2c = e. Butterflies.

Figure 1: "Marjorie Rice's pentagons of type 13. It represents an infinite family of tiling pentagons. Her pictorial notation shows which angles meet at a vertex of the tiling, and conditions on the sides of the pentagons" (Schattschneider 2018, p. 53). [39]

1.5 A working definition

In conclusion, mathematics may be considered recreational when

- no academic goal is being pursued (e.g., a publication of an article),
- it involves an element of play,
- it is accessible to everyone, and

- it has educational value; what matters is the confidence and the skills acquired on the journey to a solution, not only the solution itself.

Many of the elements listed above were present in Gardner's "Mathematical Games" column, which he wrote for Scientific American and which inspired many amateur mathematicians to fall in love with this subject (Schattschneider, 1981) [38].

2 How did recreational mathematics reach fame in the 20th century?

2.1 Gardner's youth

Gardner, a man who "has turned dozens of innocent youngsters into math professors and thousands of math professors into innocent youngsters," was a behemoth in the field of recreational mathematics, and therefore his life deserves a closer look (Gardner, 2015, p. 17) [13]. In his final book *Undiluted Hocus-Pocus: The Autobiography of Martin Gardner*, he tells his fascinating story.

Gardner initially had nothing to do with recreational mathematics, having studied philosophy at the University of Chicago (Gardner, 2015) [13]. Gardner (2015) remembers, however, that his passion for mathematics was first ignited in his childhood when he read Loyd's *Cyclopedia of Puzzles*. Still, Gardner (2015) recalls high school as a mere waste of time, except for mathematics and physics. In his spare time during college, the future writer was very enthusiastic about his hobbies and enjoyed playing chess and inventing new magic tricks, but surprisingly never performed in front of a broad audience (Gardner, 2015). Interestingly, the tricks he learned and exchanged with fellow magicians at a cafeteria in New York came in handy for inspiring many of the articles of his column "Mathematical Games" that he later wrote for *Scientific American* (Gardner, 2015). At first, Gardner (2015) intended to study physics at Caltech, an influential private university, but to get accepted, he had to meet the requirement of first attending a liberal arts college for two years. So as not to waste any more time, the future mathematician decided to apply to the University of Chicago, because at the time, a radical change had been implemented that would let him take exams without attending classes and drive his studies rapidly forward (Gardner, 2015). While young Gardner's initial intention was to finish his prerequisites as soon as possible and transfer to Caltech afterward, he fell in love with philosophy and, after all, decided to major in this field (Gardner, 2015). After obtaining his degree in philosophy and serving in the U.S. Navy during World War II, Gardner returned to the University of Chicago and attended graduate school for a year but did not finish because he feared he could not make a living as a practicing philosopher and decided to seek a career in writing instead (Gardner, 2015). He began writing for various magazines and published popular articles on magic, which were later issued into a book titled *Mathematics, Magic and Mystery* (Gardner, 2015). Although his book was successful, Gardner is mainly remembered for his magazine column, which turned out to be a milestone transforming the world of mathematics forever.

2.2 Beginnings of "Mathematical Games"

One day, a friend of Gardner who was also a magician, Royal V. Heath, showed Gardner an entertaining mathematical toy - a hexahexaflexagon that could be flexed and folded repeatedly, revealing different sides and colors each time it was folded (Gardner, 2015) [13]. Fascinated by this invention, Gardner met with the experts on flexagons, studied the mathematical principles behind them, and wrote an article on this topic, which he eventually published in *Scientific American* (Gardner, 2015). The publisher contacted Gardner and asked him if he could write a monthly column (Gardner, 2015). Gardner did not hesitate to say yes, although very few books on recreational mathematics were in print at the time (Gardner, 2015). "Mathematical Games" was born - a column that, for the first time, introduced simple as well as complex mathematical subjects to a broad audience.

Petković (2013, p. 8) [36] has deemed Gardner "perhaps the most significant twentieth-century popularizer of mathematics and mathematical recreations." Nevertheless, Gardner always remained down-to-earth. For example, in his autobiography, Gardner (2015) [13] humbly described himself as a self-taught mathematician, pointing out that, by analyzing his columns from oldest to newest, one can observe his progression on a mathematical level. Still, Gardner's "Mathematical Games" column ran in *Scientific American* over a lengthy two-decade period, which is rather curious, considering that Gardner had not attended any math courses at university and was studying the topics he was writing about as he went - learning by doing, one could say (Gardner, 2015). As it happened, his lack of mathematical background turned out to be one of his strengths and the reason his column grew incredibly popular. Since he was struggling to analyze and digest the tough mathematical topics, he sliced and diced them in an understandable way and therefore succeeded in serving them in small manageable pieces to the average reader - not a simple task, considering the abstract level of mathematics for the uninitiated (Gardner 2015). Gardner (2015) thoroughly researched every topic he wrote about, and many well-known mathematicians contributed to his column, such as Solomon W. Golomb, Raymond Smullyan, John Conway, and many others. Through "Mathematical Games," Gardner managed to turn mathematics, which was intimidating to many, into something entertaining by introducing countless intriguing brainteasers that inspired thousands of amateurs to develop an interest in solving mathematical problems. Petković (2018) [35] has concurred that the best way to overcome the opinion that mathematics is not only difficult but also too abstract and boring for many is to popularize mathematics through interesting contributions to mathematics that are understandable to a wide range of readers - exactly what Gardner did. Yam (2010) [49] has confirmed that Gardner's success in inspiring people was due to his playfulness in teaching all kinds of mathematical problems. His unique ability to enable readers to enhance their mathematical skills by making mathematics appealing helped him turn cognitive strain into cognitive ease - a phenomenon frequently discussed by renowned psychologists and economists.

3 What are the benefits of recreational mathematics?

3.1 Heuristics and biases and how they relate to mathematics

Israeli American psychologist and economist Daniel Kahneman (2012) [27] has argued that individuals tend to avoid cognitive strain, explaining that human brains prefer taking mental shortcuts, which predisposes people to cognitive biases. Kahneman's *Thinking Fast and Slow*, which won the National Academies Communication Award, provides insight into this topic, throwing light on cognitive biases, heuristics, and illusions, thus showing the kinds of limitations the human mind faces. The author also explains the dual-process theory underlying the two brain "systems," using the metaphor of two agents, "System One" and "System Two."

Thinking Fast and Slow is about two systems known for taking different approaches to reach the same goal: decision-making. System One does the fast thinking and refers to being quick, relying on intuition (Kahneman, 2012) [27]. The author clarifies that this system engages in everyday life automatically, with almost no effort. For example, turning on the lights or answering a simple mathematical equation results from an automatic response without any intervention on the part of the individual. Evolved to meet the needs of quick decision-making, System One preferably uses the concept of cognitive ease and is keen on taking mental shortcuts, also known as heuristics (Kahneman, 2012). Kahneman (2012) explains that a heuristic is a vital strategic tool often used to save resources. When not all information is available and time is of the essence, making it much more difficult to weigh every factor and make a rational decision, System One activates and simplifies the complex situation (Inglis and Simpson, 2005) [25].

Meanwhile, System Two is rooted in slow and rational cognitive processes and will take over when individuals face complex challenges requiring concentration (Kahneman, 2012) [27]. The author (2012, p. 44) exemplifies System Two with the aid of the following brainteaser:

"A bat and a ball together cost $1.10. The bat costs one dollar more than the ball. How much does the ball cost?"

If answered intuitively without pen and paper, most people will go with 10 cents, which is the wrong answer (Kahneman, 2012) [27]. However, Kahneman (2012) explains that if more time is spent on the puzzle, which allows System Two to get involved, it is easier to arrive at the correct answer, which is 5 cents. According to the dual-process theory, this is a "perfect world" example of how System Two gets involved and provides support as problems prove too hard for System One. System Two has the additional function of controlling System One and thus influencing the behavior of individuals by occasionally suppressing the impulsive side and thereby increasing the proportion of rational decision-making (Kahneman, 2012). As a result of its complex functions, System Two requires considerably more energy than System One to accurately process and analyze thoughts (Kahneman, 2012).

But, if we have these two systems that help us handle both simple and complex problems in our lives, how can decision-making possibly go wrong? On the one hand, System One relies heavily on intuition or, as some would call it, "gut feeling," which can be helpful, but does not necessarily lead to correct answers or right decisions (Kahneman, 2012) [27]. Kahneman (2012) clearly defines intuition as System One's way of connecting the dots between memorized knowledge and lived experiences to make a decision. On the other hand, for an individual to use System Two is mentally draining (Kahneman, 2012). That is why System Two often acts as a "production slacker," choosing to save energy and preferring to let System One do the hard work instead. Although System One is prone to biases, Kahneman argues that both systems are valuable to humanity and have been necessary for the survival of the human race, as, similar to animals, humans' chances of survival increased by orienting and adapting quickly to threats or prospects.

With the dual-process theory, Kahneman (2012) [27] criticizes the theory of homo economicus, characterized by agents always making rational decisions and maximizing their utility. Kahneman explains that while human brains have evolved amazingly, the decision-making process is not always rational, as described in the theory of an economic man. On the contrary, we often make irrational choices because of manipulation - keyword advertisement. Kahneman introduces evidence that people's decision making is especially affected by commercials when they are tired or stressed since this is when System Two will be less responsive. According to the author then, the view of the rational homo economicus is deeply flawed.

Problems arise when people use the biased System One instead of System Two or when System One, which is inherently more active, puts System Two in a negative emotional state, impairing rationality (Kahneman, 2012) [27]. In this way, biases can quickly occur especially when one faces more complex problems. The following biases addressed in the book *Thinking Fast and Slow* challenge the idea of homo economicus.

3.1.1 Law of small numbers

The bias of the law of small numbers arises because, inevitably, when faced with statistics, our minds attempt to make sense of the information they are confronted with (Kahneman, 2012) [27]. Kahneman (2012) explains the dysfunctional relationship between the mind and statistics and blames it on the automatic processing of System One. As the average person does not study statistics and, therefore, does not fully understand the cause and effect relationship, System One will continuously be on the lookout for the reasons behind the information it sees (Kahneman, 2012). By default, it will assume a causal connection between the results, even when none exists (Kahneman, 2012). What adds to this problem is that people have erroneous intuitions about random sampling (Tversky and Kahneman, 1971) [46]. So, according to Tversky and Kahneman (1971), people often jump to conclusions by viewing the findings based on a small sample and generalizing them to the whole population. Consequently, when presented with a study consisting of a small sample, individuals might think they know why the effects occurred, even though the observed effect may merely be due to statistical specifics, such as the small size of the sample, and not reflect any causal relationship (Kahneman, 2012). That is why, Kahneman (p. 111) explains, "extreme outcomes (both high and low) are more likely to be found in small than in large samples.

3.1.2 Anchoring effect

An anchoring effect bias occurs when people's judgments are affected by an uninformative number (Kahneman, 2012) [27]. Kahneman (2012) explains that any random number presented to an individual would change their solution to an estimation problem, meaning that the estimate would remain close to the number presented - hence the image of an anchor. Kahneman speaks about how this psychological trick is often misused in financial decision-making, such as investing in real estate, making donations, or even going on a quick shopping spree. For example, when purchasing a home, people are significantly impacted by the asking price when deciding how much they are willing to pay (Kahneman, 2012). A higher listing price would make the same house appear more expensive than a lower one (Kahneman, 2012). Due to the mere presence of a pre-offered price or a price range, System One will prime the brain toward this number, making anyone vulnerable to manipulation (Kahneman, 2012).

3.1.3 Loss aversion

When explaining loss aversion, Kahneman (2012) [27] presents the example of a bet where it is possible to lose $100 or win $150. When thinking about this gamble's attractiveness, Kahneman argues that most people (skilled traders excluded) would reject it. This is because most people are averse to losses, even though the expected value is positive in this case. For most people, the fear of losing $100 is more powerful than the hope of earning $150 (Kahneman, 2012). While System Two is responsible for the acceptance or rejection of this gamble, it is System One that produces the strong emotional reaction (Kahneman, 2012). Kahneman points out that the "loss aversion ratio," meaning the benefit compared to the loss, is generally in the range of 1.5 to 2.5 - so, for most people, the prospect of winning at least $200 would be necessary to combat the potential loss of $100.

3.1.4 Matching bias

First discovered through the famous Wason selection task, the matching bias may not be explicitly mentioned in the book *Thinking Fast and Slow* but is nevertheless one of the most puzzling heuristic biases (Inglis and Simpson, 2005) [25]. According to Inglis and Simpson (2005), in one version of the task, participants were shown four cards placed on the table, each of which had a letter on one side and a number on the other side. There was only one rule: "If a card has a D on one side, then it has 3 on the other" (Inglis and Simpson, 2005 p. 179). Participants had to decide which cards to turn without turning any unnecessary cards to confirm whether the rule was true or false (Inglis and Simpson, 2005).

Figure 2: Wason selection Task: four cards placed on the table (Inglis and Simpson 2005) [25].

This sounds simple, but there is a reason that this test is "the single most investigated experimental paradigm in the psychology of reasoning" (Manktelow, 1999, p. 8) [30]. Surprisingly, according to Inglis and Simpson (2005) [25], 90% of participants picked the combination D and 3, which they referred to as the "standard mistake." What is more, 65% of participants selected only D (Inglis and Simpson, 2005). In comparison, the authors stated that only 10% of participants selected the correct answer, D and 7. This experiment reveals that System One tends to choose items explicitly named in the rule (Inglis and Simpson, 2005). Therefore, the high number of participants who selected D and 3 is no surprise, even though picking 3 does nothing to disprove or prove the rule. Additionally, regardless of how the rule was modified, participants predominantly picked the card combination explicitly mentioned in the task rule (Inglis and Simpson, 2005). However, the authors proposed a solution to preventing this kind of bias, and it is mathematics.

3.2 Overriding heuristics with mathematical skills

The discussion of the biases from the book *Thinking Fast and Slow* and the matching bias from Inglis and Simpson's paper illustrates that an average person's decision-making is heavily biased because it relies mostly on intuition. Surprisingly, students in their first economics class will learn, though, that actors behave rationally. However, the world beyond economic models is different. According to Kahneman (2012) [27], individuals cannot exercise the best judgment because the brain likes taking mental shortcuts that impair rationality. The logical question here is how to prevent biases and improve decision-making, whether about investing, selecting insurance, or refraining from self-destructive behavior such as gambling. Kahneman acknowledges the challenge of attempting to avoid biases while still emphasizing the importance of trying to do so. Could mathematics possibly be of any help? It may, according to Inglis and Simpson (2005) [25]. The authors have clarified that individuals who have obtained an advanced mathematical education can override System One (as in the case of the effect of matching bias).

Mathematics is a mandatory subject at school for a reason - it helps develop rational and logical reasoning skills (Inglis and Simpson, 2005) [25]. As Inglis and Simpson (2005) have already illustrated, failure in abstract reasoning is attributed to System One heuristics. According to the authors, the logical conclusion would be that those successful in mathematics careers should be less affected by System One heuristics. To confirm this hypothesis, the authors conducted a study using the Wason selection task and assessing the performance of those successful in mathematics versus a well-educated general population. The sample of those successful in mathematics included undergraduate students from a top-notch university in mathematics, and the sample representing a well-educated general public comprised of trainee teachers from a leading university with no specialization in mathematics. The findings showed that 13% of mathematicians chose the correct answer, whereas only 4% of the general population did. Mathematics students and the general public committed the standard mistake at rates of 24% and 45%, respectively. Therefore, the authors concluded that mathematics students performed significantly better than the general population and therefore stated that those who have studied mathematics to an advanced stage have a better chance of accessing System Two, which, in turn, will correct the System One heuristics. The authors further stressed the importance of overriding deceptive System One because outside of the study, abstract reasoning can be applicable in many situations, such as understanding electricity bills or following instructions for tax returns.

As mathematics can help improve a person's general reasoning abilities, it may also assist in overriding System One regarding other biases. Imagine recreational mathematics as an appealing way of presenting people with problem-solving challenges and thus succeeding in sparking interest in mathematics in general. This could lead to a global paradigm shift causing individuals to increasingly rely on rational thinking as opposed to reasoning based on mere intuition. However, further research is required to confirm this hypothesis.

3.3 Developing critical thinking through recreational mathematics

Many psychologists, apart from Kahneman, have directed their attention to biases in decision-making. Although their work has differed, they have all agreed that people are inclined to take mental shortcuts and therefore are susceptible to biases (Butler, Pentoney, and Bong, 2017) [9]. Butler (2017) [8] has explained, however, that with consistent effort, one can avoid or even eliminate those biases. Avoiding them, though, is only feasible if System Two is sufficiently trained to correct System One (Kahneman, 2012) [27]. Also, involving System Two in decision-making is more likely to happen by enhancing critical thinking ability - that is, the ability to reason in a skillful, practical manner and to produce logical solutions when needed (Hussen et al., 2020).

Critical thinking is also an essential skill that every person in today's digitalized world should cultivate (Klymchuk and Sangwin, 2020) [29]. Klymchuk and Sangwin (2020) have pointed out that with rapid technological advancement, conspiracy theories and digital deception known as "deep fakes" are becoming incredibly common. Consequently, it is more crucial than ever to prepare students to distinguish facts from fake news and to train them to make evidence-based decisions (Klymchuk and Sangwin, 2020). One of the successful approaches to teaching critical thinking to students, according to the authors, involves incorporating "provocative questions" in teaching and assessment in mathematics. Klymchuk and Sangwin have explained that these types of questions are closely linked to recreational mathematics, describing them as looking like routine mathematics questions but explaining that they contain a catch that is meant to distract from the right solution. What is so unique about such questions when preparing a mathematical assessment is that they, contrary to conventional tasks, require the student to recognize mistakes and challenge the question, enforcing the habit of first and foremost analyzing and not taking things for granted (Klymchuk and Sangwin, 2020).

Mathematical formulas, rules, and theorems can only be applied if certain conditions and constraints are satisfied (Klymchuk and Sangwin, 2020) [29]. The problem with mathematics assessment, Klymchuk and Sangwin have claimed, is that even they are still being designed in such a manner that conditions or constraints are already fulfilled beforehand and therefore a student does not need to question or test them. Teaching mathematics in this traditional manner shifts the focus from the desirable critical analysis and in-depth understanding of mathematical concepts to merely applying knowledge in the form of formulas and theorems without challenging the question (Klymchuk and Sangwin, 2020). The skill of critically analyzing tasks can be useful not only in mathematics but also in everyday life (Klymchuk and Sangwin, 2020). A variety of benefits for everyday life can result from mastering critical thinking: becoming aware of recurring mistakes, misleading information, and impossible cases; adopting a skeptical perspective; and practicing rational thinking (Klymchuk and Sangwin, 2020). However, to pass this skill on to students, educators need to be accordingly trained themselves (Klymchuk and Sangwin, 2020). Few nations have grasped the importance of critical thinking and the necessity of conveying it in the classroom. Positive examples, then, are the exception, as an experimental study conducted by Klymchuk (2014) [28] illustrates.

Mathematics teachers from Germany, Hong Kong, New Zealand, Ukraine, and Australia - in total, 127 participants - were presented with a mini-test featuring seven "provocative questions" related to recreational mathematics and a post-test questionnaire for assessing their feelings and reasons for their performance (Klymchuk 2014; Klymchuk and Sangwin, 2020) [28] [29]. Although the researchers warned the educators about questions containing a catch, most of the teachers performed poorly. Klymchuk (2014) has pointed out that the test's purpose was not the testing of mathematical knowledge but the assessment of critical thinking.

An example of one of the questions is as follows:

"Find the area of the right-angled triangle if its hypotenuse is 10 cm and the height dropped on the hypotenuse is 6 cm (Klymchuk 2014 p. 65; Klymchuk and Sangwin, 2020 p. 2,351). [28] [29]"

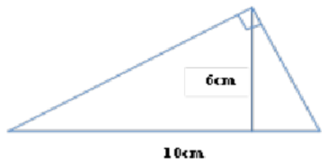


Figure 3: "The diagram for a "provocative question" (Klymchuk and Sangwin, 2020, p. 2,351). [29]"

Interestingly, this is the question where the educators performed the worst. Out of all the participants, only six gave the correct answer. The rest immediately applied the formula A=ah/2 to calculate the right-angled triangle area, even before challenging the task (Klymchuk and Sangwin, 2020) [29]. If there is no right-angled triangle based on the information provided, then there is no need to determine the area of one. The maximum height based on the hypotenuse is 5 cm, "as the hypotenuse in a right-angled triangle is a diameter of its semicircle" (Klymchuk and Sangwin, 2020, p. 5). However, after reviewing their performance, the teachers showed interest in provocative questions (Klymchuk and Sangwin, 2020). All the participants from Germany and New Zealand expressed a desire to incorporate provocative questions in their teaching practice and to encourage and reward constructive critical thinking in their classrooms, whereas about 50% of the educators from Hong Kong, Ukraine, and Australia stated that they would not make any improvements to their teaching methods (Klymchuk and Sangwin, 2020). The most robust rejection of provocative questions came from the Australian group. They reflected that students should not deviate from the script, as their primary aim should be to study for the West Australian Certificate of Education and that, therefore, there is no need to teach them beyond the already established process (Klymchuk and Sangwin, 2020). Klymchuk and Sangwin (2020) assumed that these educators prime their students to carry out assignments and apply knowledge without testing any limitations or rules. "Teaching to the test," though, is a highly outdated method, as research shows that critical thinkers make fewer harmful decisions in their personal life (Butler, Pentoney, and Bong, 2017) [9]. Based on these findings, Klymchuk and Sangwin (2020) have recognized the need to introduce provocative mathematical exercises to challenge students and encourage the development of skills beyond the script, thus providing training for the real world. In the end, life has no single-choice solution, as is still taught by traditional teaching methods (Klymchuk and Sangwin, 2020).

Other authors, such as Hussen et al. (2020) [24], have explicitly proposed recreational mathematics as one of the many techniques that educators should use to improve not only critical thinking but also its disposition. "Critical thinking disposition" refers to the personal tendency to use these skills (Hussen et al., 2020). The authors used a field of recreational mathematics, so-called opposite corner problems, for conducting a small study with first-year bachelor's degree mathematics students and found out that this type of assessment method could trigger the student to use critical thinking and thus to come up with new logical ideas. The research also confirmed that the implementation of recreational mathematics in teaching is a promising approach to increasing critical thinking and its disposition (Hussen et al., 2020). The authors have praised the variety that recreational mathematics has to offer - almost any problem or concept can be expanded or modified to fit the current knowledge level of the students. Hussen et al. have also stated that recreational mathematics provides the best introduction to more challenging mathematical topics.

Another long-time proponent of the idea that recreational math should be incorporated into the standard curriculum, Gardner (2015) [13] has told the story of Stanford statistician Susan Holmes using Gardner's books *Aha! Insight* and *Aha! Gotcha* as supporting teaching material for master's degree students in psychology who failed at math and had to pass comprehensive exams. However, despite all his efforts in promoting the employment of recreational mathematics, seemingly no other educators have been responsive at that time.

Usually, the problem at school lies here: when mathematics becomes abstract and individuals reach the bottom of their mathematical knowledge, they will give up (Trigg, 1978) [45]. Subsequently, mathematics stops being enjoyable at school. This brings to mind the story of Rice, an ordinary person inspired by recreational mathematics to the extent of becoming a luminary in the field. What if we learned from her case and redesigned education to be more engaging, thus keeping students' interest? Butler (2017) [8] has criticized the common tendency to be concerned about intelligence but not about critical thinking. What is more, the author has emphasized that anyone with consistent training can improve their critical thinking. If recreational mathematics is a successful approach to introducing critical thinking early on, should it not be part of every country's school curriculum?

4 Does recreational mathematics play an essential role in school curricula?

Even the famous popularizer of recreational mathematics, Gardner (2015, p. 23) [13], had a miserable experience with math at school: "I was trying to determine which side has the win in tic-tac-toe. Is it the first or the second player, or is it a tie if both sides make their best moves? (Answer: it's a tie.) Pauline [Gardner's teacher] snatched away the sheet on which I was scribbling and said sternly, 'When you're in my class, I expect you to work on mathematics and nothing else!'" The popular game tic-tac-toe can present a great opportunity to introduce students to the world of combinatorial mathematics, game theory, symmetry, and probability theory, Gardner (2015) has claimed. However, most educators ignore the chance to reorganize education and persist in teaching the usual way (Gardner, 2015). This results in students fearing mathematics, experiencing anxiety before going to the blackboard, or giving up when it gets too hard. Keeping students interested and engaged in math is necessary for establishing a positive attitude toward the subject. Thankfully, some countries have recognized the prospect of teaching mathematics through playful learning and try to employ recreational mathematics in the school curriculum to that end.

Sumpter (2015, pp. 121-135) [42] has explored in "what way recreational mathematics can play a part in school mathematics" and "how the concept appears in educational policy documents." She reviewed various educational policy documents, such as educational standards, curricula, and syllabi from several countries, and analyzed them by looking for references such as "pleasure," "enjoyment," or any other "positive affective dimension" related to recreational mathematics and concerned with the relationship between emotions and the individual during the learning process (Sumpter, 2015, p. 124). Below follows a summary of the thought-provoking results from her analysis.

4.1 Review of Sumpter's report for mathematical practice in different countries

4.1.1 China

The Chinese mathematics program for children aged 6-14 follows two aims: conveying mathematical knowledge and fostering positive affective dimensions - students' attitudes, emotions, and values such as creativity, curiosity, positive emotions toward the subject, and eagerness to learn (Sumpter, 2015) [42]. For that purpose, according to Sumpter (2015), Chinese educational standards instruct the teaching staff to not only monitor students' learning progress but also closely watch their emotional and behavioral development. Optimally, students will learn to think critically, apply problem-solving skills, and gain confidence in solving mathematical problems (Sumpter 2015). Sumpter (2015, p. 125) took a critical stand and noted that as students advance in school and need to meet each grades requirements, the affective dimension of the curriculum could be less about the individual's mathematics experience and more about learning to display a positive attitude: "it seems that students should show joy but it is not necessary that they feel joy." Whether the students' demonstration of joy is more important than its actual experience is not transparent (Sumpter, 2015).

4.1.2 England

In England's curriculum, mathematics is presented from a historical perspective and described as a creative and highly interconnected discipline. However, Sumpter (2015) [42] pointed out that creativity mentioned is inherent to mathematics but the curriculum does not elaborate on how the individual relates to that aspect. The curriculum highlights the importance of high-quality mathematics education and even praises the subject as a foundation for understanding the world. Also, some keywords regarding recreational mathematics, such as enjoyment and curiosity, are present. In summary, the author concluded that the focus lies on the subject material and mathematics as a tool for students.

4.1.3 Finland

In contrast to China's and England's, the Finnish curriculum highlights the individual's creative and thought-provoking development and focuses on students, their progress in mathematics, and their personal growth (Sumpter, 2015) [42]. Sumpter (2015) noted after reviewing the various goals for grades 1-9 that the progression from joy to security is observable. The author further explained that in the early stages, students are supposed to experience satisfaction and positive feelings from problem-solving, then progress to feeling competent in mathematics, and in the final stage, develop confidence in solving mathematical tasks.

4.1.4 India

The Indian mathematics curriculum first focuses on conveying mathematical knowledge but additionally intends to develop the student's ability to think and reason mathematically (Sumpter, 2015) [42]. According to Sumpter (2015), the program further highlights the importance of maintaining positive feelings toward the subject, experiencing joy, and eliminating fear. What is more, Sumpter explained that the curriculum also emphasizes empowering children to openly express thoughts and feelings and to avoid sticking to the number - literally reproducing what teachers have already said in class. Therefore, students must learn to "develop their own voice" (Sumpter, 2015, p. 9). As an excellent example of incorporating recreational mathematics into educational policy, the Indian curriculum explicitly mentions the use of mathematical games, puzzles, and stories as tools for increasing positive experiences and eliminating the fear of math.

4.1.5 Japan

The only reference to the affective side of learning mathematics listed in the Japanese curriculum is that students should "recognize the joy of mathematical activities" (Sumpter, 2015, p. 9) [42]. Instead the curriculum focuses on deepening students' understanding of mathematics, developing their fluency, and encouraging its use (Sumpter, 2015).

4.1.6 Singapore

Singapore's curriculum introduces mathematics as a tool for improving individuals' logical reasoning, spatial visualization, and abstract thinking (Sumpter, 2015) [42]. Additionally, Sumpter (2015) identified mentions of positive feelings such as enjoyment, excitement, joy, and creativity that refer to the affective side of learning and should further inspire students to pursue mathematics outside the classroom. The framework's focus is visibly on the emotional side of studying mathematics, on generating positive feelings, and on developing an appreciation for the subject (Sumpter, 2015).

4.1.7 Sweden

The Swedish curriculum states that students should acquire interest and confidence in being able to apply mathematics in many settings (Sumpter, 2015) [42]. Sumpter (2015) reported that the focus lies on the students and their intrinsic motivation. The school organization also addresses the beauty of mathematical exercises and values how individuals experience these aesthetic qualities along their educational journey (Sumpter, 2015).

4.1.8 U.S.

Looking at the Common Core Standards for mathematics and the various standards for different grades in the U.S., Sumpter (2015) [42] concluded that none of them address any connection to recreational mathematics or the affective side of learning mathematics. The focus is purely on mathematics as an instrument.

4.1.9 Summary of Sumpter's report

Sumpter (2015) [42] found that most of the countries analyzed have somewhat touched on the subject of recreational mathematics through the use of keywords that reflect joy, creativity, and positive attitude toward mathematics when defining goals. The U.S., however, does not seem to address any of the keywords regarding the emotional component of the student-subject relationship (Sumpter, 2015). India, on the contrary, as a perfect example for incorporating recreational mathematics into educational policy, directly mentions its use, as can be confirmed in the Facilitator's Manual from the Central Board of Secondary Education in India (2019) [34] (Sumpter, 2015). This manual (2019) provides educators with innovative ways of teaching mathematics, including an extra session on recreational mathematics and a variety of examples of classroom materials. In countries such as India, Gardner's aspirations for incorporating recreational mathematics into the standard curriculum have become a reality.

4.2 Review of educational standards for mathematical practice in Austria and Serbia

Sumpter's report awakened my curiosity, so I decided to conduct similar research on the mathematical educational constellation in Austria and Serbia. I grew up in these two countries as a bilingual person and have had first-hand experience with both educational systems. For the purposes of this thesis, I have focused on primary and secondary school. For Austria, I researched the standards for primary schools (Volksschule), new secondary lower schools/secondary lower schools (Neue Mittelschule/Mittelschule), and gymnasiums (Allgemeine Höhere Schule [AHS]). For Serbia, I researched primary school, which consists of eight grades, four lower and four higher, making it comparable to the Austrian education system (Eurydice, n.d.) [10].

4.2.1 Methodology

I followed an approach similar to Sumpter's and located the educational standards for mathematics education in Austria and Serbia on the internet. As no materials were available in English, I used the original standards in German and Serbian and translated them into English. In the first step, I analyzed the mathematical educational material by searching for any reference related to recreational mathematics or another positive affective dimension. Then I summarized the key ideas of the outlined goals. Below, I present the results.

4.2.2 Primary school in Austria

Compulsory education in Austria starts with primary school attendance, which lasts four years and begins at the age of six. The curriculum for primary school thematizes from its very beginning that mathematics lessons should allow space for creativity (Bundesministerium für Bildung, 2003) [4]. Individuals should engage in creative activities through playfulness and by actively exploring and discovering phenomena (Bundesministerium für Bildung, 2003). The focus is on the individual and their emotions accompanying the learning process, so, for example, intrinsic motivation acts as a driver for creative thinking. Another defined goal is that students should learn to think rationally through basic arithmetic (Bundesministerium für Bildung, 2003). Emphasis is placed on problem-solving and logical thinking (Bundesministerium für Bildung, 2003). Moreover, the curriculum (2003) explicitly states that mathematics should ensure spatial representation development. It also sets the goal of making students aware of the significance of mathematics by presenting them with a variety of child-friendly exercises from the fields of business, technology, and culture (Bundesministerium für Bildung, 2003).

The education standard (2003) [4] then goes on to list the teaching material for each grade from one to four and includes specific guidelines for a playful approach to numbers and numerical operations. Various recreational mathematics examples are explicitly mentioned, such as dice games, number puzzles, strategy games, and magic squares (Bundesministerium für Bildung, 2003). Another example refers to children exploring geometric shapes and areas through free crafting and design (Bundesministerium für Bildung, 2003). Furthermore, pupils should learn how to incorporate mathematics into their reality, such as through role-play (Bundesministerium für Bildung, 2003). In conclusion, in Austria, children's education starts by encouraging a positive attitude toward mathematics and stimulating an intrinsic interest through creativity and playfulness, and then incorporates using mathematics as a tool for developing analytical thinking skills. After such a promising start, I was even more curious to find out if recreational mathematics runs through later grades as well.

4.2.3 Secondary lower school/new secondary lower school in Austria

After primary school, one possibility for 10- to 14-year-olds to continue with the second part of compulsory education was the new secondary lower school (Neue Mittelschule). From the 2020/21 academic year on, the secondary lower school (Mittelschule) replaced the new secondary lower school (Neue Mittelschule) (Bundesministerium für Bildung, 2018; Bundesministerium für Bildung, 2020) [5] [7]. The schools' educational standard has been updated but did not change much concerning mathematics. The curriculum for secondary lower school (2020) states that as far as possible, students should establish connections between actions and concepts on the one hand and with various ideas on the other hand and thus experience mathematics as a diverse field of activity. This instruction is rather vaguely formulated and open to interpretation. Then, the standard (2020) proceeds to state the aim of experiencing mathematics' utility in various life and knowledge fields. The second part of the curriculum (2020) is concerned with the skills the student should acquire, such as critical thinking to help make rational decisions and verify assumptions (Bundesministerium für Bildung, 2020). Additionally, the standard states that individuals should develop problem-solving skills beyond mathematics and obtain insight into the development of mathematical concepts and methods (Bundesministerium für Bildung, 2020). The only possible mention of the affective side of learning is that constructive relationships between students and mathematics should be promoted (Bundesministerium für Bildung, 2020). On a different note, the document (2020) specifies that pupils should learn about influential mathematicians in history and the significance of the subject in the development of Western culture. The importance of mathematics for the present should also not be neglected in teaching (Bundesministerium für Bildung, 2020). Keeping pupils engaged is guaranteed through various teaching methods, such as partner work, group work, and project-oriented teaching (Bundesministerium für Bildung, 2020). All in all, according to the curriculum for secondary lower school (2003), mathematics should be viewed as a dynamic science, but the only reference to recreational mathematics that I could identify is about developing a constructive relationship with the subject. Unfortunately, recreational mathematics hardly appears in the secondary lower school curriculum.

4.2.4 Gymnasium (Allgemeine Höhere Schule) in Austria

After primary school, another possible option for children to continue compulsory education is attending a gymnasium (Bundesministerium für Bildung - AHS, 2020) [6]. The standard (2020) is identical to one for secondary lower schools/new secondary lower school. Therefore, there is hardly any mention regarding the affective side of the student-mathematics relationship.

4.2.5 Summary of the review for mathematical practices in Austria

Despite a promising start in incorporating recreational mathematics into the primary school curriculum, in practice, there is little room for experiencing joy through playful teaching methods in further education, such as both secondary lower schools and gymnasium. Seemingly, in Austria's educational standards, recreational mathematics is dismissed as frivolous and seen as only useful for children in lower education, such as primary school.

4.2.6 Primary school (osnovna škola) in Serbia

Serbia's compulsory education begins with a preschool preparatory program and follows with primary school, which begins at age six or seven and lasts for eight years (Eurydice, n.d.) [10]. The Serbian Ministry of Education, Science and Technological Development has issued a standard for each grade of primary school that I reviewed and summarized in terms of recreational mathematics.

The first-grade standard lays out the mathematics objectives at the very beginning. Mathematical concepts, knowledge, and skills should form the foundation of abstract and critical thinking and nurture a positive attitude toward mathematics (Službeni glasnik Republike Srbije - prvi razred, 2018) [21]. Students should also learn to apply acquired knowledge and problem-solving skills in further education and everyday life (Službeni glasnik - prvi razred, 2018). One interesting example in the standard (2018) is practicing the addition table or using ordinal numbers in the course of sports and health education classes (e.g., doing $6 + 5$ squats, jumping two times after the third squat). Different teaching methods and forms of work are combined to contribute to a more efficient teaching process, promote students' intellectual activity, and make the lessons more exciting and functional (Službeni glasnik - prvi razred, 2018). Another objective stated in the standard (2018) is to contribute to developing intellectual abilities, forming a scientific worldview, and comprehensively developing the student's personality.

The second-grade standard lays out the goal of developing the student's ability to observe, perceive, and think logically, critically, creatively, and abstractly, promoting mathematical curiosity (Službeni glasnik - drugi razred, 2018) [14]. The standard (2018) also mentions mathematics as a tool for building up the positive qualities of a student's personality, such as truthfulness, perseverance, order, accuracy, responsibility, and independence. Additionally, students are encouraged to check if conditions or constraints apply before solving mathematical tasks (Službeni glasnik - drugi razred, 2018).

The third and fourth grades do not differ much from the second grade and thus do not make any further references to the positive affective side of learning (Službeni glasnik - treći razred, 2018; Službeni glasnik - čcetvrti razred, 2018) [17] [19]. The standard regarding the fifth grade further indicates that various daily life problems should be addressed during class, such as home budget planning; therefore, the focus is on the practical application of mathematics (Službeni glasnik - peti razred, 2018) [15].

The sixth-grade standard states that apart from mastering mathematical concepts and developing intellectual abilities, students should form a scientific view of the world (Službeni glasnik - šesti razred, 2018) [20]. Additionally, it attaches great importance to building cultural, professional, and ethical habits and awakening students' mathematical curiosity (Službeni glasnik - šesti razred, 2018). The focus is on the overall growth of pupils' personalities.

In addition to what was included in the earlier grades, the standard for seventh grade additionally aims for students to solve problems and tasks in new and unfamiliar situations, express their opinions and explain them to others, and develop motivation and interest in learning the subject's content (Službeni glasnik - sedmi razred, 2018) [18]. The eighth-grade standard does not mention any additional information about the affective side of learning mathematics (Službeni glasnik - osmi razred, 2018) [16].

4.2.7 Summary of the review for mathematical practices in Serbia

In Serbia, recreational mathematics is a helpful tool for facilitating the transition from preschool to the first grade of primary school and for increasing students' interest in the subject. The Serbian curriculum directly incorporates examples of various playful teaching activities and recognizes recreational mathematics as an additional motivational factor for taking on mathematical content. The curriculum explains the reason behind this approach: as play is the basis of mathematical cognition in the preschool period, this desire for play will not disappear at the beginning of the first grade of primary school, meaning students must have the opportunity to continue developing mathematical thinking via playful activities (Službeni glasnik - prvi razred, 2018) [21]. These activities in turn will significantly contribute to the development of interest in mathematics and a positive attitude toward the subject (Službeni glasnik - prvi razred, 2018). Throughout grades one to eight, the focus is consistently on the individual's experience with mathematics. The standards do not focus on the mere application of formulas but rather encourage a playful approach to mathematics that fosters positive characteristics in the individual, even in higher primary school grades.

4.3 The curriculum and reality

Some standards discussed here specifically rely on recreational mathematics as a serious tool for increasing students' engagement, while others seem to merely touch on this subject. However, some authors have argued that mathematics' creative side is not sufficiently taken into account in teaching and that educators are not putting enough effort toward reshaping the technical, rigid, and abstract image of the subject (Anderton and Wright, 2012) [1]. Mathematics is the "language of science," but its usefulness does not necessarily make it stimulating (Anderton and Wright, 2012, p. 93). Anderton and Wright (2012) have also claimed that math is an art and that it has an imaginative and artistic side that learners mostly do not get to experience because of how it is conveyed. Petković (2018) [35], too, has explained that recreational mathematics undeniably represents the main part of popular science since the imagination, creativity, beauty, and variety of mathematical ideas are not limited by anything, which is not the case in other scientific disciplines. Anderton and Wright have asserted that students at all levels, even the university level, should learn the creative side of mathematics. According to them, a dramatic shift in students' approach to mathematics and in the general perception of what it means to be good at math could take place if mathematics were taught the right way.

That conclusion leads to the next chapter, which analyzes what educators can do to bring about the kind of joy and satisfaction that mathematicians often experience in their practice.

5 How can recreational mathematics successfully be implemented in the classroom?

As is the case for any subject, retaining students' interest by exploring mathematics in different ways is essential. Thankfully, recreational mathematics offers a variety of means, as outlined below.

5.1 Video games

Trigg (1978) [45] mentioned the accessibility factor, explaining that a pen and paper suffice for engaging in recreational mathematics. However, since 1978, technology has changed immensely and is still continually and rapidly evolving. People are moving forward with digitalization: schools, businesses, and individuals are transitioning from paper to digital formats, and the next generations are growing up as digital natives. In an updated definition of recreational mathematics, "pen and paper" could equal "mouse and screen." One could even say that video games used as teaching materials are next-level recreational mathematics. In an increasingly digitized world, the time has come for digital math learning to be more widely used, and research shows that this is already becoming a reality.

Many researchers have called into question the common belief that playing games is a waste of time (Moyer-Packenham et al., 2019) [32]. In fact, according to Moyer-Packenham et al. (2019), video games can be a vivid and enjoyable source of learning for children. In a research project, Moyer-Packenham et al. examined how 12 digital math games and their design features affected children's learning. "Design features" refer to the different attributes of those games such as "accuracy feedback, unlimited/multiple attempts, information tutorials and hints" that support learning (Moyer-Packenham et al., 2019, p. 321). One hundred ninety-three children recruited from different schools and aged 8-12 participated in this study (Moyer-Packenham et al., 2019). Each grade group performed a pre- and post-test evaluation of their knowledge of mathematical topics aligned with the Common Core State Standards (Moyer-Packenham et al., 2019). The authors demonstrated that digital math games with design features that promote learning can help pupils better grasp mathematical concepts and experience substantial learning gains compared to a non-game environment. As children's early start in mathematics determines their success in middle and high school, it is crucial to find child-friendly ways to make mathematics more exciting (Moyer-Packenham et al., 2019). Below, I present two game-based platforms that have been successfully implemented in school curricula.

5.1.1 Sumdog

Sumdog is an online learning service specifically developed to help pupils advance rapidly in mathematics and spelling through video-game-based learning (Adaptive Learning Platform, 2020) [41]. Its purpose is to reach slow learners and, most importantly, to eventually close the attainment gap (Adaptive Learning Platform, 2020). Independently of pupils' skill levels, the learning engine uses user-adapting, engaging, and interactive game-based learning that automatically advances students through the curriculum, tracks results, and sends reports to teachers and parents (Adaptive Learning Platform, 2020). Sumdog provides exercises for children aged 5-11, promising to capture each child's interest (Adaptive Learning Platform, 2020).

The platform offers schools and parents subscription model bundles in a price range between £2.60 and £3.50 per pupil that include 25 math games, progress tracking, analysis, and a time-saving set-up for teachers to tailor development goals across all grades and plan their lessons more efficiently (Adaptive Learning Platform, 2020) [41]. The learning service can be used on the phone, tablet, or computer (Adaptive Learning Platform, 2020).

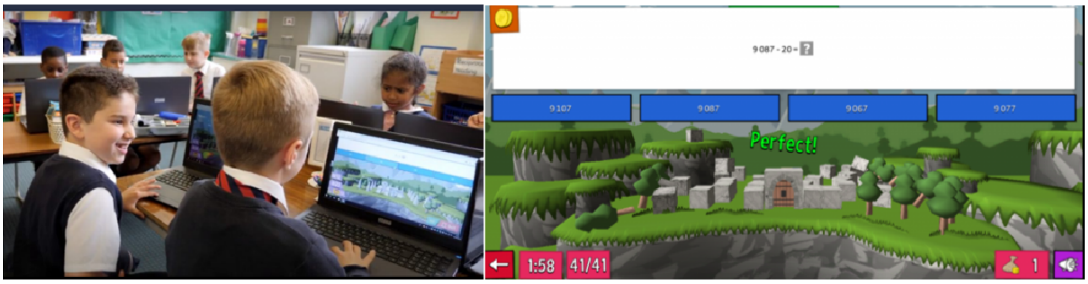


Figure 4: Pupils playing Sumdog in a class (Adaptive Learning Platform, 2020) [41]

If used regularly for at least 30 minutes a week, Sumdog is proven to almost double mathematical fluency, which involves the ability to recall and apply knowledge rapidly and accurately (Humphries, Gallacher, and Macrae, 2020) [23]. According to Humphries, Gallacher, and Macrae (2020), this high impact is equivalent to a six-month learning advantage. Sumdog strengthens mathematical fluency by alternating and revisiting topics (Adaptive Learning Platform, 2020) [41]. What is unique about its design is that the engine takes into consideration children with less advanced mathematical skills and lets them compete and share a gaming experience with more advanced pupils (Adaptive Learning Platform, 2020). This fundamental principle breeds confidence and encourages greater engagement with learning (Adaptive Learning Platform, 2020). The adaptive learning design offers a personalized learning experience at a difficulty level adapted to individual needs, thus preventing possible loss of interest (Adaptive Learning Platform, 2020). Sumdog's reward system is another factor that keeps pupils motivated (Adaptive Learning Platform, 2020). For hard work, they are awarded virtual coins that they can use to shop for items within Sumdog and decorate their virtual house or their avatar, or to teach their digital pet a new trick (Adaptive Learning Platform, 2020).

Testimonials demonstrate that Sumdog is incredibly popular among schools in the U.K. and the U.S. (Adaptive Learning Platform, 2020) [41]. This is particularly worth noting since the U.S. did not include any reference to recreational mathematics in the Common Core State

Standards (Sumpter, 2015) [42]. Many teachers have praised the significant progress they have achieved with Sumdog (Adaptive Learning Platform, 2020).

5.1.2 Mathletics

Like Sumdog, Mathletics is an online math platform designed by teachers who understand the effectiveness of learning through video games (Nguyen, 2018) [33]. Although it has the same main goal as Sumdog, Mathletics has different design features and is used by older students as well as younger students. It differentiates between early learners aged 4-11 and more mature students aged 12-14 by using different content, colors, and styles to keep every age group engaged in learning mathematics (Nguyen, 2018). Students can engage in augmented reality, storytelling tasks, video teaching material, e-books, math quizzes, video games, and even live real-time challenges and competitions with other students worldwide, known as "Live Mathletics" (Nguyen, 2018). Mathletics, like Sumdog, uses user-adapting technology and considers the individual's mathematical level (Nguyen, 2018). For rewarding engagement, Mathletics combines a mix of intrinsic and extrinsic rewards consisting of points, certificates, credits, and a "Hall of Fame" (Nguyen, 2018). Through points awarded for correct answers, one can earn certificates or a spot in the Hall of Fame (Nguyen, 2018). Certificates help develop a sense of achievement and offer the opportunity to boast about one's knowledge in front of family and friends (Nguyen, 2018). The Hall of Fame is a continuously updated leaderboard showing top students and classes worldwide (Nguyen, 2018.). As with Sumdog, Mathletics offers subscriptions for schools, but no information is provided on the website, as prices are revealed on request (Nguyen, 2018). The online tool enables teachers to plan and organize their lessons in a time-efficient way by generating reports that help with tracking students' progress, strengths, and weaknesses (Nguyen, 2018). An Oxford University study showed that completing a minimum of three curriculum activities per week, which equates to 114 activities per year, significantly impacted students' math results (Ingram, Strand, and Sarazin, 2015) [26]. According to the Mathletics website, the platform already has more than 3 million students and more than 200,000 teachers worldwide, and more than 14,000 schools have made it an integral part of math lessons (Nguyen, 2018).

While video games have their benefits, some other educational companies want to help students to reduce screen time and interact more with their peers and educators while learning math. This is where board games/puzzles come into play.

5.2 Puzzles and board games

Solving a puzzle can not only make a person's day but can also change their life. The success experienced in playing puzzle games not only breeds confidence, but also helps students develop the skills they need to enter professions such as engineering. Thomas, Badger, Ventura-Medina, and Sangwin (2013) [44] have argued that, even at the university level, mathematics can be taught to engineers in the form of puzzle-based learning. The authors have demonstrated that students were able to develop their problem-solving skills as well as intrinsic motivation for studying math using puzzle-based learning. Some authors have acknowledged that puzzle-based learning has an impact on analytical and creative skills, which are essential in mathematics degree programs (Rowlett, Webster, Bradshaw, and Hind, 2019) [37].

An Indian professor examined an eighth-grade standard student group via a pre- and post-test assessment and demonstrated that the group that used puzzle-based learning achieved more in mathematics and showed a significant difference concerning the "knowledge, understanding, and application objectives" (Thiyagu, 2012, p. 8) [43]. Thiyagu (2012) confirmed that incorporating puzzles and riddles in teaching mathematics proved effective among eighth-grade students.

The use of board games in higher education has also been found to develop strategic skills, such as critical thinking, problem-solving, discussion skills, and other abilities that are essential for education and employment. (Smith and Golding, 2018) [40]. Smith and Golding (2018) have further noted that gameplay is a fruitful approach compared to conventional teaching methods and helps students to relax and be creative. For instance, when playing board games, students can analyze their next move, identify which strategies give them the highest success rate, and build mathematical understanding (Smith and Golding, 2018). However, Smith and Golding have noted that relatively few games are used in the classroom as students move on to higher education. While recreational mathematics might be popular in elementary school, educators seem to replace games with more conventional teaching methods during higher education (Smith and Golding, 2018). The authors have explained that one reason for this trend may be the complexity of higher education topics, which can prove challenging to combine with games, although there is no evidence to support this statement. Despite these circumstances, one professor decided to swim against the current and launch a tradition of board games and puzzles at the University of Greenwich.

5.2.1 Maths Arcade

Noel-Ann Bradshaw created Maths Arcade, a groundbreaking math initiative, in 2010 at the University of Greenwich; two years later, eight other universities in the U.K. had adopted it (Rowlett, Webster, Bradshaw and Hind, 2019) [37]. Bradshaw designed the game-based project as an optional drop-in session in an informal environment at U.K. universities, giving students the chance to play strategy board games and puzzles with peers and members of staff. The aim was to thereby strengthen staff-student relationships, build mathematical thinking skills such as problem-solving and strategic thinking, support struggling learners, and challenge more confident learners (Rowlett, Webster, Bradshaw, and Hind, 2019). Additionally, as educators were also present during the sessions, students had access to tutorial assistance, which was very well received (Bradshaw, 2011) [3].

In terms of the selection of games, the aim was to use strategic games that can be played quickly, that have simple rules, and that do not highlight competition between students (Rowlett, Webster, Bradshaw, and Hind, 2019) [37]. During a session, although chess is available, the focus is on rather unusual board games that at first seem unrelated to mathematics, such as "Hex, Solomon's Stones, Quixo, Quoridor, Quarto, Pentago, and Pylos." (Rowlett, Webster, Bradshaw, and Hind, 2019 p. 4). While new attendees are excited to test out new games, over time the emphasis moves toward strategy analysis. Students explore different scenarios, analyze what happens when the rules are bent, or plot how to counter different moves (Rowlett, Webster, Bradshaw, and Hind, 2019). According to Bradshaw, Maths Arcade has made it possible to involve students with different knowledge bases and backgrounds and to improve their mathematical development. Student feedback has also been overwhelmingly

Figure 5: "Students playing Quarto at Maths Arcade" (Bradshaw, 2011, p. 27) [3]

positive, and aside from criticism about the timetable, Maths Arcade has made a difference in students' social environment and educational progress (Bradshaw, 2011) [3].

5.2.2 Luma World

After learning what Luma World stands for, one cannot help but be delighted with its revolutionary vision of incorporating board games into the Indian school curriculum. Luma World is an edutainment (educational entertainment) enterprise that focuses on game-based skill development through a unique board-gaming experience. Educators or parents can use the games to foster engagement and improve the conceptual learning of children (Educational Games and Activity Based Learning, n.d.; Iyer, 2020, personal communication, 27 November) [48]. The co-founder and chief operating officer (COO) Venkat Iyer kindly provided me with information on the company's objectives and innovative games and helped me look behind the scenes of this pioneering enterprise (Iyer 2020, personal communication, 27 November).

Luma World is based on the belief that the learning process should not be dependent on selective outcomes as with written assessments but should rather be based on curiosity, discovery, and exploration (Iyer 2020, personal communication, 27 November). By combining board games with curriculum-based learning, they aim to retain the strengths of traditional education, such as human interaction, teamwork, and mentor-driven processes, yet introduce new advancements in learning through gameplay (Iyer 2020, personal communication, 27 November). By replicating the euphoria and adrenaline rush of today's video games, Iyer explains, the company's board games, apart from being enjoyable, reduce screen time for children and simultaneously allow subliminal learning and application of learned topics in a supportive environment (Iyer 2020, personal communication, 27 November). This kind of game-based learning leads to a boost in confidence, especially when, for example, "a conceptual laggard triumphs over a straight A student" (Iyer 2020, personal communication, 27 November). Reinforcing topics through board games is easier than with traditional teaching methods because children tend to remember a game more than a textbook; also, they do not consider the process of playing as a standard academic assessment (Iyer 2020, personal communication, 27 November). Luma World currently has 15 games in its range of products (Educational Games and Activity Based Learning, n.d.) [48]. The curriculum-based academic concepts that are present in its games include numeracy, decimals, percentages, measurements, and geometry, along with life skills such as creativity, teamwork, communication, problem-solving, and financial literacy (Educational Games and Activity Based Learning, n.d.). The games are cleverly designed as "open cards," meaning children see the opponents' cards and so feel the need to do the math and calculate the moves other players will make even when it is not their turn (Iyer 2020, personal communication, 27 November). Similarly, Iyar explained that, for educators, the games are a great way to immediately recognize students' weaknesses and provide corrective action based on where a student is reluctant or hesitant during the application of a specific topic (e.g., a student who is comfortable with adding, subtracting, or multiplying but shies away from division). The COO explained that the enterprise wants to encourage mainstream adoption of games in learning and implement games as part of school curricula for kids under the age of 12. Below, I list a few popular games.

5.2.2.1 Terra Loop

Figure 6: Terra Loop (Educational Gams and Activity Based Learning, n.d.) [48]

Terra Loop is a fantasy treasure hunt board game in which one can travel all over the world while gathering precious ancient currency. The creators have stated that it can be used to introduce children to operations with four-digit numbers and the concepts of money in terms of its importance and saving (Educational Games and Activity Based Learning, n.d.) [48]. This game can also improve many other skills such as logical reasoning, critical thinking, social skills, and decision-making (Educational Games and Activity Based Learning, n.d.). When played in the classroom, children also improve their social skills by interacting with peers (Educational Games and Activity Based Learning, n.d.). Besides financial literacy, this board game teaches values such as sportsmanship, the general importance of human relationships, and helping and caring for others (Educational Games and Activity Based Learning, n.d.).

5.2.2.2 Galaxy Raiders

Galaxy Raiders is a galactic conquest game that exposes children to a new way of studying while taking them on an adventure collecting planets and moons with the help of their mathematical capabilities (Educational Games and Activity Based Learning, n.d.) [48]. In comparison to Terra Loop, this game takes players to the next level and helps with arithmetic with six-digit numbers. Children can explore different ways of calculating and estimating numbers (Educational Games and Activity Based Learning, n.d.). Apart from mental math, this game also incorporates skills in problem-solving, reverse engineering, quick decision-making,

Figure 7: Galaxy Raiders (Educational Gams and Activity Based Learning, n.d.) [48]

and critical thinking (Educational Games and Activity Based Learning, n.d.). The game also fosters a variety of values: empathy, fair play, and patience, to mention just a few (Educational Games and Activity Based Learning, n.d.).

5.2.2.3 Guess the Fence

Figure 8: Guess Fence (Educational Gams and Activity Based Learning, n.d.) [48]

This game is an innovative way to introduce children to geometry while also setting their imagination and creativity free (Educational Games and Activity Based Learning, n.d.) [48]. With the playful approach of building fences to protect a farm, children learn about properties of 2D and 3D shapes; discover patterns; and master tangrams, tessellation, symmetry, and map-reading (Educational Games and Activity Based Learning, n.d.). Furthermore, the creators have praised this game for helping children develop spatial visualization, logical reasoning, strategic thinking, problem-solving, and many other important skills for a child's mental progress (Educational Games and Activity Based Learning, n.d.). This game, too, teaches values: honesty, patience, fair play, and teamwork (Educational Games and Activity Based Learning, n.d.).

5.2.2.4 Alpha Steel

Alpha Steel is a child-friendly fantasy-themed game in which children can control a robot's special powers with math; it triggers creativity and improves a range of skills such as problem-solving, strategic thinking, and logical reasoning (Educational Games and Activity Based

Figure 9: Alpha Steel (Educational Gams and Activity Based Learning, n.d.) [48]

Learning, n.d.) [48]. While playing this game in the classroom, children can learn operations with nine-digit numbers and basic percentages (Educational Games and Activity Based Learning, n.d.). Furthermore, they learn how to handle money while learning the concepts of profit/loss and cost/revenue (Educational Games and Activity Based Learning, n.d.). Alpha Steel also introduces practical life skills such as resource allocation and sustainable living (Educational Games and Activity Based Learning, n.d.). The creators have claimed that this board game helps children distinguish between physical and mental strength, realize the importance of health over wealth, and understand the concepts of giving and sharing (Educational Games and Activity Based Learning, n.d.).

5.2.2.5 Mystical Arts

Figure 10: Galaxy Raiders (Educational Gams and Activity Based Learning, n.d.) [48]

This strategy board game is India's first tabletop game on measurements and conversion of units and decimals (Educational Games and Activity Based Learning, n.d.) [48]. The creators have stated that children will be enchanted by the fantasy world of magic and spells and thus enjoy learning math and improve their decision-making, planning, critical thinking, and focus (Educational Games and Activity Based Learning, n.d.). The game also teaches values such as patience, honesty, embracing failure, and respect for opponents (Educational Games and Activity Based Learning, n.d.).

5.2.3 The Last Gameboard

Although Maths Arcade and Luma World captivated my attention, some creators are going even further. The gaming industry is changing rapidly. The Last Gameboard aims to revolutionize tabletop gaming, being a hybrid between a video game and a board game (Metha and Wyatt, 2020) [31]. It makes use of the electronics of a tablet; however, it can fully interact with physical pieces, thus combining the advantages of modern innovation with the lengthy and widespread tradition of board games (Metha and Wyatt, 2020). The smartboard can be linked with other boards to form a larger board and a variety of games are available in the digital game store (Metha and Wyatt, 2020). This product offers next level functionality and matches the spirit of today's digital era. (Metha and Wyatt, 2020).

Although The Last Gameboard is not specifically advertised as an educational tool, I think it will not be long before it revolutionizes the academic setting. It could be a better, digital version of the Maths Arcade project, with students meeting after class in an informal setting and playing games on the smartboard with peers and educators. Additionally, the digital store offers endless possibilities for playing educational games and so eliminates the problem of buying hundreds of games to keep students interested. What better way to do homework than playing? Or, a game marathon could include random elements of the best board games by mixing them in inpredictable ways and presenting players with a challenge of a new kind with every single draw.

Summing up the topic of board games and puzzles, with a generous portion of humor, one can say that regardless of how high the number rolled, the overall benefits of engaging in board games are undeniable, or however you slice it or dice it, you always win.

6 Conclusion

This thesis has aimed to answer the question of how teachers can use recreational mathematics to spark students' interest in mathematics and contribute to the educational process. The significance of mathematics is undeniable, as it teaches fundamental life skills such as critical thinking and rational decision-making, and so helps in avoiding biases and identifying fake news. However, math is not an easy subject, so students often tend to lose interest with increasing levels of abstractness. This thesis shows that recreational mathematics offers a variety of innovative approaches for keeping students' attention and fostering intrinsic motivation for learning math. The most common examples of recreational mathematics that have already been implemented in the classroom include video games and puzzles/board games.

Mathletics and Sumdog are prominent representatives of video games for children aged 5-11 and 4-14, respectively, because of their user-adapting engine and elaborate motivating technologies, which keep children engaged and let them learn curriculum-based topics through gameplay.

Maths Arcade, a successful project at the University in Greenwich, shows that board games and puzzles promote interest and support learning even in higher education. Luma World, on the other hand, focuses on incorporating a gaming experience in the form of board games into the Indian school curriculum. These are only some of the many possibilities for using recreational mathematics in teaching. Modern fast-paced technologies are leading the way into new and uncharted territories and radically changing the way we teach and learn. Or, in the words of Noel-Ann Bradshaw (2011, p. 28) [3], the creator of Maths Arcade at the University of Greenwich, "I would encourage you to try a new idea. It may not work but it just might end up benefitting students and staff in many institutions."

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